The theoretical prediction for the muon $g-2$: the important role of hadronic contributions

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A*Midex Initiative d'excellence Aix-Marseille

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Introduction

- The muon is an elementary particle
- Same charge but 200 heavier than the electron
- Spin $1/2$ particle
- The magnetic moment of the muon is proportional to the spin $\vec{\mu} = g \left(\frac{Qe}{2m} \right) \vec{s}$

$$a_\mu = \frac{g-2}{2}$$

Why is it interesting ?

1) can be measured very precisely

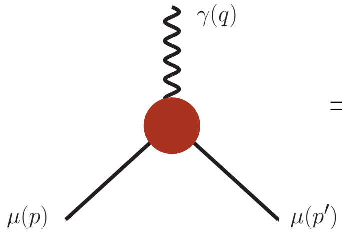
2) can also be predicted very precisely in the SM

3) sensitive to new physics

Introduction

- Corrections to the vertex function : Dirac and Pauli form factors

Assuming Lorentz invariance and P and T symmetries, the vertex function can be decomposed into 2 form factors

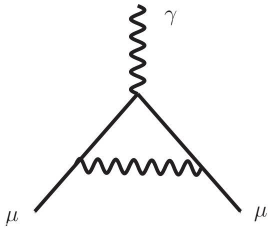


$$= -ie\,\overline{u}(p',\sigma')\Gamma_\mu(p',p)u(p,\sigma)$$
$$= -ie\,\overline{u}(p',\sigma')\left[\gamma_\mu F_1(q^2) + \frac{i\sigma_{\mu\nu}q_\nu}{2m}F_2(q^2)\right]u(p,\sigma)$$

$F_1(0) = 1$ (charge conservation)

$$F_2(0) = a_\mu = \tfrac{g-2}{2}$$

- Classical result : $g = 2$ for elementary fermions (Dirac equation)



Quantum field theory : $a_\mu = \frac{g-2}{2} \neq 0$

$\hookrightarrow$ Generated by quantum effects

$$a_\mu^{(1)} = \frac{\alpha}{2\pi}$$ [Schwinger '48]

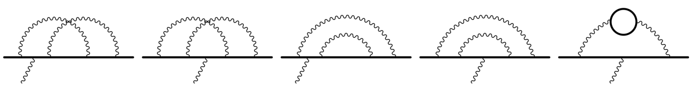
Standard model contributions : QED

- **QED accounts for more than 99.99% of the final result** [Aoyama et al. '12 '19]

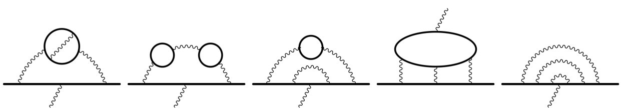
$$a_\mu^{\mathrm{QED}} = \left(\frac{\alpha}{\pi}\right) a_\mu^{(1)} + \left(\frac{\alpha}{\pi}\right)^2 a_\mu^{(2)} + \left(\frac{\alpha}{\pi}\right)^3 a_\mu^{(3)} + \cdots$$

$\rightarrow$ 5-loop contributions are known !

Order α^4 (7 diagrams)

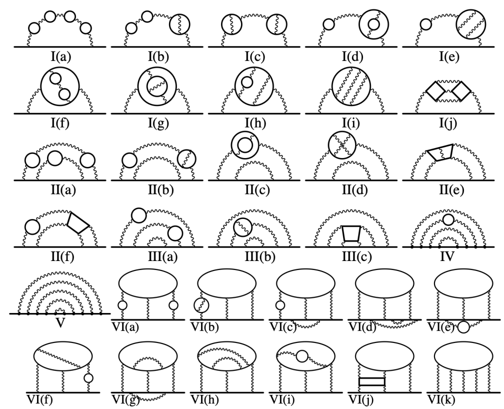


Order α^6 (72 diagrams)



Order α^8 (891 diagrams) ...

Order α^{10} (12 672 diagrams)



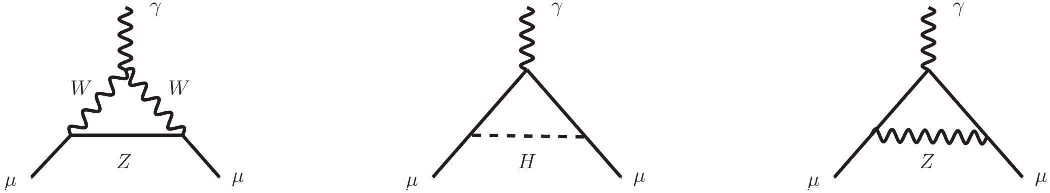
n	$a_\mu^{(1)} \times 10^{11}$	n	$a_\mu^{(1)} \times 10^{11}$
1	116 140 973.321(23)	4	381.004(17)
2	413 217.6258(70)	5	5.0783(59)
3	30 141.90233(33)		

$\rightarrow$ Uncertainty far below Δa_μ. Strong test of QED.

$$a_\mu^{\mathrm{QED}} = 116\ 584\ 718.931(104) \times 10^{-11}$$
$$a_\mu^{\mathrm{SM}} = 116\ 591\ 810(43) \times 10^{-11}$$

Standard model contributions

- **Electroweak corrections** [Czarnecki '02] [Gnendiger '13]



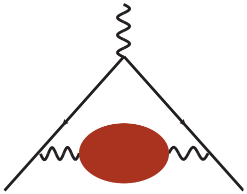
→ Two-loop contributions are known : $a_\mu^{\mathrm{EW}} \times 10^{11} = 153.6(1.0)$

→ Contributes to only 1.5 ppm ($4 \times$ exp. error) $\Rightarrow$ under control

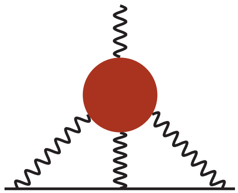
- **QCD corrections**

→ Quarks and gluons do not directly couple to the muon : contribution via loop diagrams

→ The two relevant contributions (to reduce the error) are



Hadronic Vacuum Polarisation (LO-HVP, α^2)

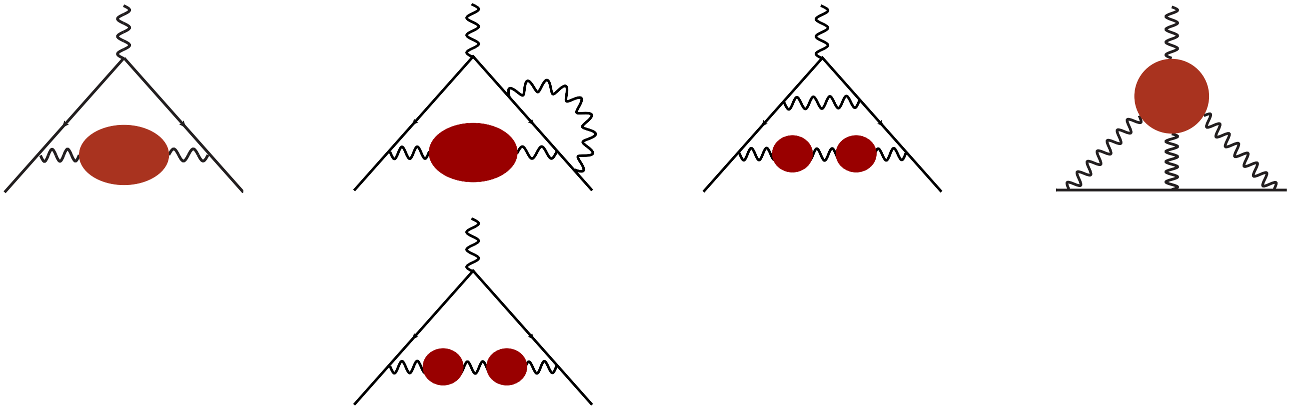


Hadronic Light-by-Light scattering (HLbL, α^3)

- **Contribution from unknown particles / interactions (?)** $a_\ell^{\mathrm{NP}} = \mathcal{C}\,\frac{m_\ell^2}{\Lambda^2}$

Other hadronic contributions

LO HVP | **NLO HVP** | **NNLO HVP** | **HLbL**



- NLO HVP and NNLO HVP differ by the QED kernel functions
- Not negligible, but error under control (the required relative precision is smaller)

Theory status in the white-paper

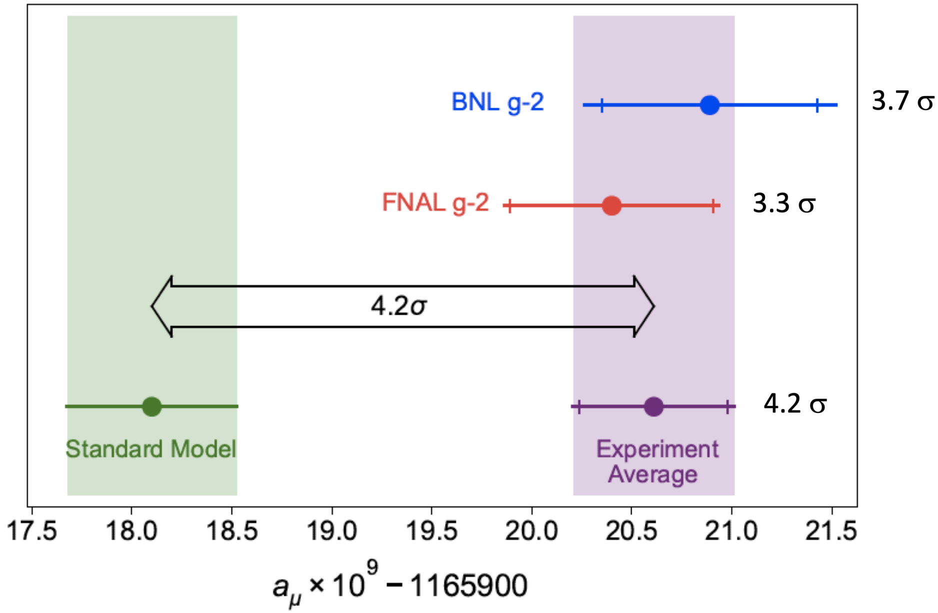
The Muon $g-2$ Theory Initiative :

- website : https ://muon-gm2-theory.illinois.edu/
- Organized 7 workshops between 2017-2021
- White Paper posted 10 June 2020

  The anomalous magnetic moment of the muon in the Standard Model [Phys.Rept. 887 (2020) 1-166]

Contribution	$a_\mu \times 10^{11}$	
- QED (leptons, 10^{th} order)	$116\ 584\ 718.95 \pm 0.08$	[Aoyama et al. '12]
- Electroweak	153.6 ± 1.0	[Gnendiger et al. "13]
- Strong contributions		
HVP (LO)	$6\ 931 \pm 40$	[DHMZ 19, KNT 20]
HVP (NLO)	-98.3 ± 0.7	[Hagiwara et al. 11]
HVP (NNLO)	12.4 ± 0.1	[Kurtz et al. '14]
HLbL	94 ± 19	[See WP]
Total (theory)	$116\ 591\ 810 \pm 43$	

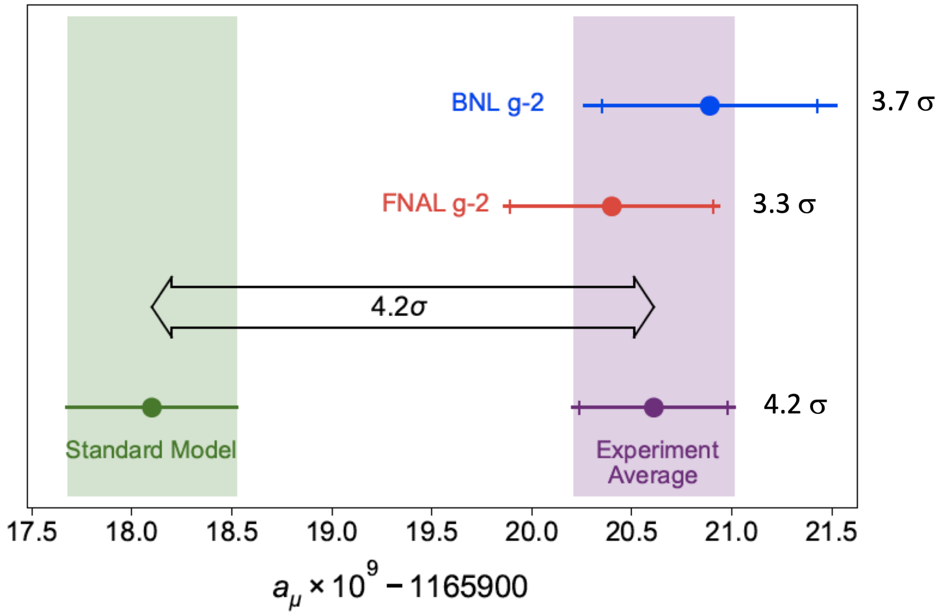
Status after the first run of the E989 experiment at Fermilab



- Remarkable confirmation of the Brookhaven result (2004)
- Similar precision for both theory and experiment
- This is a large discrepancy ($2\times$ electroweak contribution !)
- Theory error is dominated by hadronic contributions

  $\rightarrow$ reduction of the theory error by a factor of 3-4 needed to match upcoming experiments

Status after the first run of the E989 experiment at Fermilab

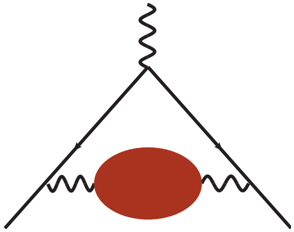


BUT :

- does not include the most recent lattice results
  - → new complete lattice calculation of the HLbL contribution by Mainz
  - → first sub-percent calculation of the HVP contribution by BMW

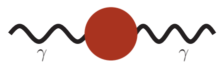
Outline of the talk : hadronic contributions

- **Hadronic Vacuum Polarisation** (HVP, α^2)



- Blobs : all possible intermediate hadronic states (ρ, $\pi\pi$, $\cdots$)
- Precision physics (Goal : precision < 0.3%)

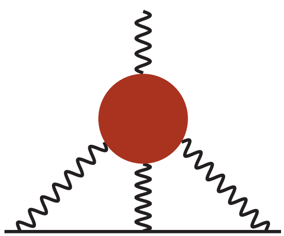
$\Pi_{\mu\nu}(Q) =$



$$= (Q_\mu Q_\nu - \delta_{\mu\nu} Q^2)\, \Pi(Q^2)$$

$$= \int \mathrm{d}^4x\, e^{iQ\cdot x}\, \langle V_\mu(x) V_\nu(0) \rangle$$

- **Hadronic Light-by-Light scattering** (HLbL, α^3)



Hadronic light-by-light tensor $\Pi_{\mu\nu\lambda\sigma}(p_1, p_2, p_3)$

- Small but contributes to the total uncertainty !
- 4-point correlation function
- More difficult, but need 10% precision

Standard model prediction of hadronic contributions

- Perturbative QCD can not be used : we need non-perturbative methods
- Two first-principle approaches :

The dispersive framework (data-driven)

→ based on analyticity, unitarity ...

→ ... but relies on experimental data (needs a careful propagation of exp. uncertainties)

→ several group have published results for the HVP [Davier et al. '19] [Keshavarzi et al. '20]

→ more difficult for the LbL
(analytic structure of the 4-point function more difficult, exp. data sometimes missing)

Lattice QCD

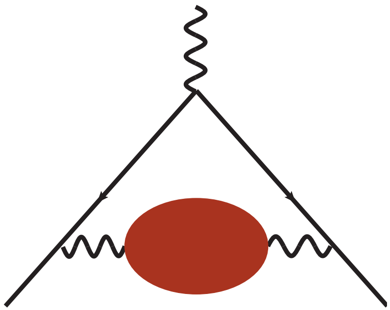
→ ab-initio calculations

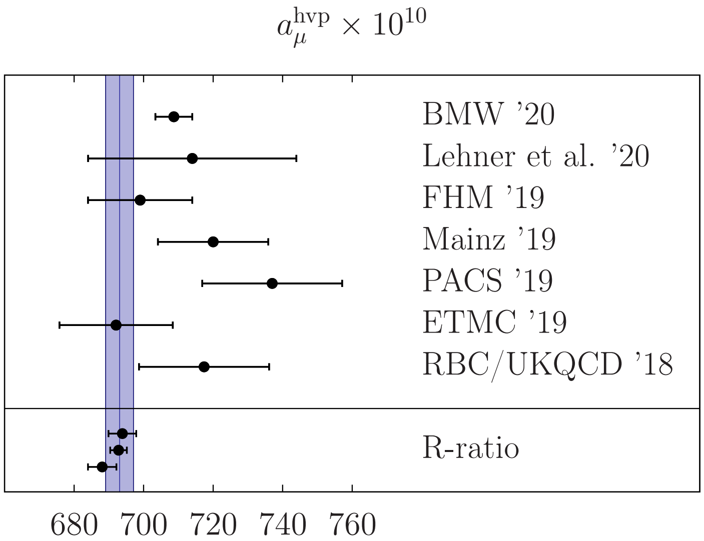
→ but we need to control all sources of error (huge challenge at this level of precision)

→ many groups are working on this subject : cross-checks possible

- It provides two completely independent determinations

Hadronic vacuum polarization





→ Many lattice collaborations (with different systematic errors)

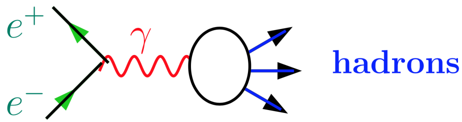
→ Precision of about 2% for lattice, 0.6% for the data driven approach

→ Recent lattice calculation below 1% by the Budapest-Marseille-Wuppertal collaboration

Hadronic vacuum polarization : dispersive framework

- Use analyticity + optical theorem

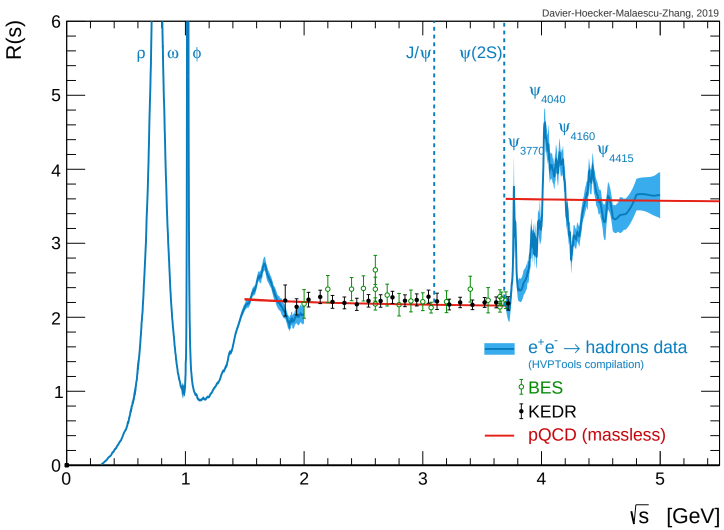
$$R_{\rm had}(s) = \frac{\sigma^0(e^+e^- \to \gamma^* \to \text{hadrons})}{(4\pi\alpha^2/3s)}$$



- $K(s)$ is a known kernel function

$$a_\mu^{\rm HVP} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \left\{ \int_{m_\pi^2}^{E_{\rm cut}^2} {\rm d}s \frac{R_{\rm had}^{\rm data}(s)\widehat{K}(s)}{s^2} + \int_{E_{\rm cut}^2}^{\infty} {\rm d}s \frac{R_{\rm had}^{\rm pQCD}(s)\widehat{K}(s)}{s^2} \right\}$$

- Compilation of experimental data from many experiments



Hadronic vacuum polarization : dispersive framework

Compilation of results :

- DHMZ19 [Eur. Phys. J. C80, 241 (2020)]

$$a_\mu^{\rm hvp} = 694.0(1.0)_{\rm stat}(2.5)_{\rm syst}(0.7)_{\rm QCD}(2.8)_{\rm KLOE/BABAR} \times 10^{-10} \quad [0.58\%]$$

- KNT19 [Phys. Rev. D101, 014029 (2020)]

$$a_\mu^{\rm hvp} = 692.78(1.21)_{\rm stat}(1.97)_{\rm syst}(0.21)_{\rm vp}(0.7)_{\rm fsr} \times 10^{-10} \quad [0.35\%]$$

- White paper average

$$a_\mu^{\rm hvp} = 693.1(2.8)_{\rm stat}(0.7)_{\rm DV+QCD}(2.8)_{\rm KLOE/BABAR} \times 10^{-10} \quad [0.58\%]$$

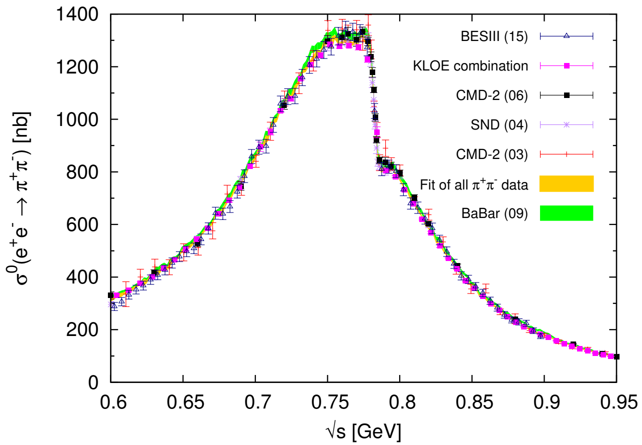
→ obtained with the same experimental data sets

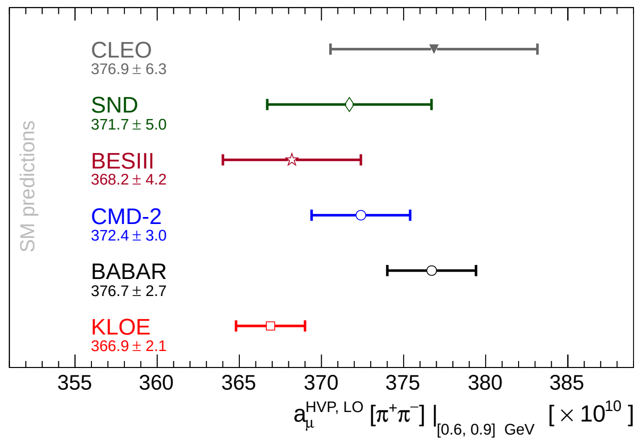
→ error dominated by the « KLOE/BABAR » discrepancy

→ good agreement (... for the sum over all contributions)

Hadronic vacuum polarization and dispersive theory

- Most precise determination so far
- Subject to experimental uncertainties : careful propagation of experimental uncertainties
  - → Groups with ≠ methodologies are in good agreement [Davier et al. '19] [Keshavarzi et al. '20]
  - → But local discrepancies (tensions already there in the experimental data) !
  - → Problematic for the dominant $\pi\pi$ channel

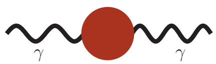




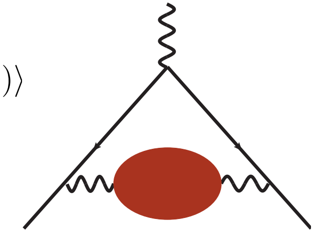
Difference between BABAR and KLOE : $\Delta a_\mu = 9.8(3.4) \times 10^{-10}$

Difference pheno / exp for the $g-2$: $\Delta a_\mu = 28(8) \times 10^{-10}$

Lattice QCD approach to the hadronic vacuum polarization (HVP)

$\Pi_{\mu\nu}(Q) =$ 

$= \left(Q_\mu Q_\nu - \delta_{\mu\nu} Q^2\right) \Pi(Q^2) = \int \mathrm{d}^4 x \, e^{iQ\cdot x} \, \langle V_\mu(x) V_\nu(0) \rangle$



EM current : $V_\mu(x) = \frac{2}{3}\overline{u}(x)\gamma_\mu u(x) - \frac{1}{3}\overline{d}(x)\gamma_\mu d(x) - \frac{1}{3}\overline{s}(x)\gamma_\mu s(x) + \frac{2}{3}\overline{c}(x)\gamma_\mu c(x) + \cdots$

- Integral representation over Euclidean momenta

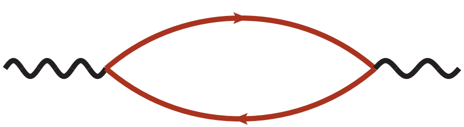
$$a_\mu^{\mathrm{HVP}} = 4\alpha^2 \int_0^\infty \mathrm{d}Q^2 \ f(x_0) \ \left(\Pi(Q^2) - \Pi(0)\right)$$

- Time-momentum representation [Blum '02] [Bernecker, Meyer '11]

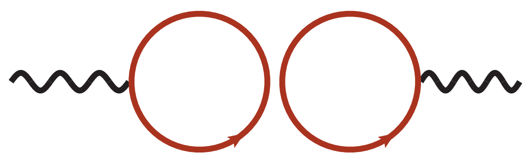
$$a_\mu^{\mathrm{HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \mathrm{d}x_0 \ K(x_0) \ G(x_0) \ , \qquad G(x_0) = -\frac{1}{3} \sum_{k=1}^{3} \sum_{\vec{x}} \langle V_k(x) V_k(0) \rangle$$

- Iso-symmetric QCD : two sets of Wick contractions

Connected contribution



(quark) disconnected contribution



$\mathrm{O}(1-2\%)$

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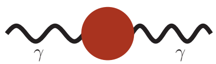
$\Pi_{\mu\nu}(Q) =$ 

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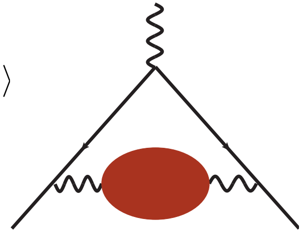

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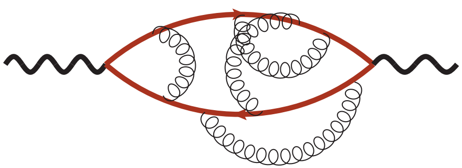
- Iso-symmetric QCD : two sets of Wick contractions

Connected contribution



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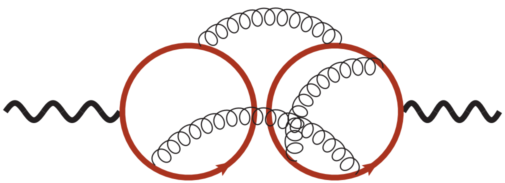
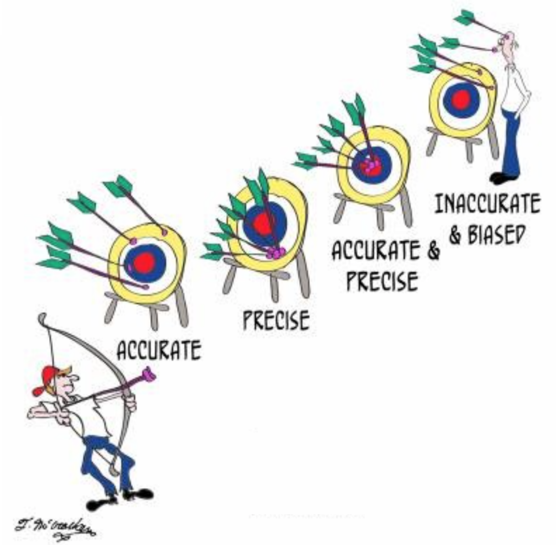


$\mathrm{O}(1-2\%)$

Systematic errors in lattice QCD

Statistical error

- Monte-Carlo algorithm : statistical error $\to \sim 1/\sqrt{N_{\mathrm{meas}}}$
  $\to$ noise/signal $\propto \exp((m_V - m_\pi)t)$



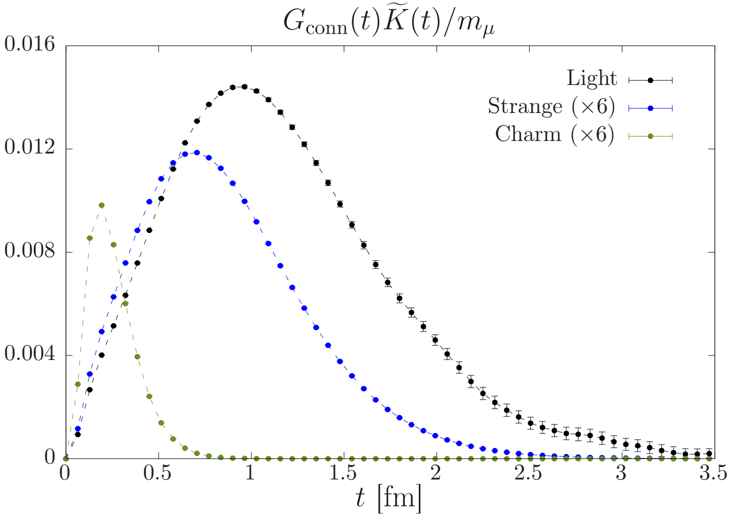
Systematic errors

- Finite lattice spacing : $a \neq 0$
  $\to$ need several (small !) lattice spacings
- Finite volume V
  $\to$ one should take the infinite volume limit
  $\to$ χPT can help in some cases (pion dominates FSE)
- Unphysical quark masses
  $\to$ All collaborations now have at least one physical pion mass ensemble.
- Isospin-breaking corrections
  $\to$ Need to be included at this level of precision

Lattice actions

- Different lattice actions are used : Staggered, Wilson-Clover, Twisted mass, Domain wall
- They are all equivalent in the continuum limit ($\to$ QCD !)
- But they have different features at finite value of the lattice spacing

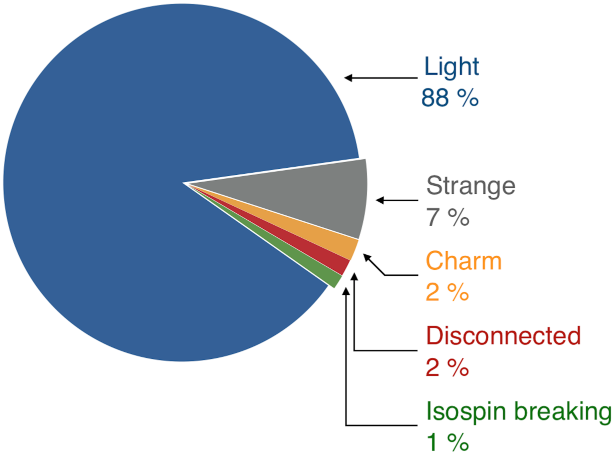
Light, strange and charm quark contributions at the physical pion mass



- Physical pion mass with $a = 0.065$ fm
- Flavor decomposition :

$$G(t) = G_l(t) + G_s(t) + G_c(t) + G_{\rm disc}(t)$$

$$a_\mu^{{\rm HVP},f} = \left(\frac{\alpha}{\pi}\right)^2 \sum_{t=0}^{\infty} K(t)\ G_f(t)$$



Challenges for a high-precision calculation :

- Light contribution dominates
  → **noise/signal** increases exponentially with t
  → **finite-size effects** O(3%) at physical point
- Disconnected diagrams of the order of O(2-3%)
- **Continuum extrapolation**
- **QED + strong isospin breaking corrections : O(1%)**

Solution to the noise problem

Signal / noise problem

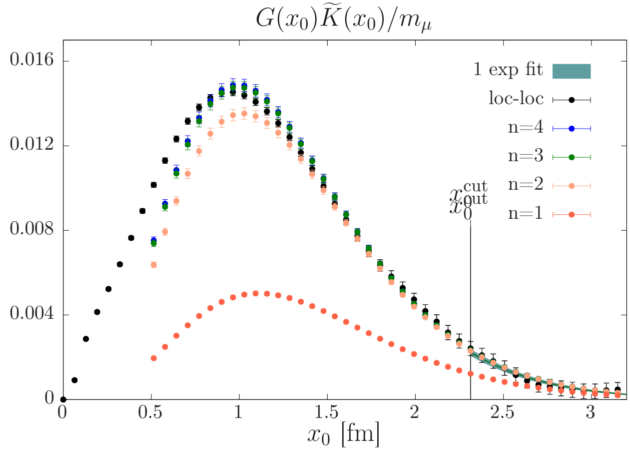
Solution to the noise problem

The vector correlators admits a spectral decomposition :

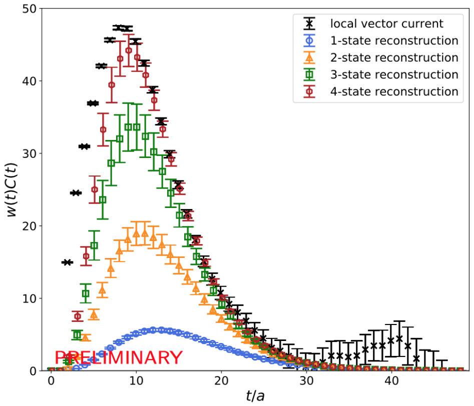
$$\langle V(x_0)V(0)\rangle = \sum_n \langle 0|V|n\rangle \, \frac{1}{2E_n} \, \langle n|V(0)|0\rangle \; e^{-E_n x_0}$$

- $|n\rangle$ are the eigenstates in finite volume
- E_n and $\langle 0|V|n\rangle$ can be computed on the lattice using sophisticated spectroscopy methods

[Mainz and RBC/UKQCD Collaborations]



[A. Gerardin et al, Phys.Rev. D100 (2019), 014510]



[Plot by A. Meyer (RBC/UKQCD) @ Lattice 2019]

→ Only a few number of states are needed (but more states needed at the physical pion mass)

→ Noise now grows linearly with x_0 (not exponentially)

→ Other algorithmic improvement : low-mode averaging (exact all-to-all propagator)

Corrections for finite-size effects

Finite volumes effects

Corrections for finite-size effects

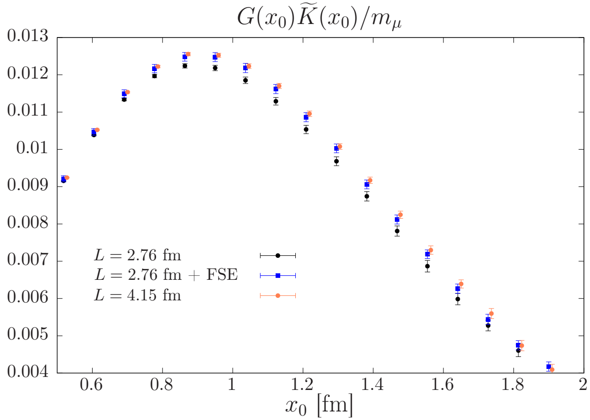
- Chiral perturbation theory : NLO not enough, NNLO corrections are quite large [Aubin et al. '20]

  [C. Aubin et al, arXiv :1905.09307], [J. Bijnens et al, JHEP 1712 (2017) 114]

- Correction based on the time-like pion form factor [H. Meyer, Phys.Rev.Lett. 107 (2011)]

$$G^{I=1}(x_0,\infty) = \frac{1}{48\pi^2}\int_{2m_\pi}^{\infty} \mathrm{d}\omega\, \omega^2\, \rho(\omega^2)\, e^{-\omega x_0}\,, \quad \rho(\omega^2) = \left(1-\frac{4m_\pi^2}{\omega^2}\right)^{3/2} |F_\pi(\omega)|^2$$

$$G^{I=1}(x_0,L) = \sum_i |A_i|^2\, e^{-E_i x_0}\,, \quad A_i : \text{obtained from } F_\pi$$



[Phys. Rev. D 100, 014510 (2019)]

On a typical lattice $L = 6$ fm :

$\Delta a_\mu = 22.7 \times 10^{-10}$ [$\sim 3\%$]

→ similar results for ETMC

→ and RBC-UKQCD [C. Lehner, Talk at Lattice 2019]

- Direct lattice calculation in very large volume : 11 fm [Budapest-Marseille-Wuppertal '21]

  → Finite size effects correction $18.1(2.5)(1.4) \times 10^{-10}$ compared to a 6 fm box

  → This correction is now well understood

- New Hamiltonian approach in [M. Hansen, A. Patella, arXiv :1904.10010]

Isospin-breaking corrections

QED + strong isospin-breaking effects

Isospin-breaking corrections

- Most lattice simulations are performed with QCD only and in the isospin symmetric limit

$m_u \neq m_d$: $\mathrm{O}(\frac{m_u - m_d}{\Lambda_{\mathrm{QCD}}}) \approx 1/100$	Strong isospin breaking
$Q_u \neq Q_d$: $\mathrm{O}(\alpha_{\mathrm{em}}) \approx 1/100$	Electromagnetic isospin breaking

$\rightarrow$ Separation of strong IB and QED effects is prescription dependent

$\rightarrow$ Furthermore, the definition of the iso-symmetric theory is also scheme dependent.

- Small effects O(1%) - but challenging to compute

- Strategy 1 : expand the path integral in $(m_u - m_d)$ and α_{em}

[RM123, JHEP 1204 (2012) 124] [RM123, Phys.Rev. D87 (2013), 114505]

$\rightarrow$ Non-compact QED in finite volume : dynamical variable is the gauge potential $A_\mu(x)$

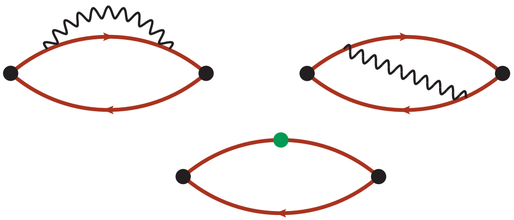
$\rightarrow$ Finite-size effect : $1/L^2$ absent, $1/L^3$ (might be negligible at our level of precision)

[J. Bijnens et al, Phys.Rev. D100 (2019), 014508]

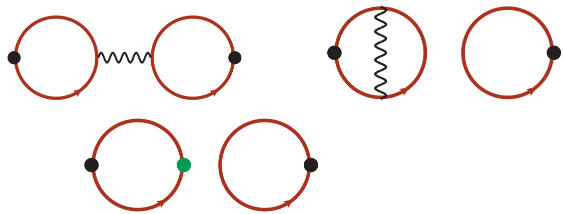
- Strategy 2 : generate gauge configurations for QED+QCD theory (usually electro-quenched)

Isospin-breaking corrections

- Corrections to the connected part :



- Corrections to the disconnected part :

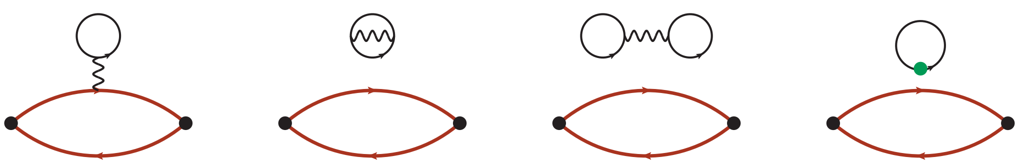


- Three collaborations have published (partial) results so far

		Conn.	Disc.
ETMC	6.0(2.3)	1.1(1.0)	$\times$
RBC-UKQCD	10.6(4.3)	5.9(5.7)	$-6.9(2.1)$
HPQCD-Fermilab-MILC	9.0(2.3)	$\times$	$\times$

(in units of 10^{-10})

- Challenging : beyond the electro-quenched approximation (diagrams are $1/N_c$ suppressed)



- BMW '21 : first calculation that includes all diagrams. About 1% of the full contribution.

Determination of the lattice spacing

Lattice spacing and the continuum extrapolation

Determination of the lattice spacing

- The results of a lattice simulations is in lattice units $(am_\pi,\, af_\pi)$
- The conversion in physical requires the knowledge of the lattice spacing in physical units
- But a_μ is a dimensionless quantity : so why does it matter ?

$\rightarrow$ Because the muon mass is an external scale

$\rightarrow$ We need to know the muon mass in lattice units !

$\rightarrow$ Error propagation [DellaMorte '17] :

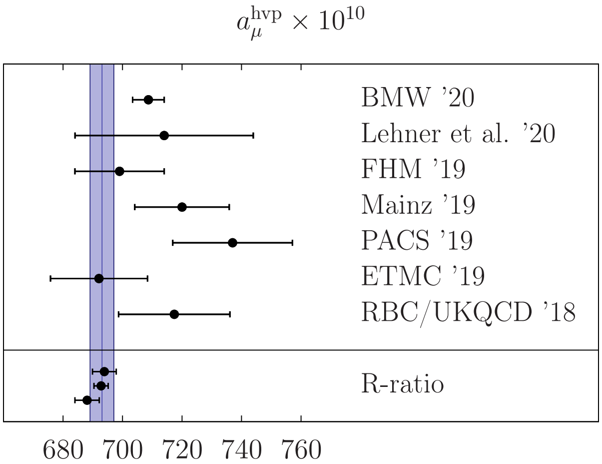
$$\Delta a_\mu^{\rm hvp} = \left| a \frac{{\rm d}a_\mu^{\rm hvp}}{{\rm d}a} \right| \cdot \frac{\Delta a}{a} = \left| m_\mu \frac{{\rm d}a_\mu^{\rm hvp}}{{\rm d}m_\mu} \right| \cdot \frac{\Delta a}{a} \approx 2\frac{\Delta a}{a}$$

We want few permil precision : one of the biggest challenge for lattice

$\rightarrow$ The BMWc uses the Ω Baryon mass to set the scale (mass well known experimentally) : few permil. Other possibility : pion decay constant (but QED corrections are more involved)

- Also important for the continuum extrapolation

HVP : conclusion



- First sub-percent lattice calculation by BMWc (competitive with data-driven approach)
- If confirmed, would reduce the discrepancy with experiment to $< 2\sigma$
- Need confirmation by other lattice groups (expected within 1 year)
- Other cross-checks are important (windows, running α)
- **Goal** : 0.2%
  - average between lattice and dispersive might help ...
  - ... but only if they agree
  - It is probably too soon to quote a "SM estimate of the LO-HVP" with $\sim 0.5\%$ precision

Cross-checks : window quantities

→ Current situation : two incompatible determinations (R-ratio vs Lattice)

→ Before concluding on the presence or not of NP : we need to agree on the SM estimate !

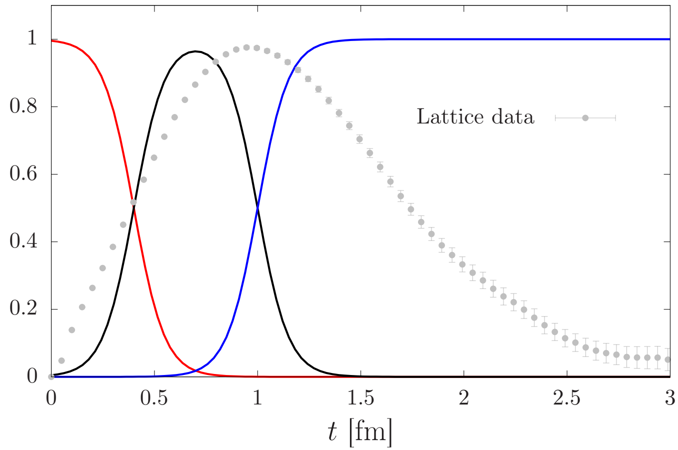
What could go wrong on the lattice ?

Given the importance of this quantity, it would be preferable to have a second observable :

- based on the same input (vector correlator) + easy to calculate
- less affected by systematic error (and statistical error)
- that allows for cross checks between different collaborations ...
- ... and also with R-ratio data

Cross-checks : window quantities

$$a_\mu^{\rm win} = \left(\frac{\alpha}{\pi}\right)^2 \sum_t G(t)\, K(t)\, W(t; t_0, t_1)$$



- the sum over the 3 windows gives the full contribution
- each window has different systematic errors

Short-distance	Intermediate-distance	Long-distance
stat. precise	stat. precise	noise problem
discretization effects	small finite volume effect	finite volume corrections
		large taste breaking (staggered)

Cross-checks : window quantities

- on the lattice : flavor decomposition for internal cross checks
- focus on the light quark contribution (or iso-vector contribution)

  $\rightarrow$ contributes to most of the signal
- Intermediate window : most systematics are significantly reduced

  $\rightarrow$ the continuum extrapolation is still needed !

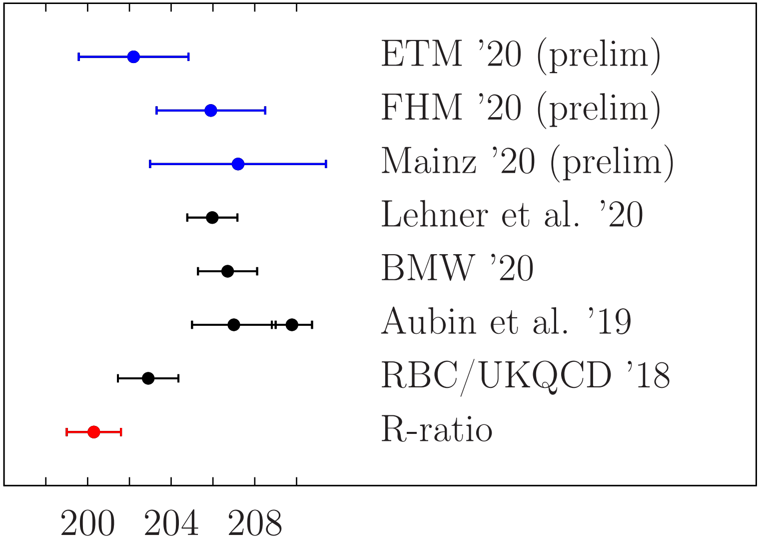
  $\rightarrow$ difficult for sub-percent precision

  $\rightarrow$ Importance of different lattice results with difference discretiations

Cross-checks : window quantities

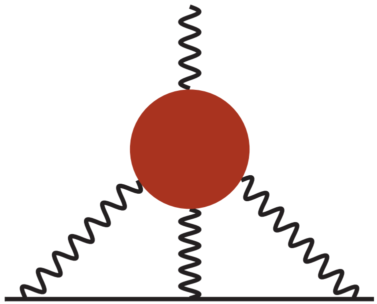
- workshop organized in November to discuss those issues [https ://indico.cern.ch/event/956699/]

$$\left(a_\mu^{\rm hvp}\right)^{{\rm win},l} \times 10^{10}$$



- lattice results systematically above the R-ratio
- agreement between lattice calculations not yet satisfactory : need to be improved !
- not discussed here : the running of electromagnetic coupling α

Hadronic light-by-light scattering contribution



$O(\alpha^3)$

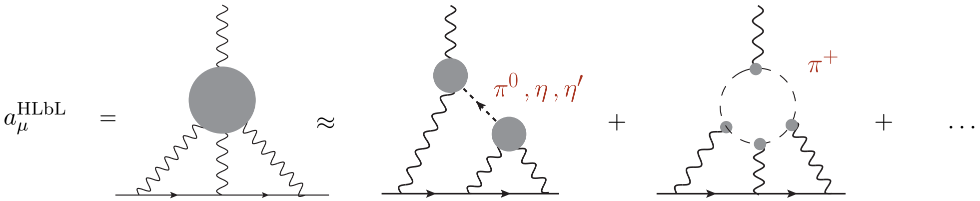
- Glasgow consensus ('09) $\rightarrow$ dispersive framework

	$a_\mu \times 10^{11}$	$a_\mu \times 10^{11}$
π^0, η, η'	114 ± 13	93.8 ± 4
pion/kaon loops	-19 ± 19	-16.4 ± 0.2
S-wave $\pi\pi$	'	-8 ± 1
axial vector	15 ± 10	6 ± 6
scalar + tensor	-7 ± 7	-1 ± 3
q-loops / short. dist. cstr	2	15 ± 10
charm + heavy q		3 ± 1
total HLbL	105 ± 26	92 ± 19
LO HVP		6931 ± 40

$$\Delta a_\mu^{\rm exp} = 28 \times 10^{-10} \approx 3 \times a_\mu^{\rm hlbl}$$

[J. Prades, E. de Rafael, A. Vainshtein '09]
[White paper '20]

Status before 2014 : model calculations



[de Rafael '94]

1) Chiral counting

2) N_c counting

[extracted from A. Nyffeler's slide], units : $a_\mu \times 10^{11}$

Contribution	BPP	HKS, HK	KN	MV	BP, MdRR	PdRV	N, JN
π^0, η, η'	85±13	82.7±6.4	83±12	114±10	–	114±13	99 ± 16
axial vectors	2.5±1.0	1.7±1.7	–	22±5	–	15 ± 10	22 ± 5
scalars	−6.8±2.0	–	–	–	–	−7±7	−7±2
π, K loops	−19±13	−4.5±8.1	–	–	–	−19±19	−19±13
π,K loops +subl. N_C	–	–	–	0±10	–	–	–
quark loops	21±3	9.7±11.1	–	–	–	2.3 (c-quark)	21±3
Total	83±32	89.6±15.4	80±40	136±25	110±40	105 ± 26	116 ± 39

BPP = Bijnens, Pallante, Prades '95, '96, '02; HKS = Hayakawa, Kinoshita, Sanda '95, '96; HK = Hayakawa, Kinoshita '98, '02; KN = Knecht, AN '02; MV = Melnikov, Vainshtein '04; BP = Bijnens, Prades '07; MdRR = Miller, de Rafael, Roberts '07; PdRV = Prades, de Rafael, Vainshtein '09; N = AN '09, JN = Jegerlehner, AN '09

1) Pseudoscalar contributions dominate numerically : transition form factors $\pi, \eta, \eta' \to \gamma^*\gamma^*$ as input

2) Glasgow consensus : $a_\mu^{\rm HLbL} = (105 \pm 26) \times 10^{-11}$

3) Results are in good agreement but errors are difficult to estimate (model calculations)

Hadronic light-by-light : data-driven approach

- HVP : single scalar function

$$\Pi_{\mu\nu}(Q) = \left(Q_\mu Q_\nu - \delta_{\mu\nu} Q^2\right) \Pi(Q^2) = \int \mathrm{d}^4 x \, e^{iQ\cdot x} \, \langle V_\mu(x) V_\nu(0) \rangle$$

- HLbL [Colangelo, Hoferichter, Procura, Stoffer (2015)]

  $\rightarrow$ complicated analytical structure (4-point instead of 2-point function for the HVP)

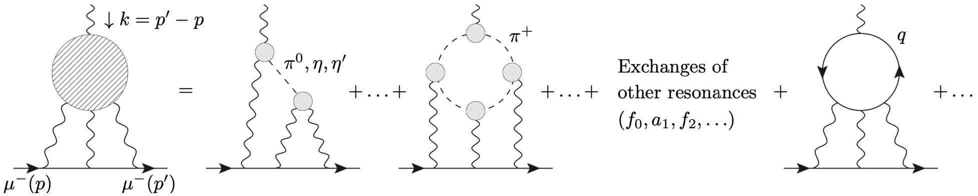
$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i$$

$$a_\mu^{\mathrm{HLbL}} = -e^6 \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \frac{\sum_{i=1}^{12} \hat{T}_i(q_1, q_2; p) \hat{\Pi}_i(q_1, q_2, -q_1 - q_2)}{q_1^2 q_2^2 (q_1 + q_2)^2 [(p + q_1)^2 - m_\mu^2][(p - q_2)^2 - m_\mu^2]}$$

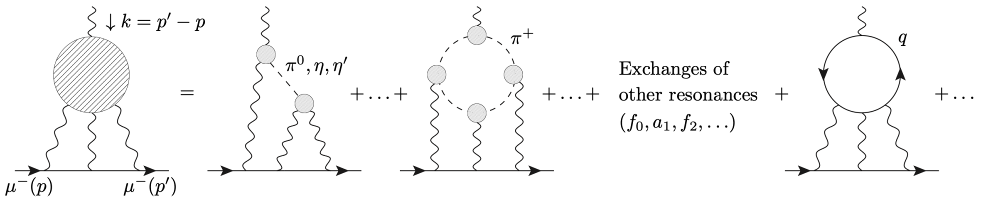
  $\rightarrow$ 7 independent scalar functions determined using dispersive relations

  $\rightarrow$ notion of large/small momenta less clear than for HVP (only one virtuality)

- Make the following decomposition rigorous :



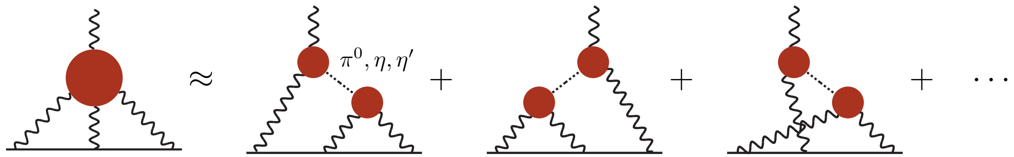
Hadronic light-by-light : data-driven approach



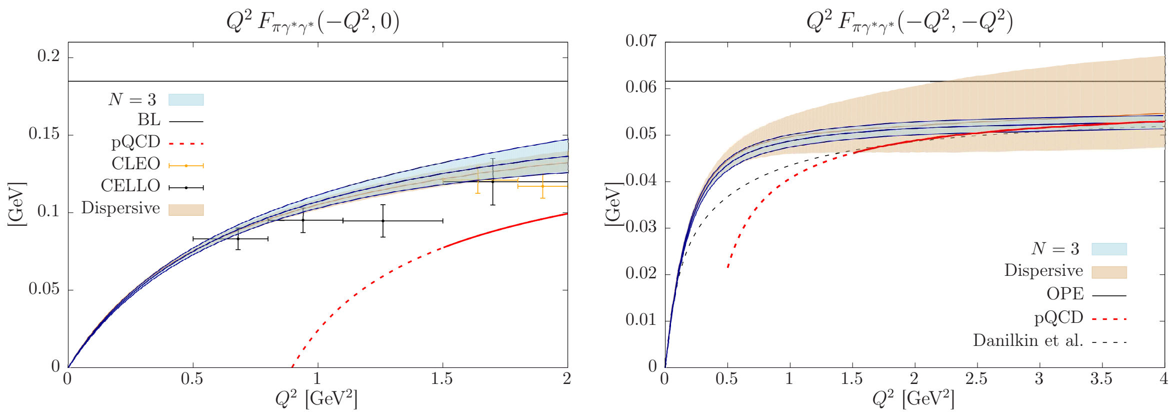
- Impressive improvement in the last few years
- The pseudoscalar-pole contributions are well-defined and dominant

	$a_\mu \times 10^{11}$	$a_\mu \times 10^{11}$
π^0, η, η'	114 ± 13	93.8 ± 4
pion/kaon loops	-19 ± 19	-16.4 ± 0.2
S-wave $\pi\pi$		-8 ± 1
axial vector	15 ± 10	6 ± 6
scalar + tensor	-7 ± 7	-1 ± 3
q-loops / short. dist. cstr	2	15 ± 10
charm + heavy q		3 ± 1
total HLbL	105 ± 26	92 ± 19

Lattice inputs for the dispersive framework : the pion-pole contribution



- Pion transition form factor



- Fully model independant

$$a_\mu^{\mathrm{HLbL};\pi^0} = (59.9 \pm 3.6) \times 10^{-11}$$

[A. G et al, Phys.Rev. D100 (2019)]

→ Compatible with the dispersive result

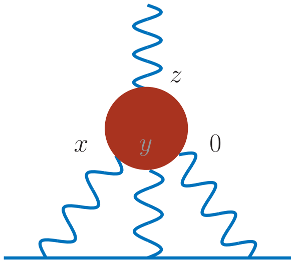
$$a_\mu^{\mathrm{HLbL};\pi^0} = 62.6^{+3.0}_{-2.5} \times 10^{-11}$$

[Hoferichter et al. '18]

- The ETM collaboration has presented very preliminary results for the η and η'

Direct lattice calculation of the Hadronic light-by-light contribution

- Two collaborations : RBC/UKQCD and Mainz, both using position space approaches
- Mainz approach : [PoS LATTICE2015 (2016) 109] [arXiv :1911.05573]



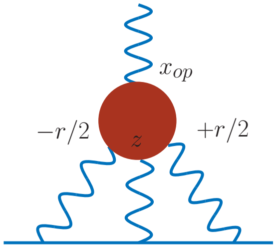
$$a_\mu^{\rm HLbL} = \frac{me^6}{3} \int \mathrm{d}^4y \int \mathrm{d}^4x \, \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(x,y) \; i\widehat{\Pi}_{\rho,\mu\nu\lambda\sigma}(x,y)$$

$$i\widehat{\Pi}_{\rho,\mu\nu\lambda\sigma}(x,y) = -\int \mathrm{d}^4z \, z_\rho \, \langle J_\mu(x) J_\nu(y) J_\sigma(z) J_\lambda(0) \rangle$$

→ $\mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$ is the QED kernel, computed semi-analytically in infinite volume

→ Avoid $1/L^2$ finite-volume effects from the massless photons

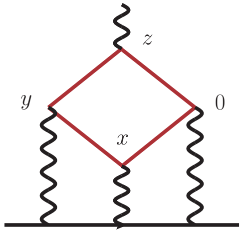
- RBC/UKQCD approach : [T. Blum et al, Phys.Rev. D93 (2016)] [arXiv :1911.08123]



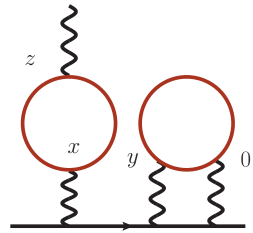
- exact photon propagators
  - → QED_L : photon in finite volume, power-law volume corrections
  - → QED_∞ : photons in infinite volume
- stochastic evaluation of sum over r

Wick contractions : 5 classes of diagrams

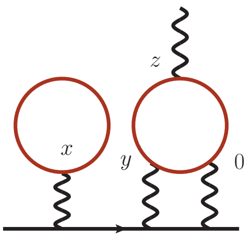
- Fully connected contribution

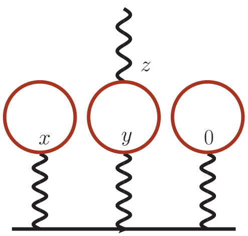


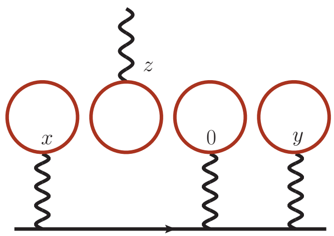
- Leading 2+2 (quark) disconnected contribution



- Sub-dominant disconnected contributions (3+1, 2+1+1, 1+1+1+1)





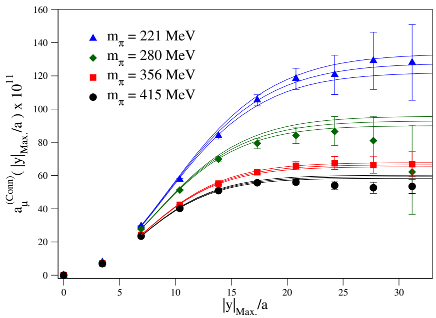


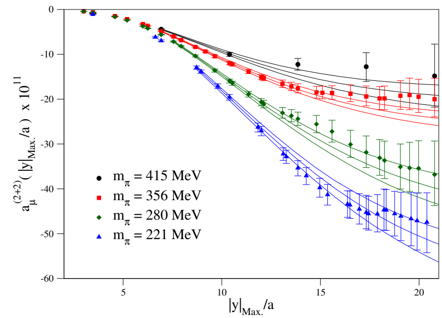
- Second set of diagrams vanish in the SU(3) limit (at least one quark loop which couple to a single photon)
  → Smaller contributions, have been shown to be irrelevant at the 10% level [Mainz '21 : 2104.02632]

- 2+2 disconnected diagrams are not negligible !
  → Large-N_c prediction : 2+2 disc ≈ - 50 % × connected [Bijnens '16]
  → Cancellation ⇒ more difficult (correlations does not seems to help in practice ...)

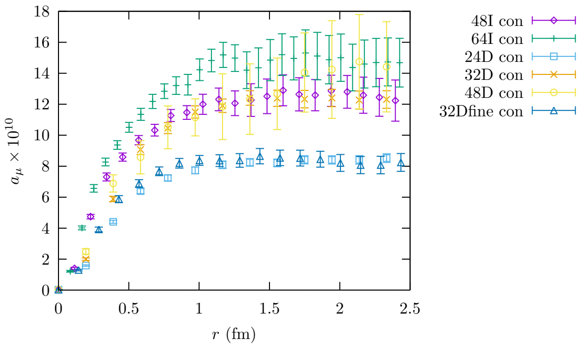
Large cancellation between the two leading contributions

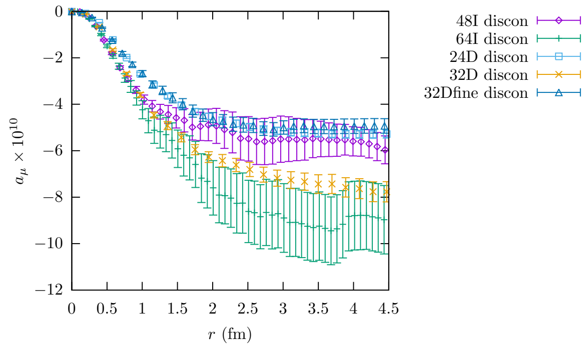
- Connected and disconnected contributions from Mainz ($m_\pi = 200$ MeV)





- Connected and disconnected contribution from RBC/UKQCD at the physical pion mass

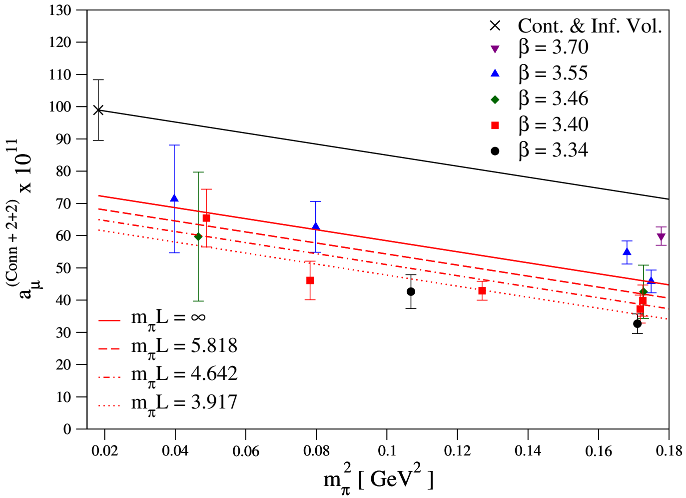




- Major difficulties : signal/noise problem, finite-size effects are important

Which errors are relevant

Mainz group [2104.02632]



- Statistical noise at long distances
- Finite-volume effects are large
- Continuum extrapolation

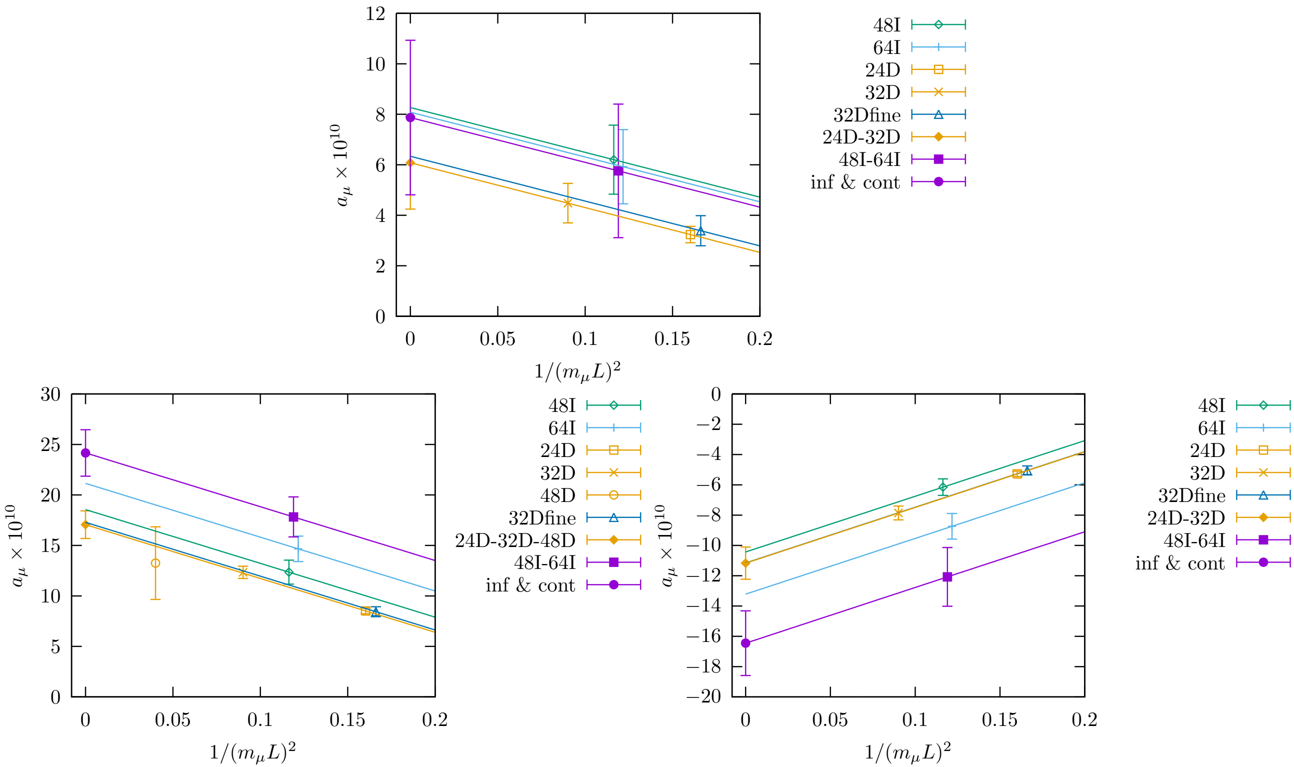
$\rightarrow$ Chiral extrapolation milder than expected (based on π^0-pole contribution)

$\rightarrow$ Sub-dominant diagrams smaller than the required precision

$\rightarrow$ Isospin-breaking corrections are not relevant here

Which errors are relevant

- Similar observation from RBC-UKQCD [Phys. Rev. Lett. 124, 132002 (2020)]



Summary of lattice results

- 2017 : RBC-UKQCD : first publication [Phys.Rev.Lett. 118 (2017)]

$$a_\mu^{\mathrm{HLbL}} = (53.5 \pm 13.5) \times 10^{-10}$$

→ errror is statistical only

→ no continuum extrapolation, no finite-size effect study

- 2020 : pdate that includes a systematic error estimate [Phys.Rev.Lett. 124 (2020)]

$$a_\mu^{\mathrm{HLbL}} = 72(40)_{\mathrm{stat}}(17)_{\mathrm{syst}} \times 10^{-10}$$

→ $\mathrm{QED_L}$ (QED in finite volume)

→ finite-size effects are large

- 2020 : Mainz : first publication that focus on systematics [Eur.Phys.J.C 80 (2020)]

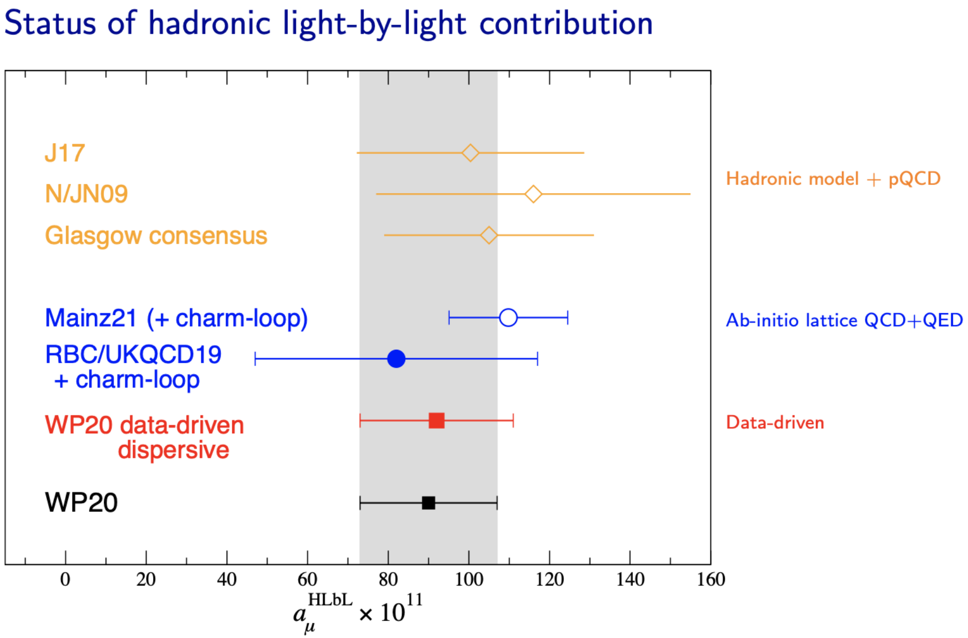
→ not yet at the physical point ($m_\pi \approx 400$ MeV)

→ finite-size correction + study of discretization effects

- 2021 : Mainz : recent update with a first complete calculation [2104.02632]

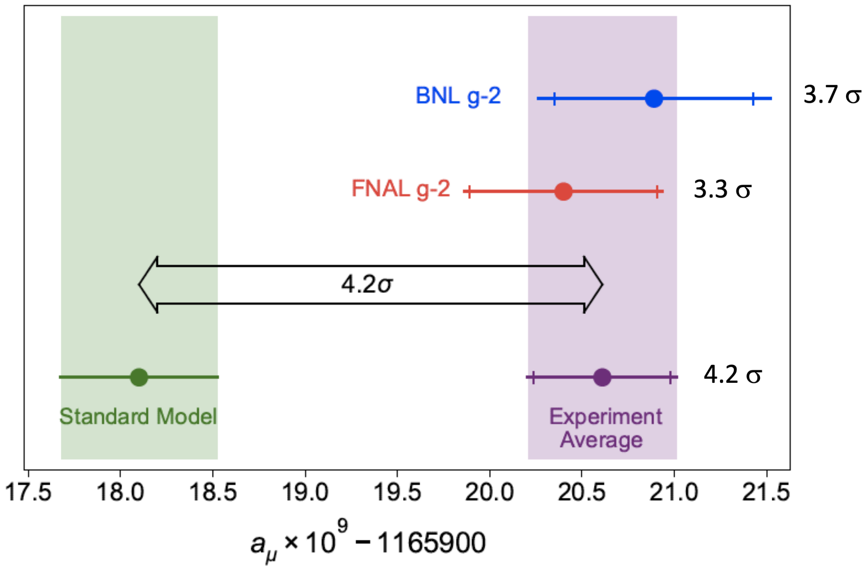
$$a_\mu^{\mathrm{HLbL}} = (107.4 \pm 11.3 \pm 9.2) \times 10^{-10}$$

Conclusion HLbL



- First lattice QCD results are now published
  - → In good agreement with the dispersive framework
  - → But systematic errors are sizeable, cross-checks would be welcome
- Lattice can also provide input to the dispersive framework
  - → pseudoscalar-pole contribution
- Close, but not yet at the target precision ($< 10\%$)

Conclusion



- First run $\approx 1/20$ of the expected total statistics
  - $\rightarrow$ are we close to NP discovery ?
  - $\rightarrow$ non-perturbative hadronic contributions dominate the error
  - $\rightarrow$ The recent BMW lattice result reduces the tension !
- Rapid progress on the lattice
  - $\rightarrow$ first sub-percent lattice calculation by BMW, but in tension with R-ratio estimates
  - $\rightarrow$ first complete calculation by Mainz : confirm the size of the HLbL contribution