Helicity formalism revisited for polarised particle decays

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Phase definition of spin states

- In QM, spin operators $\hat{\boldsymbol{S}} = (\hat{S}_x, \hat{S}_y, \hat{S}_z)$ define a right-handed spin coordinate system (x, y, z).
- $|s, m\rangle$ defined as $\hat{S}^2$ and $\hat{S}_z$ simultaneous eigenstates
- Orthogonal spin components $\hat{S}_x, \hat{S}_y$ determine phase definition: for a z-axis rotation:

$$R_z(\alpha)\,|s, m\rangle = e^{-i\alpha\hat{S}_z}\,|s, m\rangle = e^{-i\alpha m}\,|s, m\rangle\,, \tag{1}$$

→ *x-y axis must be correctly defined when phases matter (interference effect)*

Example: phase definition of helicity states

- Helicity coordinate system for particle with momentum $\boldsymbol{p}$ obtained via an Euler rotation $R(\phi, \theta, \psi)$:

$$\cos\theta = (p_z/|p|)\,, \\ \phi = \mathrm{atan2}\,(p_y, p_x)\,, \tag{2}$$

- ψ angle, associated to a rotation around $\boldsymbol{p}$, determines the choice of the orthogonal spin coordinate axes
- ψ angle choice purely conventional (usually $\psi = 0$, $\psi = -\phi$)

$\rightarrow$ ψ *angle choice defines helicity state phase*

Phase issue for fermion states

- Spin states transform differently from spin coordinate systems
- $SU(2)$ spin s vs $SO(3)$ vector representations
- For fermions, spin state sign is not defined by coordinate system: under 2π rotations spin state change sign

$$R_z(2\pi)\,|s,m\rangle = e^{-2i\pi\hat{S}_z}\,|s,m\rangle = (-1)^{2s}\,|s,m\rangle\,. \tag{3}$$

→ *Fermion states defined by different rotation sequences leading to the same coordinate system may differ by a sign*

→ *To be took into account when matching fermion states for different decay chains*

Helicity formalism

- Helicity formalism (Jakob, Wick, Annals Phys. 7 (1959) 404; Richman's guide)

$$|p, \theta, \phi, \lambda_1, \lambda_2\rangle = \sum_{J,M} \sqrt{\frac{2J+1}{4\pi}} D^J_{M,\lambda_1-\lambda_2}(\phi, \theta, 0) \times |p, J, M, \lambda_1, \lambda_2\rangle \,, \tag{4}$$

- Only one *D* rotation matrix but two different helicity systems involved
- Particle 1 helicity coordinate system defined by $R(\phi, \theta, 0)$ Euler rotation
- Particle 2 helicity coordinate system defined by mirroring the *z* axis: orthogonal axes undefined

$\rightarrow$ *Phase of particle 2 spin states undefined*

Helicity formalism revisited

- Most trivial solution: use particle 1 helicity system to express particle 2 spin states
- Particle 2 states become opposite helicity states, with opposite helicity defined as

$$\hat{\bar{\lambda}} = -\hat{\boldsymbol{S}} \cdot \frac{\boldsymbol{p}}{p}, \tag{5}$$

- Eq. (4) become

$$\begin{aligned} \left|p, \theta, \phi, \lambda_1, \bar{\lambda}_2\right\rangle = \sum_{J,M} \sqrt{\frac{2J+1}{4\pi}} D^{J}_{M,\lambda_1+\bar{\lambda}_2}(\phi, \theta, 0) \\ \times \left|p, J, M, \lambda_1, \bar{\lambda}_2\right\rangle . \end{aligned} \tag{6}$$

→ *Apparently identical to Eq.* (4)*, but ensuring a definition of the particle 2 spin state phase*

Helicity amplitudes for 3-body decay

- $A \to R(\to 1,2), 3$ decay chain amplitude

$$\mathcal{A}^{A\to R,3\to 1,2,3}_{m_A,\lambda_1^R,\bar\lambda_2^R,\bar\lambda_3^R}(\Omega) = \left\langle \{\boldsymbol{p}_i\}, \lambda_1^R, \bar\lambda_2^R, \bar\lambda_3^R \,\middle|\, \hat{T} \,\middle|\, s_A, m_A \right\rangle$$
$$= \sum_{\lambda_R} \mathcal{H}^{R\to 1,2}_{\lambda_1^R,\bar\lambda_2^R} D^{*s_R}_{\lambda_R,\lambda_1^R+\bar\lambda_2^R}(\phi_1^R, \theta_1^R, 0)$$
$$\times \mathcal{H}^{A\to R,3}_{\lambda_R,\bar\lambda_3^R} D^{*s_A}_{m_A,\lambda_R+\bar\lambda_3^R}(\phi_R, \theta_R, 0). \tag{7}$$

- $A \to S(\to 1,3), 2$ decay chain amplitude

$$\mathcal{A}^{A\to S,2\to 1,2,3}_{m_A,\lambda_1^S,\bar\lambda_2^S,\bar\lambda_3^S}(\Omega) = \sum_{\lambda_S} \mathcal{H}^{S\to 1,3}_{\lambda_1^S,\bar\lambda_3^S} D^{*s_S}_{\lambda_S,\lambda_1^S+\bar\lambda_3^S}(\phi_1^S, \theta_1^S, 0)$$
$$\times \mathcal{H}^{A\to S,2}_{\lambda_S,\bar\lambda_2^S} D^{*s_A}_{m_A,\lambda_S+\bar\lambda_2^S}(\phi_S, \theta_S, 0). \tag{8}$$

Matching of final particle spin states

- Different final particle spin state definitions: need to define common spin states for each decay chain
- Spin states must have exactly the same definition:

$\rightarrow$ *Spin coordinate systems must coincide, for z and x, y axes*

$\rightarrow$ *For fermions, the sign issue must be took into account*

- First condition partly addressed in the literature (I did not find a general prescription for generic ≥ 4 body decays)
- Second condition never addressed in the literature

The rotation sequence in the helicity formalism

- The description of a particle decay in the helicity formalism can be seen as a binary tree of spin states
  - Nodes are particle states, edges are spin rotations
- The transition between mother and daughter particle spins, defined in different rest frames, is allowed by the use of helicity states
  - Boost effects are transformed into spin rotations
- The ancestor particle spin state is of arbitrary choice
  - All the daughter spin states are defined relatively to their mother states

$\rightarrow$ *Different decay chains are characterised by different rotation sequences*

Matching of final particle spin states

- To define common spin states for each decay chain, for fermions, one needs:
  - That each rotation sequence, for each final particle, leads from the ancestor spin coordinate system to the same final particle one
  - That different rotation sequences do not differ by relative 2π rotations
- How to define the rotation sequences accordingly?

Matching of final particle spin states

- One possible solution: rotate the final particle spin states inverting the rotation sequence step-by-step
- So that by construction:

- *The rotation sequence comes back to its starting point*
- *No loops are allowed in the rotation sequence* $\rightarrow$ *no relative* 2π *rotations can be introduced*

- Applicable to any multi-body decay topology

Helicity amplitudes for 3-body decay

- Final particle spin states for a 3-body decay are rotated as

$$\begin{aligned} |s_1, m_1\rangle &= \sum_{m_1'} D^{s_1}_{m_1', m_1}(0, -\theta_R, -\phi_R)\,|s_1, m_1'\rangle \\ &= \sum_{m_1'} D^{s_1}_{m_1', m_1}(0, -\theta_R, -\phi_R) \\ &\times \sum_{\lambda_1^R} D^{s_1}_{\lambda_1^R, m_1'}(0, -\theta_1^R, -\phi_1^R)\,|s_1, \lambda_1^R\rangle\,. \end{aligned} \tag{9}$$

Matching of final particle spin states

- The $A \to R(\to 1,2),3$ decay chain amplitude becomes

$$\begin{aligned}
\mathcal{A}^{A\to R,3\to 1,2,3}_{m_A,m_1,m_2,m_3}(\Omega) = \sum_{m_1',\lambda_1^R} & D^{*s_1}_{\lambda_1^R,m_1'}(0,-\theta_1^R,-\phi_1^R) \\
& \times D^{*s_1}_{m_1',m_1}(0,-\theta_R,-\phi_R) \\
\times \sum_{m_2',\lambda_2^R} & D^{*s_2}_{\lambda_2^R,m_2'}(0,-\theta_1^R,-\phi_1^R) \\
& \times D^{*s_2}_{m_2',m_2}(0,-\theta_R,-\phi_R) \\
\times \sum_{\lambda_3^R} & D^{*s_3}_{\lambda_3^R,m_3}(0,-\theta_R,-\phi_R) \\
& \times \mathcal{A}^{A\to R,3\to 1,2,3}_{m_A,\lambda_1^R,\bar{\lambda}_2^R,\bar{\lambda}_3^R}(\Omega).
\end{aligned} \tag{10}$$

Polarised decay rate

- The polarised decay rate can be obtained introducing initial and final particle spin density matrices

$$
\begin{aligned}
p(\hat{\rho}^A, \hat{\rho}^{\{i\}}; \Omega) &= \mathrm{tr}\left[\hat{\rho}^A \hat{T} \hat{\rho}^{\{i\}} \hat{T}^\dagger\right] \\
&= \sum_{m_A, m'_A} \sum_{\{m_i\},\{m'_i\}} \hat{\rho}^A_{m_A, m'_A} \hat{\rho}^{\{i\}}_{\{m_i\},\{m'_i\}} \\
&\quad \times \mathcal{A}_{m_A,\{m_i\}}(\Omega) \mathcal{A}^*_{m'_A,\{m'_i\}}(\Omega) \qquad (11)
\end{aligned}
$$

Effects of an unphysical phase in spin state definition

Summary of Sec.6 of arXiv:1911.10025

- Studied the effect of an unphysical phase due to improper spin state definition

$\rightarrow$ *It messes up interference terms between different decay chains*

- Considering fits to experimental data:

$\rightarrow$ *For zero or z-axis polarisation components, the unphysical phase might be absorbed into the helicity couplings*

  - This will depend on the specific decay under consideration

$\rightarrow$ *For generic polarisation, it prevents the simultaneous extraction of helicity couplings and polarisation components*

Example: amplitude model for $\Lambda_c^+ \to pK^-\pi^+$

- As an example let's consider the $\Lambda_c^+ \to pK^-\pi^+$ decay
- Two possibilities to write its amplitude model including Λ_c^+ polarisation vector:

- Starting directly from a given Λ_c^+ polarisation frame
  - Amplitude written as in the previous slides

- Decomposing invariant mass and decay plane orientation angles, via Dalitz plot decomposition proposed in arXiv:1910.04566
  - Now will focus on this equivalent but simpler approach

Dalitz plot decomposition for $\Lambda_c^+ \to pK^-\pi^+$

- Separated invariant mass dependence from decay plane orientation angles ϕ_p, θ_p, χ, the Euler angles rotating the initial Λ_c^+ spin coordinate system to the Λ_c^+ decay plane coordinate system defined as

$$\hat{\boldsymbol{z}}_{\mathrm{DP}} \parallel \boldsymbol{p}(p) \qquad \hat{\boldsymbol{x}}_{\mathrm{DP}} \parallel \boldsymbol{p}(K^-) - \left[\boldsymbol{p}(K^-) \cdot \hat{\boldsymbol{p}}(p)\right] \hat{\boldsymbol{p}}(p) \qquad \hat{\boldsymbol{y}}_{\mathrm{DP}} = \hat{\boldsymbol{z}} \times \hat{\boldsymbol{x}} \tag{12}$$

- Spin rotation equal for all the decay chains $\to \Lambda_c^+$ spin state phase in the decay plane well defined

$$\mathcal{A}_{m_{\Lambda_c^+},\lambda_p}(\Omega) = \sum_{\nu_{\Lambda_c^+}} D^{*1/2}_{m_{\Lambda_c^+},\nu_{\Lambda_c^+}}(\phi_p, \theta_p, \chi)\mathcal{A}_{\nu_{\Lambda_c^+},\lambda_p}(m^2_{pK^-}, m^2_{K^-\pi^+}), \tag{13}$$

- Amplitude written applying the “revised” helicity formalism starting from the Λ_c^+ decay plane coordinate system

Helicity amplitudes for $\Lambda_c^+ \to pK^-\pi^+$

- $\Lambda_c^+ \to pK^*$
  - Using the decay plane system, no spin rotation needed

$$\mathcal{A}^{\Lambda_c^+ \to pK^*}_{\nu_{\Lambda_c^+}, \lambda_p, \bar{\lambda}_{K^*}} = \mathcal{H}^{\Lambda_c^+ \to K^* p}_{\lambda_p, \bar{\lambda}_{K^*}} \; \delta_{\nu_{\Lambda_c^+}, \lambda_p + \bar{\lambda}_{K^*}} \tag{14}$$

- $K^* \to K^-\pi^+$

$$\mathcal{A}^{K^* \to K^-\pi^+}_{\bar{\lambda}_{K^*}}(\theta_K) = \mathcal{H}^{K^* \to K^-\pi^+}_{0,0} d^{J_{K^*}}_{\bar{\lambda}_{K^*},0}(\theta_K) \mathcal{R}(m^2_{K^-\pi^+}), \tag{15}$$

  - θ_K being kaon momentum polar angle in the K^* opposite helicity coordinate system

Helicity amplitudes for $\Lambda_c^+ \to pK^-\pi^+$

- $\Lambda_c^+ \to \Lambda^*\pi^+$

$$\mathcal{A}^{\Lambda_c^+ \to \Lambda^*\pi^+}_{\nu_{\Lambda_c^+},\lambda_{\Lambda^*}}(\theta_{\Lambda^*}) = \mathcal{H}^{\Lambda_c^+ \to \Lambda^*\pi^+}_{\lambda_{\Lambda^*},0} d^{1/2}_{\nu_{\Lambda_c^+},\lambda_{\Lambda^*}}(\theta_{\Lambda^*}), \tag{16}$$

- $\Lambda^* \to pK^-$

$$\mathcal{A}^{\Lambda^* \to pK^-}_{\lambda_{\Lambda^*},\lambda_p^{\Lambda^*}}(\theta_p^{\Lambda^*}) = \mathcal{H}^{\Lambda^* \to pK^-}_{\lambda_p^{\Lambda^*},0} d^{J_{\Lambda^*}}_{\lambda_{\Lambda^*},\lambda_p^{\Lambda^*}}(\theta_p^{\Lambda^*})\mathcal{R}(m^2_{pK^-}), \tag{17}$$

- Analogous expressions for Δ^{*++} chain

Proton spin rotation

- K^* chain proton spin states taken as reference
- Rotation of Λ^* amplitudes

$$\mathcal{A}^{\Lambda_c^+\to\Lambda^*(\to pK^-)\pi^+}_{\nu_{\Lambda_c^+},\lambda_{\Lambda^*},\lambda_p}(m^2_{pK^-}, m^2_{K^-\pi^+}) = \sum_{\lambda_p^{\Lambda^*}=\pm 1/2} d^{1/2}_{\lambda_p^{\Lambda^*},\lambda_p}(-\theta_{\Lambda^*} - \theta_p^{\Lambda^*}) \times \mathcal{A}^{\Lambda_c^+\to\Lambda^*(\to pK^-)\pi^+}_{\nu_{\Lambda_c^+},\lambda_{\Lambda^*},\lambda_p^{\Lambda^*}}(m^2_{pK^-}, m^2_{K^-\pi^+}) \tag{18}$$

- Note the importance of the minus signs

Polarised decay rate of $\Lambda_c^+ \to pK^-\pi^+$

- Density matrix for the Λ_c^+ polarisation

$$\rho^{\Lambda_c^+} = \frac{1}{2}(\mathbb{I} + \boldsymbol{P} \cdot \boldsymbol{\sigma}) = \frac{1}{2}\begin{pmatrix} 1 + P_z & P_x - iP_y \\ P_x + iP_y & 1 - P_z \end{pmatrix}, \tag{19}$$

- Decay rate becomes

$$p(\Omega, \vec{P}) \propto \sum_{\lambda_p = \pm 1/2} \Big\{ (1 + P_z)|\mathcal{A}_{1/2,\lambda_p}(\Omega)|^2 + (1 - P_z)|\mathcal{A}_{-1/2,\lambda_p}(\Omega)|^2 + 2Re\left[(P_x - iP_y)\mathcal{A}^*_{1/2,\lambda_p}(\Omega)\mathcal{A}_{-1/2,\lambda_p}(\Omega)\right]\Big\}. \tag{20}$$

Summary

- Phase of spin states must be consistently defined in the helicity formalism
  - Match of full spin coordinate system + sign issue for fermion states
- Rewriting of the helicity formalism ensuring a consistent phase definition for spin states
- Proposed method to match final particle spin states, including phases, applicable to generic multibody decays
  - By inverting the spin rotation sequence for each decay chain and each final particle
- Effects of phase mismatches can not be neglected when interference effects between different decay chains are important