Schlessinger Point Method: Theory and Applications

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Schlessinger Point Method (SPM)

- **SPM interpolating fraction**
  Given a finite set of N data points (x_i, f_i) the SPM constructs the rational interpolant $p(x)/q(x)$ given by
  Schlessinger, PR 167 (1968)

$$C_N(x) = \frac{p(x)}{q(x)} = \frac{f_1}{1+}\frac{a_1(x-x_1)}{1+}\frac{a_2(x-x_2)}{1+}\cdots\frac{a_{N-1}(x-x_{N-1})}{1}$$

- **Coefficients a_i obtained recursively**

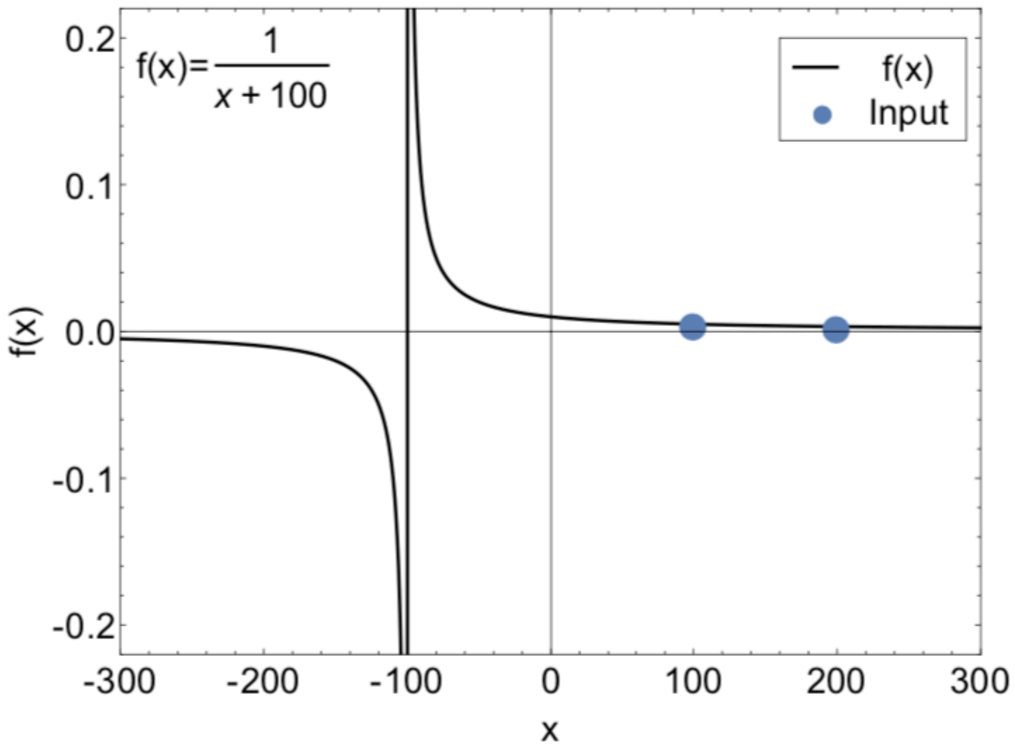
$$a_1 = \frac{f_1/f_2 - 1}{x_2 - x_1}$$

$$a_i = \frac{1}{x_i - x_{i+1}}\left[1 + \frac{a_{i+1}(x_{i+1} - x_{i-1})}{1+}\frac{a_{i-2}(x_{i+1} - x_{i-2})}{1+}\cdots\frac{a_1(x_{i+1} - x_1)}{1 - f_1/f_{i+1}}\right]$$

- $[p(x), q(x)]$ **order**
  - $[N/2 - 1, N/2]$
    for an even number of input points
  - $[(N-1)/2, (N-1)/2]$
    for an odd number of input points

SPM examples

- $f(x) = 1/(x + \mu)$
  - **Only 2 points** needed to reproduce these curves
  - **“First guess”** of the method



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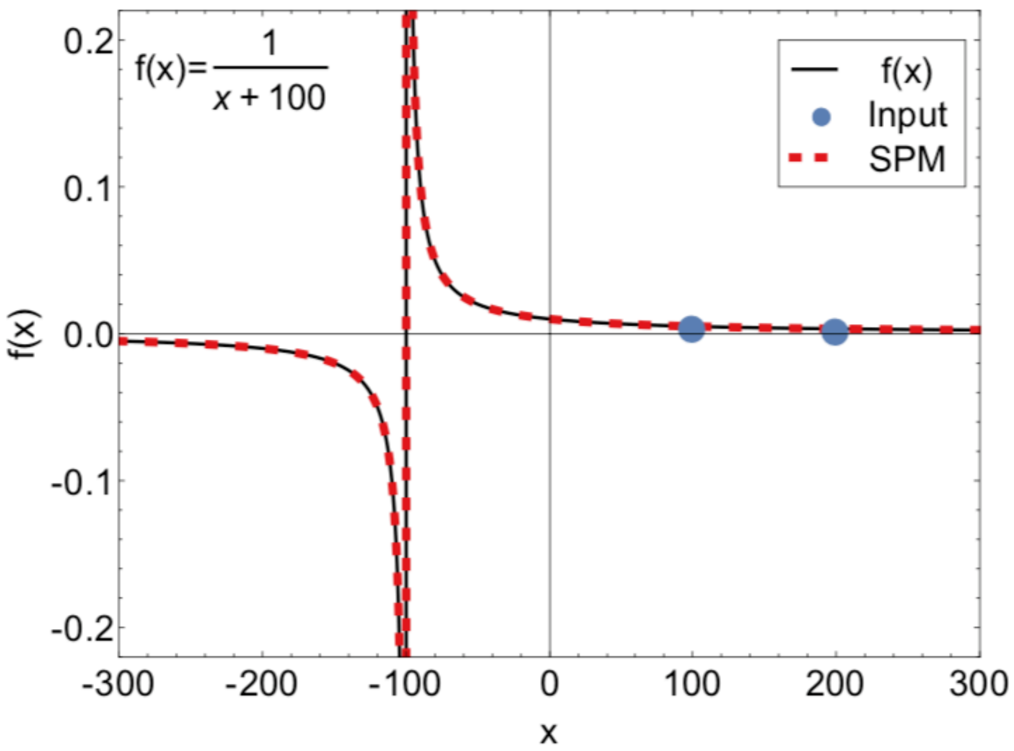

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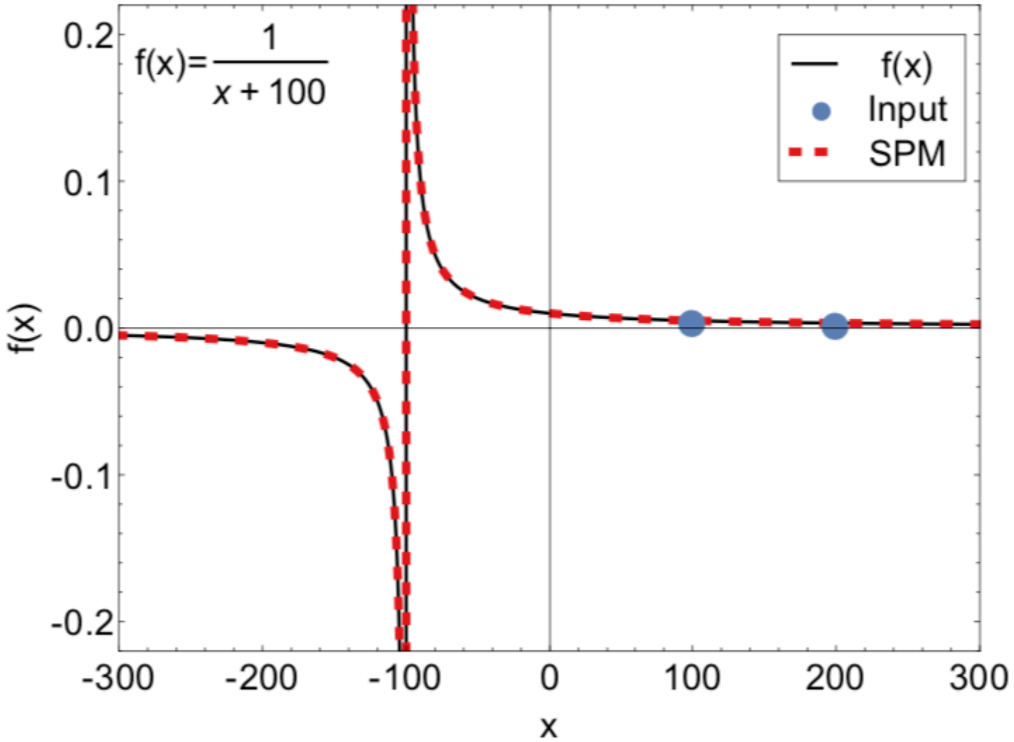
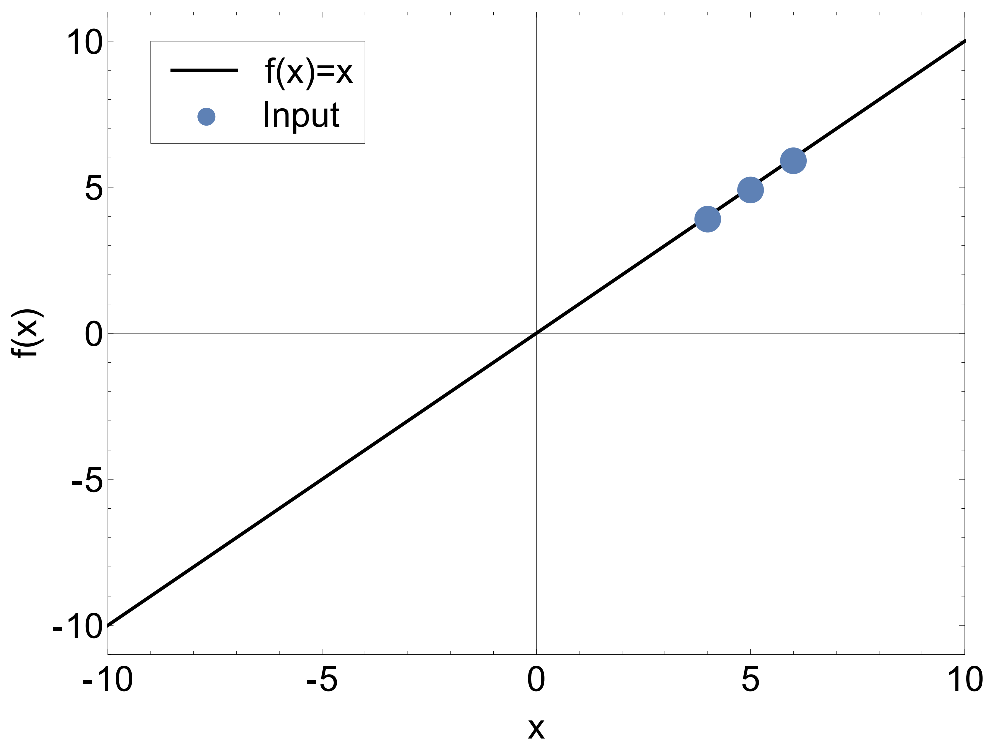


- $f(x) = x$
  - **Use 3 points**
    to reproduce a straight line
  - **15 digits precision**
    yields

$$f(x) = \frac{22 + 1.8 \times 10^{15} x}{1.8 \times 10^{15} - x} \approx x$$



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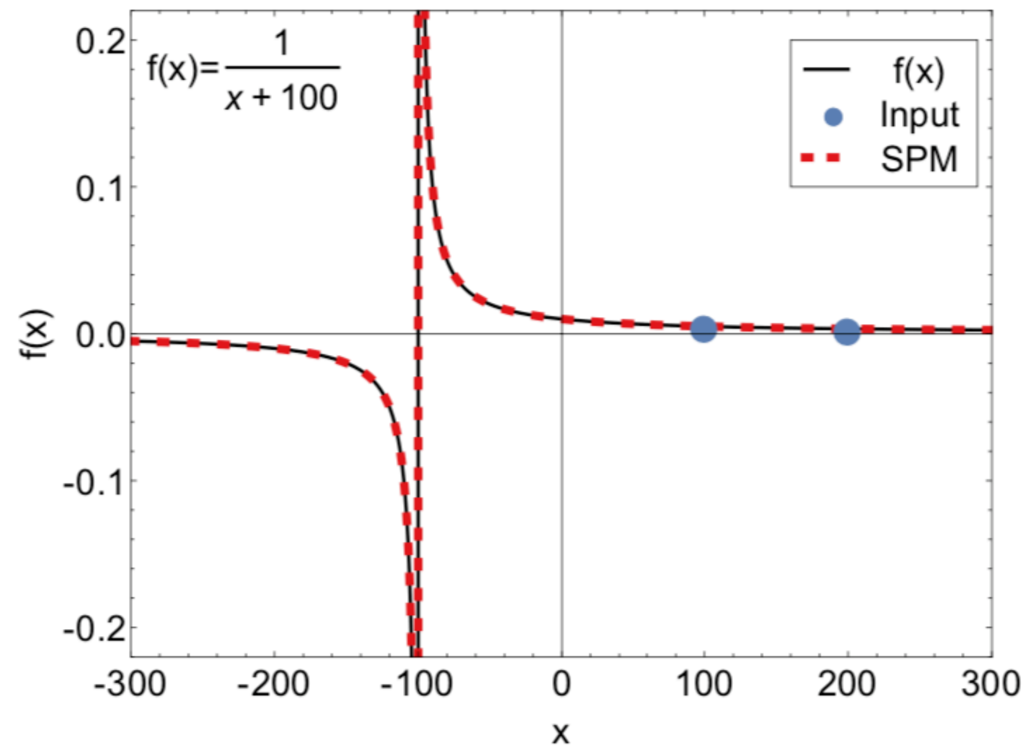

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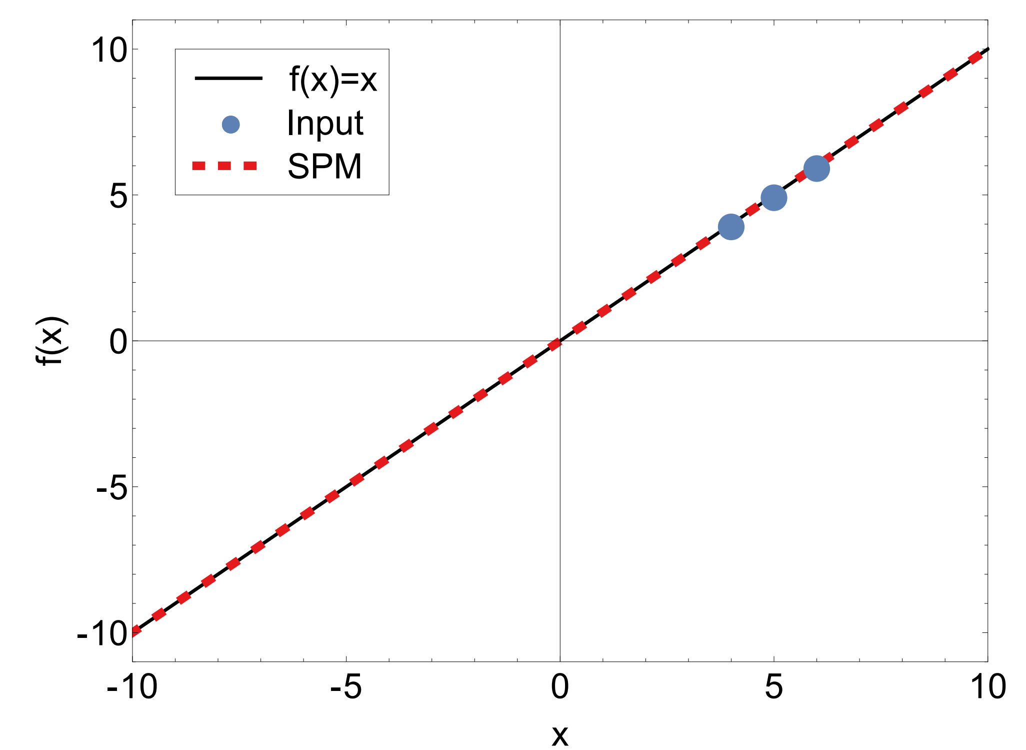
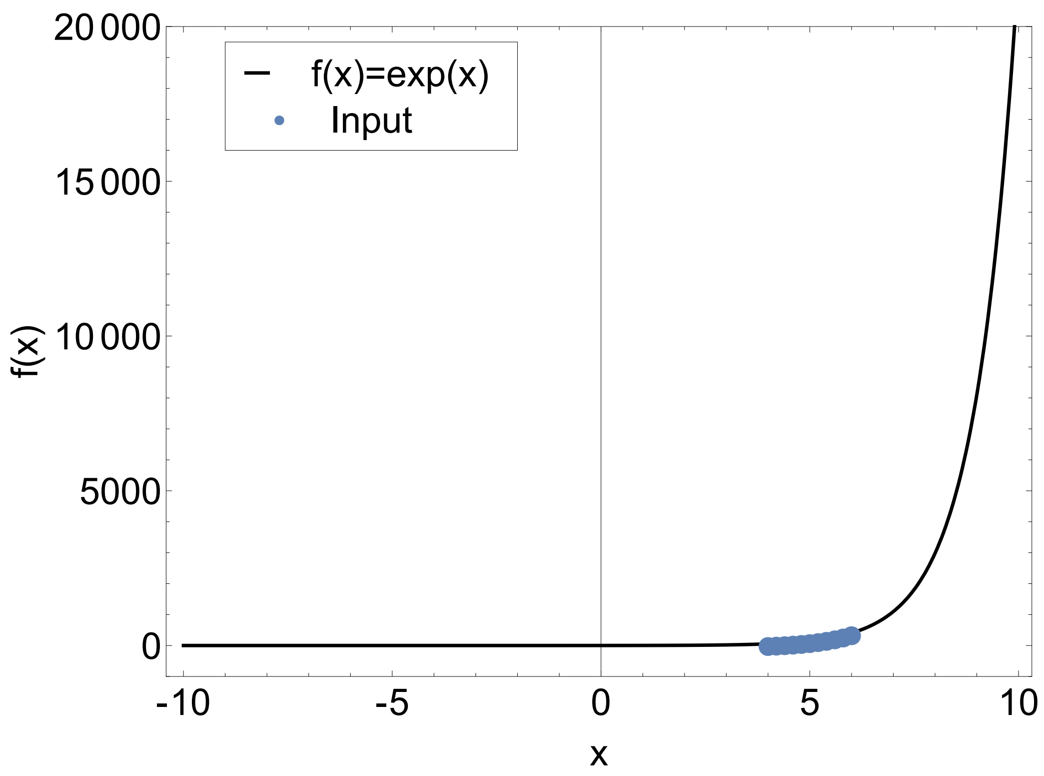


SPM examples

- $f(x) = e^x$
  - **Use 11 points**
  - **15 digits precision**
    yields

$$f(x) = \frac{263504 + 170536x + 46451x^2 + 10389x^3 + 756x^4 + 148x^5}{265568 - 98809x + 15473x^2 - 1274x^3 + 55x^4 - x^5}$$



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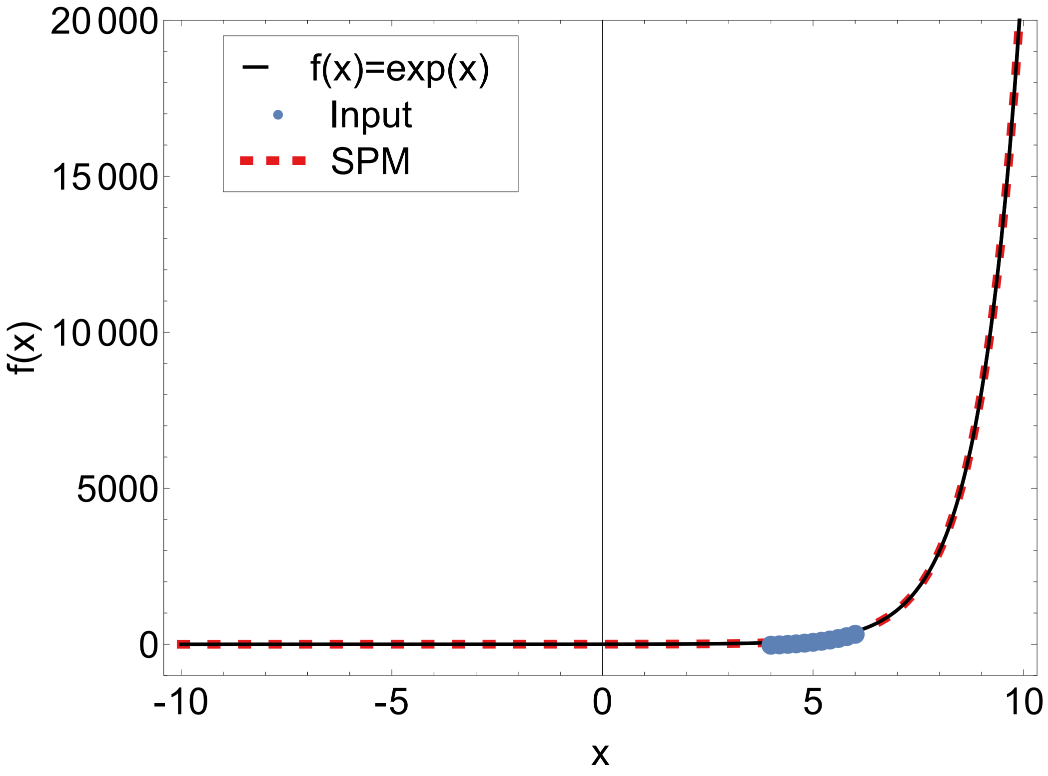

SPM examples

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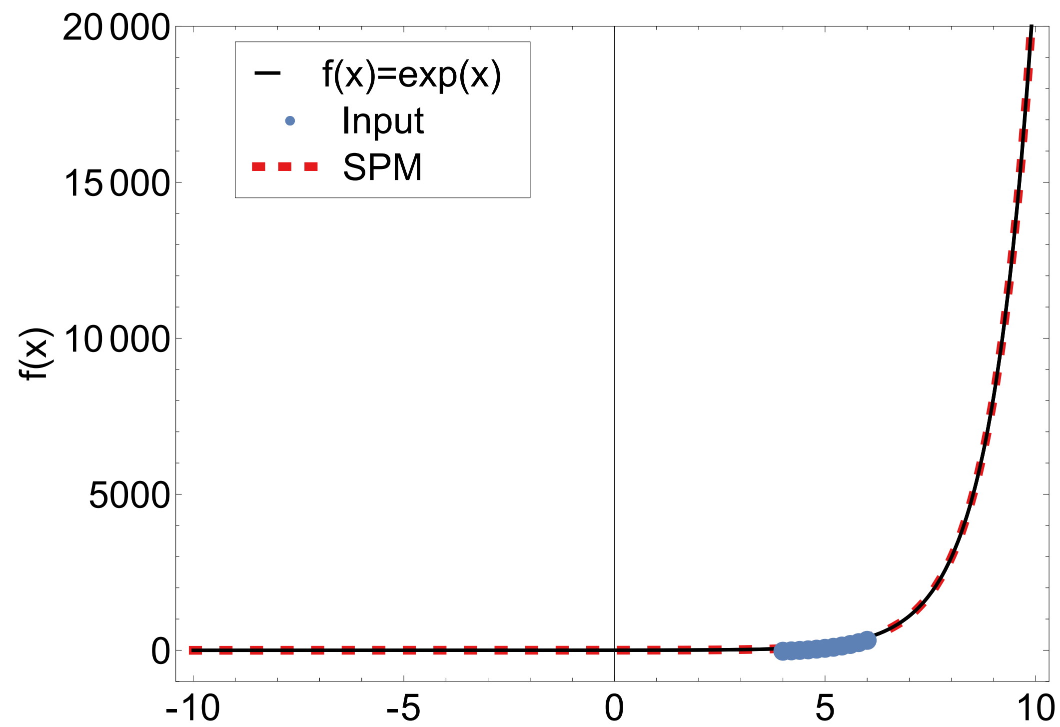
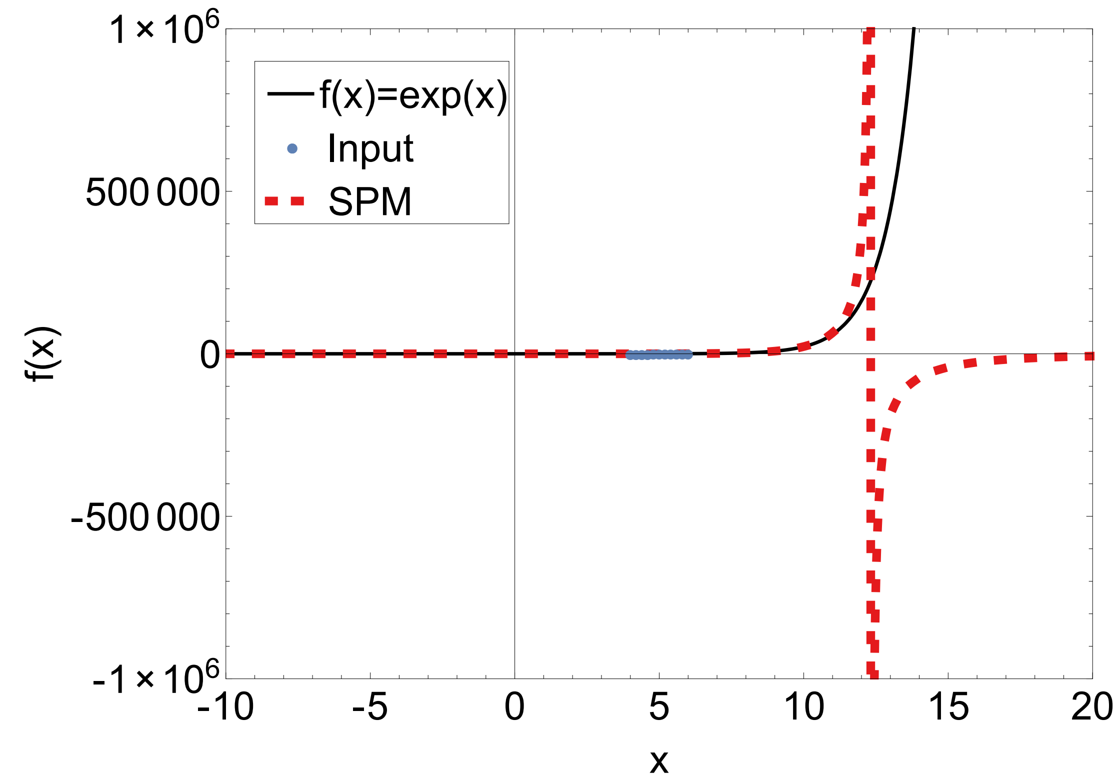
  - **Essential singularities**
    will not be reproduced everywhere





SPM and analytic continuation

- **Analytic continuation**
  Can be obtained by setting $x = \alpha e^{i\theta}$ in $C_N(x)$
  - **Pole singularities**
    can be exactly reproduced
  - **Branch cuts**
    can be approximately reproduced by a series of poles
  - **Rational fractions can have only one sheet**
    many sheeted functions can only be reconstructed on a single sheet

- **(Generalized) spectral functions**
  perfect application domain

$$D(p_0) = \int_{-\infty}^{\infty} \mathrm{d}\omega \, \frac{2\omega\rho(\omega)}{\omega^2 + p_0^2} + \sum_{j=1}^{n} \frac{Z_j}{p_0^2 - z_j}$$

$$\rho(\omega) = 2 \ \mathrm{Im} \ D(p_0 \to -i(\omega + i0^+))$$

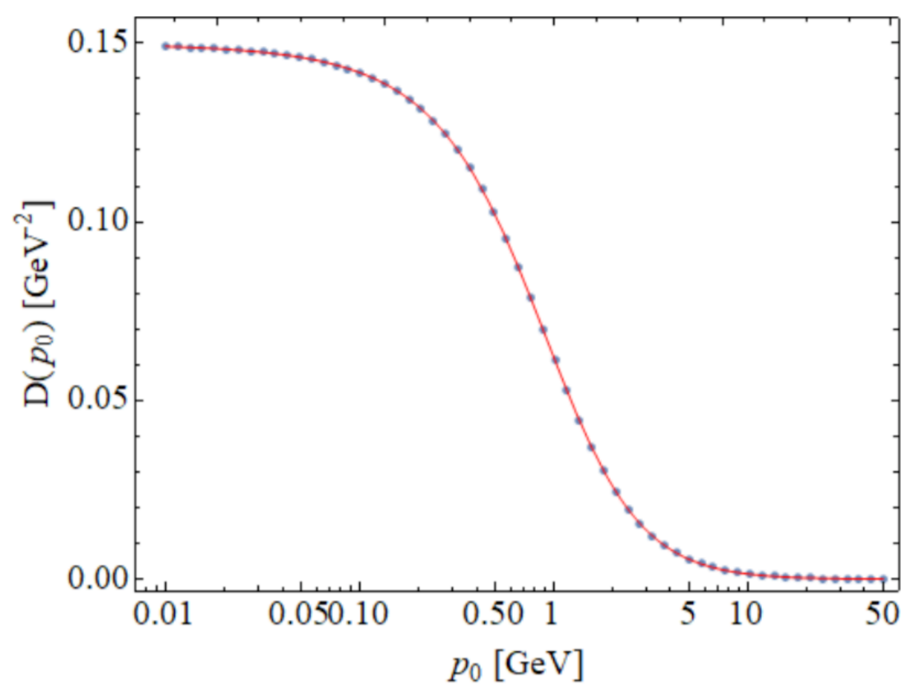
Breit-Wigner (BW) propagator

- **SPM test**
  Simple Breit Wigner propagator

$$\rho(\omega) = 2\ \mathrm{Im}\ D(p_0 \to -i(\omega + i0^+)) = \frac{1}{\pi}\frac{2\Gamma\omega}{(\omega^2 - \Gamma^2 - M^2)^2 + 4\Gamma^2\omega^2}$$

$$D(p_0) = \int_{-\infty}^{\infty} \mathrm{d}\omega\, \frac{2\omega\rho(\omega)}{\omega^2 + p_0^2} = \frac{1}{2\pi}\frac{1}{(p_0 + \Gamma)^2 + M^2}$$

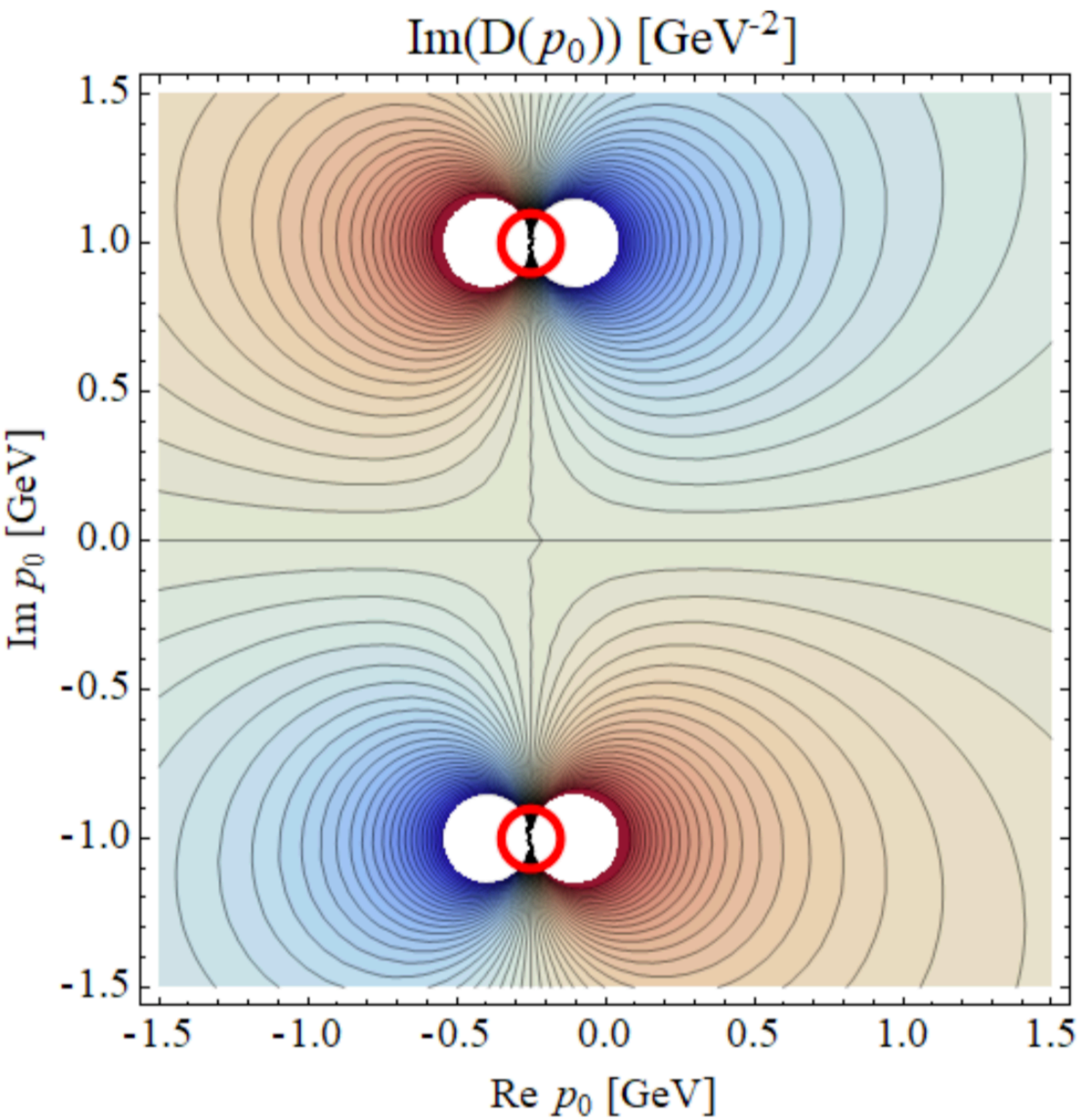
- **Procedure**
  - **Choose parameters**
    $M = 4\Gamma = 1\ \mathrm{GeV}$
  - **Generate Euclidean propagator data**
    100 points between 0.01 and 50 GeV
  - **Apply SPM**
    on your favourite 60 input points
  - **Compare exact/reconstructed analytic structure**
    in the p_0 and p_0^2 complex plane
  - **Construct the spectral function**
    compare it with the exact one
  - **Calculate the propagator**
    compare it with the exact one



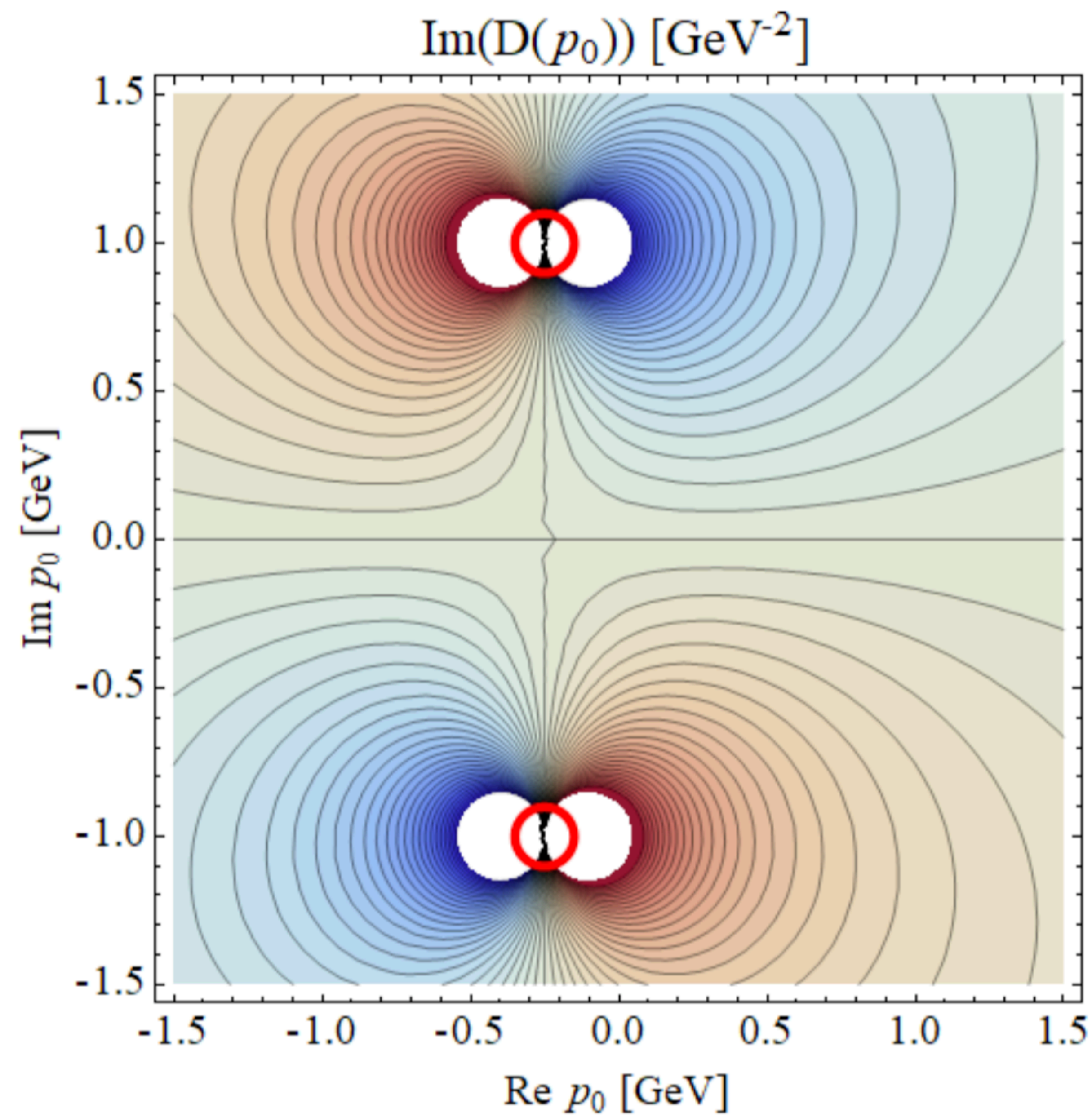
BW propagator I: p_0 plane

- **Poles:**
  perfectly reconstructed

- **Exact**

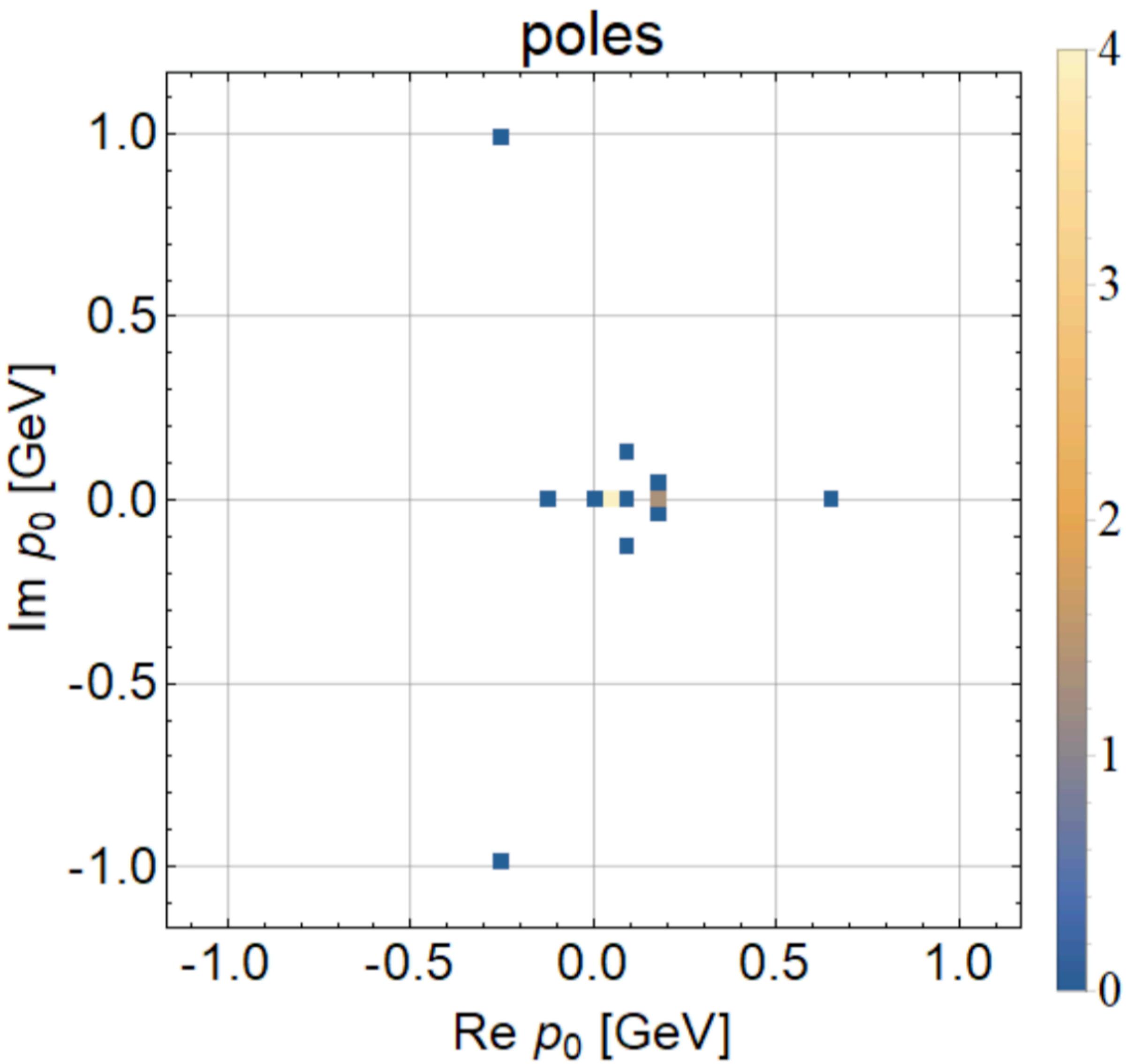


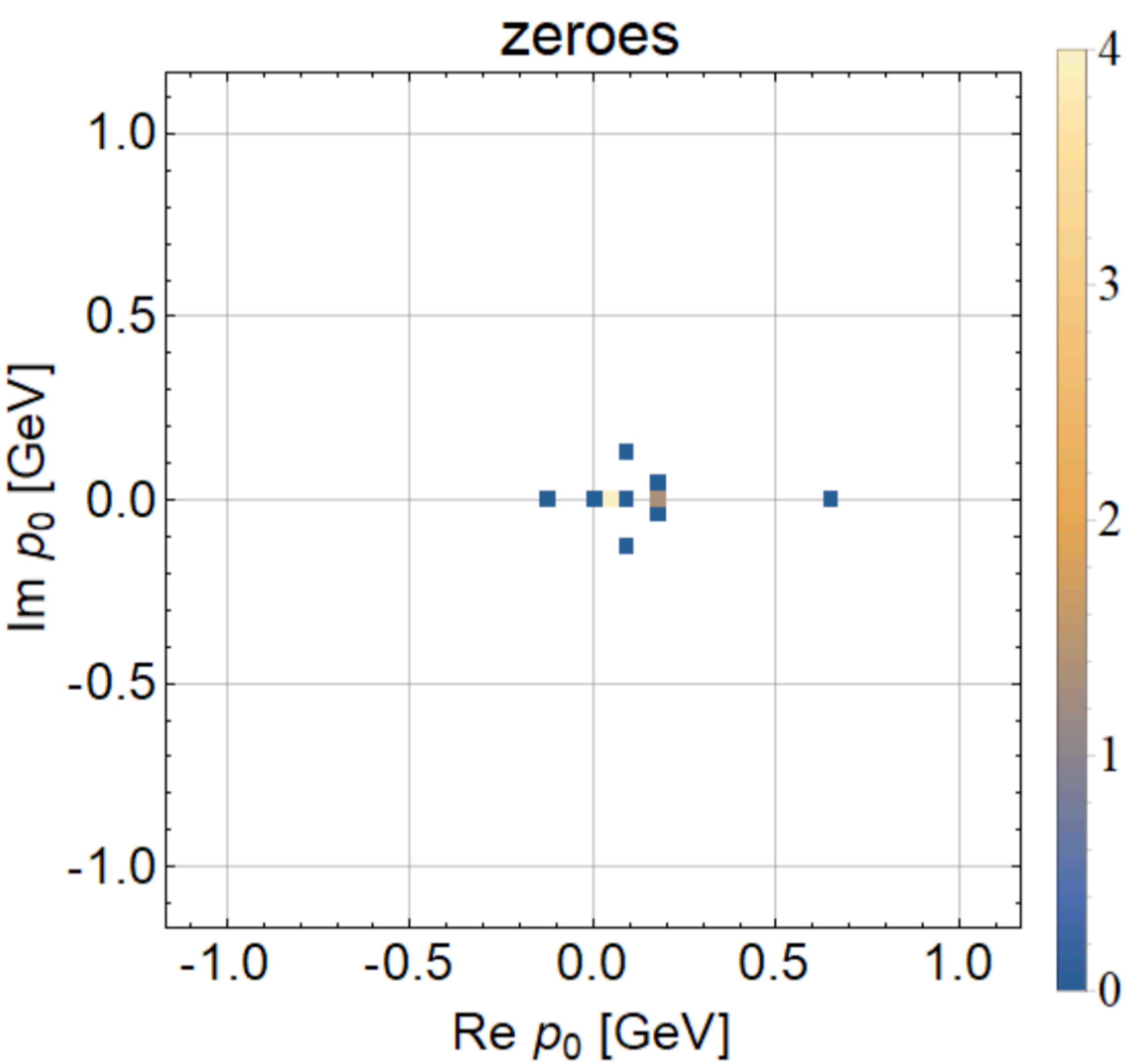
- **Reconstructed**



BW propagator I: p_0 plane

- **How?**
  Near cancellation between poles and zeros
  - **For** $N = 60$
    there are 30 poles and 29 zeros
  - **Near cancellation**
    leaves poles with very small residue

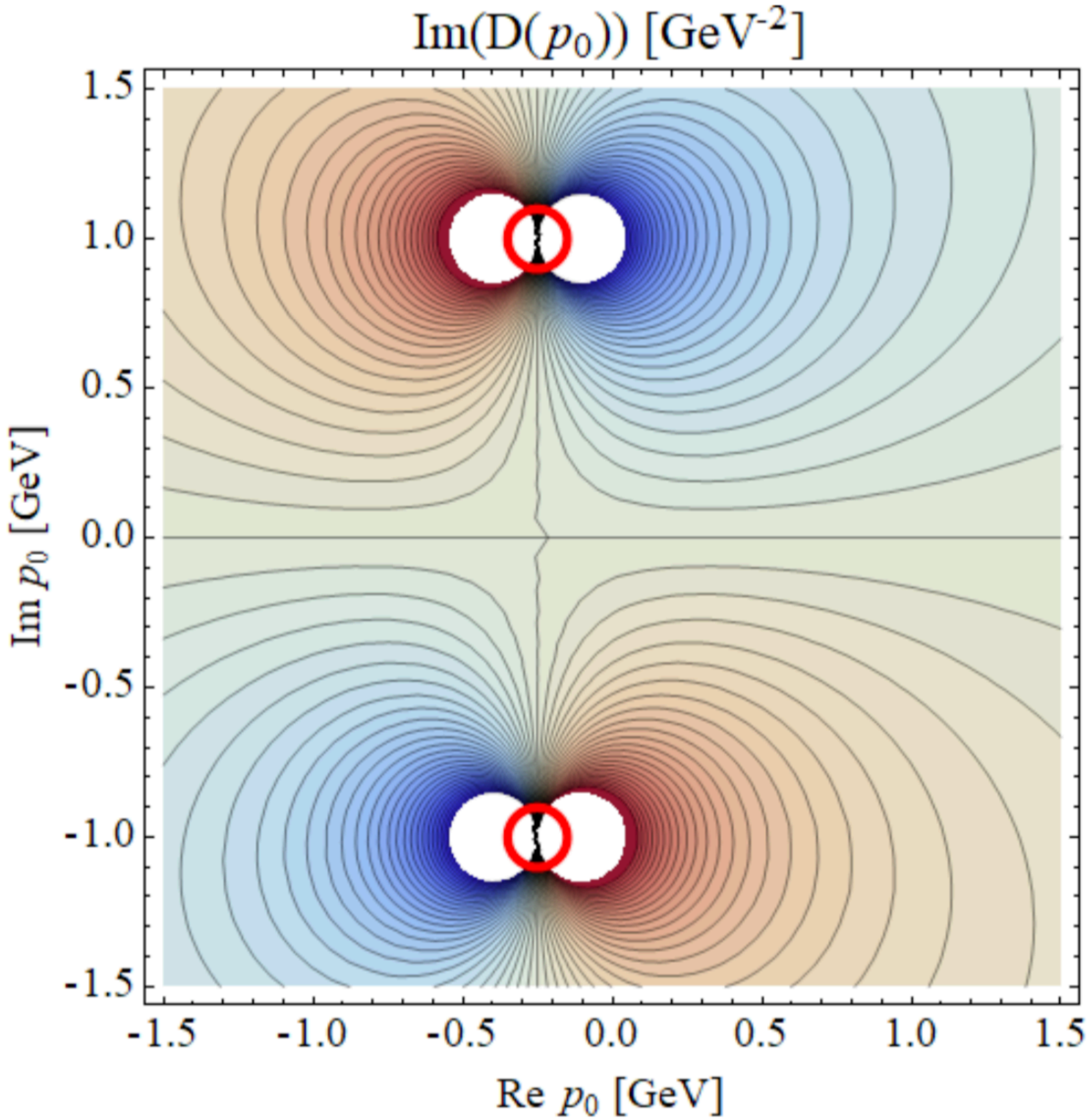




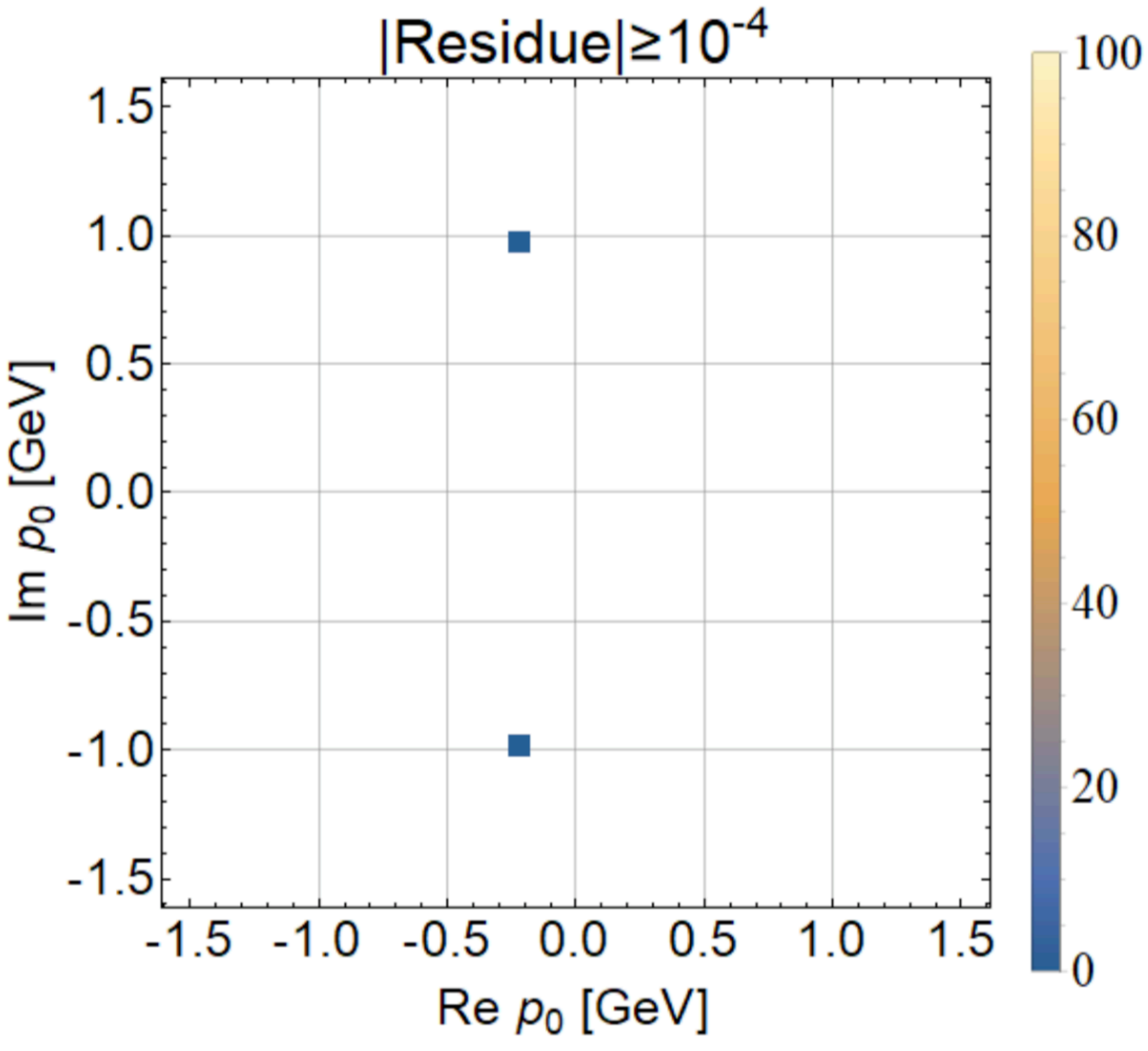
BW propagator I: p_0 plane

- **Filtering**
  physical poles can be identified by using a threshold for the residues

- **Exact**



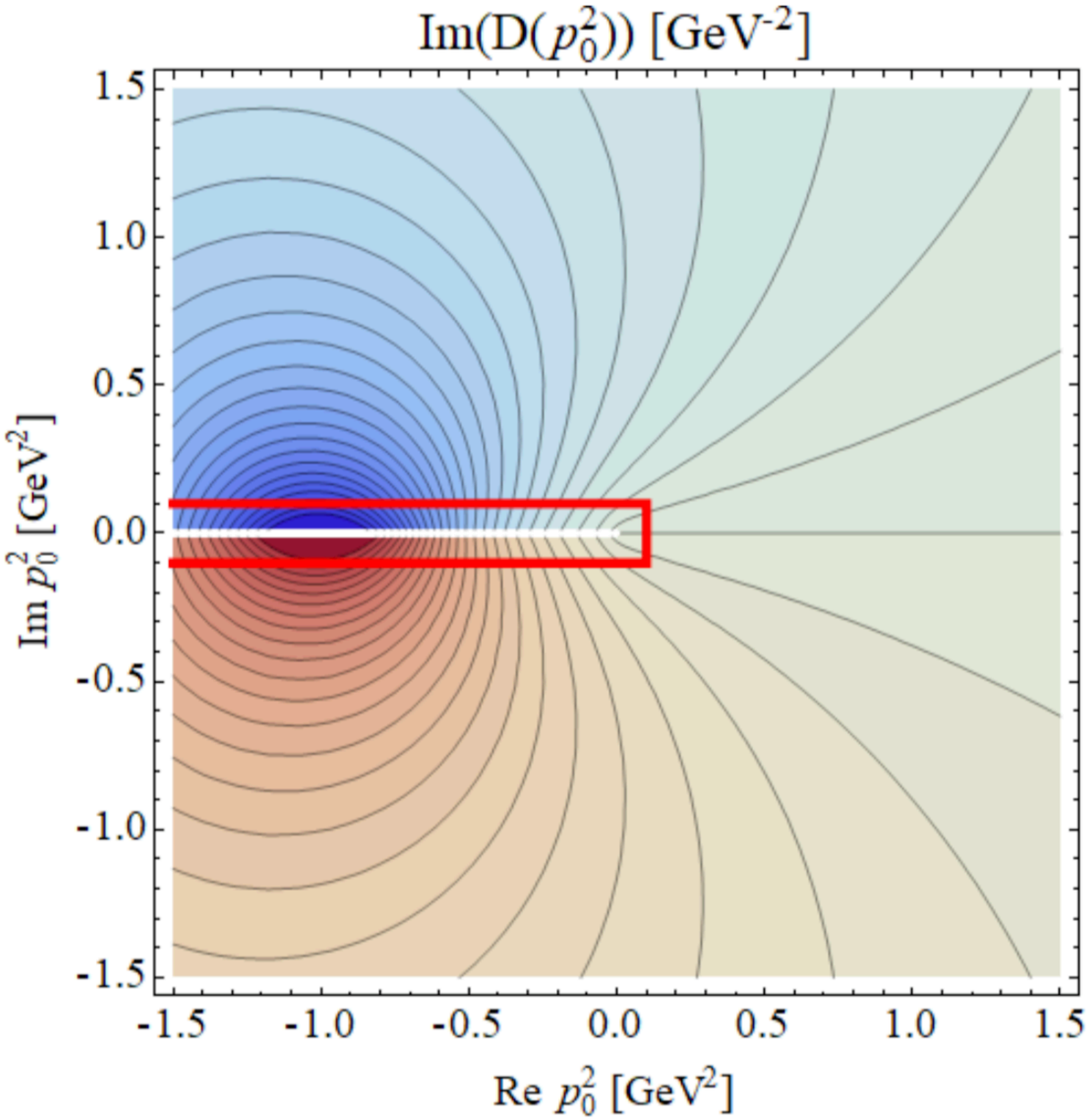
- **Reconstructed**



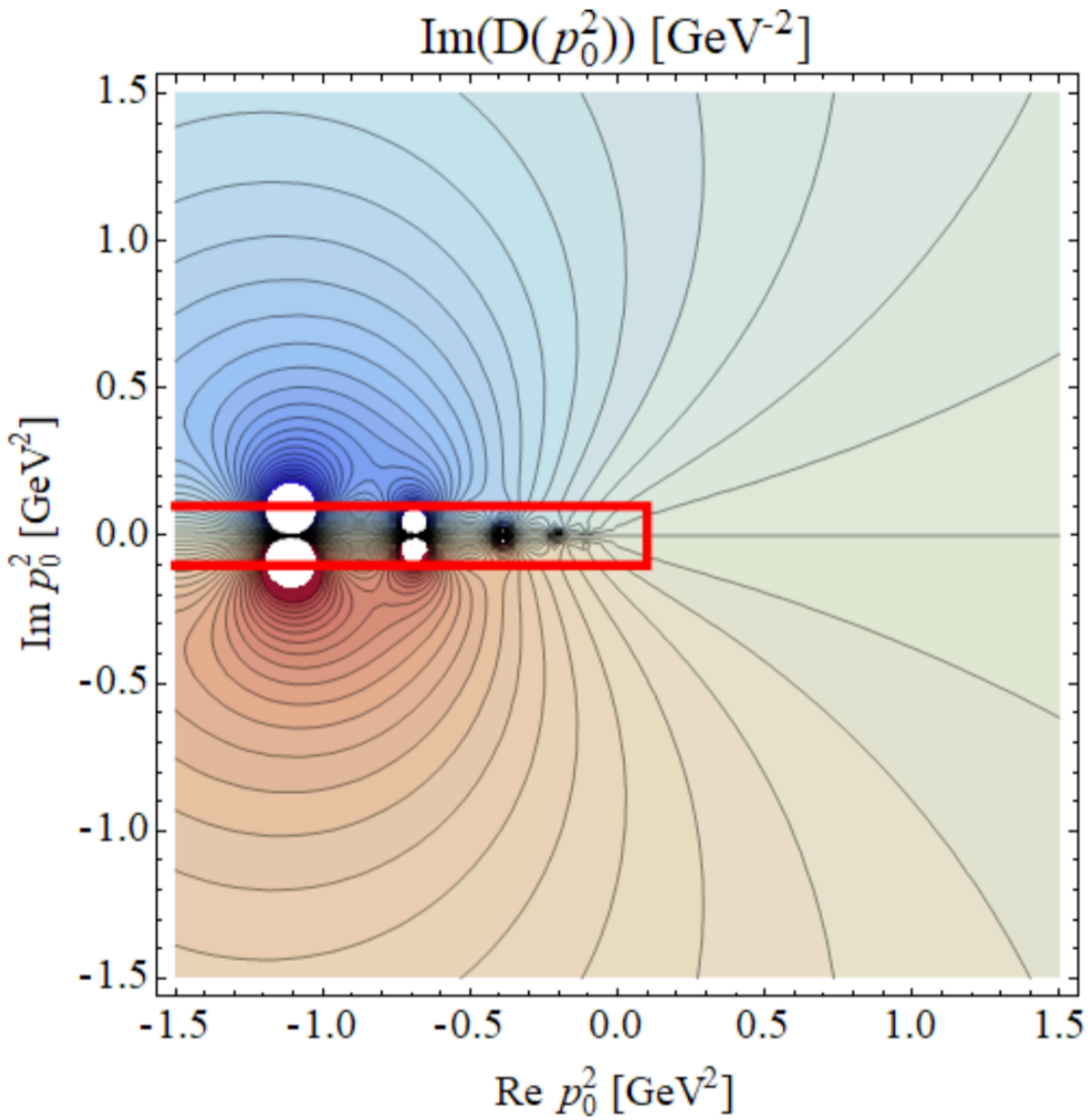
BW propagator II: p_0^2 plane

- **Branch cut:**
  visualized as a series of poles

  - **Exact**

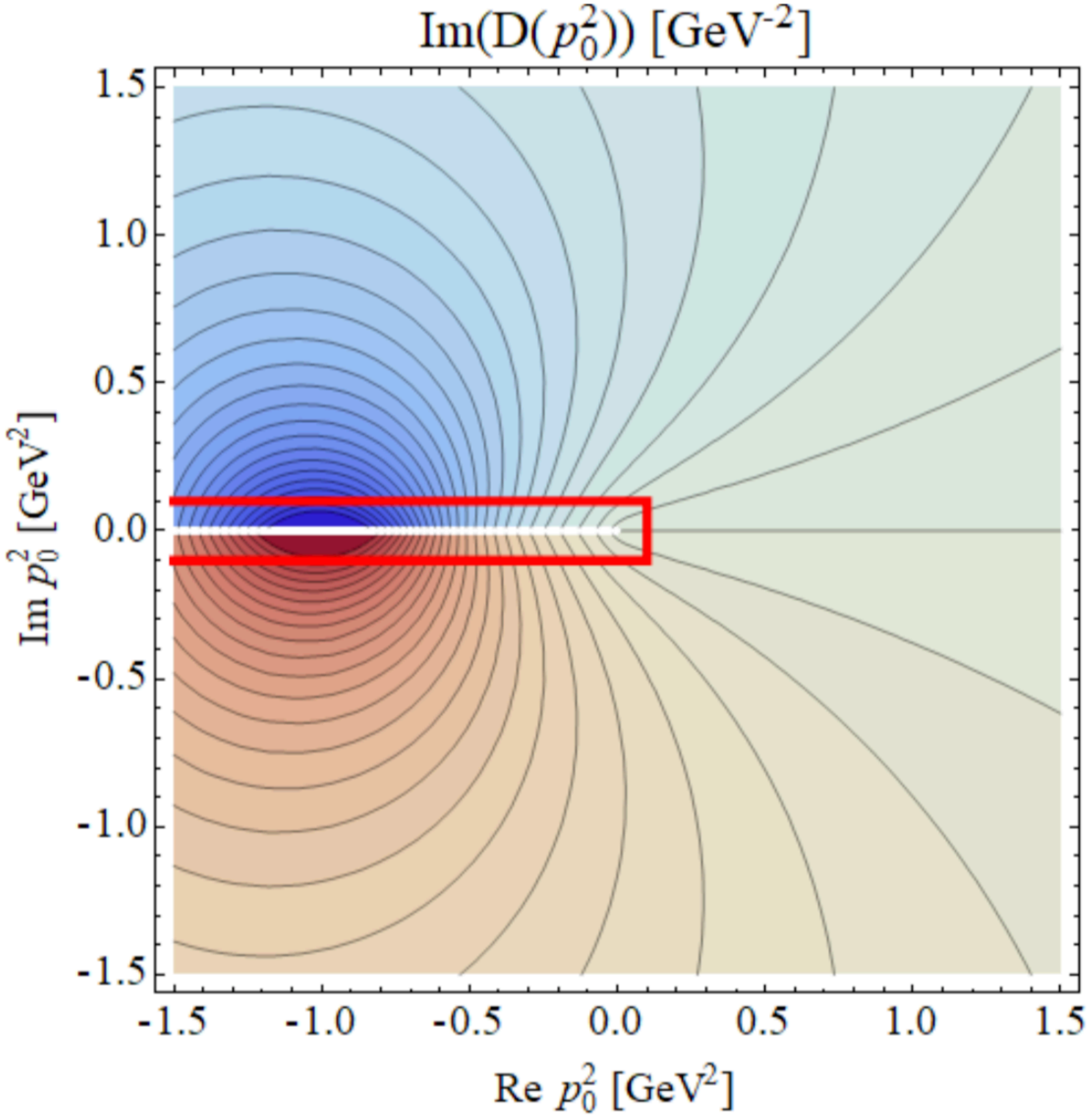


  - **Reconstructed**

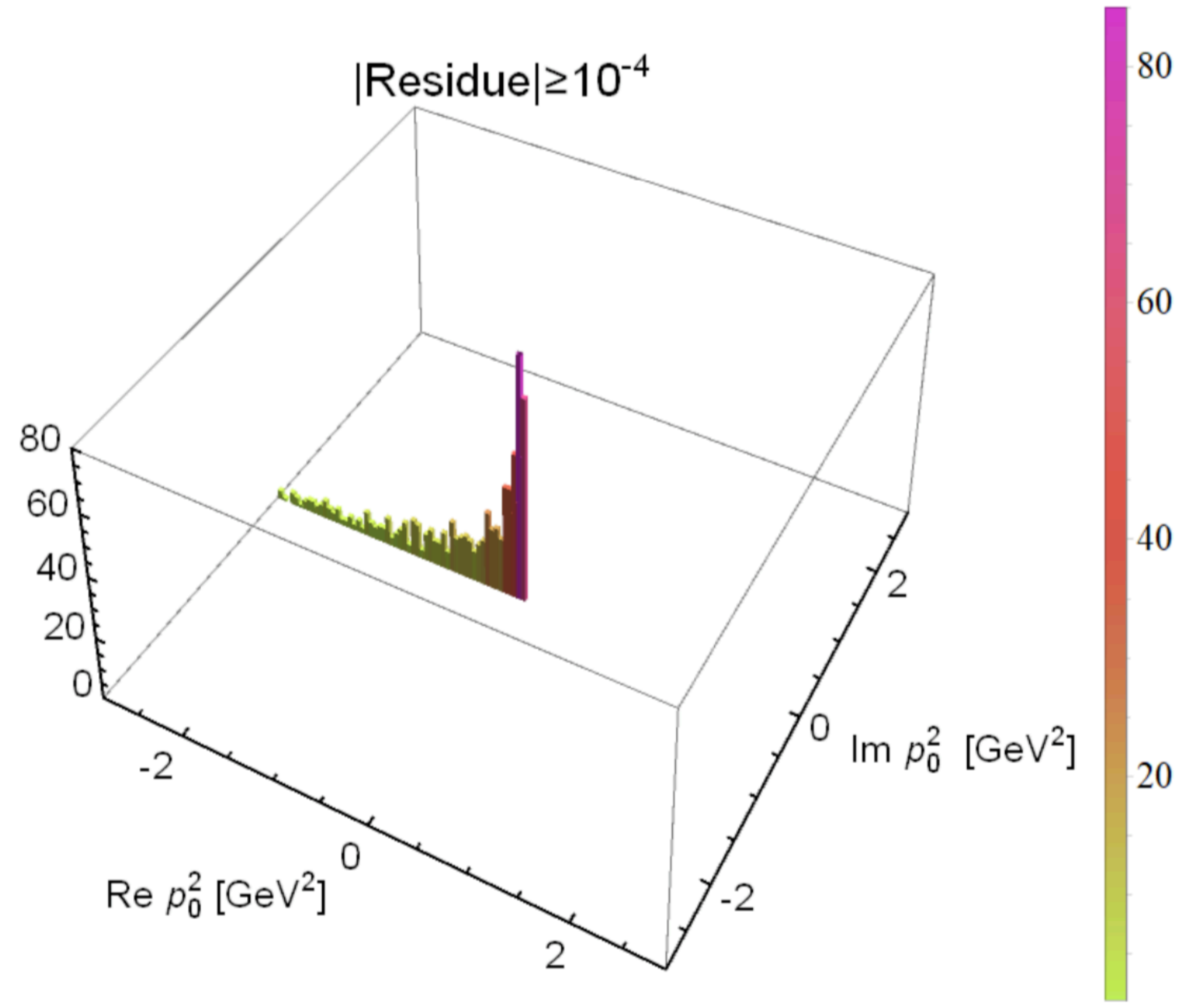


BW propagator II: p_0^2 plane

- **Branch cut:**
  more clearly visible in a histogram, showing the location of the poles for 100 random subsets of the 60 input points
  - **Exact**



  - **Reconstructed**



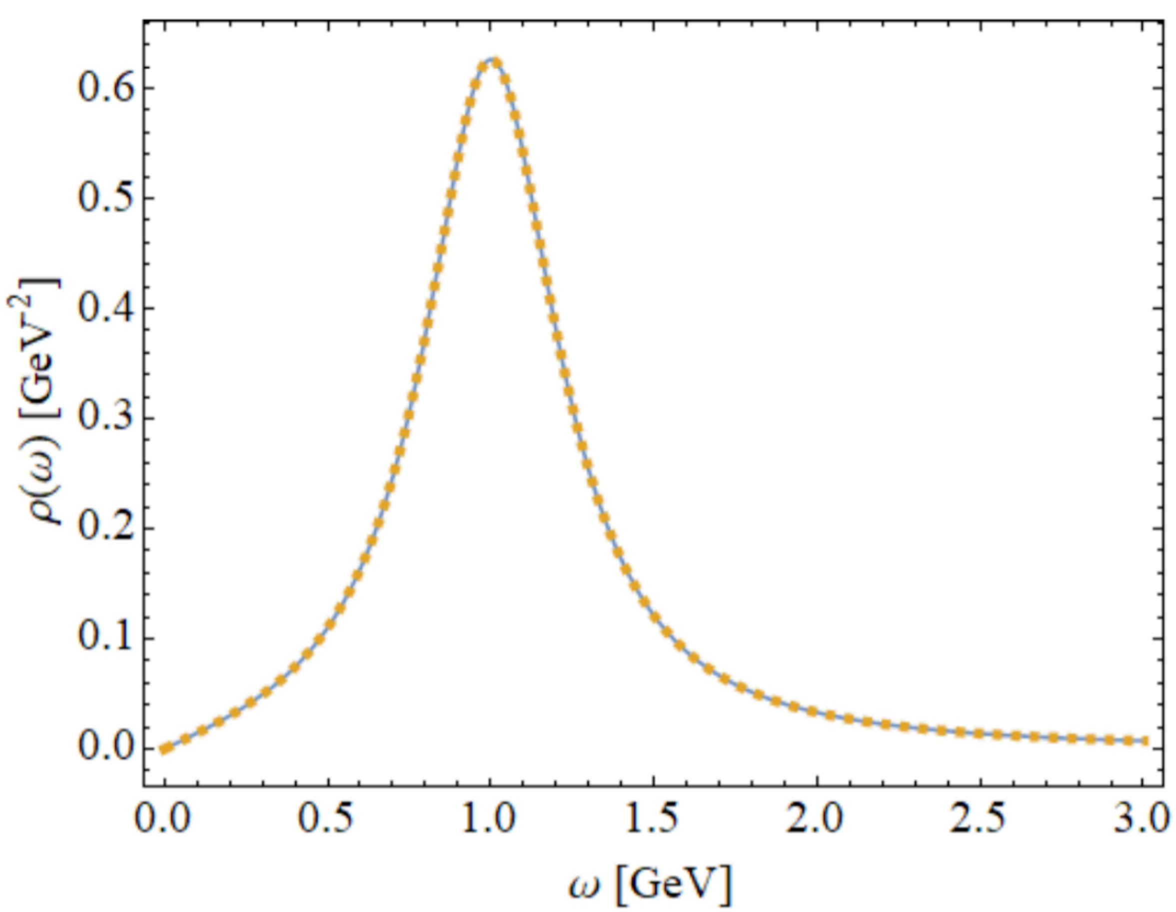
BW spectral function

- **Spectral function**
  Obtained as:

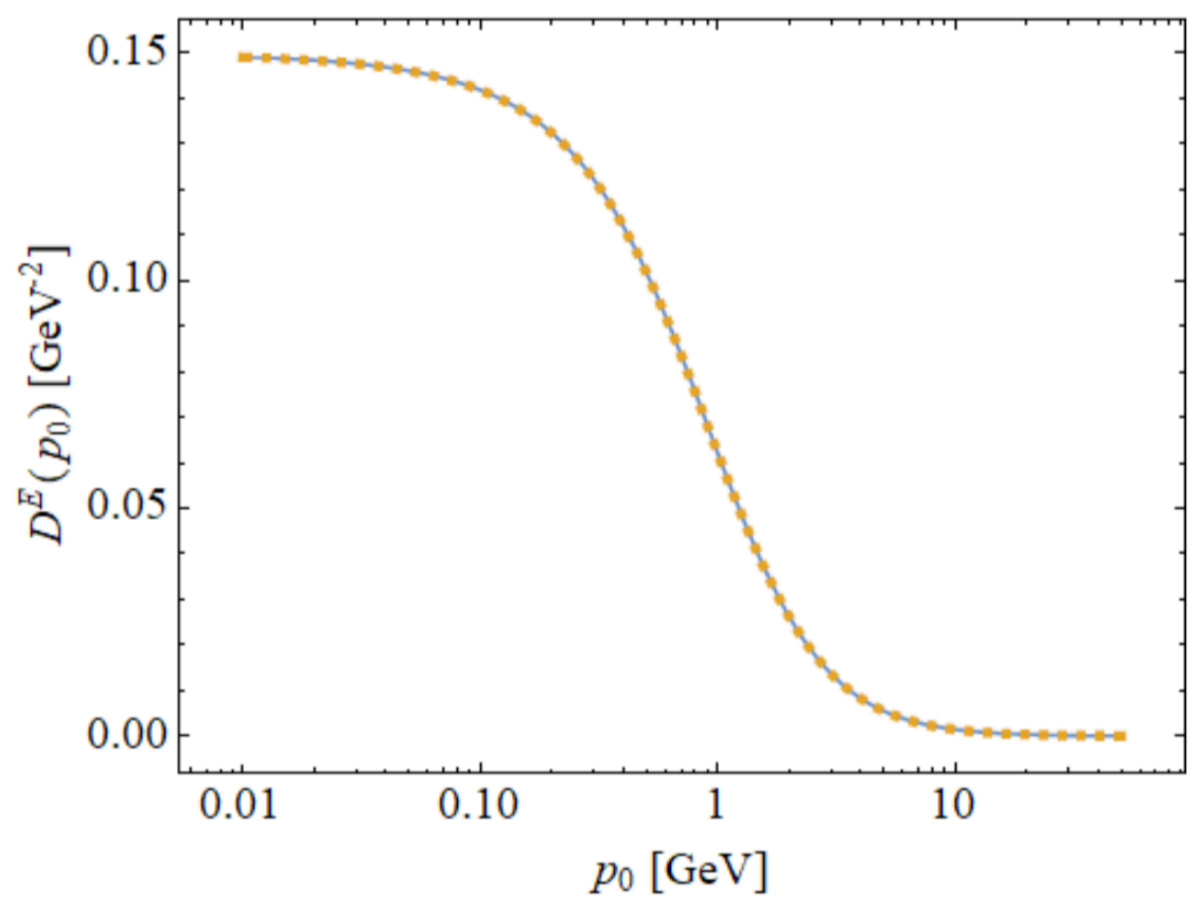
$$\rho(\omega) = 2\ \mathrm{Im}\ D(p_0 \to -i(\omega + i0^+))$$

$$D(p_0) = \int_{-\infty}^{\infty} \mathrm{d}\omega\, \frac{2\omega\rho(\omega)}{\omega^2 + p_0^2}$$

- **Reconstructed spectral function**



- **Reconstructed propagator**



BW propagator plus poles

- **Add complex conjugated poles** to BW propagator

$$D(p_0) = \frac{1}{2\pi}\frac{1}{(p_0+\Gamma)^2+M^2} + \sum_{j=1}^{n}\frac{Z_j}{p_0^2 - z_j}$$
$$= \int_{-\infty}^{\infty} \mathrm{d}\omega\, \frac{2\omega\rho(\omega)}{\omega^2+p_0^2} + \sum_{j=1}^{n}\frac{Z_j}{p_0^2 - z_j}$$

$$\rho(\omega) = 2\ \mathrm{Im}\ D(p_0 \to -i(\omega + i0^+))$$
$$= \frac{1}{\pi}\frac{2\Gamma\omega}{(\omega^2-\Gamma^2-M^2)^2+4\Gamma^2\omega^2}$$

- **Procedure**
  - **Choose parameters**
    $M = 4\Gamma = 1$ GeV, $Z_1 = Z_2 = 1$, $z_{1,2} = (-1 \pm i)$ GeV^2
  - **Generate Euclidean propagator data**
    100 points between 0.01 and 50 GeV
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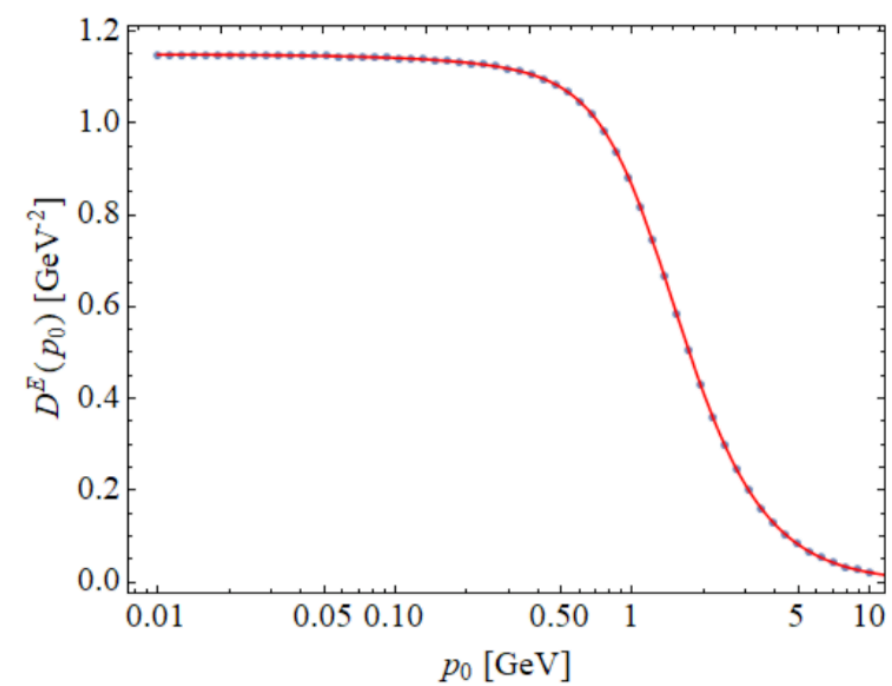
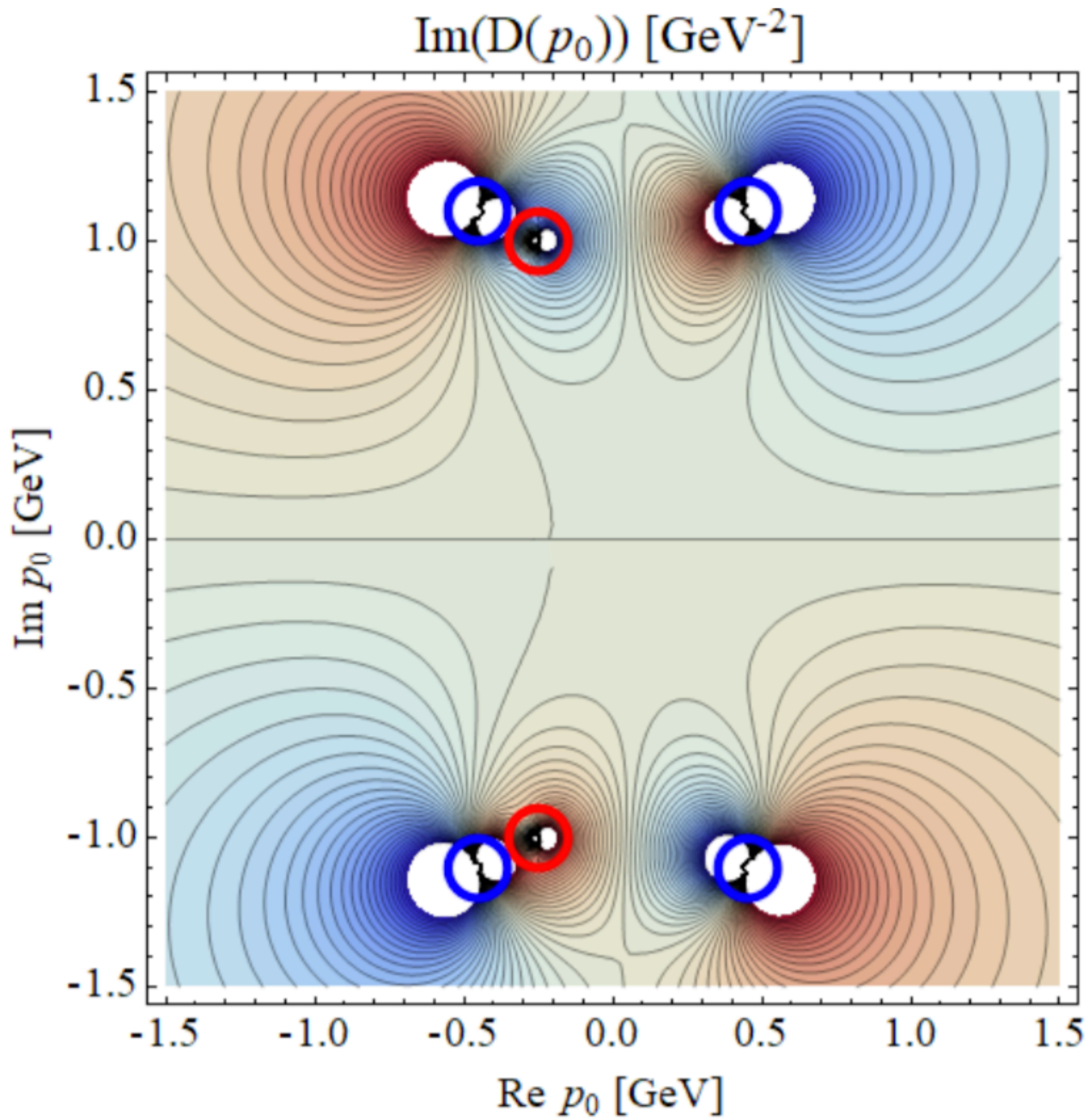


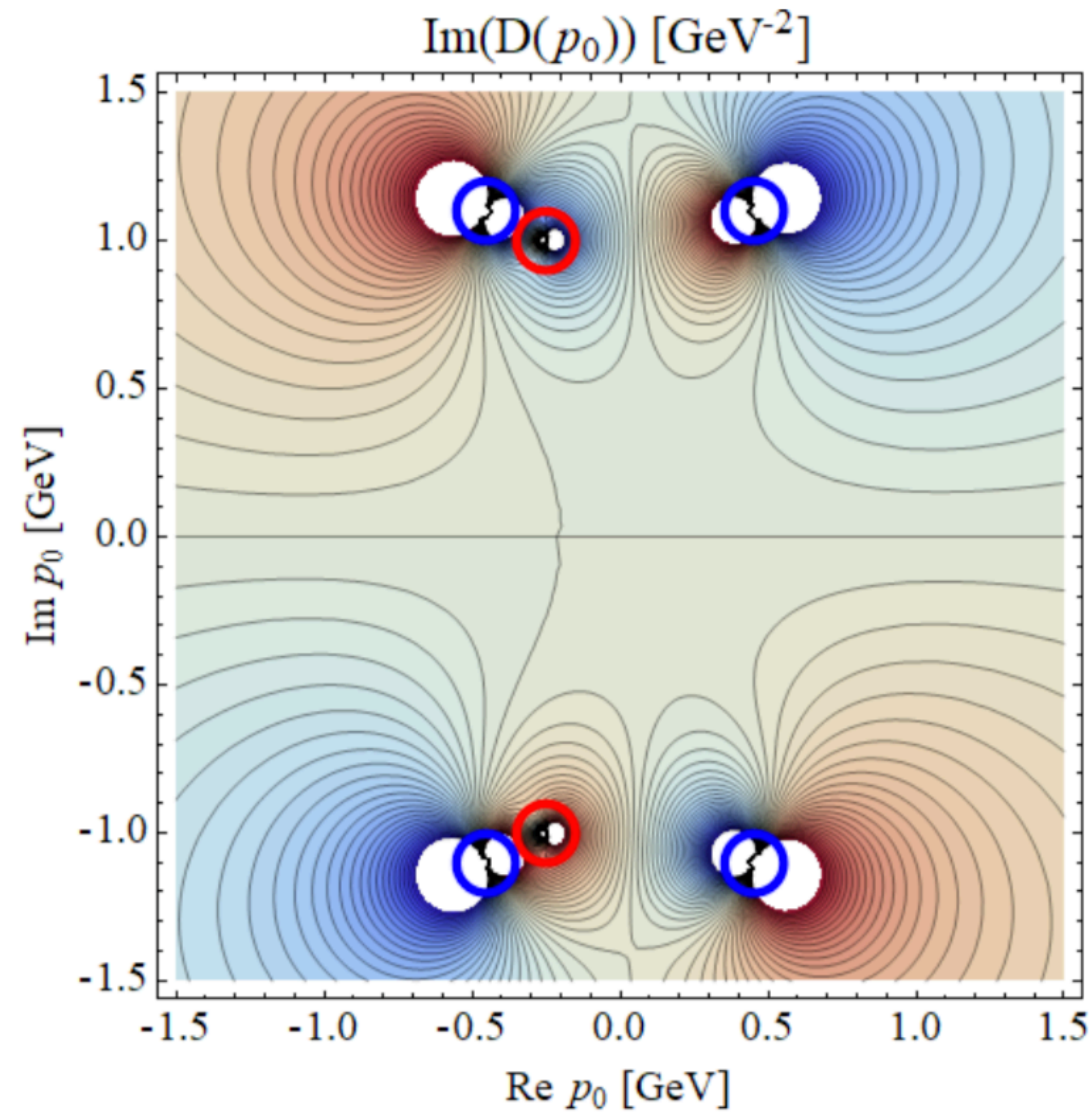
BW propagator plus poles I: p_0 plane

- **Poles:**
  perfectly reconstructed

- **Exact**

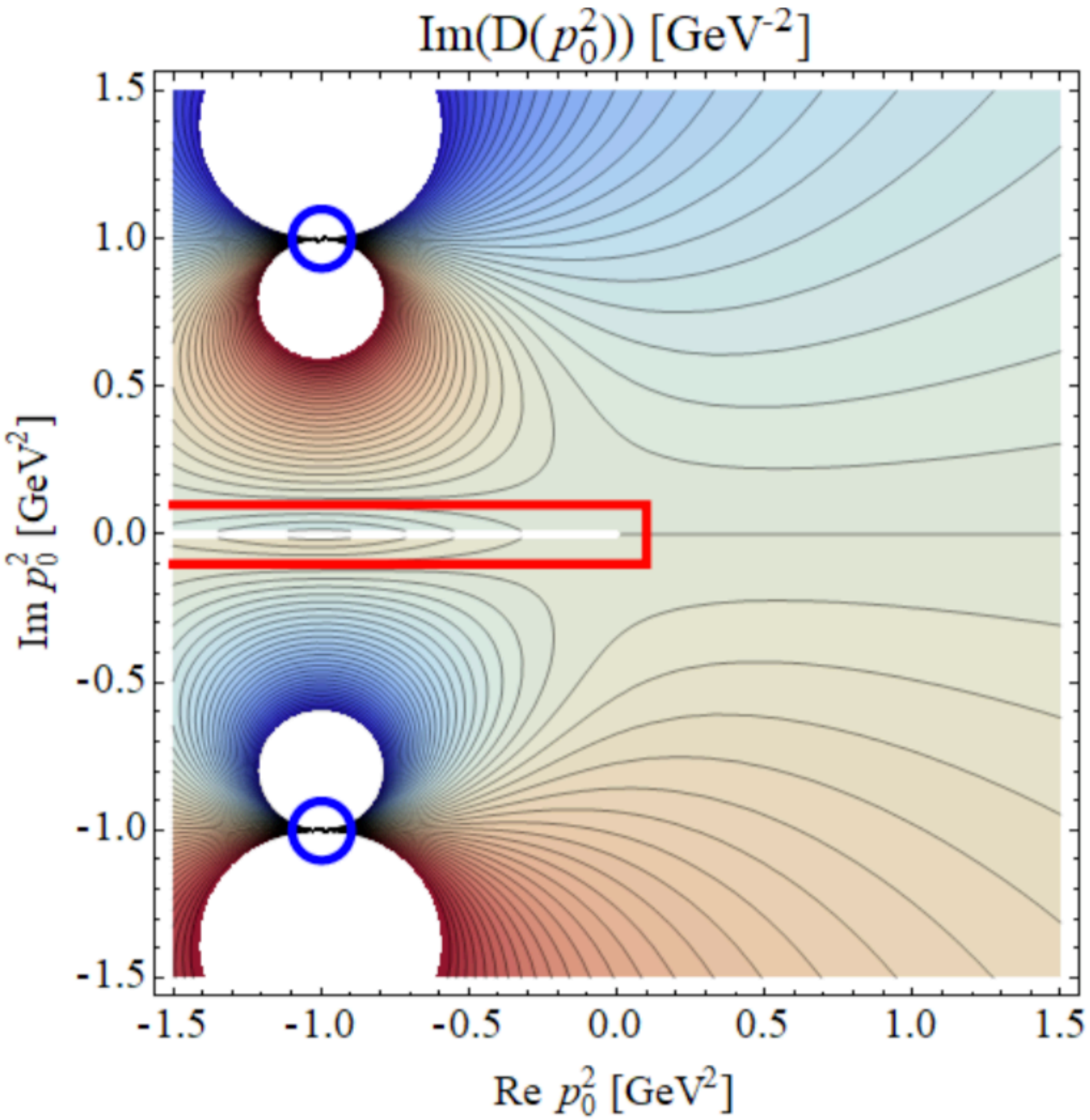


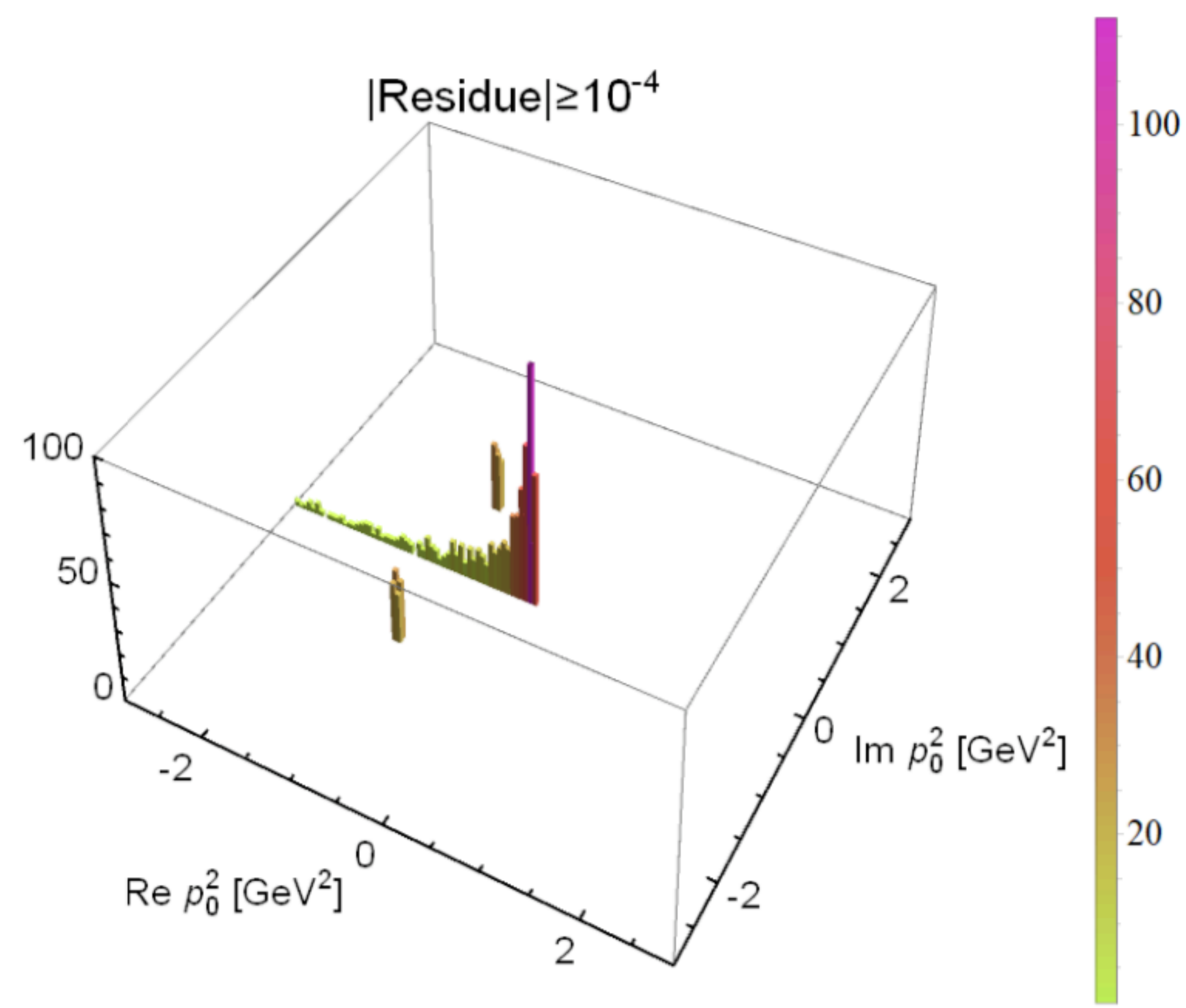
- **Reconstructed**



BW propagator plus poles II: p_0^2 plane

- **Branch cut:**
  more clearly visible in a histogram, showing the location of the poles for 100 random subsets of the 60 input points
  - **Exact**
  - **Reconstructed**





BW plus poles spectral function

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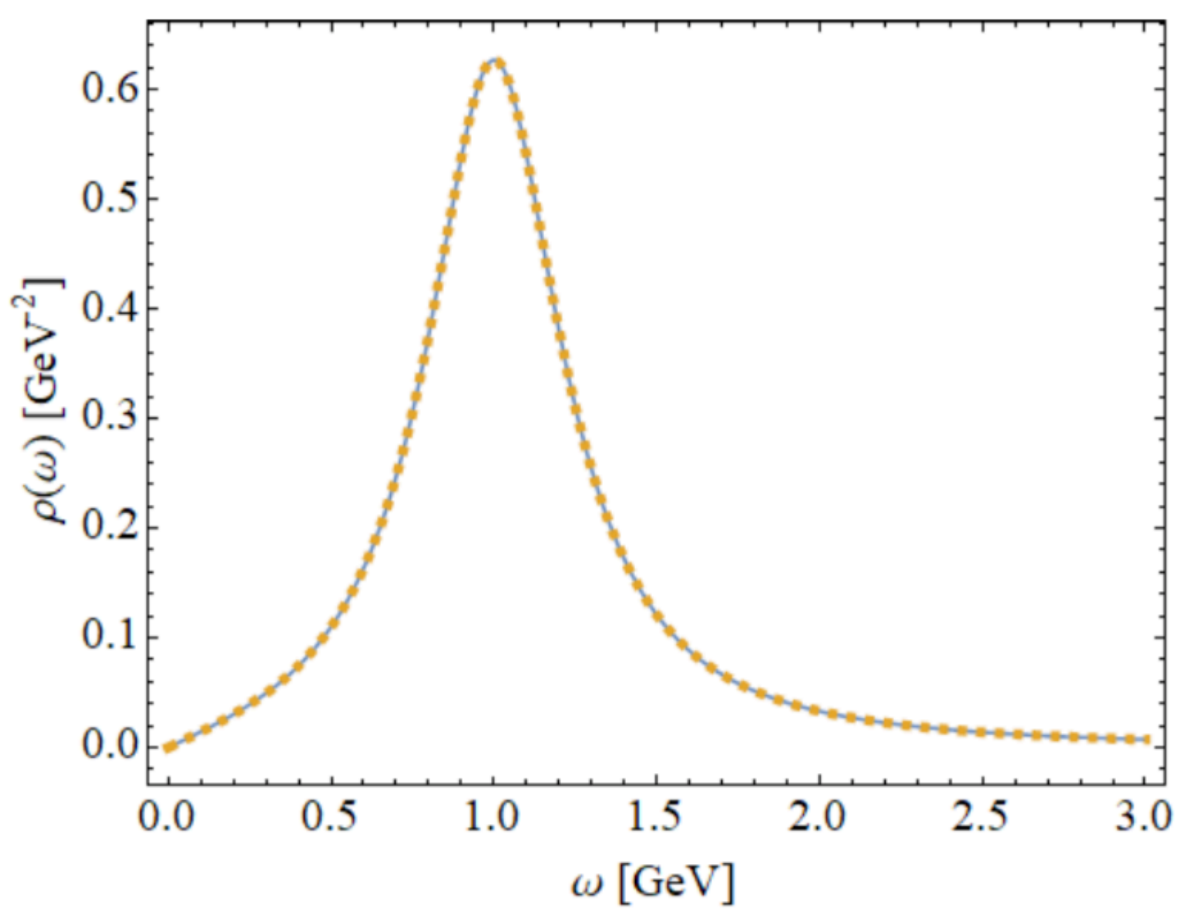
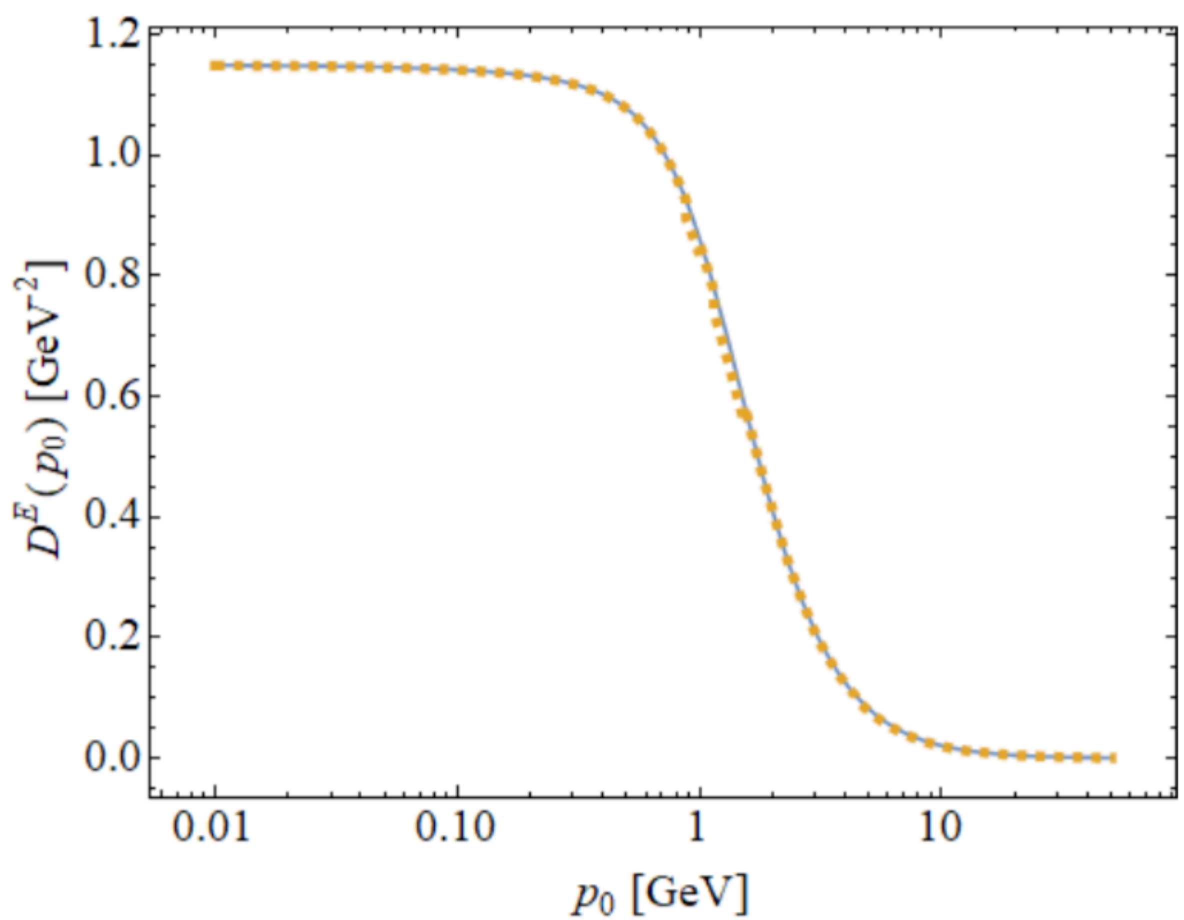
- **Reconstructed spectral function**



- **Reconstructed propagator**



Adding noise

- **Spice up life with some noise!**
  Set $D(p_{0i}) \to D(p_{0i})(1 + \varepsilon r_i)$ with $\varepsilon = 10^{-3}$, r_i a random number drawn from a normal distribution with zero mean and unit standard deviation