Symmetric Mass Generation in Lattice Gauge Theory

Simon Catterall (Syracuse)

Fermions in Flatland ECT* July 5-9 2021

Plan

- Mass without symmetry breaking (SMG)
- ’t Hooft anomalies and SMG in continuum.
- Anomalies with Kähler–Dirac fermions on and off the lattice
- Numerical results: SMG for staggered fermions in 3 and 4D
- SMG using gauge interactions

Fermion masses

Typically fermions acquire mass by breaking symmetries:

- Explicitly eg Dirac mass breaks axial symmetry.
- Spontaneously eg. $<\overline{q}q>\neq 0$ in QCD.
- Via anomalies eg η'

Does this exhaust the possibilities ?

No!
Fermions masses can arise without breaking symmetries
provided **all** 't Hooft anomalies vanish

Symmetric Mass Generation (SMG)

Example of 't Hooft anomaly constraints

Consider $SU(5)$ gauge theory with global $G = U(1)$

L Weyl fields: $\chi_{\alpha\beta}(1)$ and $\psi^{\alpha}(-3)$

chiral gauge theory

No invariant mass term but gauge anomaly cancels $A(\bar{\mathbf{5}}) = -A(\mathbf{10})$

Imagine weakly gauging $G \to$ 't Hooft anomaly:
$\sum_a Q_a^3 = 5 \times (-3)^3 + 10 \times (1)^3 = -125$

Key observation

't Hooft anomalies must be same in IR and UV

Options in IR:

- Composite (gauge singlet) massless fermions
- Goldstone bosons from breaking G

Symmetric mass generation

One obvious color singlet composite fermion in I.R
$\overline{\xi} = \chi_{ab}\psi^a\psi^b$ – $U(1)$ charge -5
Precisely what is needed for the anomaly $(-5)^3 = -125$!
Can satisfy 't Hooft anomaly if $\overline{\xi}$ remains massless!.

SMG - a small twist

To make all states **massive** in I.R must cancel 't Hooft anomaly in U.V
Just add a singlet $\xi(+5)$!
Can now couple $\xi, \overline{\xi}$ with Dirac mass
$G\overline{\xi}\xi = G\chi_{ab}\psi^a\psi^b\xi \leftarrow \text{four fermion term}$
Preserves G!

Another observation

Notice sixteen Weyl fermions needed...

Discrete anomalies

Magic fermion numbers arise from cancellation of 't Hooft anomalies arising from discrete symmetries

D=1	Time reversal	8 Majorana
D=2	chiral fermion parity	8 Majorana/Weyl
D=3	Time reversal	16 Majorana
D=4	Spin-Z_4 symmetry	16 Majorana/Weyl

eg. four dimensions: $n_L - n_R = 0 \mod 16$

n_L, n_R number of L/R Weyl fermions

$\psi_L \to -i\psi_L \quad \psi_R \to +i\psi_R \quad \leftarrow$ spin-Z_4 symmetry

Arises from $G = U(1) \to Z_4$ due to four fermion term

How are these anomalies manifested ?

$$Z(A) = \int D\psi e^{iS(\psi,A)}$$
$A \equiv$ gauge or scalar field

Classical symmetry $A \to A'$.
Global anomaly when $Z(A') = e^{2\pi i\eta} Z(A)$

eg. Witten's global $SU(2)$ anomaly corresponds to $\eta = \frac{1}{2}$
Implies:
$$Z = \int DA\, Z(A) = 0$$

Hard to compute η in general

Dai-Freed method: calculate the dependence of Z for free fermions on background geometry

Consequences for SMG

Take home points ...

- SMG requires that **all** anomalies cancel in the U.V – gauge plus 't Hooft anomalies associated with global symmetries including **discrete**
- One method to see anomalies is to examine Z in a curved background. Cancelling anomalies amounts to demanding that Z be independent of background.
- Anomaly cancellation **necessary** condition for SMG. But dynamics is also important...
- Four fermion terms offer one path to SMG but structure of those terms is largely unknown for $D > 1$.

Why might a lattice theorist be interested in SMG ?

Chiral lattice gauge theories are hard ...

- Nielsen-Ninomiya forces one to start with (lattice) Dirac fermions
- Two main approaches:
  - Separate L and R modes in 5th dimension (DWF)
  - Separate L and R modes in p-space. Use strong interactions to give cut-off scale masses to (say) the R handed states (mirrors)

In both approaches ...

- Need to use four fermion operators to decouple L and R. Yields theories with discrete global symmetries.
- Cancelling all 't Hooft anomalies **necessary** condition to gap mirrors $\rightarrow$ fermions must come in multiples of 16 ..
- Explicit four fermion terms needed for SMG only known in low dimensions (Kitaev ..)

(Reduced) staggered fermions and SMG

$$S = \sum_{x,\mu} \chi^a(x)\eta_\mu(x) D^S_\mu \chi^a(x) - \frac{G^2}{8}\sum_x \epsilon_{abcd}\chi^a(x)\chi^b(x)\chi^c(x)\chi^d(x)$$

$\chi^a(x)$: 4 single component Grassmanns

$\eta_\mu(x) = (-1)^{\sum_{i=1}^{\mu-1} x_i}$ and $\xi_\mu(x) = (-1)^{\sum_{i=\mu+1}^{d} x_i}$

Symmetries

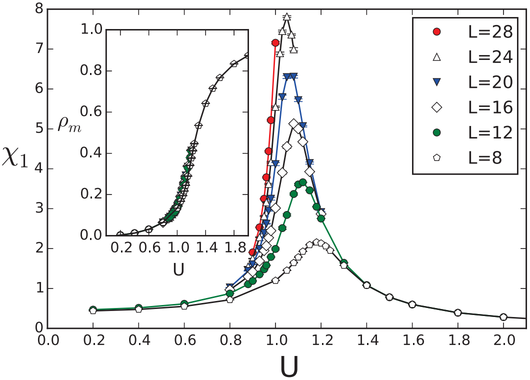
- $SU(4)$
- shift: $\chi(x) \to \xi_\mu(x)\chi(x+\mu)$
- Z_4: $\chi^a(x) \to i\epsilon(x)\chi^a(x)$

Symmetries prohibit all bilinear terms in $\Gamma(\chi)$

Phase diagram $D = 3$

- $G \to \infty < \chi^1\chi^2\chi^3\chi^4 > \neq 0$. Fermions massive. But condensate breaks no symmetries
- $G \to 0$. Massless fermions.

Must be at least 1 phase transition



S. Chandrasekharan PRD93 (2016) 081701

(RH)MC simulations

- Implement four fermion via Yukawa term with **real** scalar $G\sigma_+^{ab}\chi^a\chi^b$ i.e replace $SU(4) \to SO(4)$

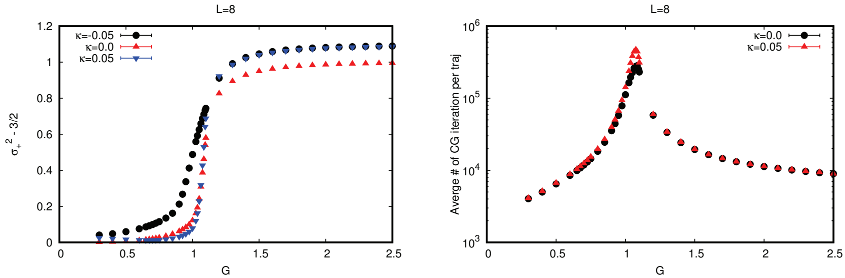
$$\sigma_+^{ab} = \frac{1}{2}\left(\sigma^{ab} + \frac{1}{2}\epsilon^{abcd}\sigma^{cd}\right)$$

  σ_+ adj rep of $SU_+(2) \in SO(4) = SU_+(2) \times SU_-(2)$
- Fermion operator $M = \eta.\Delta + G\sigma_+$ is antisymmetric and invariant under $SU_-(2)$. Hence eigenvalues come in quartets $\lambda, \lambda, -\lambda, -\lambda$.
- $\mathrm{Pf}(M) = \left(M^\dagger M\right)^{\frac{1}{4}}$ with no sign problem.
- In $D = 4$ need additional scalar kinetic term:

$$\kappa \sum_x \sigma_+(x) - \Box\sigma_+(x) \leftarrow \text{marginal in } D = 4$$

Phase diagram $D = 4$

(with David Schaich and Nouman Butt)



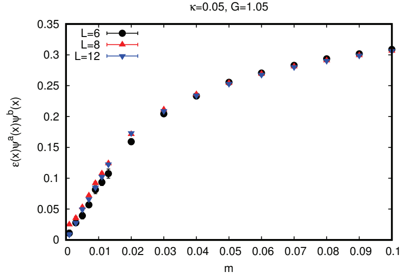
Notice: four fermi as Dirac-like term

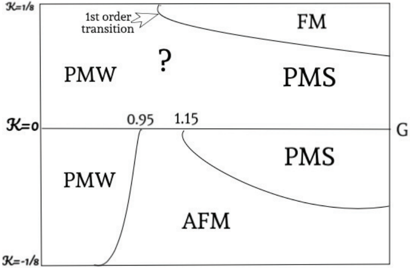
$$\chi^a \Omega^a$$

where $\Omega^a = \epsilon^{abcd} \chi^b \chi^c \chi^d$

No symmetry breaking $D = 4$

Important to check that fermion bilinears do not form spontaneously:





Evidence for direct, continuous phase transition between massless and massive phases with no symmetry breaking (S.C et al. PRD98 (2018) 114514)

Notice: continuum limit describes $4 \times 4 = 16$ Majorana fermions !

Lattice anomalies ?

Simulations indicate that fermions gapped in continuum - 16 Majorana fermions as required by continuum arguments
Is there any anomaly argument based directly in lattice for this ?
Yes !

A gravitational anomaly for staggered fermions

- Staggered fermions may be generalized to random lattices of arbitrary topology by replacing them by Kähler–Dirac fermions
- Massless free Kähler–Dirac fermions have an exact $U(1)$ symmetry – analog of $U_\epsilon(1) = e^{i\alpha\epsilon(x)}$ for staggered.
- This symmetry is **anomalous even for finite lattices**:

$$Z \to e^{2i\alpha\chi} Z \quad \text{where } \chi = \text{Euler character} = \sum_{i=0}^{d} N_i \, (-1)^i$$

Example of QM anomaly for finite number dof !

Kähler–Dirac fermions

Generalization of staggered fermions

- Staggered fermions best understood as a discretization of Kähler-Dirac (KD) fermions
- KD equation alternative to Dirac equation. In locally flat backgrounds describes $2^{D/2}$ degenerate Dirac spinors.

Kähler-Dirac equation

$$(K - m)\Omega = \left(d - d^{\dagger} - m\right)\Omega = 0$$

Note: $K^2 = -\Box$. Thus K alternative to $\gamma.D$.

Ω - collection of forms.

More on Kähler-Dirac

From Kähler-Dirac field $\Omega = (\omega_0, \omega_1, \ldots, \omega_D)$ form matrix

$$\Psi = \sum_{p=0}^{D} \omega_{n_1 \ldots n_p}(x) \gamma_1^{n_1} \gamma_2^{n_2} \cdots \gamma_p^{n_p}$$

Can show that the Kähler-Dirac equation:

$$(d - d^\dagger - m)\Omega = 0$$

in flat space equivalent to:

$$(\gamma^\mu \partial_\mu - m)\Psi = 0$$

In $D = 4$:

Four copies of Dirac equation where Dirac spinors correspond to columns of Ψ.

Lattice Kähler–Dirac fermions

- Approximate continuum by (oriented) triangulation T
- Place p-forms on p-simplices $\Omega \to \Omega_L$
- Replace $d, d^\dagger$ by $\delta, \overline{\delta}$ where

$$\delta(a_0 \ldots a_p) = \sum_{i=0}^{p} (-1)^i (a_0 \ldots \overline{a_k} \ldots a_p)$$

- Lattice Kähler–Dirac equation:

$$(\delta - \overline{\delta} - m)\Omega_L = 0$$

No fermion doubling! Zero mode structure reproduced on lattice.

Yields staggered operator on hypercubic lattices with topology of torus.
Valid for any (oriented) random triangulation of any topology.

An anomalous symmetry for Kähler–Dirac

Linear operator $\Gamma : \omega_p \to (-1)^p \omega_p \quad \text{with } \{\Gamma, K\} = 0$
Generates exact $U(1)$ symmetry of massless equation:

$$\Omega \to e^{i\alpha\Gamma}\Omega$$

$$\overline{\Omega} \to \overline{\Omega} e^{i\alpha\Gamma}$$

$$\Delta S_{\mathrm{KD}} = 0$$

Measure ?

$$\prod_p d\omega_p \to e^{2iN_0\alpha} e^{-2iN_1\alpha} .. e^{2i(-1)^d N_d\alpha} \prod_p d\omega_p = e^{2i\chi\alpha} \prod_p d\omega_p$$

Partition function transforms by a phase which depends only on topology of lattice !

S.C et al JHEP 2010 (2018) 013.

Cancelling the anomaly

- Consider the sphere: $\chi(S^{2n}) = 2$. Phase=$e^{4i\alpha}$. Gravitational anomaly breaks $U(1) \to Z_4$.
- Cancelled if we have $4n$ flavors of Kähler–Dirac (staggered) field. Equal to $16n$ Dirac in flat space limit.
- SMG is then possible using four fermion terms built from 4 flavors of Kähler–Dirac /staggered fermion.

But the simulations (and continuum anomaly cancellation) suggest minimum number of Dirac is 8 (i.e 16 Majorana).
Can we find an extension of this argument that gives this result ?

DWF setup

Consider $\mathcal{M} = S^d \times [0, L_{d+1}]$ with $\chi_{\mathcal{M}} = 0$

$$D_{d+1} = D_d^{KD} + \Gamma \frac{\partial}{\partial x_{d+1}} - M$$

Localizes **reduced** Kähler–Dirac fermions $\Psi_\pm$ on boundaries.

$$\Gamma \Psi_\pm = \pm \Psi_\pm \quad \Psi_\pm = \frac{1}{2}(1 \pm \Gamma)\Psi$$

These boundary modes are gapped $m \sim e^{-ML_{d+1}}$ and yield 4 Majorana per bulk Kähler–Dirac field as $L_{d+1} \to \infty$

To cancel Z_4 anomaly on each boundary **separately** need 4 flavors of bulk Kähler–Dirac field.
Each boundary then carries 16 Majorana fermions

Summary and what's next ...

Generalizations of staggered fermions have gravitational anomalies that can be computed exactly in lattice
Cancellation of **all** 't Hooft anomalies to achieve SMG **requires** that we cancel off these anomalies
Get results consistent with continuum arguments

- Can we use Kähler–Dirac /staggered fermions +SMG to formulate a chiral lattice gauge theory ? DWF construction ? .. or directly in bulk – see PRD104 (2021) 014503.
- Can we replace irrelevant four fermion operators by renormalizable gauge interactions ?

Rewriting the $SO(4)$ model

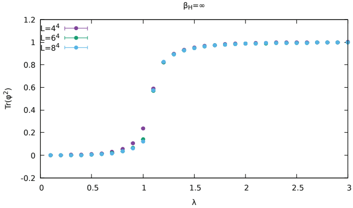
(with Goksu Can Toga and Nouman Butt) Consider

$$S = \sum_{x,\mu} \eta_\mu(x) \mathrm{Tr}\left(\psi^\dagger \Delta_\mu \psi\right) - \lambda^2 \mathrm{Tr}\left(\psi^\dagger \psi \psi^\dagger \psi\right)$$

where $\psi \to G\psi H^\dagger$ under $SU_-(2) \times SU_+(2)$.

Impose reality condition: $\psi^\dagger = \sigma_2 \psi^T \sigma_2$
Satisfied if $\psi = \sum_{A=1}^4 \sigma_A \chi_A$ with $\sigma_A = (I, i\sigma_i)$ and χ real
$\rightarrow$ action and symmetries identical to $SO(4)$ model eg. $\mathrm{Tr}(\psi^\dagger \psi) = 0$

Quick check:



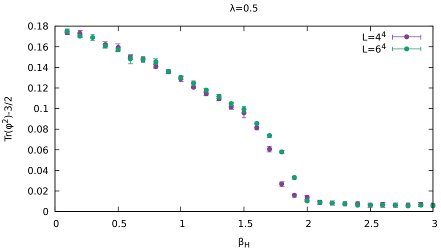
Gauging the $SU_+(2)$

The idea ...

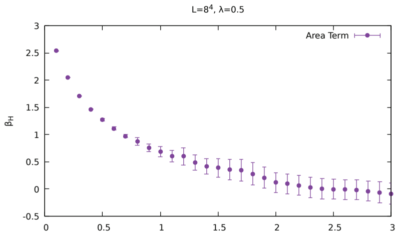
Use the $SU_+(2)$ gauge interaction to generate an effective strong four fermi term in the I.R even as $\lambda \to 0$.

$$S_{\mathrm{kin}} = \sum_{x,\mu} \frac{1}{2} \eta_\mu(x) \mathrm{Tr} \left[\psi^\dagger(x) \left(\psi(x+\mu) V_\mu^\dagger(x) - \psi(x-\mu) V_\mu(x-\mu) \right) \right]$$

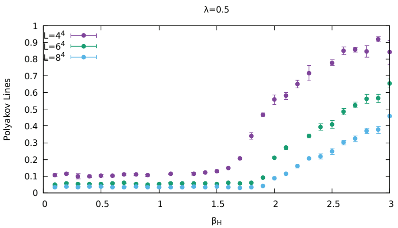
Four fermion condensate vs β_H



Phases



String tension



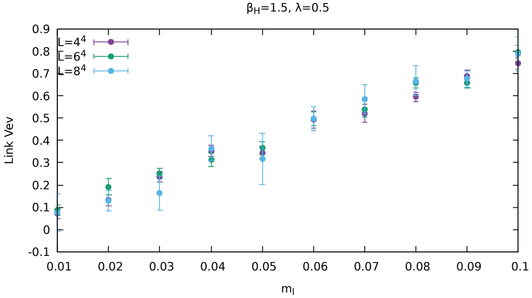
Polyakov line

- Four fermion condensate forms when $SU_+(2)$ confines (even for small Yukawa).
- Theory admits exact Z_2 center symmetry: $V_\mu(x) \to -V_\mu(x)$ and $\psi(x) \to \epsilon(x)\psi(x)$ with $\epsilon(x) = (-1)^{\sum_i x_i}$. $|P|$ good order parameter for deconfinement. $|P| \to 0$ as $V \to \infty$ suggests no phase transition.

No symmetry breaking

Add explicit shift symmetry breaking link bilinear

$$O_1 = m_1 \sum_{x,\mu} \epsilon(x)\xi_\mu(x)\chi(x)\chi(x+\mu)$$



Summary

- SMG allows fermions to acquire masses without breaking symmetries and leads to new phase transitions for strongly interacting fermions.
- $D = 3, 4$ there are well studied lattice examples which use staggered/Kähler–Dirac fermions
- Necessary condition for SMG is cancellation of all 't Hooft anomalies. Leads to magic numbers of fermions (8 Weyl in 2d, 16 Majorana in 3d, 16 Weyl 4d). These numbers can be gotten from **gravitational** anomalies of staggered fermions.
- SMG may offer new mechanism to gap out mirrors in efforts to construct chiral lattice gauge theories
- Standard method uses four fermion interactions. May be possible to circumvent irrelevant ops using gauge interactions.

Backups