Analyticity at the service of spectroscopy

College of William and Mary / Jefferson Lab,

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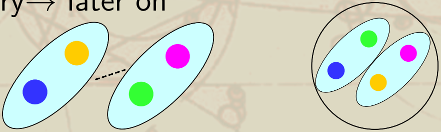
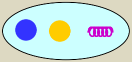
Motivation

- In this talk: Amplitude analyses of meson-meson interactions
- Ordinary hadrons → first part of the talk

- Not so ordinary→ later on

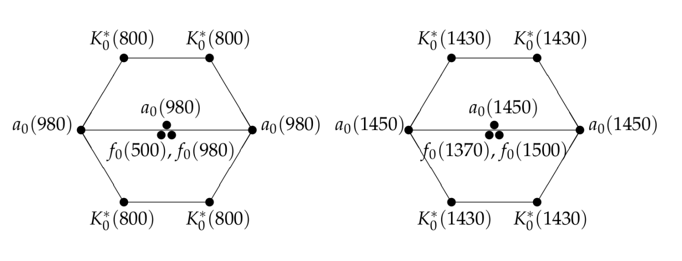


- Hybrid→ later on



- Glueball → last part

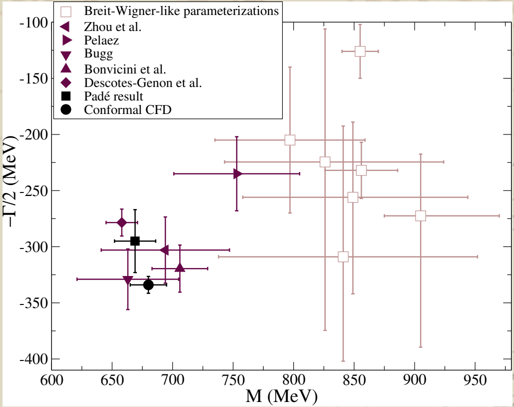
Spectroscopy for strange states



- Precise determination using model independent techniques.
- We can study more than 6 resonances appearing in πK.
- Another 4 appearing in $\pi\pi \to K\bar{K}$ scattering.
- Used to determine the $f_0(500)/\sigma$, the $K_0^*(700)/\kappa$, etc...

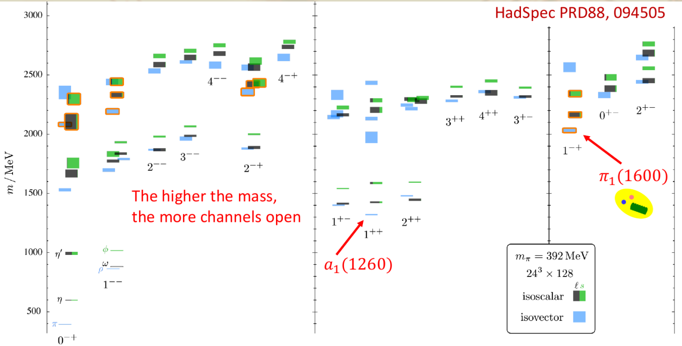
Motivation

- The $K_0^*(700)/\kappa$ meson still needs confirmation according to the PDG.
- Most of its determinations $\rightarrow$ simple models, Breit-Wigners, etc...
- Relevant for the identification of the scalar nonet.

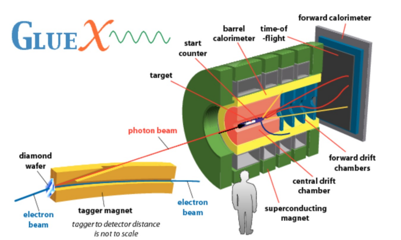


Motivation

- Light spectrum

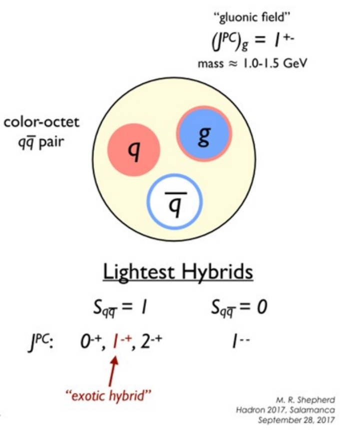


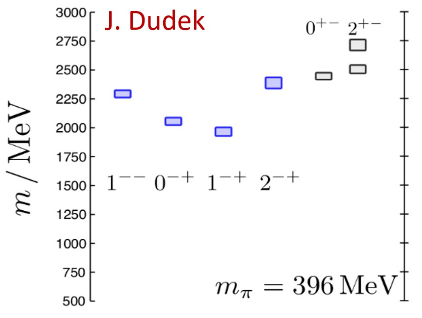
- Hybrids→ GlueX



Motivation

- Only one hybrid expected.
- $J^{PC} = 1^{-+} \rightarrow$ lightest hybrid candidate.





PDG status

- Glueball expected at around 1.5-2 GeV.
- Three different candidates measured close by.
- Only two of them should exist if there was ni glueball.

$f_0(1370)$ [j] $I^G(J^{PC}) = 0^+(0^{++})$

Mass m = 1200 to 1500 MeV
Full width Γ = 200 to 500 MeV

$f_0(1370)$ DECAY MODES	Fraction (Γ_i/Γ)	p (MeV/c)
$\pi\pi$	seen	672
4π	seen	617
$4\pi^0$	seen	617
$2\pi^+2\pi^-$	seen	612
$\pi^+\pi^-2\pi^0$	seen	615
$\rho\rho$	dominant	†
$2(\pi\pi)_{S\text{-wave}}$	seen	–
$\pi(1300)\pi$	seen	†
$a_1(1260)\pi$	seen	35
$\eta\eta$	seen	411
$K\overline{K}$	seen	475
$K\overline{K}n\pi$	not seen	†
6π	not seen	508
$\omega\omega$	not seen	†
$\gamma\gamma$	seen	685
e^+e^-	not seen	685

$f_0(1500)$ [n] $I^G(J^{PC}) = 0^+(0^{++})$

Mass $m = 1504 \pm 6$ MeV (S = 1.3)
Full width $\Gamma = 109 \pm 7$ MeV

$f_0(1500)$ DECAY MODES	Fraction (Γ_i/Γ)	Scale factor	p (MeV/c)
$\pi\pi$	(34.9±2.3) %	1.2	740
$\pi^+\pi^-$	seen		739
$2\pi^0$	seen		740
4π	(49.5±3.3) %	1.2	691
$4\pi^0$	seen		691
$2\pi^+2\pi^-$	seen		686
$2(\pi\pi)_{S\text{-wave}}$	seen		–
$\rho\rho$	seen		†
$\pi(1300)\pi$	seen		143
$a_1(1260)\pi$	seen		217
$\eta\eta$	(5.1±0.9) %	1.4	515
$\eta\eta'(958)$	(1.9±0.8) %	1.7	†
$K\overline{K}$	(8.6±1.0) %	1.1	568
$\gamma\gamma$	not seen		752

$f_0(1710)$ [t] $I^G(J^{PC}) = 0^+(0^{++})$

Mass $m = 1723^{+6}_{-5}$ MeV (S = 1.6)
Full width $\Gamma = 139 \pm 8$ MeV (S = 1.1)

$f_0(1710)$ DECAY MODES	Fraction (Γ_i/Γ)	p (MeV/c)
$K\overline{K}$	seen	706
$\eta\eta$	seen	665
$\pi\pi$	seen	851
$\omega\omega$	seen	360

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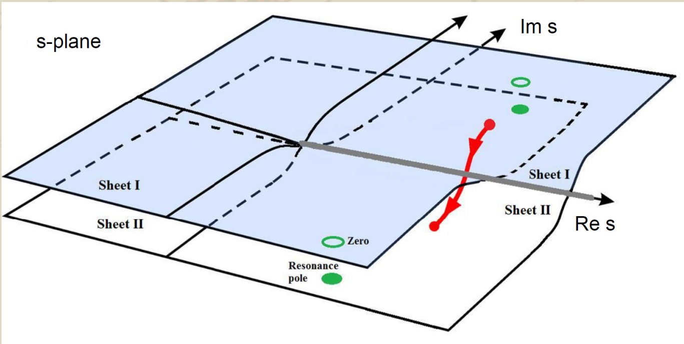
S-matrix principles: Unitarity

- UNITARITY both right and left branch cuts
  $SS^\dagger = I \Rightarrow T - T^\dagger = iTT^\dagger$.
- Due to elastic unitarity

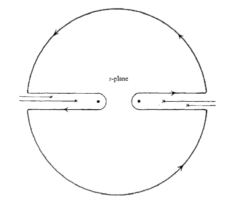
$$S^{II}(z) = \frac{1}{S^{I}(z)}.$$

- Looking for a zero of the scattering matrix in the first sheet.



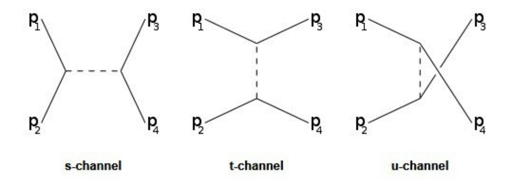
S-matrix principles: Analiticity and Crossing

- CAUSALITY⇒ANALITICITY No poles in the first Riemann sheet⇒ Cauchy theorem.



$$T(s,t) = \frac{1}{\pi}\int_{s_{th}}^{\infty} ds' \frac{\operatorname{Im} T(s',t)}{s'-s} + LHC.$$

- Possible subtractions.
- Using Cauchy theorem one can obtain Dispersion Relations (DR).
- Together with crossing → closed set of dispersion relations.



- There is a plethora of techniques and models → all of them accompanied by their drawbacks:
- Breit-Wigners → good for narrow, isolated resonances.
- K-matrix Good for overlapping resonances and inelasticities → could be poles in the FIRST Riemann sheet.
- Conformal mapping Good and stable in the ELASTIC region
- UChPT → SU(2) ChPT good (nice convergence and hierarchy) → not unitary. Unitarizing ChPT is not a systematic approach, as it also assumes and approximates several contgributions → SU(3) bad convergence, SU(4), SU(5) really?

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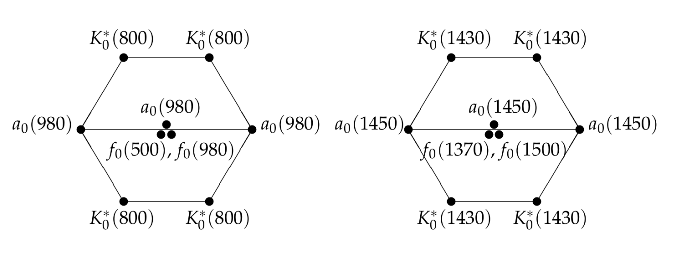
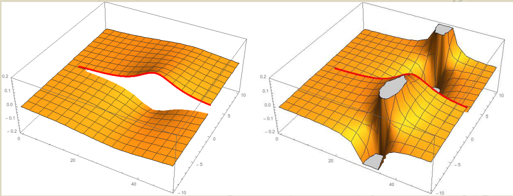
Meson Spectroscopy

Eur.Phys.J. C77 91

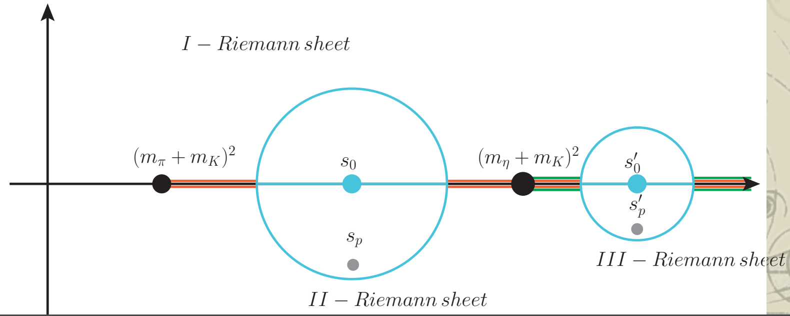


- Resonances→poles in unphysical sheets
- Analytic continuation is usually model dependent→ precise and model independent determination using S-matrix principles.



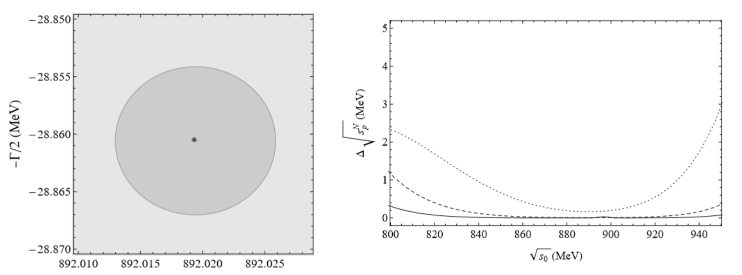
- High L or broad resonance parameters not stable when using simple models. Customary $(q(s)/q(s_r))^L$ and $B_L(q, q_r) \Rightarrow$ systematic effects.
- Rigorous dispersive techniques cannot get the poles at higher energies.
- Partial wave is described by a Padé approximant.

$$t_l(s) \simeq P_1^N(s, s_0) = \sum_{k=0}^{N-1} a_K (s - s_0)^k + \frac{a_N (s - s_0)^N}{1 - \frac{a_{N+1}}{a_N}(s - s_0)}.$$

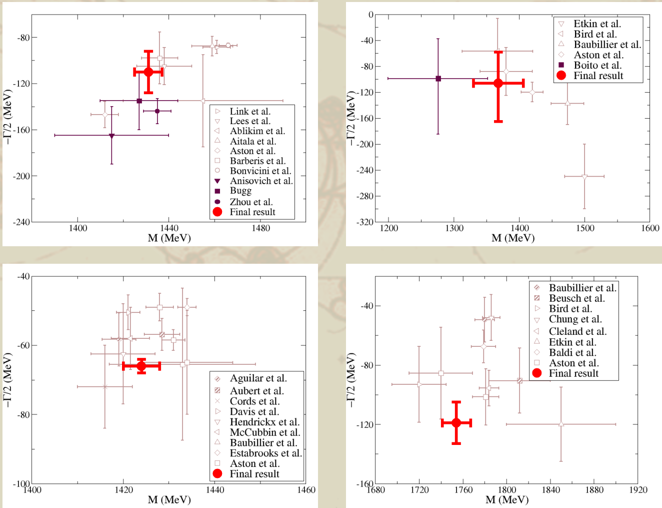


Meson Spectroscopy Eur.Phys.J. C77 91

- We stop at a N ($N+1$ derivatives) where the systematic uncertainty is smaller than the statistical one (usually $N=4$ is enough).
- s_0 fixed $\rightarrow$ gives the minimum difference between N and $N+1$.
- Run a Montecarlo for every fit to calculate the parameters and errors of each resonance.
- Different fitting functions (all dr constrained) included as systematics.



Meson Spectroscopy Eur.Phys.J. C77 91

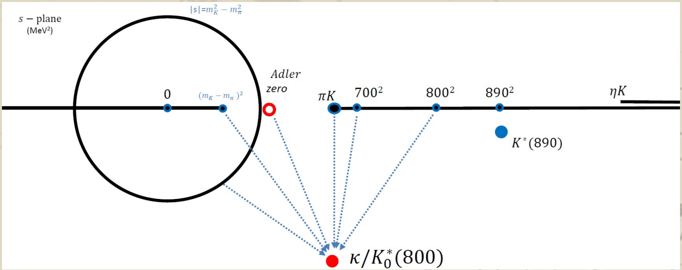


- Determination of $K_0^*(1430)$, $K_1^*(1410)$, $K_2^*(1430)$ and $K_3^*(1780)$ vs PDG averaged values.

Spectroscopy for the κ particle

Eur.Phys.J.C77 91

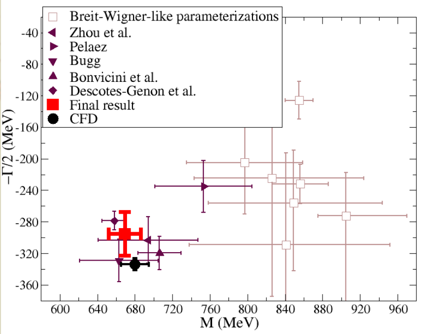
- Too broad to be determined using simple models.
- Threshold behavior (ChPT), Adler Zero and LHC play a role in its parameters.
- Problem shared by Lattice. Phys.Rev.Lett.123

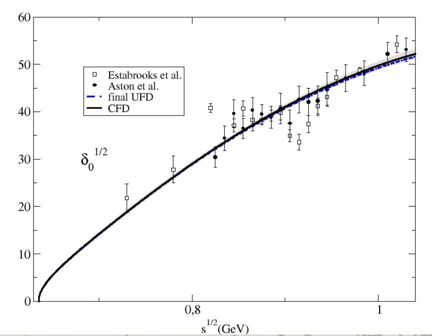


Spectroscopy for the κ particle Eur.Phys.J.C77 91

- $K_0^*(700)$ Padé → triggered the change of name from $K_0^*(800)$.

$$\sqrt{s_p} = (670 \pm 18) - i(295 \pm 28)MeV$$
$$\sqrt{s_p} = (682 \pm 29) - i(274 \pm 12)MeV\,(PDG) \quad (1)$$

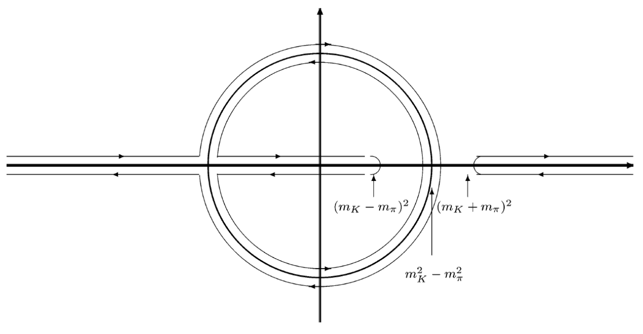




Spectroscopy for the κ particle

To be submitted

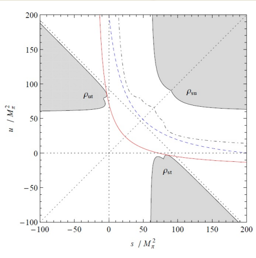
- Several different models and methods used to determine its parameters.
- Clear convergence with the use of analytic techniques.
- Model dependent determinations not suitable for this scenario.
- Model independent:$\rightarrow$ Padé (before), HDR dispersion relations (next) $S^{II}(s) = \frac{1}{S^{I}(s)}$.

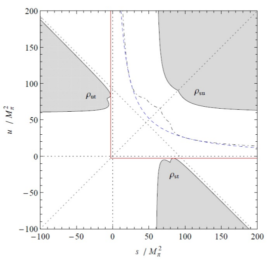


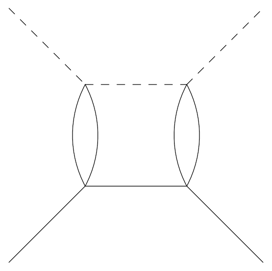
HDR

1-Eur.Phys.J.C 78 897, 2-Invited to Phys.Rep.

- Dispersion relations obeying $(s-a)(u-a)=b$. Most previous works $\rightarrow a=0$.
- This work: a used to maximize applicability region $\rightarrow$ next slide
- Relations between $\pi K, \pi\pi, K\bar{K}$ to determine scattering lengths.
- Sub-threshold expansion $\rightarrow$ Universal band not so universal.
- First describe all partial waves $\rightarrow$ impose HDR (Steiner), FDR and Roy eqs. on them for a total of 18 dispersion relations.







HDR both πK and $\pi\pi \to K\bar{K}$ Invited to Phys.Rep.

- HDR used for both πK and $\pi\pi \to K\bar{K}$ channels

$$f_0^{\pm}(s) = a_0^{\pm} + \frac{1}{\pi}\sum_l \int_{s_{th}}^{\infty} ds' K_{0l}^{\pm}(s,s') Im f_l^{\pm}(s')$$

$$+\frac{1}{\pi}\sum_l \int_{4m_\pi^2}^{\infty} dt' G_{0(2l-2),(2l-1)}^{\pm}(s,t') Im g_{(2l-2),(2l-1)}^{0,1}(t')$$

$$g_0^0(t) = \frac{\sqrt{3}}{2} m_+ a_0^+ + \frac{t}{\pi}\int_{4m_\pi^2}^{\infty} \frac{Im g_0^0(t')}{t'(t'-t)} dt'$$

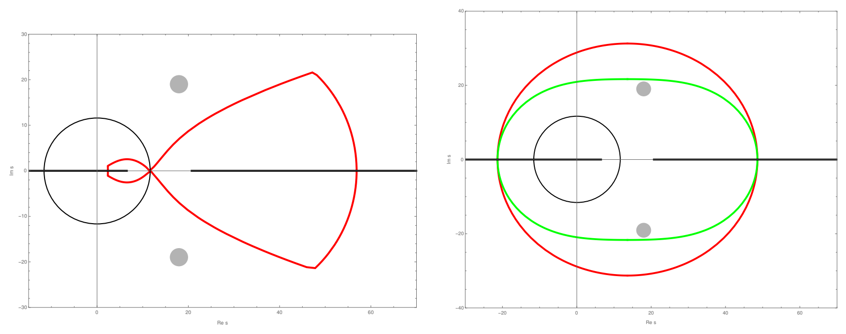
$$+\frac{t}{\pi}\sum_l \int_{4m_\pi^2}^{\infty} \frac{dt'}{t'} G_{0,2l-2}^{0}(t,t') Im g_{2l-2}^{0}(t') + \sum_l \int_{m_+^2}^{\infty} ds' G_{0,l}^{+}(t,s') Im f_l^{+}(s').$$

- πK inputs dominate their own partial waves
- Both channels are iterated until the result converges
- Suitable for calculating the $K_0^*(700)/\kappa$ resonance pole.

HDR

1-Eur.Phys.J.C 78 897, 2-Invited to Phys.Rep.

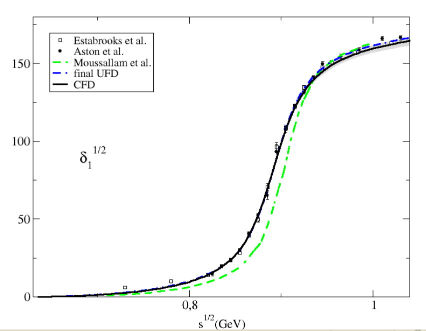
- Tension between FDR and Lattice.
- There is an universal band in the a_0^-, a_0^+ plane for $a_0^+ \Rightarrow$ no unique solution [Ananthanarayan et al. (2001)].
- Scarcity of πK data $\rightarrow$ SL poorly determined.
- $K_0^*(700)$ pole out of FDR/fixed-t range of validity.



Preliminary: $K^*(892)$ pole

To be submitted

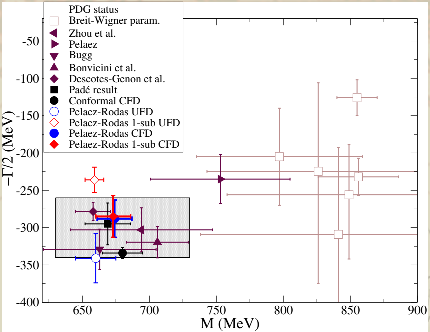
- Data description still compatible.
- Model independent continuation to the complex plane.



$$\sqrt{s_p} = (891 \pm 1) - i(56 \pm 2)/2 \text{ MeV} \quad \text{HDR}$$
$$\sqrt{s_p} = (892 \pm 1) - i(57 \pm 2)/2 \text{ MeV} \quad \text{CFD Fit}$$
$$\sqrt{s_p} = (896 \pm 1) - i(48 \pm 1)/2 \text{ MeV} \quad \text{PDG (Breit-Wigner)}$$

Preliminary: CFD $K_0^*(700)/\kappa$ pole To be submitted

- Compatible with previous analysis
- All uncertainties have been taken into account



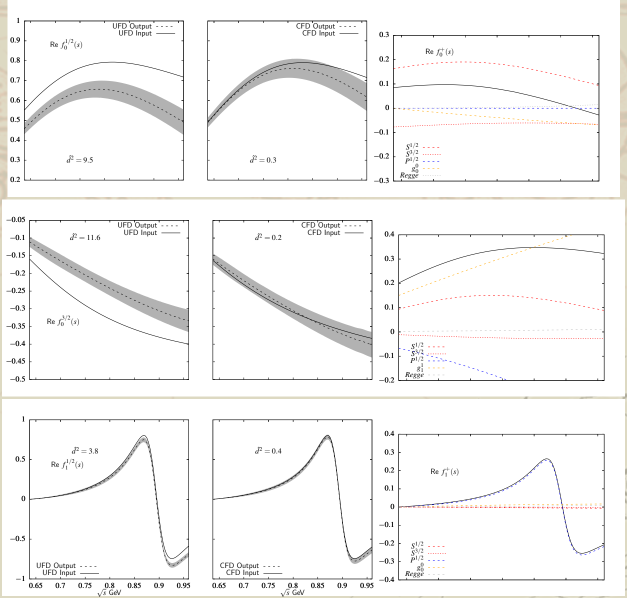
$$\sqrt{s_p} = (651 \pm 8) - i(572 \pm 32)/2 \text{ MeV}$$ HDR

$$\sqrt{s_p} = (658 \pm 13) - i(557 \pm 24)/2 \text{ MeV}$$ Descotes-Genon,Moussallam

$$\sqrt{s_p} = (680 \pm 50) - i(600 \pm 80)/2 \text{ MeV}$$ PDG

Preliminary results

Invited to Phys.Rep.



πK spectroscopy:Summary and projects

- We have extracted, in a model independent, and precise way the parameters of several strange resonances.
- For the first time the Padé method has been applied to the extraction of inelastic resonances with good convergence.
- The $K_0^*(700)/\kappa$ has been calculated using two different analytic methods.
- These results triggered the change of denomination in the PDG from $K_0^*(800)$ to $K_0^*(700)$.
- Our dispersive extraction is the only one fully accounting all uncertainties, while it is also the one describing the data.
- Future projects: $f_0(1370), f_0(1500)$ extraction.

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Hunting the $\pi_1(1600)$ Phys.Rev.Lett. 122, continuation of Phys.Lett.B 779 464-472

$\pi_1(1400)$ $I^G(J^{PC}) = 1^-(1^{-+})$

$\pi_1(1400)$ MASS	1354 ± 25 MeV (S = 1.8)
$\pi_1(1400)$ WIDTH	330 ± 35 MeV

Decay Modes

Mode		Fraction (Γ_i/Γ)
Γ_1	$\eta\pi^0$	seen
Γ_2	$\eta\pi^-$	seen
Γ_3	$\eta'\pi$	

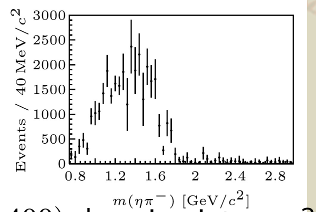
$\pi_1(1600)$ $I^G(J^{PC}) = 1^-(1^{-+})$

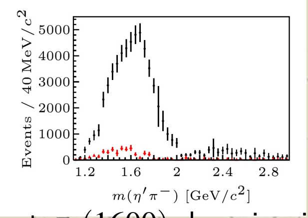
$\pi_1(1600)$ MASS	1662^{+8}_{-9} MeV
$\pi_1(1600)$ WIDTH	241 ± 40 MeV (S = 1.4)

Decay Modes

Mode		Fraction (Γ_i/Γ)
Γ_1	$\pi\pi\pi$	seen
Γ_2	$\rho^0\pi^-$	seen
Γ_3	$f_2(1270)\pi^-$	not seen
Γ_4	$b_1(1235)\pi$	seen
Γ_5	$\eta'(958)\pi^-$	seen
Γ_6	$f_1(1285)\pi$	seen

- PDG reports 2 different resonances



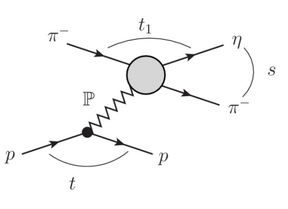


- $\pi_1(1400)$ decaying into $\eta\pi$? Different $\pi_1(1600)$ decaying into $\eta'\pi$?
- A hybrid should not decay to $\eta\pi$

Hunting the $\pi_1(1600)$

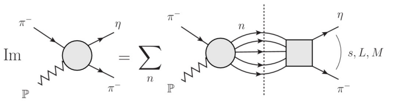
Phys.Rev.Lett. 122 042002

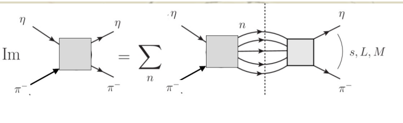
- Peripheral production $\Rightarrow$ factorization of the pomeron $\Rightarrow Ima(s) = \rho(s)t^*(s)a(s)$.



- Amplitude built using $t(s) = \frac{N(s)}{D(s)}$ method $\Rightarrow a(s) = p^2 q \frac{n(s)}{D(s)}$.

- Numerators are smooth polynomials $n(s) = \sum_j a_j w^j(s)$, where $w(s) = \frac{s}{s+s_0}$.



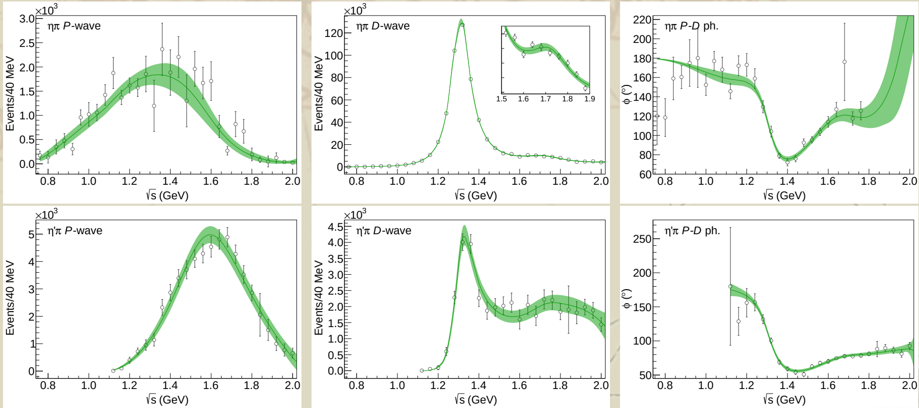


- K-matrix approach with dispersive phase space.

$$D^J(s)_{ki} = (K^J(s)^{-1})_{ki} - \frac{s}{\pi}\int_{s_k}^{\infty} ds' \frac{\rho(s')N^J_{ki}(s')}{s'(s'-s-i\varepsilon)}.$$

Hunting the $\pi_1(1600)$ Phys.Rev.Lett. 122 042002

- We use an average of 6 parameters for each figure.
- $\chi^2 \approx 1.3$, no significant deviation for any partial wave.
- 1 T-matrix pole produces 2 different peaks for the P-wave $\rightarrow$ 300 MeV distance.

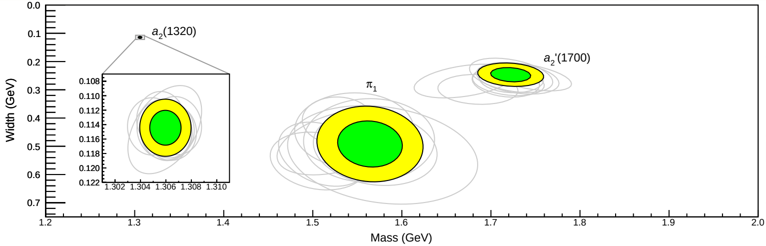


Hunting the $\pi_1(1600)$ Phys.Rev.Lett. 122 042002

- Most robust extraction of this hybrid candidate.
- Theoretical predictions and experiment reconciled.
- Statistical uncertainties $\rightarrow$ 100k sample bootstrap.

Poles	Mass (MeV)	Width (MeV)
$a_2(1320)$	$1306.0 \pm 0.8 \pm 1.3$	$114.4 \pm 1.6 \pm 0.0$
$a_2'(1700)$	$1722 \pm 15 \pm 67$	$247 \pm 17 \pm 63$
$\pi_1(1600)$	$1564 \pm 24 \pm 86$	$492 \pm 54 \pm 102$

- Systematics (diferent LHC, numerators, subtractions ...) included.



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$a_1(1260) \to 3\pi$

Phys.Rev. D98 096021

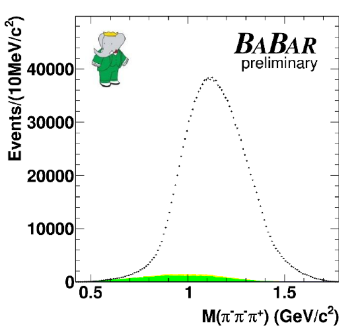
- Simple application of 3-body to an interesting resonance.

$a_1(1260)$ [k] $I^G(J^{PC}) = 1^-(1^{++})$

Mass $m = 1230 \pm 40$ MeV [l]
Full width $\Gamma = 250$ to 600 MeV

$a_1(1260)$ DECAY MODES	Fraction (Γ_i/Γ)	p (MeV/c)
$(\rho\pi)_{S-\text{wave}}$	seen	353
$(\rho\pi)_{D-\text{wave}}$	seen	353
$(\rho(1450)\pi)_{S-\text{wave}}$	seen	†
$(\rho(1450)\pi)_{D-\text{wave}}$	seen	†
$\sigma\pi$	seen	–
$f_0(980)\pi$	not seen	179
$f_0(1370)\pi$	seen	†
$f_2(1270)\pi$	seen	†
$K\bar{K}^*(892)+$ c.c.	seen	†
$\pi\gamma$	seen	608

3π final states



- Decays mostly to 3π final states.
- The τ decay extraction should be the cleanest.
- CLEO $\to$ Dominant $\rho\pi$ channel.

$a_1(1260) \to 3\pi$

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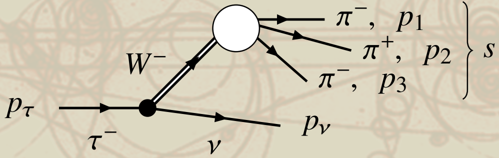
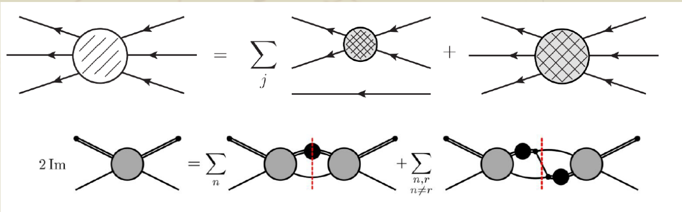


- Extracted from the decay $\tau^- \to \pi^-\pi^+\pi^-\nu_\tau$



$a_1(1260) \to 3\pi$

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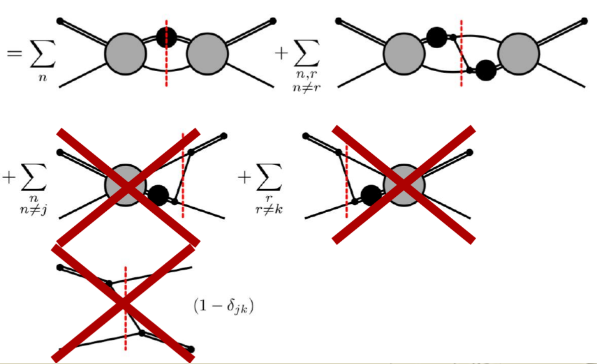
- Simplified model of 3 body interactions → disconnected diagrams neglected



- Algebraic unitarity equation

$$\mathrm{Im}\, t(s) = t(s) t^{\dagger}(s) \int d\phi_3 |\sum_j f(\sigma_j)|^2$$

$a_1(1260) \to 3\pi$ Phys.Rev. D98 096021

- Amplitude defined through:

$$t(s) = \frac{g^2}{m^2 - s - ig^2 C(s)/2},$$

- Phase space:

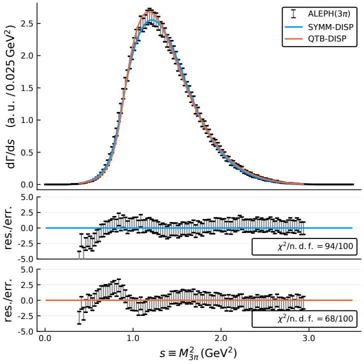
$$\rho_{\mathrm{SYMM}}(s) = \frac{1}{2}\int \mathrm{d}\Phi_3 \left| f_\rho(\sigma_1) N_0(\Omega_1,\Omega_{23}) - f_\rho(\sigma_3) N_0(\Omega_3,\Omega_{12}) \right|^2$$

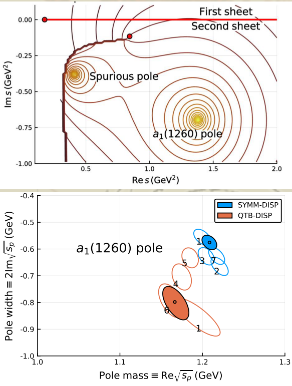
$$\rho_{\mathrm{SYMM-DISP}}(s) = l_0 + \frac{s}{\pi}\int ds' \frac{\rho_{\mathrm{SYMM}}(s')}{s'(s'-s-i\varepsilon)},$$

- The ρ line shape is implemented through its customary Breit-Wigner shape.

$a_1(1260) \to 3\pi$ Phys.Rev. D98 096021

- Clear deviations from not-symmetrized results.
- Symmetrized has way less deviations and more right physics
- Statistical errors calculated through bootstrap.
- $m = 1209 \pm 4^{+12}_{-9}$ MeV and $\Gamma = 576 \pm 11^{+89}_{-20}$ MeV.





Spectroscopy 2:Summary and projects

- We have extracted in a robust way only one hybrid meson decaying to $\eta^{(\prime)}\pi$.
- There is no statistical significance for a second light hybrid.
- We have extracted the parameters of the $a_1(1260)$ from a three body decay.
- The τ decay should be very clean, and the dispersive model has the smallest systematic effects.
- All these extractions have been included in the PDG and used for they new averages.
- Future projects: Include more $\eta^{(\prime)}\pi$ data, include $U(3)$ scattering lengths?.
- Future projects: Khuri-Treiman for particles with spin.

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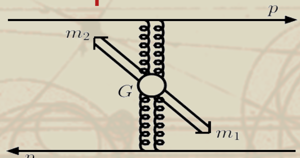
Consensus?

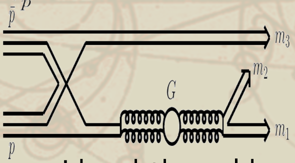
- The glueball is expected to be predominant in either the $f_0(1500)$ or the $f_0(1710)$.
- Not much of a consensus → **V. Mathieu et al. Int.J.Mod.Phys. E18 (2009) 1-49**.
- Recent years → not much of an improvement.
- $f_0(1500) \rightarrow 0.89|gg\rangle$ **Giacosa et al. Phys.Rev. D72 (2005) 094006**.
- $f_0(1710) \rightarrow 0.93|gg\rangle$ **Albaladejo-Oller Phys.Rev.Lett. 101 (2008) 252002**.

Data: Glueball “rich" experiments

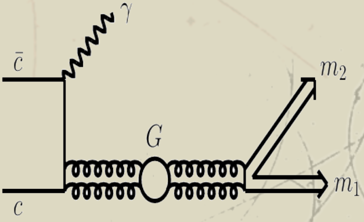
- Pomeron collisions



- $p\bar{p}$ anihilation

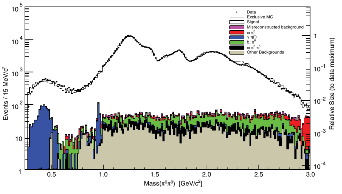


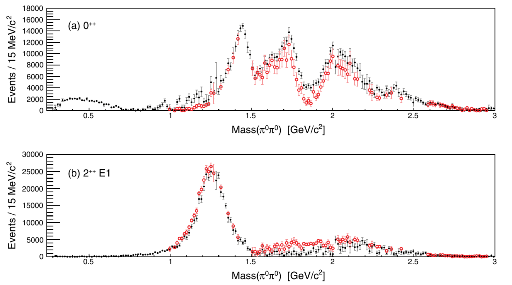
- J/ψ radiative decays considered the golden channel for glueballs.



Data: BESIII $J/\psi \to \gamma\pi\pi$

- Data on $J/\psi \to \gamma\pi\pi$ half a million events.

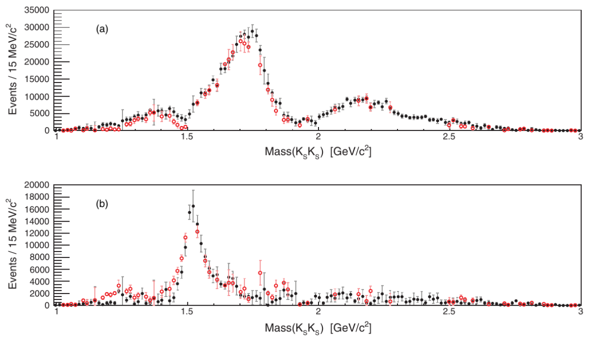




- 3 prominent f_0's with similar couplings.
- The $2^{++}E1$ partial wave is dominated by the $f_2(1270)$.

Data: BESIII $J/\psi \to \gamma K\bar{K}$

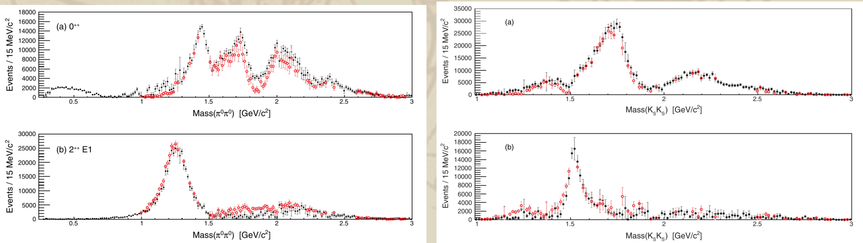
- Another 3 prominent f_0's



- The couplings are fairly different, with a way more prominent $f_0(1710)$.
- The $2^{++}E1$ partial wave is dominated by the $f_2'(1525)$.

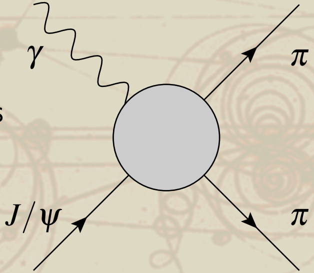
Data: BESIII J/ψ

- How many f_0 do we have here?

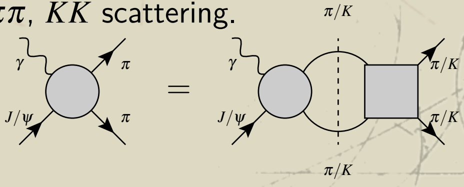


- Is the coupling of the $f_0(1710)$ greater$\rightarrow$ glueball hint?

$J/\psi \to \gamma\, m_1\, m_2$

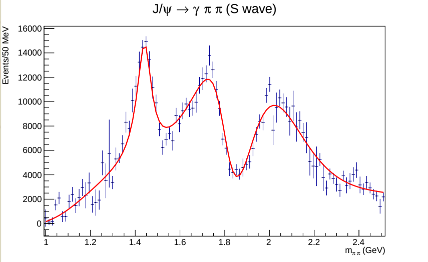


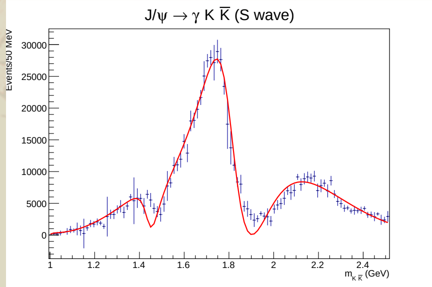
- Slightly different kinematics
- Left hand cut $\to s = 0$ GeV.
- $Ima(s) = \rho(s)t(s)^{*}a(s)$
- $t(s) \to \pi\pi,\, K\bar{K}$ scattering.



Coupled-channel scenario

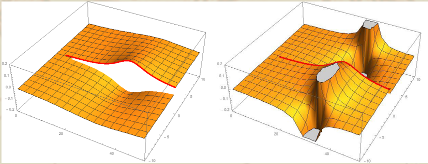
- Fit from 1 GeV to 2.5 GeV, $\chi^2 \approx 1.7$.
- Interested in the f_0.
- Coupled channel between just $\pi\pi$ and $K\bar{K}$.





Complex plane

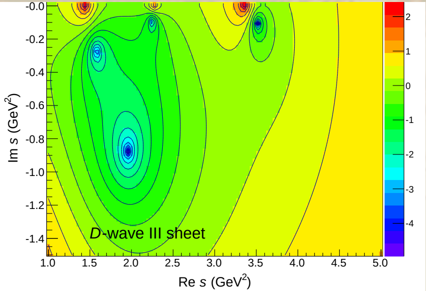
- We use the analytical properties of the parameterization → complex plane continuation.

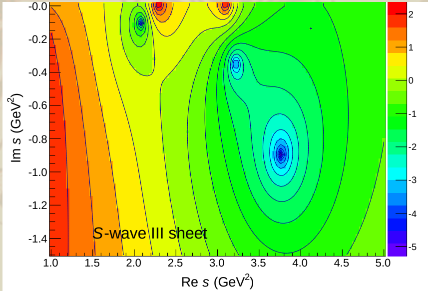


- $m(f_0(1500)) = 1460$ MeV $\quad \Gamma(f_0(1500)) = 85$ MeV.
- $m(f_0(1710)) = 1800$ MeV $\quad \Gamma(f_0(1710)) = 190$ MeV.
- $m(f_0(210)) = 1970$ MeV $\quad \Gamma(f_0(1710)) = 490$ MeV.

Scalar poles

- Complex plane plots





- Few “spurious” poles, all far from real axis
- HOWEVER, there is ONE FIRST sheet pole.

- Testing new ideas
- Another f_0 above 2 GeV?
- Decoupling between numerator and denominator
- Several Chew-Mandelstam terms
- Future tests:
- A third inelastic threshold
- Three body effects?

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- We have shown how powerfull analytic techniques are to the extraction of relevant resonance poles.
- Precise analytical and dispersive determinations of the $K_0^*(700)/\kappa$ meson.
- We have extracted in a model independent way the parameters of 7 strange resonances.
- We have determined the existence of one hybrid meson.
- We have started extracting parameters of resonances decaying to more than two mesons.
- Several ongoing works divided between many JPACers.

Thank you for your attention!

Spare slides!

Omnès-Muskhelishvili equations

- Omnès-Muskhelishvili DR with as less subtractions as possible
- S-channel and T-channel coupled in a complicated non-linear way

$$g_0^0(t) = \Delta_0^0(t) + \frac{t\Omega_0^0(t)}{t_m - t}\left[\alpha + \frac{t}{\pi}\int_{4m_\pi^2}^{t_m} dt' \frac{(t_m - t')\Delta_0^0(t')\sin\phi_0^0(t')}{\Omega_{0,R}^0(t')t'^2(t'-t)}\right.$$
$$\left. + \frac{t}{\pi}\int_{t_m}^{\infty} dt' \frac{(t_m - t')|g_0^0(t')|\sin\phi_0^0(t')}{\Omega_{0,R}^0(t')t'^2(t'-t)}\right],$$
$$g_1^1(t) = \Delta_1^1(t) + \Omega_1^1(t)\left[\frac{1}{\pi}\int_{4m_\pi^2}^{t_m} dt' \frac{\Delta_1^1(t')\sin\phi_1^1(t')}{\Omega_{1,R}^1(t')(t'-t)}\right.$$
$$\left. + \frac{1}{\pi}\int_{t_m}^{\infty} dt' \frac{|g_1^1(t')|\sin\phi_1^1(t')}{\Omega_{1,R}^1(t')(t'-t)}\right].$$

- If more subtractions$\Rightarrow$ scalar and vector partial waves coupled in a non-linear way.

Regge poles: single channel

- The contribution of a single pole to a partial wave is

$$f(J,s) = f_{background} + \frac{\beta(s)}{J-\alpha(s)} \approx \frac{\beta(s)}{J-\alpha(s)}$$

- $\alpha(s)$ is the position of the pole, whereas $\beta(s)$ is the residue.
- Unitarity condition on the real axis implies

$$Im\alpha(s) = \rho(s)\beta(s)$$

- The analytical properties of $\beta(s)$ implies

$$\beta(s) = \frac{\hat{s}^{\alpha(s)}}{\Gamma(\alpha(s)+3/2)}\gamma(s)$$

- The trajectory and residue should satisfy these integral equations:

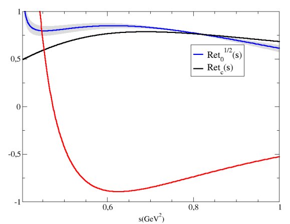
$$\mathrm{Re}\,\alpha(s) = \alpha_0 + \alpha' s + \frac{s}{\pi} PV \int_{4m^2}^{\infty} ds' \frac{\mathrm{Im}\alpha(s')}{s'(s'-s)},$$

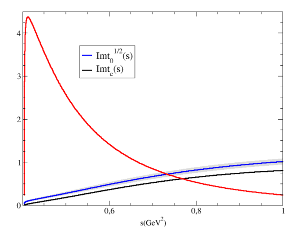
$$\mathrm{Im}\,\alpha(s) = \frac{\rho(s) b_0 \hat{s}^{\alpha_0+\alpha' s}}{|\Gamma(\alpha(s)+\frac{3}{2})|} \exp\Bigg(-\alpha' s[1-\log(\alpha' s_0)] + \frac{s}{\pi} PV \int_{4m^2}^{\infty} ds' \frac{\mathrm{Im}\alpha(s')\log\frac{\hat{s}}{\hat{s}'} + \arg\Gamma\left(\alpha(s')+\frac{3}{2}\right)}{s'(s'-s)}\Bigg),$$

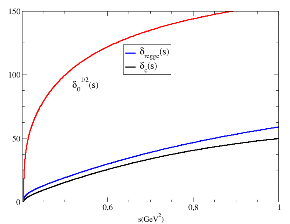
$$\beta(s) = \frac{b_0 \hat{s}^{\alpha_0+\alpha' s}}{\Gamma(\alpha(s)+\frac{3}{2})} \exp\Bigg(-\alpha' s[1-\log(\alpha' s_0)] + \frac{s}{\pi} \int_{4m^2}^{\infty} ds' \frac{\mathrm{Im}\alpha(s')\log\frac{\hat{s}}{\hat{s}'} + \arg\Gamma\left(\alpha(s')+\frac{3}{2}\right)}{s'(s'-s)}\Bigg),$$

- Constants fixed by forcing the amplitude to have THE POLE AND RESIDUE OF THE DESIRED RESONANCE

κ resonance: ordinary vs non-ordinary







- If we impose a linear Regge trajectory the result does not describe the data.

Hunting the $\pi_1(1600)$: formula

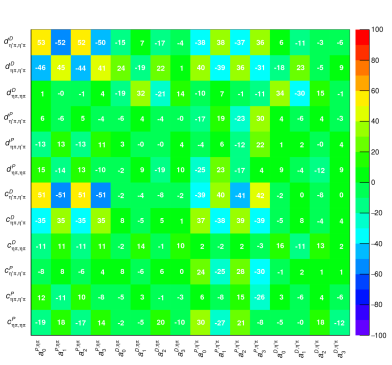
- $\eta^{(\prime)}\pi$ coupled channel up to 2 GeV.
- $\rho\pi$ cannot be included without including big systematic contribution (Deck).
- We use a K-matrix approach with a Chew-Mandelstam phase space.

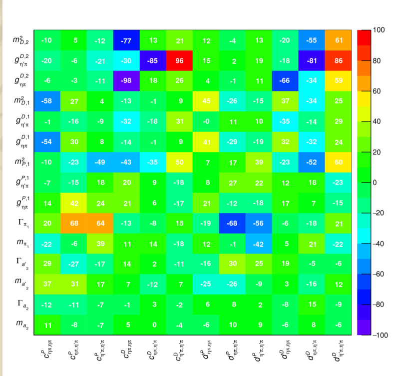
$$
\begin{aligned}
D(s)_{ij} &= (K^{-1})_{ij}(s) - \frac{s}{\pi}\int_{s_{th}}^{\infty} ds' \frac{\rho(s')N(s')}{s'(s'-s)}, \\
K(s)_{ij} &= \sum_R \frac{g_i^R g_j^R}{m_R^2 - s} + c_{ij} + d_{ij}s.
\end{aligned}
$$

- Just 1 K-matrix pole for the P-wave, 2 for the D-wave.

Hunting the $\pi_1(1600)$: correlation

- Numerator and denominator parameters uncorrelated
- P and D waves not correlated





Hunting the $\pi_1(1600)$: systematics

- Numerator

Systematic	Poles	Mass (MeV)	Deviation (MeV)	Width (MeV)	Deviation (MeV)
Variation of the numerator function $n(s)$					
Polynomial expansion	$a_2(1320)$	1305.9	-0.1	114.7	0.3
	$a_2'(1700)$	1723	1	249	2
	$\pi_1(1600)$	1563	-1	479	-13
Systematic assigned	$a_2(1320)$		0.0		0.0
	$a_2'(1700)$		0		0
	$\pi_1(1600)$		0		0
$t_{\text{eff}} = -0.5\,\text{GeV}^2$	$a_2(1320)$	1306.8	0.8	114.1	-0.3
	$a_2'(1700)$	1730	8	259	13
	$\pi_1(1600)$	1546	-18	443	-49
Systematic assigned	$a_2(1320)$		0.8		0.0
	$a_2'(1700)$		0		0
	$\pi_1(1600)$		0		0

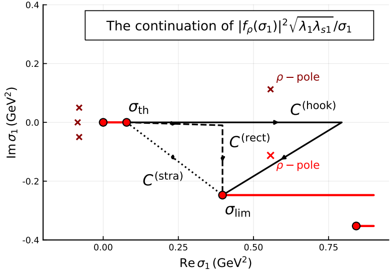
Denominator

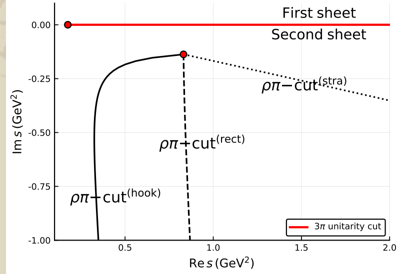
Systematic	Poles	Mass (MeV)	Deviation (MeV)	Width (MeV)	Deviation (MeV)
Variation of the function $\rho N(s')$					
$s_L = 0.8\,\mathrm{GeV}^2$	$a_2(1320)$	1306.4	0.4	115.0	0.6
	$a_2'(1700)$	1720	-3	272	26
	$\pi_1(1600)$	1532	-33	484	-8
$s_L = 1.8\,\mathrm{GeV}^2$	$a_2(1320)$	1305.6	-0.4	113.2	-1.2
	$a_2'(1700)$	1743	21	254	7
	$\pi_1(1600)$	1528	-36	410	-82
Systematic assigned	$a_2(1320)$		0.0		0.0
	$a_2'(1700)$		21		26
	$\pi_1(1600)$		36		82
$\alpha = 1$	$a_2(1320)$	1305.9	-0.1	114.7	0.3
	$a_2'(1700)$	1685	-37	299	52
	$\pi_1(1600)$	1506	-58	552	60
Systematic assigned	$a_2(1320)$		0.0		0.0
	$a_2'(1700)$		37		52
	$\pi_1(1600)$		58		60
$Q_J, \alpha = 1$	$a_2(1320)$	1304.9	-1.1	114.2	-0.2
	$a_2'(1700)$	1670	-52	269	22
	$\pi_1(1600)$	1511	-53	528	36
$Q_J, \alpha = 1.5$	$a_2(1320)$	1306.0	0.1	115.0	0.6
	$a_2'(1700)$	1717	-5	272	25
	$\pi_1(1600)$	1578	14	530	39
$Q_J, \alpha = 2$	$a_2(1320)$	1306.2	0.2	114.7	0.3
	$a_2'(1700)$	1723	1	261	15
	$\pi_1(1600)$	1570	6	508	16
Systematic assigned	$a_2(1320)$		1.1		0.0
	$a_2'(1700)$		52		25
	$\pi_1(1600)$		53		0

$a_1(1260)$

- Integration path complex

$$\rho_{QTB} \propto \int_{4m_\pi^2}^{(\sqrt{s}-m_\pi)^2} d\sigma_1 f^{II}(\sigma_1) f^{I}(\sigma_1) \frac{\sqrt{\lambda_1 \lambda_{s1}}}{\sigma_1}$$

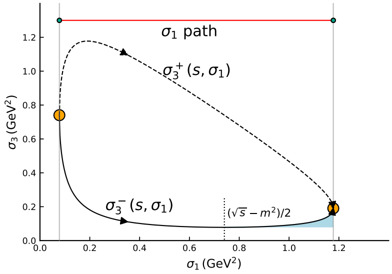


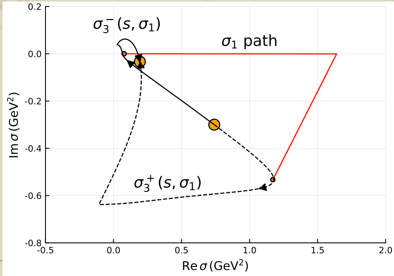


$a_1(1260)$

- The endpoints of integration move along the lines

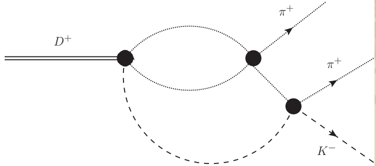
$$\rho_{\rm INT}(s) = \frac{1}{2\pi(8\pi)^2 s}\int_{4m_\pi^2}^{\sigma_{\rm lim}} {\rm d}\sigma_1 \int_{\sigma_3^-(\sigma_1,s)}^{\sigma_3^+(\sigma_1,s)} {\rm d}\sigma_3 \frac{f_\rho^{(II)}(\sigma_1)}{\sqrt{\sigma_1-4m_\pi^2}} \frac{f_\rho^{(I)}(\sigma_3)}{\sqrt{\sigma_3-4m_\pi^2}} \times \frac{W(\sqrt{s},\sqrt{\sigma_1},\sqrt{\sigma_3})}{((\sqrt{s}+\sqrt{\sigma_1})^2-m_\pi^2)((\sqrt{s}+\sqrt{\sigma_3})^2-m_\pi^2)}.$$



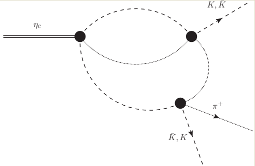


Future Project: Khuri-Treiman analysis

- Already done for $D^+ \to \pi^+\pi^+K^-$ [Niecknig, Kubis (2015)].
- Using πK as input and predicting D^+ decay.



- Can we make it work the other way around η_c decay $\Rightarrow \pi K$ scattering.

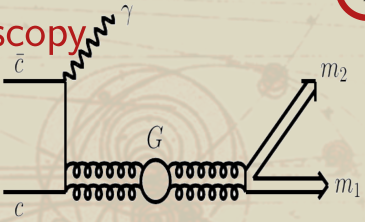


- New high precision LCHb/BELLE 2 data?

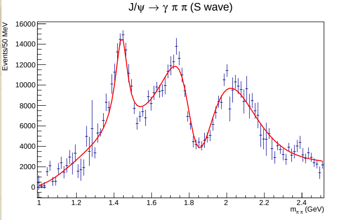
Future project: New HDR

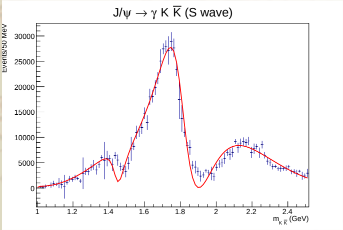
- It's been shown (cita) that symmetric variables under s, t, u exchanges offer the biggest convergence in the complex plane.
- Maximum energy in the real axis $\rightarrow$ 1.7 GeV.
- It offers two possibilities:
- 1- Select between incompatible data sets above 1.4 GeV.
- 2- Determine if the $f_0(1370), f_0(1500)$ appear in this proccess $\rightarrow$ glueball related .

Future projects: More spectroscopy



- $J/\psi \to \gamma\pi\pi$
- How many f_0 do we have here?





- Khuri-Treiman $\to$ particles with spin.

Future projects: Regge with spin

- What about resonances decaying to resonances?
- Their spin can be accommodated through a set of coupled integral eqs.
- What about inelastic resonances $\rightarrow$ coupled channel Regge physics.

$a_1(1260) \to 3\pi$ Phys.Rev. D98 096021

- Phase space:

$$\rho_{\mathrm{SYMM}}(s) = \frac{1}{2}\int \mathrm{d}\Phi_3 \left| f_\rho(\sigma_1) N_0(\Omega_1, \Omega_{23}) - f_\rho(\sigma_3) N_0(\Omega_3, \Omega_{12}) \right|^2$$

$$\rho_{\mathrm{SYMM-DISP}}(s) = l_0 + \frac{s}{\pi}\int ds' \frac{\rho_{\mathrm{SYMM}}(s')}{s'(s' - s - i\varepsilon)},$$

- The ρ line shape is implemented through its customary Breit-Wigner shape.

$$f(\sigma) = \mathcal{N} \frac{p(\sigma)R}{\sqrt{1 + (p(\sigma)R)^2}} \frac{1}{m_\rho^2 - \sigma - i m_\rho \Gamma_\rho(\sigma)}$$

$$\Gamma(\sigma) = \Gamma_\rho \times \frac{p^3(\sigma)}{\sqrt{\sigma}\sqrt{1 + (p(\sigma)R)^2}} \frac{p\left(m_\rho^2\right)}{m_\rho \sqrt{1 + \left(p\left(m_\rho^2\right) R\right)^2}}$$