EXACT FORMULAE FOR WIGNER FUNCTIONS AT EQUILIBRIUM

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Based on *arXiv:2007.08249 + work in progress,*
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RESULT

We present the first exact results at general global equilibrium with both acceleration and rotation.

Exact Wigner function for fermions
at general global equilibrium

$$\mathcal{W}_{1/2}(x,k) = \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3 p}{2\varepsilon}\sum_{n=1}^{\infty}(-1)^{n+1}e^{-n\widetilde{\beta}(n\phi)\cdot p}.$$

$$\left[e^{-in\frac{\phi:\Sigma}{2}}(m+\not{p})\delta^4\left(k-\frac{\Lambda^n p+p}{2}\right)+(m-\not{p})e^{in\frac{\phi:\Sigma}{2}}\delta^4\left(k+\frac{\Lambda^n p+p}{2}\right)\right]$$

Exact solutions give all quantum corrections in relativistic fluids!

How to compute them?

Throughout the presentation,
the key ideas are in blue boxes!

Outline:

1) Equilibrium in quantum-relativistic systems
2) Field theories and Wigner functions
3) Exact mean values

MEAN VALUES AT EQUILIBRIUM

The system is described by a density operator. Mean values are:

$$\langle \widehat{O} \rangle = \mathrm{Tr}\left(\widehat{\rho}\widehat{O}\right)$$

General global equilibrium in special relativity

$$\widehat{\rho} = \frac{1}{Z} e^{-b_\mu \widehat{P}^\mu + \frac{1}{2}\varpi_{\mu\nu}\widehat{J}^{\mu\nu}}$$

The four-temperature is, in general: $\beta^\mu = \dfrac{u^\mu}{T}$ $\quad \beta^2 = \dfrac{1}{T^2}$

At global equilibrium (b and ϖ constant):

$$\beta^\mu(x) = b^\mu + \varpi^{\mu\nu} x_\nu$$

General decomposition of the thermal vorticity

$$\varpi^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} w_\rho u_\sigma + (\alpha^\mu u^\nu - \alpha^\nu u^\mu)$$

At global equilibrium (constant thermal vorticity):

$$w_\mu = \frac{\omega_\mu}{T} \qquad \alpha_\mu = \frac{a_\mu}{T}$$

THERMAL EXPECTATION VALUES

How to compute mean values in a field theory in general global equilibrium?

Use the similarity of the density operator with a Poincaré transformation (without solving equations in curvilinear coordinates).

IDEA I:
Analytic continuation and factorization of the density operator

Complex thermal vorticity: $\varpi^{\mu\nu} \to -i\phi^{\mu\nu}$

Factorization (Baker-Campbell-Haussdorff):

$$\widehat{\rho} = \frac{1}{Z} e^{-b_\mu \widehat{P}^\mu - i\frac{1}{2}\phi_{\mu\nu}\widehat{J}^{\mu\nu}} = \frac{1}{Z} e^{-\widetilde{b}_\mu(\phi)\widehat{P}^\mu} e^{-\frac{i}{2}\phi_{\mu\nu}\widehat{J}^{\mu\nu}}$$

The tilde-transform is defined as:

$$\widetilde{b}^\mu(\phi) = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \underbrace{\left(\phi^{\mu\nu_1}\phi_{\nu_1\nu_2}\dots\phi^{\nu_{k-1}\nu_k}\right)}_{k-\text{times}} b_{\nu_k}$$

Now the density operator is a Poincaré transformation, we can use group theory!

We deal with unphysical quantities. We must continue the results back to real thermal vorticity.

$$\widehat{\rho} = \frac{1}{Z} e^{-\widetilde{b}(\phi)\cdot\widehat{P}} e^{-\frac{i}{2}\phi:\widehat{J}} = \frac{1}{Z}\widehat{T}\left(-i\widetilde{b}(\phi)\right)\widehat{\mathcal{D}}\left(\Lambda(\phi)\right)$$

FIELDS AND WIGNER FUNCTIONS

Free scalar and Dirac fields in Minkowski space-time:

$$\widehat{\Phi}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3 p}{2\varepsilon} \widehat{a}(p) e^{-ip\cdot x} + \widehat{b}^\dagger(p) e^{ip\cdot x}$$

$$\widehat{\Psi}(x) = \frac{1}{(2\pi)^{3/2}} \sum_\sigma \int \frac{\mathrm{d}^3 p}{2\varepsilon} \widehat{a}_\sigma(p) u_\sigma(p) e^{-ip\cdot x} + \widehat{b}^\dagger_\sigma(p) v_\sigma(p) e^{ip\cdot x}$$

(Anti)commutation rules: $[a_\sigma(p), a^\dagger_\tau(p')]_\pm = 2\varepsilon \delta^3(\boldsymbol{p} - \boldsymbol{p'}) \delta_{\sigma\tau}$

For each field exists a **Wigner function and equation.**

Scalar field:

$$\mathcal{W}_0(x,k) = \frac{2}{(2\pi)^4}\int \mathrm{d}^4y\, e^{-ik\cdot y}\langle : \widehat{\Phi}^\dagger\,(x+y/2)\,\widehat{\Phi}\,(x-y/2) :\rangle$$

$$\left(m^2 - k^2 + \frac{\hbar^2}{4}\Box\right)\mathcal{W}_0 = 0$$

Dirac field:

$$\mathcal{W}_{1/2}(x,k) = -\frac{1}{(2\pi)^4}\int \mathrm{d}^4y\, e^{-ik\cdot y}\langle : \widehat{\Psi}\,(x-y/2)\,\widehat{\overline{\Psi}}\,(x+y/2) :\rangle$$

$$\left(m - \not{k} - \frac{i\hbar}{2}\not{\partial}\right)\mathcal{W}_{1/2} = 0$$

$$\mathrm{Tr}\left(\widehat{\rho}\,\widehat{a}^{\dagger}_{\sigma}(p)\widehat{a}_{\tau}(p')\right) = \frac{1}{Z}\mathrm{Tr}\left(\widehat{T}\left(-i\widetilde{b}(\phi)\right)\widehat{\mathcal{D}}\left(\Lambda(\phi)\right)\,\widehat{a}^{\dagger}_{\sigma}(p)\widehat{a}_{\tau}(p')\right)$$

(Anti)commutation relations and the transformation rules under Poincaré group:

$$\begin{aligned}\langle a^{\dagger}_{\sigma}(p)a_{\tau}(p')\rangle =&(-1)^{2S}\sum_{\alpha}W^{(S)}(\Lambda,p)_{\alpha\sigma}e^{-\widetilde{b}\cdot\Lambda p}\langle a^{\dagger}_{\alpha}(\Lambda p)a_{\tau}(p')\rangle+\\ &+2\varepsilon e^{-\widetilde{b}\cdot\Lambda p}W^{(S)}(\Lambda,p)_{\tau\sigma}\delta^{3}(\boldsymbol{\Lambda p}-\boldsymbol{p'})\end{aligned}$$

W is the **Wigner rotation:** $W^{(S)}(\Lambda,p) = D^{(S)}\left([\Lambda p]^{-1}\Lambda[p]\right)$

Vanishing thermal vorticity: equation for Bose-Einstein and Fermi-Dirac distributions.

IDEA II:
Iterative solution

$$\langle a^\dagger_\sigma(p) a_\tau(p')\rangle \sim 2\varepsilon e^{-\widetilde{b}\cdot\Lambda p} W^{(S)}_{\tau\sigma}(\Lambda, p)\delta^3(\boldsymbol{\Lambda p} - \boldsymbol{p'})$$

$$\Downarrow$$

$$\langle a^\dagger_\sigma(p) a_\tau(p')\rangle \sim 2\varepsilon(-1)^{2j} W^{(S)}_{\tau\sigma}(\Lambda^2, p) e^{-\widetilde{b}\cdot(\Lambda p + \Lambda^2 p)}\delta^3(\boldsymbol{\Lambda^2 p} - \boldsymbol{p'}) +$$
$$+ 2\varepsilon e^{-\widetilde{b}\cdot\Lambda p} W^{(S)}_{\tau\sigma}(\Lambda, p)\delta^3(\boldsymbol{\Lambda p} - \boldsymbol{p'})$$

Eventually, we find a series:

$$\langle a^\dagger_\sigma(p) a_\tau(p')\rangle = 2\epsilon' \sum_{n=1}^{\infty} (-1)^{2S(n+1)} \delta^3(\boldsymbol{\Lambda^n p} - \boldsymbol{p'}) W^{(S)}_{\tau\sigma}(\Lambda^n, p) e^{-\widetilde{b}\cdot\sum_{k=1}^{n} \Lambda^k p}$$

This is to be continued to real thermal vorticity.

The result is used to obtain **exact** Wigner functions.

Example: fermionic field

$$\mathcal{W}_{1/2}(x,k) = \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3 p}{2\varepsilon}\sum_{n=1}^{\infty}(-1)^{n+1} e^{-n\widetilde{\beta}(n\phi)\cdot p}.$$

$$\left[e^{-in\frac{\phi:\Sigma}{2}}(m+\not{p})\delta^4\left(k-\frac{\Lambda^n p+p}{2}\right)+(m-\not{p})e^{in\frac{\phi:\Sigma}{2}}\delta^4\left(k+\frac{\Lambda^n p+p}{2}\right)\right]$$

This expression **fulfills the Wigner equation** at global equilibrium with imaginary thermal vorticity.

Expanding the exponentials we obtain the **full semiclassical expansion** in ħ.

STRESS-ENERGY TENSOR FOR PURE ACCELERATION

$$T^{\mu\nu} = \rho u^\mu u^\nu - p\Delta^{\mu\nu} + \mathcal{A}\alpha^\mu\alpha^\nu \qquad \alpha^\mu = a^\mu/T$$

Massless scalar field:

$$\langle : \widehat{T}_C^{\mu\nu} : \rangle = \int \mathrm{d}^4k \; k^\mu k^\nu \mathcal{W}_0(x,k) + \frac{1}{4}\left(\partial^\mu\partial^\nu - g^{\mu\nu}\Box\right) \int \mathrm{d}^4k \, \mathcal{W}_0(x,k)$$

$$\rho = \frac{3T_0^4}{16\pi^2}\sum_{n=1}^{\infty}\frac{\phi^4}{\sinh^4\left(\frac{n\phi}{2}\right)} \qquad \phi \to \frac{ia}{T_0}$$

The series **diverges at real thermal vorticity!**

THE PROBLEM

Iterative equation in the scalar case:

Homogeneous equation

$$\boxed{\langle a^\dagger(p)a(p')\rangle = e^{-\widetilde{b}\cdot\Lambda p}\langle a^\dagger(\Lambda(p))a(p')\rangle} + 2\varepsilon\delta^3(\boldsymbol{\Lambda p}-\boldsymbol{p'})e^{-\widetilde{b}\cdot p'}$$

The general solution: $S(p,p') = H(p,p') + S_0(p,p')$

The equation may have non-analytic solutions!
Example: solution to the homogeneous equation for scalar fields with acceleration along z:

$$H(p,p') = F(\phi)e^{\frac{1}{T_0}\frac{(p_z - p'_z)}{\phi}}$$

IDEA III:
Mathematical operation to remove non-analytic part:
ANALYTIC DISTILLATION

DEFINITION

Let f(z) be a function defined in a subset D of the complex plane. Let z_0 be a point where it may not be analytic. Let f(z) have an asymptotic power series in different subdomains D_i (Stokes phenomenon).

$$f(z) \sim \sum_{n=-\infty}^{\infty} a_n^{(i)} (z - z_0)^n$$

If the series of the common coefficients for n≥0 is convergent, we define that series as the ***analytic distillate.***

$$\mathrm{dist}_{z_0}(f(z)) = \sum_{n=0}^{\infty} A_n (z - z_0)^n$$

Back to the stress-energy tensor of the massless scalar field: **full asymptotic power expansion**[1].

$$\rho \propto \mathrm{dist}_{\phi=0}\left(\sum_{n=1}^{\infty}\frac{\phi^4}{\sinh^4(n\phi/2)}\right)=\frac{8\pi^4}{45}-\frac{4\pi^2}{9}\phi^2-\frac{11}{90}\phi^4$$

$$p \propto \mathrm{dist}_{\phi=0}\left(\sum_{n=1}^{\infty}\frac{\phi^4\cosh(n\phi)}{\sinh^4(n\phi/2)}\right)=\frac{8\pi^4}{45}+\frac{8\pi^2}{9}\phi^2+\frac{19}{90}\phi^4$$

$$\mathcal{A} \propto \mathrm{dist}_{\phi=0}\left(\sum_{n=1}^{\infty}\frac{\phi^2}{\sinh^2(n\phi/2)}\right)=\frac{2\pi^2}{3}\phi^2+\frac{1}{6}\phi^4$$

Now we can continue to real thermal vorticity.

[1]D. Zagier, appendix on *Quantum Field Theory I: Basics in Mathematics and Physics*

Our results agree with computation in Rindler coordinates and with the previous literature[2] (perturbative results).

$$\alpha^2 = a^\mu a_\mu / T^2$$

$$\rho = T^4 \left(\frac{\pi^2}{30} - \frac{\alpha^2}{12} - \frac{11\alpha^4}{480\pi^2} \right)$$

$$p = T^4 \left(\frac{\pi^2}{90} + \frac{\alpha^2}{18} + \frac{19\alpha^4}{1440\pi^2} \right)$$

$$\mathcal{A} = T^4 \left(\frac{1}{12} + \frac{\alpha^2}{48\pi^2} \right)$$

Results are consistent with the **Unruh effect**, and vanish at:

$$T_U^2 = \frac{a^\mu a_\mu}{4\pi^2}$$

We find exact results also at equilibrium with rotation[3].

[2] G. Y. Prokhorov, O. V. Teryaev, and V. I. Zakharov, JHEP03, 137 (2020)
[3]V. E. Ambrus, Phys.Lett.B771, 151 (2017)

We obtain **exact results** for massless fermions[4] :

$$\rho = T^4 \left(\frac{7\pi^2}{60} - \frac{\alpha^2}{24} - \frac{17\alpha^4}{960\pi^2} \right) \qquad p = \frac{\rho}{3} \qquad \mathcal{A} = 0$$

Exact axial current at equilibrium with rotation[5]:

$$\langle J_A^\mu \rangle = T^2 \left(\frac{1}{6} - \frac{w^2}{24\pi^2} - \frac{\alpha^2}{8\pi^2} \right) \omega^\mu$$

$$w^2 = \omega^\mu \omega_\mu / T^2$$

$$\alpha^2 = a^\mu a_\mu / T^2$$

[4]G. Y. Prokhorov, O. V. Teryaev, and V. I. Zakharov, Phys.Rev. D99, 071901 (2019)
[5]V. E. Ambrus, E. Winstanley,1908.10244

SPIN DENSITY & POLARIZATION

We can compute the **exact expression of polarization at global equilibrium**. Use the spin density matrix[6]:

$$\Theta_{rs}(k) = \frac{\mathrm{Tr}\left(\widehat{\rho}\widehat{a}_s^\dagger(k)\widehat{a}_r(k)\right)}{\sum_t \mathrm{Tr}\left(\widehat{\rho}\widehat{a}_t^\dagger(k)\widehat{a}_t(k)\right)} = \frac{\int \mathrm{d}k^0 \int \mathrm{d}\Sigma_\mu k^\mu \, \overline{u}_r(k)\mathcal{W}_{1/2}(x,k)u_s(k)}{\mathrm{tr}\left(\int \mathrm{d}k^0 \int \mathrm{d}\Sigma_\mu k^\mu \, \overline{u}(k)\mathcal{W}_{1/2}(x,k)u(k)\right)}$$

Exact solution for the free Dirac field: $q_n = 2\left(\mathbb{I} + e^{n\varpi:J/2}\right)^{-1} k$

$$\Theta_{rs}^{(N)}(k) = \int \mathrm{d}\Sigma_\mu \sum_{n=1}^{\infty} \frac{k^\mu(-1)^{n+1}}{\left|\frac{\partial f}{\partial k^0}\right| \left|\det\left(\mathbb{I} + e^{n\varpi:\Sigma/2}\right)\right| \epsilon_{\boldsymbol{q_n}}} e^{-n\widetilde{\beta}(-n\varpi)\cdot q_n}$$

$$\overline{u}_r(k) e^{n\frac{\varpi:\Sigma}{2}}(\not{q}_n + m)u_s(k) \,\Big|_{k^0=k^0_+}$$

[6] F.Becattini, arXiv:2004.04050,(2020)

We obtain the exact spin polarization vector:

$$S^\mu(k) = -\frac{1}{2m}\epsilon^{\mu\nu\rho\sigma}k_\sigma \mathrm{tr}\left([k]^{-1}D^{(S)}(J_{\nu\rho})[k]\Theta(k)\right)$$

The first order in thermal vorticity reproduces the literature[7]:

$$S^\mu(k) \approx -\frac{1}{8m}\epsilon^{\mu\rho\sigma\gamma}k_\gamma \frac{\int \mathrm{d}\Sigma_\lambda k^\lambda \, n_F(1-n_F)\varpi_{\rho\sigma}}{\int \mathrm{d}\Sigma_\lambda k^\lambda \, n_F}$$

We have the solution **to all orders in vorticity!**

[7]F. Becattini, V. Chandra, L. Del Zanna, and E. Grossi,Annals Phys., vol. 338, 2013

OUTLOOK

We tested the new method comparing with exact solutions found for pure rotation and pure acceleration in curvilinear coordinates.

Our method provides exact results for general global equilibrium with **both acceleration and rotation.**

We calculated the exact results for massless fields introducing the analytic distillation. It can be **extended to the massive case.**

CONCLUSIONS

We determine **exact solutions in general global equilibrium with rotation and with acceleration**.

We have introduced a new operation on complex functions to extract their analytic part, the **analytic distillation.**

We derived expressions for the **spin density matrix** and **spin polarization vector** to **all order** in thermal vorticity.

THANK YOU FOR THE ATTENTION!