# Heavy-Flavor Transport in QCD Matter:

## transport coefficient

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# Outline

- General setup
- Sensitivity of observables to transport coefficients
- Systematic uncertainties from the *pp* baseline

# Transport theory: the Boltzmann equation

Time evolution of HQ phase-space distribution $f_Q(t, \mathbf{x}, \mathbf{p})$[1]:

$$\frac{d}{dt} f_Q(t, \mathbf{x}, \mathbf{p}) = C[f_Q]$$

- Total derivative along particle trajectory

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v} \frac{\partial}{\partial \mathbf{x}} + \mathbf{F} \frac{\partial}{\partial \mathbf{p}}$$

  Neglecting $\mathbf{x}$-dependence and mean fields: $\partial_t f_Q(t, \mathbf{p}) = C[f_Q]$

- Collision integral:

$$C[f_Q] = \int d\mathbf{k} [\underbrace{w(\mathbf{p}+\mathbf{k}, \mathbf{k}) f_Q(\mathbf{p}+\mathbf{k})}_{\text{gain term}} - \underbrace{w(\mathbf{p}, \mathbf{k}) f_Q(\mathbf{p})}_{\text{loss term}}]$$

  $w(\mathbf{p}, \mathbf{k})$: HQ transition rate $\mathbf{p} \to \mathbf{p} - \mathbf{k}$

[1] For results based on BE see contributions from other groups

# From Boltzmann to Fokker-Planck

Expanding the collision integral for *small momentum exchange*[2] (Landau)

$$C[f_Q] \approx \int d\mathbf{k} \left[ k^i \frac{\partial}{\partial p^i} + \frac{1}{2} k^i k^j \frac{\partial^2}{\partial p^i \partial p^j} \right] [w(\mathbf{p}, \mathbf{k}) f_Q(t, \mathbf{p})]$$

[2] B. Svetitsky, PRD 37, 2484 (1988)

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The Boltzmann equation reduces to the *Fokker-Planck equation*

$$\frac{\partial}{\partial t} f_Q(t,\mathbf{p}) = \frac{\partial}{\partial p^i} \left\{ A^i(\mathbf{p}) f_Q(t,\mathbf{p}) + \frac{\partial}{\partial p^j} [B^{ij}(\mathbf{p}) f_Q(t,\mathbf{p})] \right\}$$

where (verify!)

$$A^i(\mathbf{p}) = \int d\mathbf{k}\, k^i w(\mathbf{p},\mathbf{k}) \quad \longrightarrow \quad \underbrace{A^i(\mathbf{p}) = A(p)\, p^i}_{\text{friction}}$$

$$B^{ij}(\mathbf{p}) = \frac{1}{2} \int d\mathbf{k}\, k^i k^j w(\mathbf{p},\mathbf{k}) \quad \longrightarrow \quad \underbrace{B^{ij}(\mathbf{p}) = (\delta^{ij} - \hat{p}^i \hat{p}^j) B_0(p) + \hat{p}^i \hat{p}^j B_1(p)}_{\text{momentum broadening}}$$

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Problem reduced to the *evaluation of three transport coefficients*, directly derived from the scattering matrix

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# Approach to equilibrium in the FP equation

The FP equation can be viewed as a continuity equation for the phase-space distribution of the kind $\partial_t \rho(t,\vec{p}) + \vec{\nabla}_p \cdot \vec{J}(t,\vec{p}) = 0$

$$\frac{\partial}{\partial t} \underbrace{f_Q(t,\mathbf{p})}_{\equiv \rho(t,\vec{p})} = \frac{\partial}{\partial p^i} \underbrace{\left\{ A^i(\mathbf{p}) f_Q(t,\mathbf{p}) + \frac{\partial}{\partial p^j}[B^{ij}(\mathbf{p}) f_Q(t,\mathbf{p})] \right\}}_{\equiv -J^i(t,\vec{p})}$$

admitting a steady solution $f_{\rm eq}(p) \equiv e^{-E_p/T}$ when the current vanishes:

$$A^i(\vec{p}) f_{\rm eq}(p) = -\frac{\partial B^{ij}(\vec{p})}{\partial p^j} f_{\rm eq}(p) - B^{ij}(\mathbf{p}) \frac{\partial f_{\rm eq}(p)}{\partial p^j}.$$

One gets

$$A(p) p^i = \frac{B_1(p)}{T E_p} - \frac{\partial}{\partial p^j} \left[ \delta^{ij} B_0(p) + \hat{p}^i \hat{p}^j (B_1(p) - B_0(p)) \right],$$

leading to the Einstein *fluctuation-dissipation* relation

$$A(p) = \frac{B_1(p)}{T E_p} - \left[ \frac{1}{p} \frac{\partial B_1(p)}{\partial p} + \frac{d-1}{p^2} (B_1(p) - B_0(p)) \right]$$

# The relativistic Langevin equation

The Fokker-Planck equation can be recast into a form suitable to follow the dynamics of each individual quark: the Langevin equation

$$\frac{\Delta p^i}{\Delta t} = -\underbrace{\eta_D(p)p^i}_{\text{determ.}} + \underbrace{\xi^i(t)}_{\text{stochastic}} \quad ,$$

with the properties of the noise encoded in

$$\langle \xi^i(\mathbf{p}_t)\xi^j(\mathbf{p}_{t'})\rangle = b^{ij}(\mathbf{p})\frac{\delta_{tt'}}{\Delta t} \qquad b^{ij}(\mathbf{p}) \equiv \kappa_L(p)\hat{p}^i\hat{p}^j + \kappa_T(p)(\delta^{ij} - \hat{p}^i\hat{p}^j)$$

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*Transport coefficients* to calculate:

- *Momentum diffusion*;
- *Friction* term

In the following we will establish their link with the transport coefficients appearing in the Fokker-Planck equation. In particular, the momentum dependence of the noise term requires some care.

# Numerical implementation

Introduce the tensor

$$\begin{aligned} C^{ij}(\mathbf{p}) &\equiv \sqrt{\kappa_L(p)}\hat{p}^i\hat{p}^j + \sqrt{\kappa_T(p)}(\delta^{ij} - \hat{p}^i\hat{p}^j) \\ &\equiv \sqrt{\kappa_L(p)}P_L^{ij} + \sqrt{\kappa_T(p)}P_T^{ij} \end{aligned}$$

and redefine the noise variable as

$$\xi^i \equiv C^{ik}(\mathbf{p})\frac{1}{\sqrt{\Delta t}}\zeta^k \quad \text{with} \qquad \langle \zeta^k(t_n)\zeta^l(t_m)\rangle = \delta_{mn}\delta^{kl}.$$

The Langevin equation becomes then

$$\boxed{\Delta p^i = -\eta_D(p)p^i\Delta t + C^{ik}(\mathbf{p} + \xi\Delta\mathbf{p})\sqrt{\Delta t}\,\zeta^k},$$

where, for the sake of generality, the argument of the strength of the noise term can be evaluated ($\xi \in [0,1]$) at any point in the interval $[\mathbf{p}, \mathbf{p} + \Delta\mathbf{p}]$. In the following we will consider the cases $\xi = 0$ (Ito *pre-point* scheme).

# The *pre-point* Ito discretization

Actually, in the Ito *pre-point* scheme $\xi = 0$, so that the friction coeffiecients appearing in the FP and Langevin equations are the same: $A(p) = \eta_D^{\rm pre}(p)$. Furthermore, in order to approach thermal equilibrium, the Einstein relation must be satisfied:

$$\boxed{\eta_D^{\rm pre}(p) = A(p) = \frac{B_1(p)}{TE_p} - \left[\frac{1}{p}\frac{\partial B_1(p)}{\partial p} + \frac{d-1}{p^2}(B_1(p) - B_0(p))\right]}$$

NB: $A(p)$, $B_0(p)$ and $B_1(p)$ can be calculated from the scattering matrix. However, since the Einstein relation must satisfied, one has to calculate only two of them and fix the last one through the above equation: our choice is to calculate $B_0$ and $B_1$

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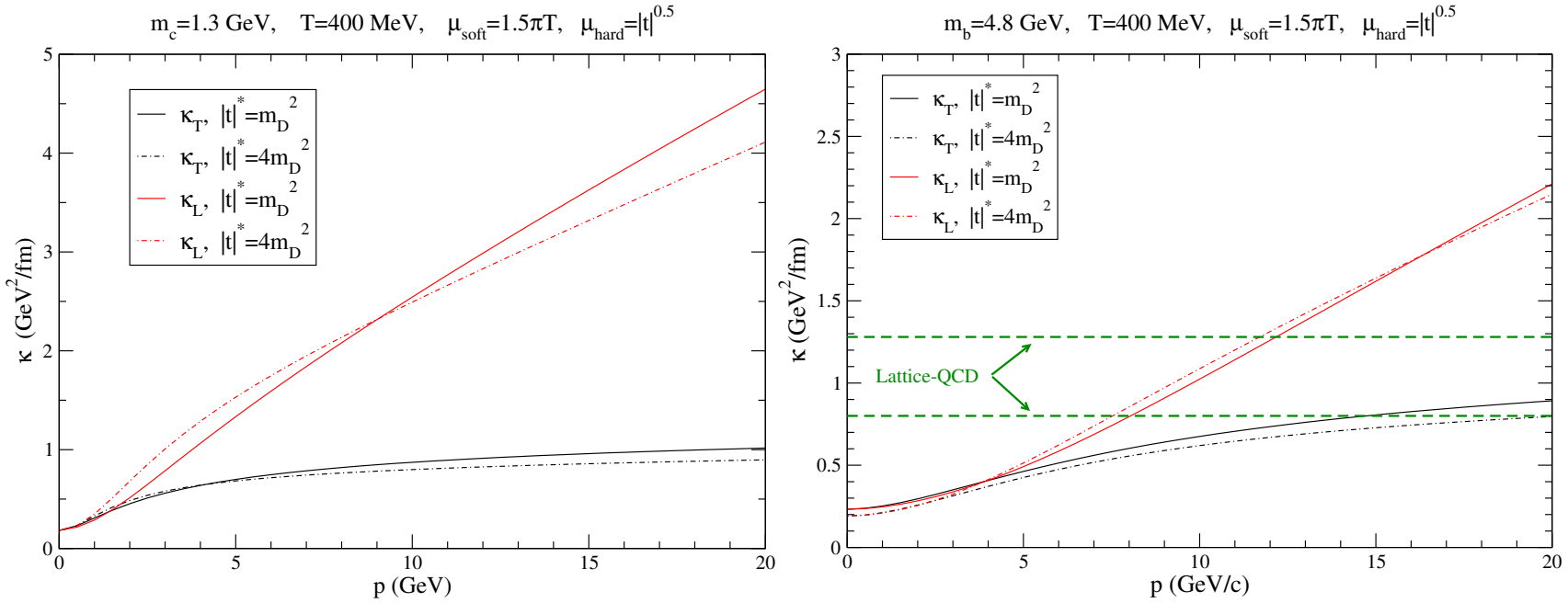
The Ito scheme is the *only discretization* in which the Langevin friction term $\eta_D$ coincides with the one in the FP equation, which can be in principle derived from the scattering matrix

# Transport coefficients: numerical results

Combining together the hard and soft contributions...




...the dependence on the intermediate cutoff $|t|^*$ is very mild!

NB1 Notice, in the case of charm, the strong momentum-dependence of $\kappa_L$, much milder in the case of beauty, for which $\kappa_L \approx \kappa_T$ up to 5 GeV

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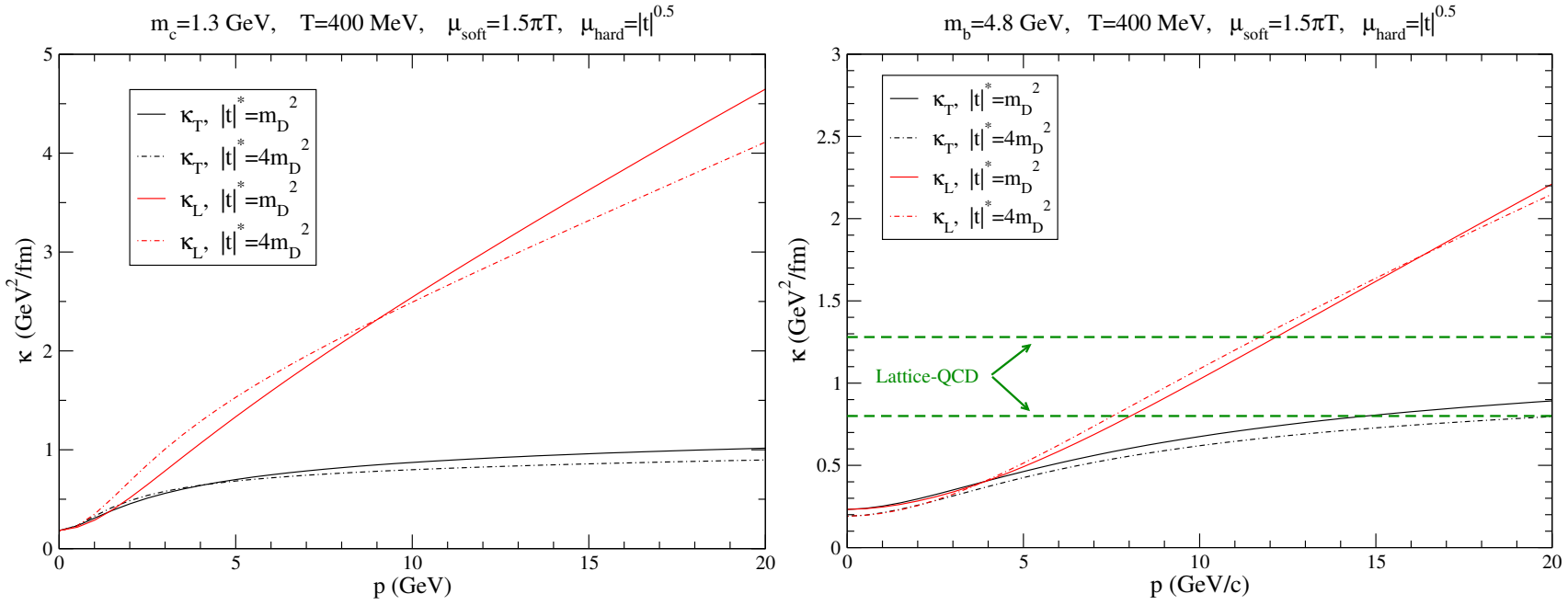


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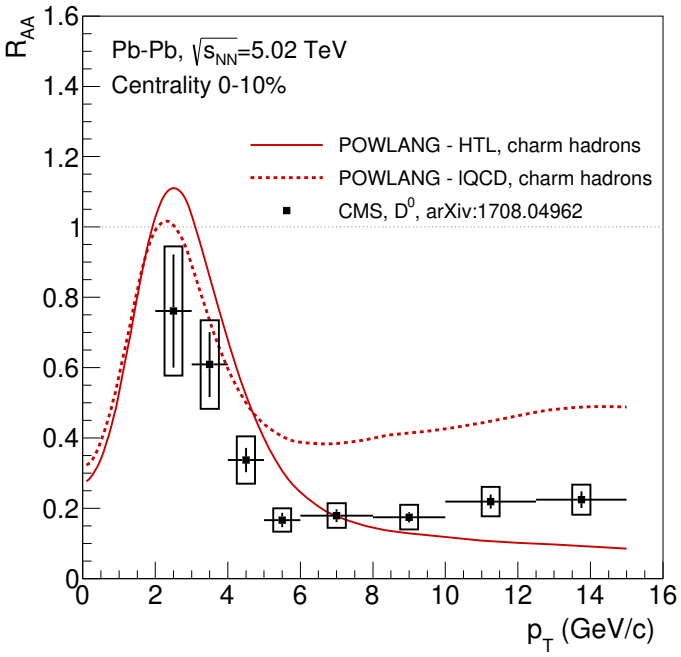
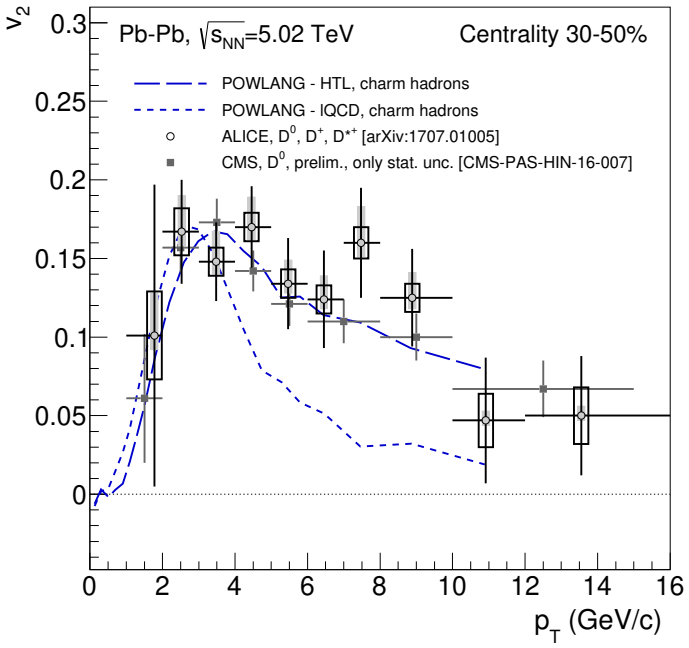
NB2 In our setup the friction coefficient is obtained from the Einstein relation $\eta_D(p) = \kappa_L(p)/2E_pT + \ldots$

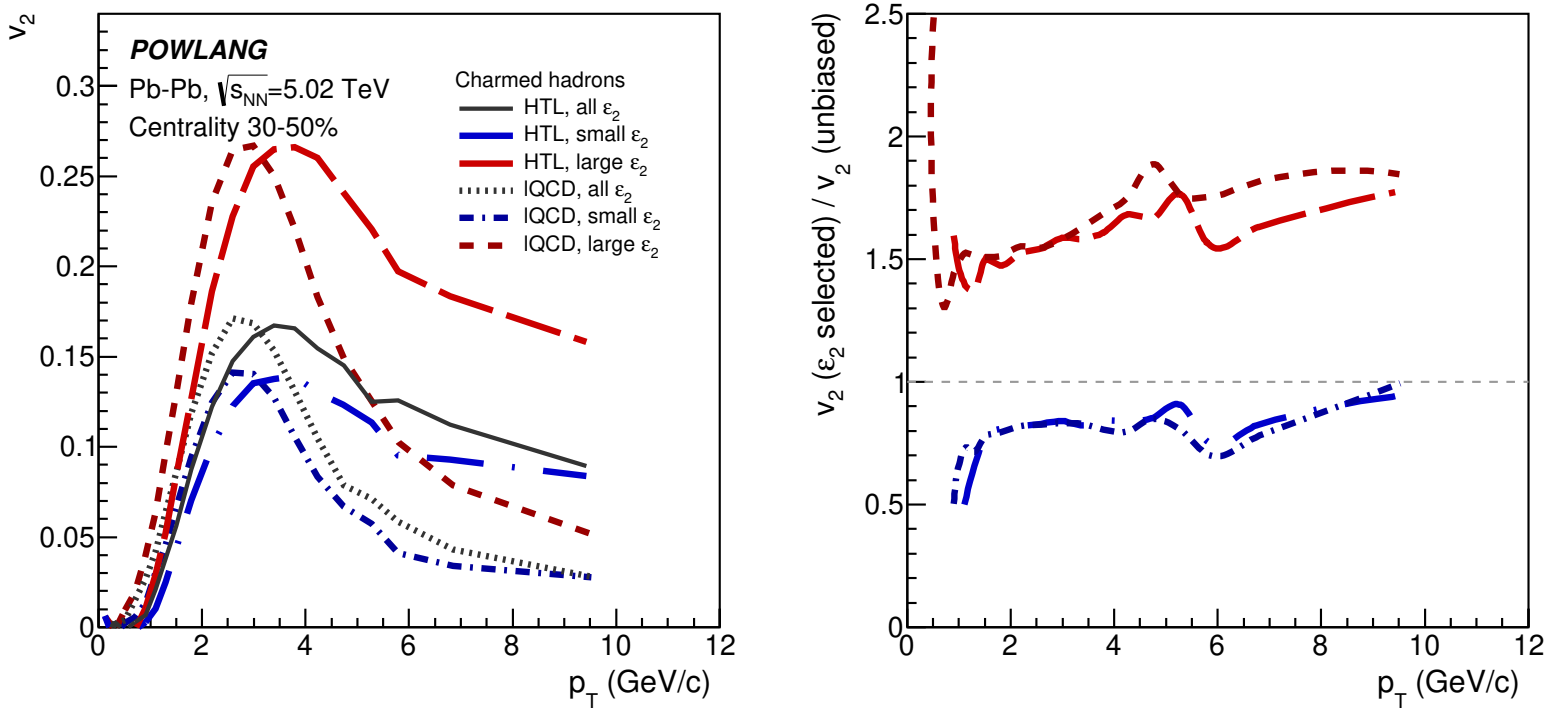
# Impact on the observables: $R_{\mathrm{AA}}$ and $v_2$



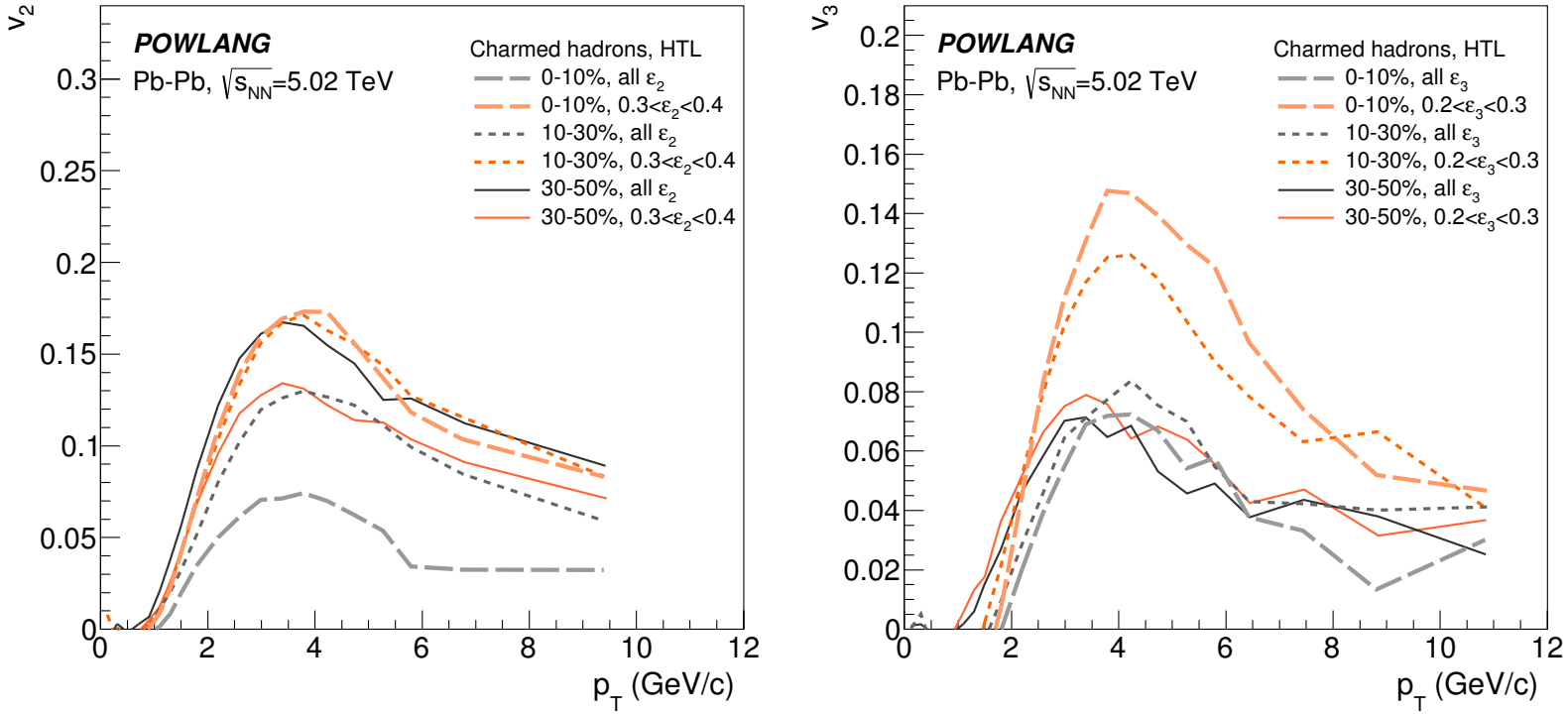

- Different momentum dependence of $\kappa_{\parallel}$ and $\kappa_{\perp}$ in the two approaches strongly affects charm results for $p_T \gtrsim 3-4$ GeV;
- Reality sits perhaps in between

# Impact on the observables: ESE




Current event-shape-engineering observables sensitive to the initial eccentricity of the fireball rather the on the transport coefficients

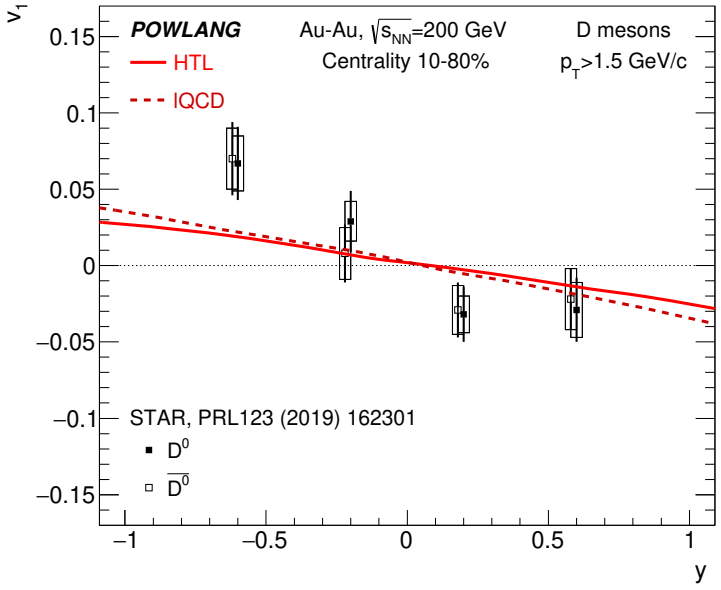
# ESE: a different perspective




A stronger sensitivity to the HQ-medium coupling could be obtained selecting events of the same eccentricity $\epsilon_n$, but belonging to different centrality classes[3]: medium effects stronger in more central events!

[3] A.B. et al., Eur.Phys.J.C 79 (2019) 6, 494

# The peculiar role of $v_1$




- $v_1^D >> v_1^\pi$ arises from the initial spatial asymmetry of the HQ distribution with respect to the tilted fireball[4];

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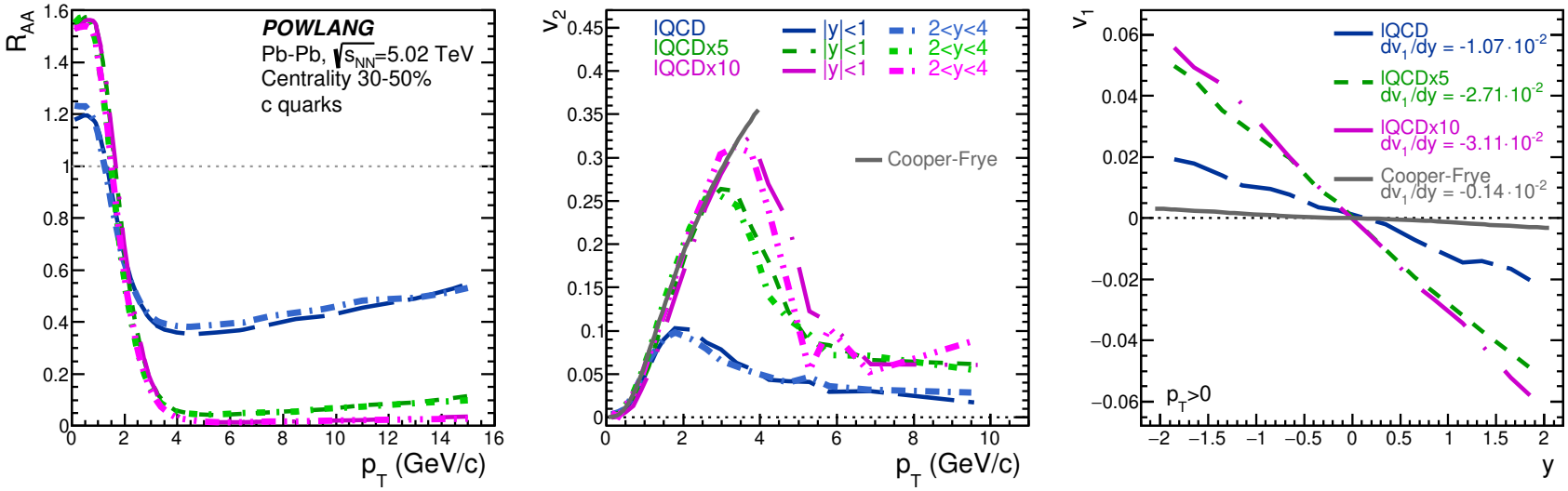
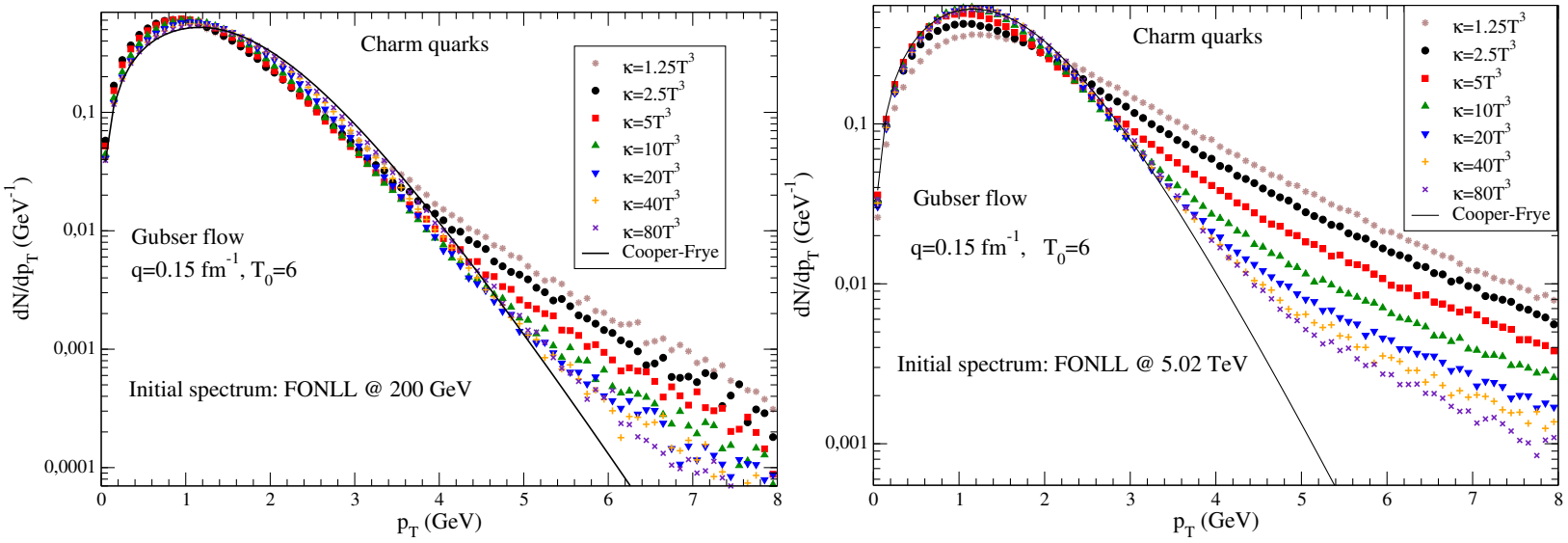



- $v_1^D >> v_1^\pi$ arises from the initial spatial asymmetry of the HQ distribution with respect to the tilted fireball[4];
- Since $D_s = 2T^2/\kappa$, for $\kappa \to \infty$ each HQ tends to flow with its original fluid-cell: $v_1^D$ does not approach the Cooper-Frye result in the strong-coupling limit

[4] A.B. et al., 2102.08064

# The influence of the initial spectrum




Different initial $Q\overline{Q}$ spectra evolved in the same hydrodynamic background (Gubser flow)

- Closer approach to thermalization with the softer initial spectrum;
- Lack of momentum dependence of $\kappa$ relevant for the tails;

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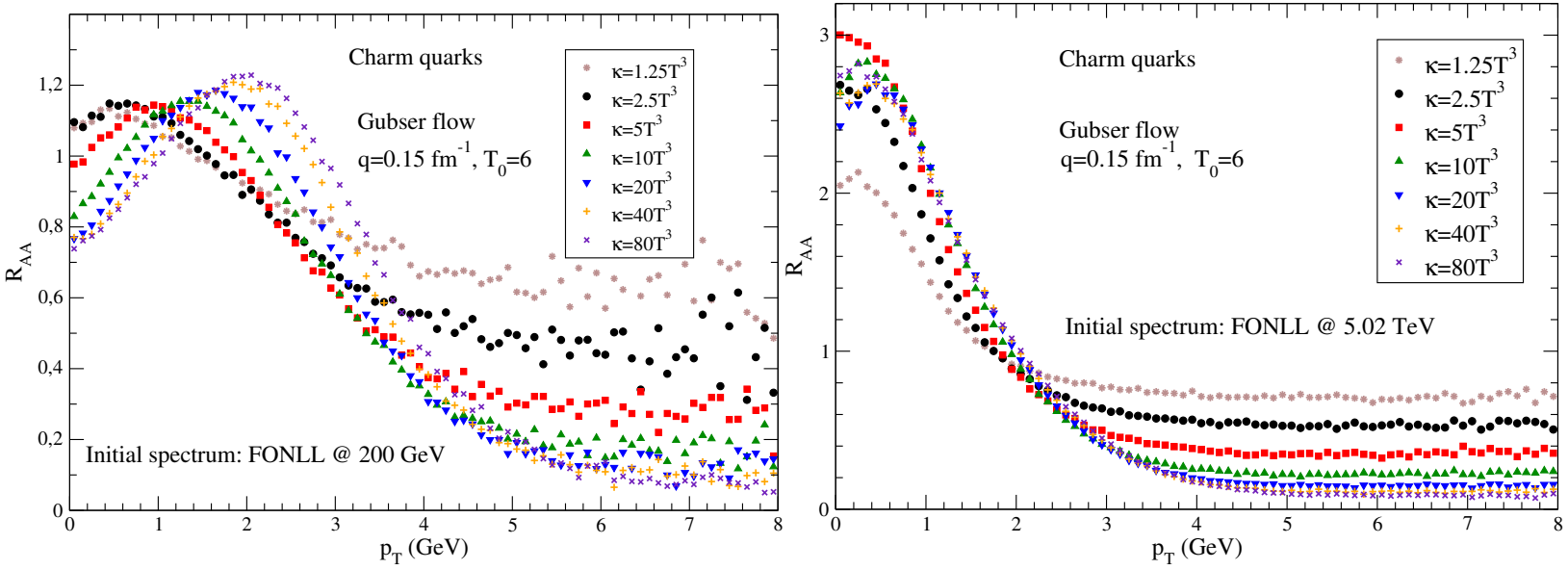


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- Closer approach to thermalization with the softer initial spectrum;
- Lack of momentum dependence of $\kappa$ relevant for the tails;
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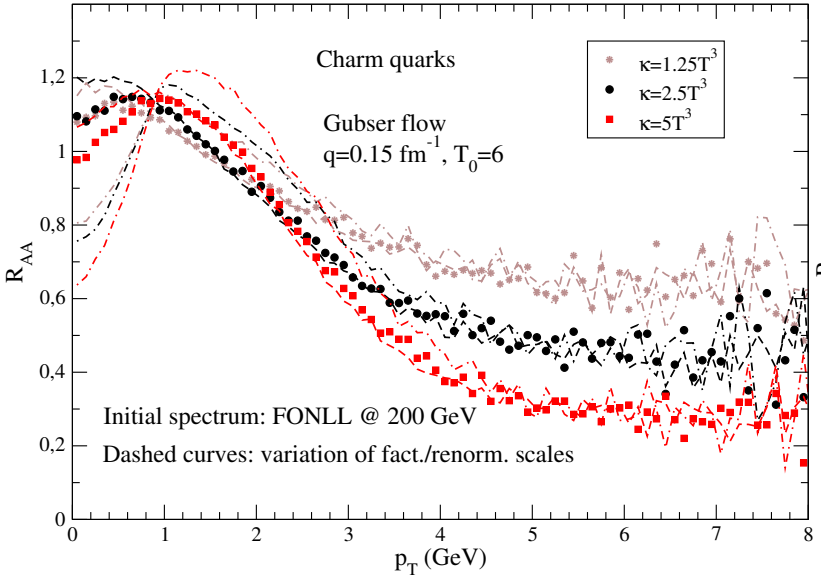


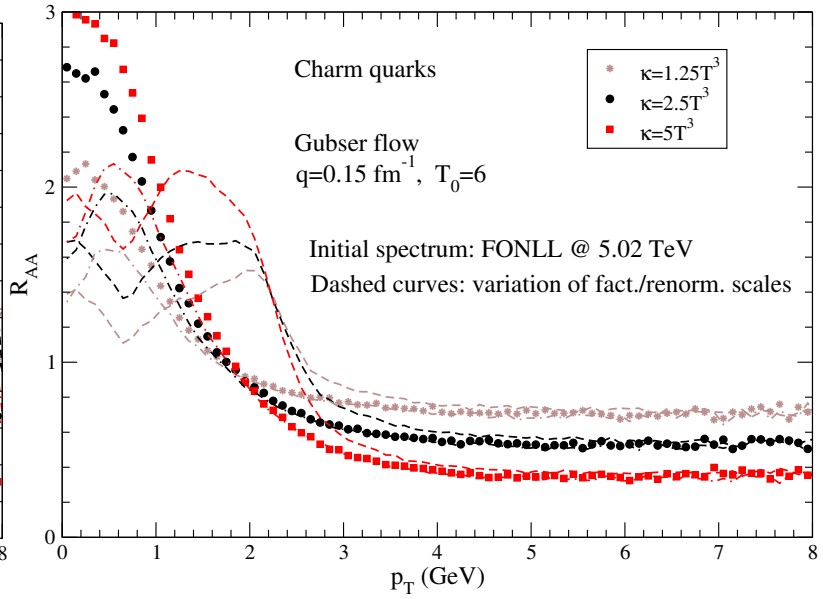


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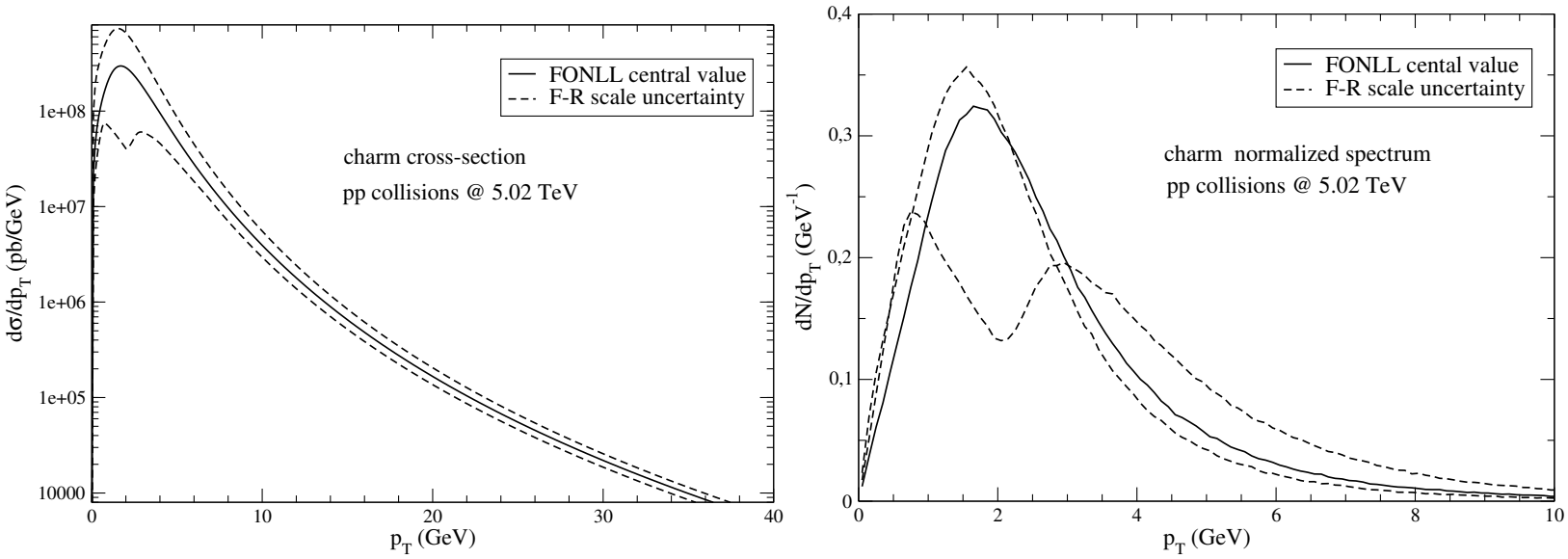


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# Back-up slides

# The link with the Fokker-Plank equation

Consider an arbitrary function of the HQ momentum $g(\mathbf{p})$ and take the expectation value over the thermal ensemble of its variation, keeping only terms up to order $\Delta t$:

$$\langle g(\mathbf{p}+\Delta\mathbf{p}) - g(\mathbf{p})\rangle = \left\langle \frac{\partial g}{\partial p^i}\Delta p^i + \frac{1}{2}\frac{\partial^2 g}{\partial p^i \partial p^j}\Delta p^i \Delta p^j \right\rangle + \dots$$

From

$$\Delta p^i = -\eta_D(p) p^i \Delta t + C^{ik}(\mathbf{p}+\xi\Delta\mathbf{p})\sqrt{\Delta t}\,\zeta^k, \quad \langle \zeta^k \rangle = 0, \quad \langle \zeta^k \zeta^l \rangle = \delta^{kl}$$

one gets:

$$\langle g(\mathbf{p}+\Delta\mathbf{p}) - g(\mathbf{p})\rangle = \left\langle \frac{\partial g}{\partial p^i}\left(-\eta_D p^i + \xi \frac{\partial C^{ik}}{\partial p^j} C^{jk}\right) + \frac{1}{2}\frac{\partial^2 g}{\partial p^i \partial p^j} C^{ik} C^{jk} \right\rangle \Delta t + \dots$$

In the above the expectation value is taken accorrding to the HQ phase-space distribution

$$\langle g(\mathbf{p})\rangle_t \equiv \int d\mathbf{p}\, g(\mathbf{p}) f(t,\mathbf{p})$$

# The link with the Fokker-Plank equation

Time evolution given be the differential equation

$$\frac{d}{dt}\langle g(\mathbf{p})\rangle_t \equiv \int d\mathbf{p}\, g(\mathbf{p})\frac{\partial}{\partial t}f(t,\mathbf{p})$$

Integrating by parts:

$$\mathrm{LHS} = \int d\mathbf{p}\, g(\mathbf{p})\left\{\frac{\partial}{\partial p^i}\left[\left(\eta_D p^i - \xi\frac{\partial C^{ik}}{\partial p^j}C^{jk}\right)f(t,\mathbf{p})\right] + \frac{1}{2}\frac{\partial^2}{\partial p^i \partial p^j}\left[(C^{ik}C^{jk})f(t,\mathbf{p})\right]\right\}$$

Comparing with the FP equation

$$\frac{\partial}{\partial t}f_Q(t,\mathbf{p}) = \frac{\partial}{\partial p^i}\left[A^i(\mathbf{p})f_Q(t,\mathbf{p})\right] + \frac{\partial}{\partial p^i \partial p^j}[B^{ij}(\mathbf{p})f_Q(t,\mathbf{p})]$$

one gets

$$A^i(\mathbf{p}) \equiv A(p)p^i = \eta_D p^i - \xi\frac{\partial C^{ik}}{\partial p^j}C^{jk}$$

$$C^{ij}(\mathbf{p}) \equiv \sqrt{\kappa_L(p)}P_L^{ij} + \sqrt{\kappa_T(p)}P_T^{ij} = \sqrt{2B_1(p)}P_L^{ij} + \sqrt{2B_0(p)}P_T^{ij}$$

# Dependence on the discretization scheme

The transport coefficients describing momentum-diffusion in the Langevin equation *always* coincide with the corresponding ones in the Fokker-Planck equation, no matter which discretization scheme is employed. In general, this is not the case for the friction term. From

$$\eta_D(p)p^i = A(p)p^i + \xi \frac{\partial C^{ik}}{\partial p^j} C^{jk}$$

one gets

$$\eta_D(p) = A(p) + \xi \left[ \frac{1}{p}\frac{\partial B_1}{\partial p} + \frac{d-1}{p^2}\sqrt{2B_0(p)}(\sqrt{2B_1(p)} - \sqrt{2B_0(p)}) \right]$$

where, furthermore, $A(p)$, $B_0(p)$ and $B_1(p)$ are related by the Einstein relation.

# The *post-point* discretization

In the *post-point* scheme $\xi = 1$, so that

$$\eta_D^{\rm post}(p)p^i = A(p)p^i + \frac{\partial C^{ik}}{\partial p^j} C^{jk}$$

Notice that $\eta_D^{\rm post}(p)$ entering in the Langevin equation *is not* the quantity which is directly evaluated from the scattering matrix!

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$$\eta_D^{\rm post}(p)p^i = A(p)p^i + \frac{\partial C^{ik}}{\partial p^j} C^{jk}$$

Notice that $\eta_D^{\rm post}(p)$ entering in the Langevin equation *is not* the quantity which is directly evaluated from the scattering matrix! Imposing the Einstein relation one has

$$\eta_D^{\rm post}(p) = \frac{B_1(p)}{TE_p} - \left[\frac{1}{p}\frac{\partial B_1(p)}{\partial p} + \frac{d-1}{p^2}(B_1(p) - B_0(p))\right] + \left[\frac{1}{p}\frac{\partial B_1}{\partial p} + \frac{d-1}{p^2}\sqrt{2B_0(p)}(\sqrt{2B_1(p)} - \sqrt{2B_0(p)})\right]$$

Notice that in the case $B_0(p) = B_1(p) = D(p)$ one has simply

$$\eta_D^{\rm post}(p) = \frac{D(p)}{TE_p}$$

However, in general this *is not* the case and, furthermore, $\eta_D^{\rm post}(p)$ does not follow from any scattering amplitude!