Anomalous Dimensions from Massive Vacuum Diagrams

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Relativistic Fermions in Flatland: theory and application
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Outline

Introduction

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  - strongly bound in 3d
  - perturbatively only accessible in 2d or 4d
- Calculate anomalous dimensions of the ren. group equations in 2d and/or 4d and extrapolate to 3d
- Calculation of anom dim very well established field in particle physics
  $\hookrightarrow$ apply these methods here

Models

- Gross-Neveu-Yukawa [Zerf,Mihaila,PM,Herbut,Scherer '17]
  - chiral Ising model
  - chiral XY model
  - chiral Heisenberg model
- QED_3-Gross-Neveu-Yukawa [Zerf,PM,Boyack,Maciejko '18]
- Néel algebraic spin liquid [Zerf,Boyack,PM,Gracey,Maciejko '19]
- Abelian Higgs model [Ihrig,Zerf,PM,Herbut,Scherer '19]
- lattice quantum electrodynamics [Zerf,Boyack,PM,Gracey,Maciejko '20]

In short: Models with interactions between scalars, fermions and photons

Outline

Calculation of anomalous dimensions

In general, one can follow the following recipe

- Start with a Lagrange density describing the model
- Derive the Feynman rules
- Calculate the relevant L-loop N-point functions
- Extract the renormalization factors (or anom dims) from the UV poles

Extraction of UV poles

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- For simplicity, make all lines massive to avoid infra-red problems
  $\hookrightarrow$ infra-red rearrangement [Misiak,Münz '94, Chetyrkin,Misiak,Münz '98, Larin,v.Ritbergen,Vermaseren '97]
- See also five-loop QCD anomalous dimensions [Luthe,Maier,PM,Schröder '17]

Infra-red rearrangement

Based on the exact decomposition
(k: loop momentum, q external momentum)

$$\frac{1}{(k+q)^2} = \frac{1}{k^2 - M^2} - \frac{q^2 + 2kq + M^2}{(k^2 - M^2)(k+q)^2}$$

Note:

- the first term on the rhs is infra-red finite
- the second term can still lead to IR divergences but the UV degree of divergence is reduced

After subtraction of sub-divergences by explicit counter terms the UV-finite but IR-divergent remainder can be dropped.

$\hookrightarrow$ need to use explicit counter terms (no multiplicative renormalization)
$\hookrightarrow$ requires the introduction of a counter term for the auxiliary mass M.

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- evaluate master integrals

Integration by parts

- individual calculation of all appearing $\mathcal{O}(10^3)$ - $\mathcal{O}(10^7)$ Feynman integrals is not feasible
- the number of integrals can be greatly reduced by applying the so-called integration-by-parts identities [Chetyrkin,Tkachov]
- Integration-by-parts identities are based on the property

$$0 = \int \mathrm{d}^d k \frac{\partial}{\partial k_i^\mu} \frac{1}{D_1^{k_1} \cdots D_n^{k_n}}$$

  which being the integral of a total derivative evaluates to a surface term and can be shown to vanish.
- allows to write all appearing integrals J_i as linear combination of $\mathcal{O}(10-100)$ basis (master) integrals M_j

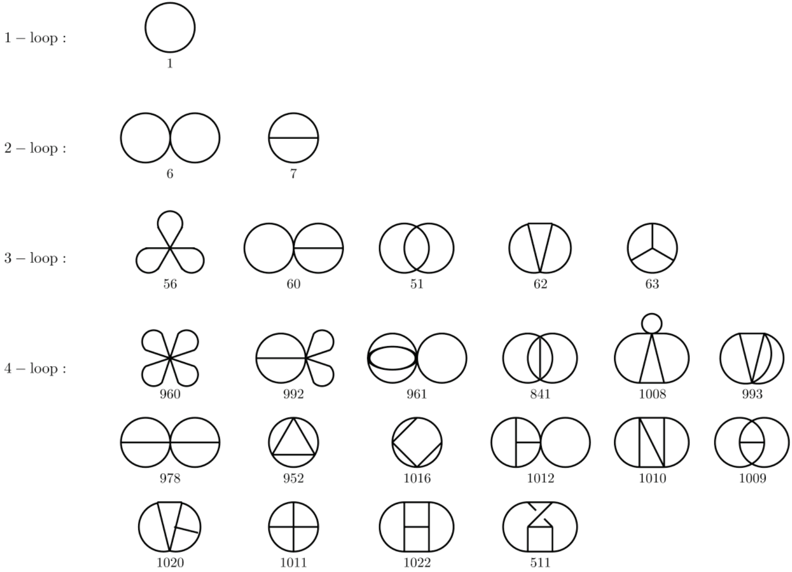
$$J_i = \sum_j C_{ij}(d) M_j$$

Integration by parts

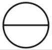
Integration-parts-relations can either be used by

- constructing a set of symbolic relations reducing the number of propagators
  `LiteRed` [Lee]
- explicitly applying the relations to a set of integrals and solving the resulting system of linear equations (Laporta's algorithm)
  (`Air`) [Anastasiou, Lazopoulos]
  `FIRE` [Smirnov]
  (`Reduze`) [v. Manteuffel, (Studerus)]
  `KIRA` [Maierhöfer,Usowitch,Uwer]

Massive Tadpoles – up to 4 loop



Master integrals – 2 loop



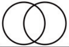
$$\frac{}{J^2} = -\frac{3(d-2)}{4(d-3)}\left\{ {}_2F_1\left(\frac{4-d}{2},1;\frac{5-d}{2};\frac{3}{4}\right) - 3^{\frac{d-5}{2}}\frac{2\pi\Gamma(5-d)}{\Gamma(\frac{4-d}{2})\Gamma(\frac{6-d}{2})}\right\}$$

$$= -\frac{3(d-2)}{4(d-3)}\left\{1 - 3^{\frac{d-3}{2}}(d-4)\int_0^{\frac{\pi}{3}} d\tau(2\sin(\tau))^{4-d} - 3^{\frac{d-5}{2}}\frac{2\pi\Gamma(5-d)}{\Gamma(\frac{4-d}{2})\Gamma(\frac{6-d}{2})}\right\}$$

$$\overset{d=n-2\epsilon}{=} -\frac{3(n-2-2\epsilon)}{4(n-3-2\epsilon)}\left\{1+3^{-\epsilon}\frac{\frac{n-4}{2}-\epsilon}{3^{\frac{3-n}{2}}}\sum_{j=0}^{\infty}\frac{(2\epsilon)^j}{j!}\mathrm{Ls}_{j+1}^{(4-n)} - 3^{-\epsilon}\frac{3^{\frac{n-5}{2}}2\pi\Gamma(5-n+2\epsilon)}{\Gamma(\frac{4-n}{2}+\epsilon)\Gamma(\frac{6-n}{2}+\epsilon)}\right\}$$

[Schröder,Vuorinen '05]

Master integrals – 3+ loop



$$\frac{\quad}{J^3} \overset{d=4-2\epsilon}{=} -2-\frac{5}{3}\epsilon-\frac{1}{2}\epsilon^2+\frac{103}{12}\epsilon^3+\frac{7}{24}(163-128\zeta_3)\epsilon^4 + \left(\frac{9055}{48}+\frac{136\pi^4}{45}+\frac{32}{3}\ln^2 2(\pi^2-\ln^2 2)-168\zeta_3-256a_4\right)\epsilon^5 + \left(\frac{63517}{96}+\frac{16}{5}\ln^4 2(4\ln 2-15)-\frac{16}{3}\pi^2\ln^2 2(4\ln 2-9)-\frac{68}{15}\pi^4(4\ln 2-3) - \frac{1876}{3}\zeta_3+1240\zeta_5-1152a_4-1536a_5\right)\epsilon^6+\mathcal{O}\left(\epsilon^7\right)$$

$$\frac{\quad}{J^3} \overset{d=4-2\epsilon}{=} -2\zeta_3\epsilon^2+\left(\frac{17\pi^4}{90}+\frac{2}{3}\ln^2 2(\pi^2-\ln^2 2)+9\mathrm{Ls}_2^2-16a_4\right)\epsilon^3+\mathcal{O}\left(\epsilon^4\right)$$

$$\frac{\quad}{J^3} \overset{d=4-2\epsilon}{=} 1+\frac{8}{3}\epsilon+\left(\frac{25}{3}-6\sqrt{3}\mathrm{Ls}_2\right)\epsilon^2+\left(\frac{76}{3}-6\zeta_3+\sqrt{3}\left(-\frac{\pi^3}{3}+6(\ln 3-2)\mathrm{Ls}_2-6\mathrm{Ls}_3\right)\right)\epsilon^3$$

[Schröder,Vuorinen ’05]

How to calculate the master integrals? Factorial Series

- The idea of the method goes back to Laporta who suggested the to calculate Feynman integrals in form of of a factorial series.
- Take an integral and raise the power of one propagator to the power x e.g. $I(1,1,1) \to I(x) = I(x,1,1)$
- Using IBP relations on can obtain a difference equation for the integral

$$\sum_{k+0}^{R} p_k(x) I(x+k) = \sum_i \sum_{k=0}^{R_i} p_{ik}(x) J_i(x+k)$$

  where J_i are integrals of simpler sectors
- Make an ansatz for $I(x)$ in terms of a factorial series (N.B. not the most general one)

$$I(x) = \sum_{s=0}^{\infty} \frac{\Gamma(x+1)}{\Gamma(x+d/2+s+1)} a_s$$

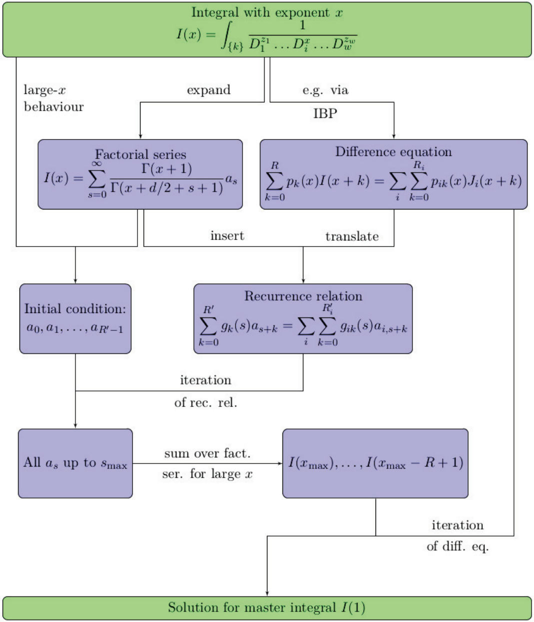
Factorial Series cont'd

- Inserting the ansatz into the difference equation results in a recurrence relation for a_s

$$\sum_{k=0}^{R'} g_k(s) a_{s+k} = \sum_i \sum_{k=0}^{R'_i} g_{ik}(s) a_{i,s+k}$$

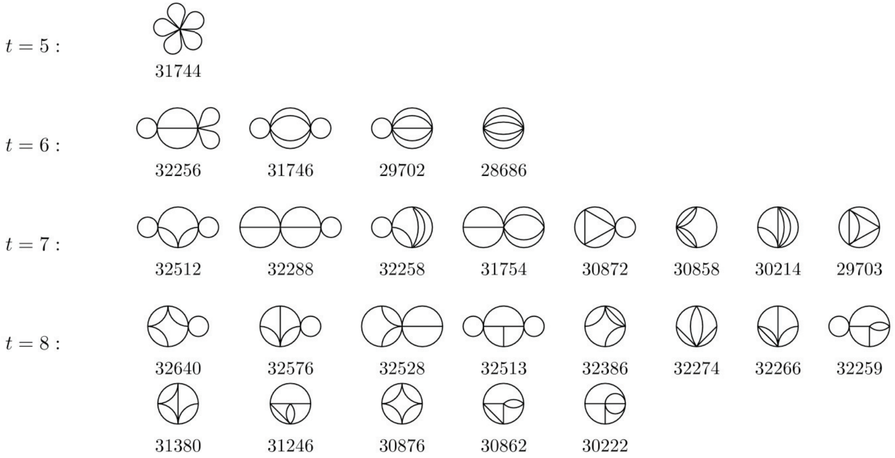
- given the initial values $a_0, a_1, \ldots$ are known, an arbitrary number of values for a_n can be calculated.
- using the obtained values for a_n $I(x)$ can be calculated

$$I(x) = \sum_{s=0}^{\infty} \frac{\Gamma(x+1)}{\Gamma(x+d/2+s+1)} a_s$$
$$= \frac{\Gamma(x+1)}{\Gamma(x+d/2+1)} \left(a_0 + \frac{a_1}{(x+d/2+1)} + \frac{a_2}{(x+d/2+1)(x+d/2+2)} + \cdots \right)$$



Integral with exponent x
$I(x) = \int_{\{k\}} \frac{1}{D_1^{z_1} \ldots D_i^{x} \ldots D_w^{z_w}}$
large-x behaviour
expand
e.g. via IBP
Factorial series
$I(x) = \sum_{s=0}^{\infty} \frac{\Gamma(x+1)}{\Gamma(x+d/2+s+1)} a_s$
Difference equation
$\sum_{k=0}^{R} p_k(x) I(x+k) = \sum_i \sum_{k=0}^{R_i} p_{ik}(x) J_i(x+k)$
insert
translate
Initial condition:
$a_0, a_1, \ldots, a_{R'-1}$
Recurrence relation
$\sum_{k=0}^{R'} g_k(s) a_{s+k} = \sum_i \sum_{k=0}^{R'_i} g_{ik}(s) a_{i,s+k}$
iteration of rec. rel.
All a_s up to $s_{\max}$
sum over fact. ser. for large x
$I(x_{\max}), \ldots, I(x_{\max} - R + 1)$
iteration of diff. eq.
Solution for master integral $I(1)$

Massive Tadpoles – 5 loop



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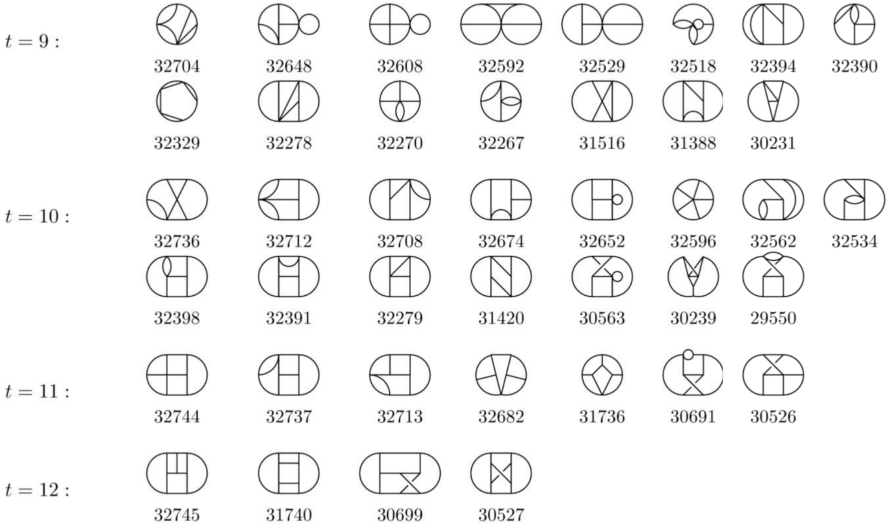
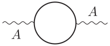
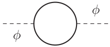
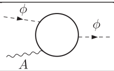
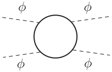


[Luthe '15]

Some statistics

Loops	1	2	3	4
	2	20	370	9, 291
	4	27	459	11, 332
	5	107	3, 078	106, 501
	32	1, 042	40, 164	1, 735, 706

from "Abelian Higgs model at four loops, fixed-point collision and deconfined criticality"[Ihrig,Zerf,PM,Herbut,Scherer '19]

Outline

Example: Abelian Higgs model

The Lagrangian is given by

$$\mathcal{L} = |D_\mu \phi|^2 + \frac{1}{4} F_{\mu\nu}^2 + r|\phi|^2 + \lambda(|\phi|^2)^2$$

with scalar fields $\phi = (\phi_1, ..., \phi_n)$.

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Go to the *renormalized* Lagrangian

$$\mathcal{L}' = Z_\phi|D_\mu\phi|^2 + Z_{\phi^2} r\mu^2|\phi|^2 + Z_{\phi^4}\lambda\mu^\epsilon(|\phi|^2)^2 + \frac{Z_A}{4}F_{\mu\nu}^2 - \frac{1}{2\xi}(\partial_\mu A_\mu)^2 .$$

with renomalized couplings

$$\alpha = e_0^2\mu^{-\epsilon}Z_A , \quad \lambda = \lambda_0\mu^{-\epsilon}Z_\phi^2 Z_{\phi^4}^{-1}$$

Example: Abelian Higgs model

β functions given by

$$\beta_i = \frac{\mathrm{d} g_i}{\mathrm{d} \ln b} = \epsilon g_i + \sum_k \beta_i^{(k\ell)}$$

$$\beta_\alpha^{(1\ell)} = -\frac{n}{3}\alpha^2 \,, \qquad \beta_\alpha^{(2\ell)} = -2n\alpha^3 \,,$$

$$\beta_\alpha^{(3\ell)} = \left(\tfrac{49}{72}n^2 - \tfrac{29}{8}n\right)\alpha^4 - \tfrac{n^2+n}{2}\alpha^3\lambda + \tfrac{n^2+n}{8}\alpha^2\lambda^2 \,.$$

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$$\beta_\lambda^{(1\ell)} = -6\alpha^2 + 6\alpha\lambda - (n+4)\lambda^2\,,$$

$$\beta_\lambda^{(2\ell)} = \left(\tfrac{14}{3}n + 30\right)\alpha^3 - \left(\tfrac{71}{6}n + \tfrac{29}{2}\right)\alpha^2\lambda - (4n+10)\alpha\lambda^2 + \left(\tfrac{9}{2}n + \tfrac{21}{2}\right)\lambda^3\,,$$

$$\begin{aligned}\beta_\lambda^{(3\ell)} = &\left(-\tfrac{7}{18}n^2 + \left[\tfrac{203}{8} - 27\zeta_3\right]n + \tfrac{367}{8} - 45\zeta_3\right)\alpha^4 \\ &+ \left(-\tfrac{5}{216}n^2 + \left[18\zeta_3 - \tfrac{989}{8}\right]n - \tfrac{889}{4} - 54\zeta_3\right)\alpha^3\lambda \\ &+ \left(\tfrac{43}{16}n^2 + \left[18\zeta_3 + \tfrac{1749}{16}\right]n + \tfrac{1093}{8} + 126\zeta_3\right)\alpha^2\lambda^2 \\ &+ \left(-\tfrac{33}{16}n^2 + \left[-15\zeta_3 - \tfrac{461}{16}\right]n - \tfrac{185}{4} - 33\zeta_3\right)\lambda^4 \\ &+ \left(\left[\tfrac{25}{2} - 6\zeta_3\right]n + \tfrac{29}{2} + 6\zeta_3\right)\alpha\lambda^3\end{aligned}$$

Fixed points

Looking for fixed points: $\beta_i(g_i^*) = 0 \, \forall \, i$

At one loop:

$$\alpha^* = 3\epsilon/n + \mathcal{O}(\epsilon^2)$$

$$\lambda_\pm^* = \frac{3(18 + n \pm \sqrt{n^2 - 180n - 540})}{2n(n+4)}\epsilon + \mathcal{O}(\epsilon^2)$$

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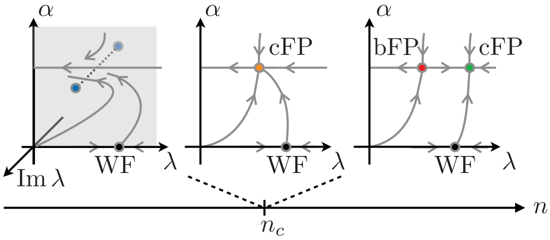
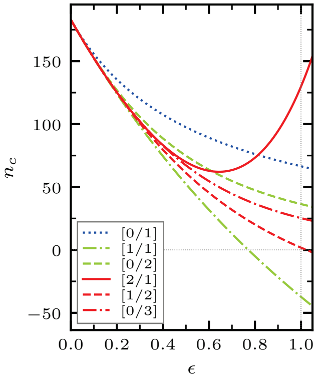


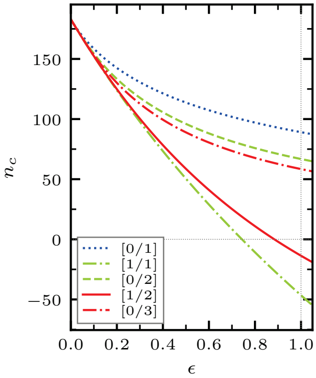
Fixed-point collision

Critical number n_c, extrapolation to $d = 3$, i.e. $\epsilon = 1$

Padé



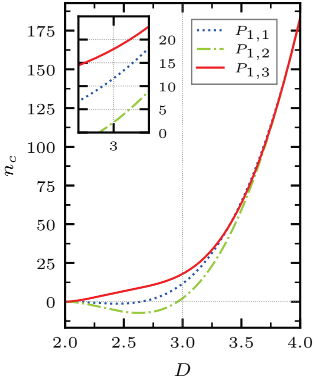
Padé-Borel

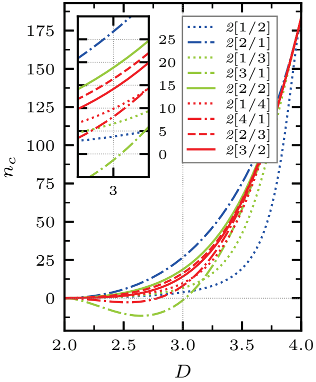


Fixed-point collision

Critical number n_c, interpolate to $d = 3$ using information at $d = 2$

Polynomial | 2-sided Padé





Outline

Conclusions

- We calculated the beta functions and anomalous dimensions for several models up to four-loop order
- From this the behaviour near critical points can be obtained
- Four-loop is rather straight forward
- Five-loop is still a challange and needs a good motivation