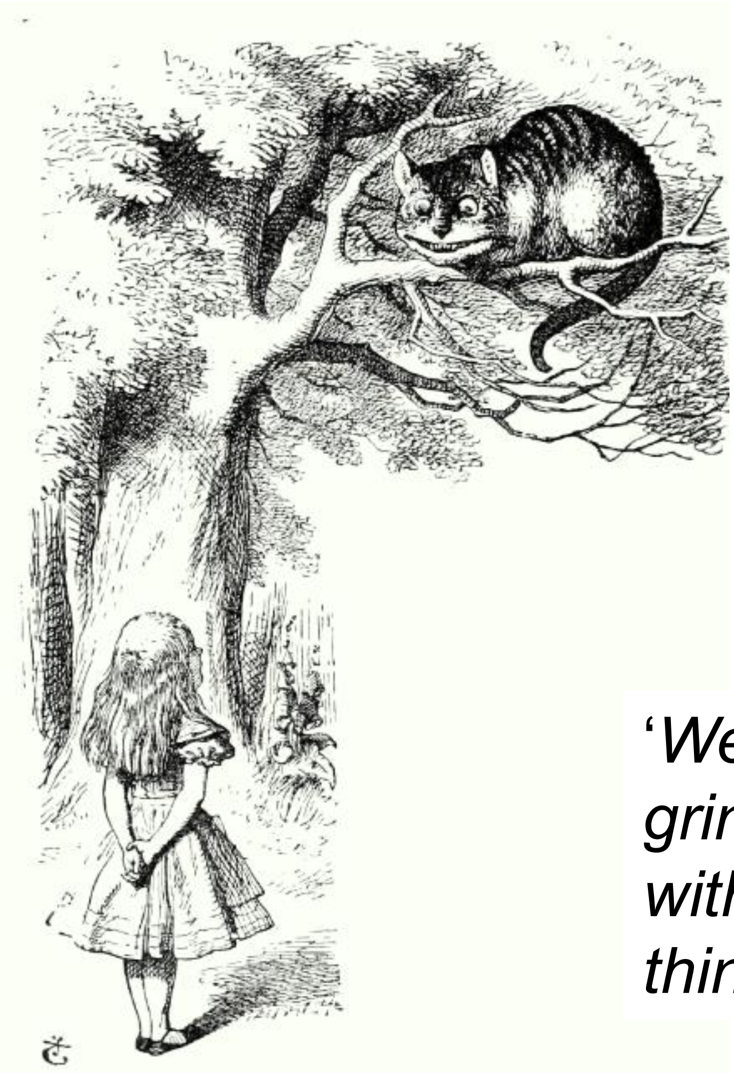
Disappearing and delocalized properties: Weak measurements and the Quantum Cheshire Cat effect

Alex Matzkin

Lab de physique théorique et modélisation
CNRS Unit 8089 @ Université de Cergy-Pontoise

4 Oct 2019
ECT Trento

Vanishing of the Cheshire cat (illustrations by J. Tenniel for the original 1865 edition)



'Well! I've often seen a cat without a grin,' thought Alice; 'but a grin without a cat! It's the most curious thing I ever saw in my life!'

Quantum Cheshire cat in interferometric setups

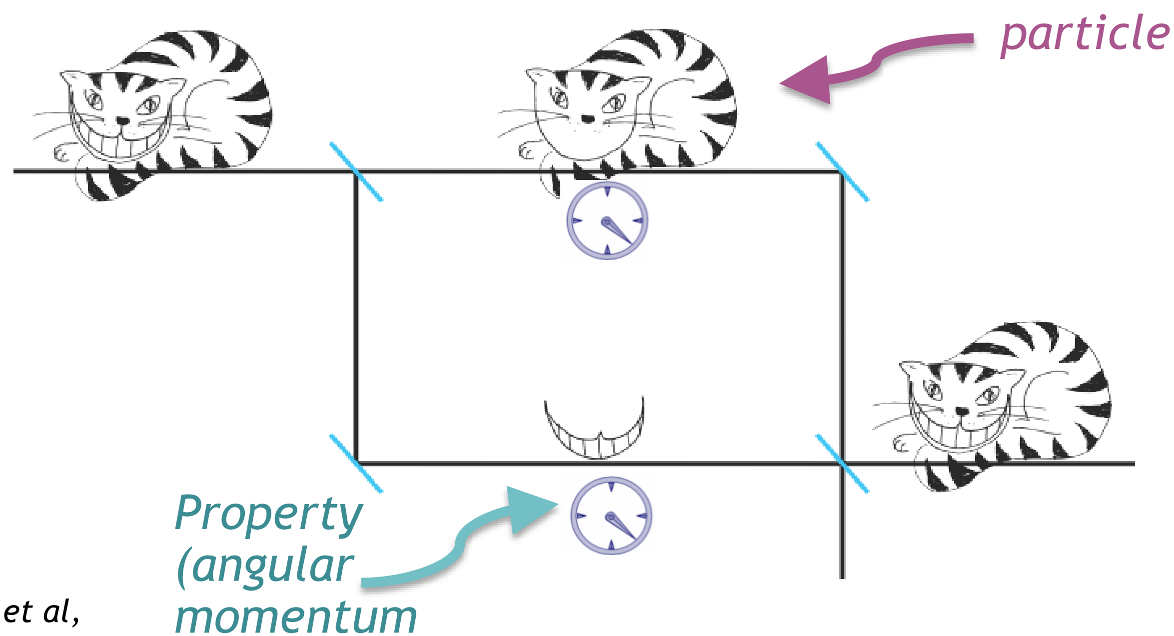


Fig. Denkmayr et al, Nature Comm. 2014

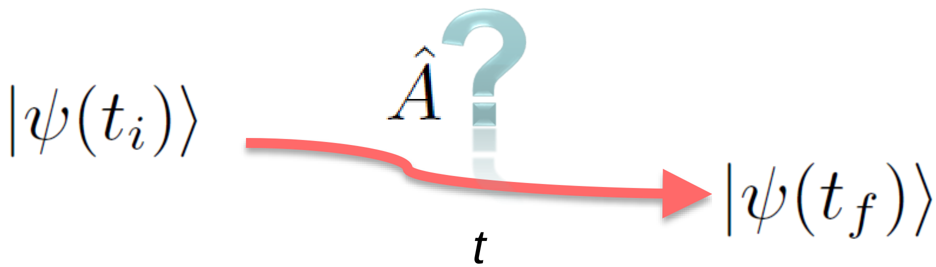
How can this be measured ?

Disappearing and delocalized properties: Weak measurements and the Quantum Cheshire Cat effect

Outline

1. Measurements in quantum mechanics, particle paths and boxes
2. Weak measurements
3. Weak values
4. Examples
5. Quantum Cheshire Cat
6. Weak values and properties of a quantum system
7. Conclusion

The question we will be interested in throughout :
what is the value of a physical property between two measurements?



System properties $\leftrightarrow$ measurements

Measurements are special in quantum mechanics

Postulates on

- Representation of physical states (vectors...)
- Time evolution of state vectors (Schrödinger eq)

Quantum measurements postulates

1. Each dynamical variable A is represented by a Hermitian operator $\hat{A}$ whose eigenvalues a_k are the possible values that the dynamical variable can take
2. Born's rule: $\mathcal{P}_\psi(a_k) = |\langle a_k|\,\psi\rangle|^2$
3. Post-measurement state: $|a_k\rangle\,\langle a_k|\,\psi\rangle$ (projection, reduction...)

Simple case: Qubit (spin ½)

Initial state $$|\psi\rangle = \alpha_+ |+\rangle + \alpha_- |-\rangle$$

spin component along z $$\sigma_z |\pm\rangle = \pm 1 |\pm\rangle$$

Premeasurement state: "action of the operator on the initial state"

$$\sigma_z |\psi\rangle = \alpha_+ |+\rangle - \alpha_- |-\rangle$$

spin z value
not defined

Projection

$$|+\rangle \langle + | \psi\rangle \quad \text{or} \quad |-\rangle \langle - | \psi\rangle$$

Probabilities

$$\mathcal{P}_\psi(\pm 1) = |\langle \pm | \psi\rangle|^2 = |\alpha_\pm|^2$$

Eigenstate-eigenvalue link

System property defined $\leftrightarrow$ Eigenstate of observable (Dirac)

➔ Initial state of the system disturbed

- by the interaction with the "pointer"
- by the projection (effective collapse)

$$|\psi\rangle = \alpha_+ |+\rangle + \alpha_- |-\rangle$$

$$\sigma_z |\pm\rangle = \pm 1 |\pm\rangle$$

Premeasurement state: "action of the operator on the initial state"

$$\sigma_z |\psi\rangle = \alpha_+ |+\rangle - \alpha_- |-\rangle$$

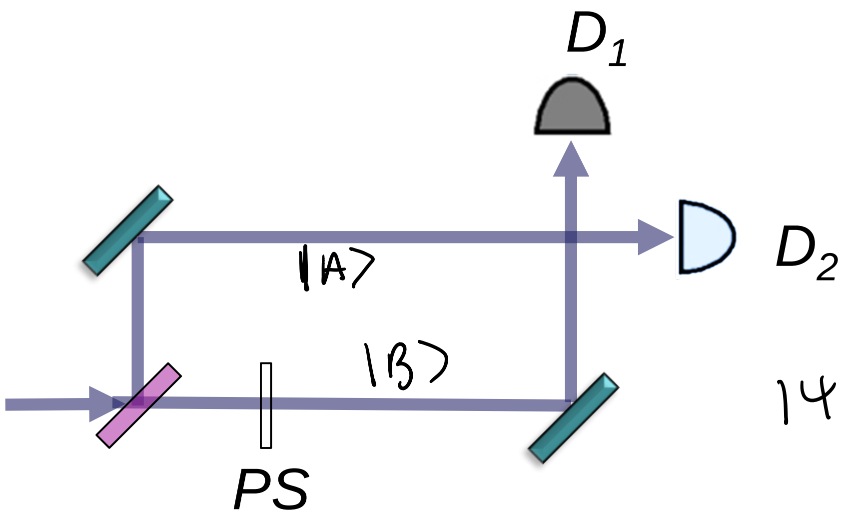
Projection

$$|+\rangle \langle +| \psi\rangle \quad \text{or} \quad |-\rangle \langle -| \psi\rangle$$

Probabilities

$$\mathcal{P}_\psi(\pm 1) = |\langle \pm | \psi\rangle|^2 = |\alpha_\pm|^2$$

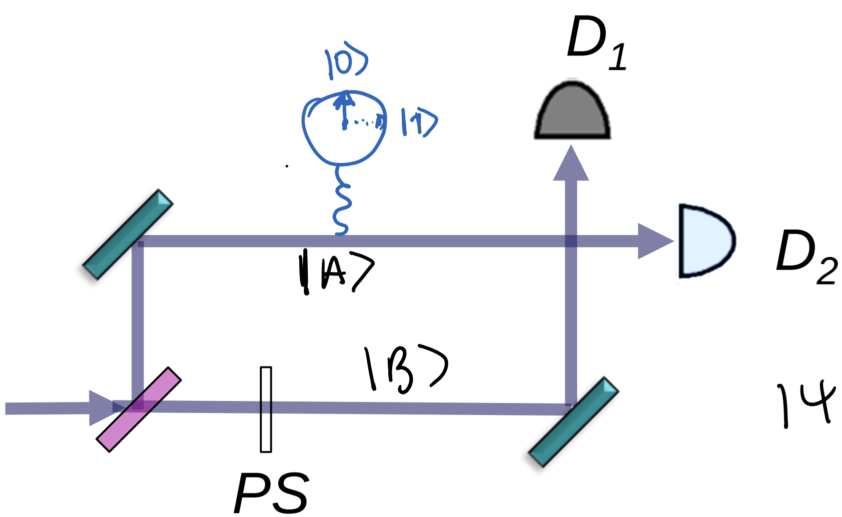
- interferometer (without BS2)



which path took a particle detected at D_2?

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|A\rangle + |B\rangle\right)$$

- interferometer (without BS2)



which path took a particle detected at D_2?

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|A\rangle + |B\rangle) \otimes |0\rangle$$

detector ready state

$$H_{int} = |A\rangle\langle A|(|0\rangle\langle 1| + |1\rangle\langle 0|)$$

$$|\psi(t_2)\rangle = \frac{1}{\sqrt{2}}(|A\rangle|1\rangle + |B\rangle|0\rangle)$$

D_2 D_1

2.3 Quantum pointers: Von Neumann model

Initially $t = t_i$ the system is prepared in state $|\psi(t_i)\rangle$.
Another quantum system (the pointer) is in state $|\varphi(t_i)\rangle$ (eg Gaussian)

$$\langle x|\,\varphi(t_i)\rangle = \frac{1}{(2\pi\delta^2)^{1/4}} \exp\left[-\frac{x^2}{4\delta^2}\right]$$

Total initial quantum state is the uncoupled state

SYSTEM

$$|\Psi(t_i)\rangle = |\psi(t_i)\rangle\,|\varphi(t_i)\rangle\,.$$

POINTER (Quantum)

We assume the system and the pointer will interact during a brief time interval τ centered around $t = t_0$ (physically corresponding to the time during which the system and the quantum pointer interact). Let the interaction Hamiltonian be specified by

SYSTEM OBSERVABLE

$$H_{int} = g(t - t_0)AP$$

POINTER VARIABLE

$g(t - t_0)$ is a smooth function non-vanishing only in the interval $t_0 + \tau/2 < t < t_0 + \tau/2$ and such that $g \equiv \int_{t_w-\tau/2}^{t_w+\tau/2} g(t)dt$ appears as the effective coupling constant.

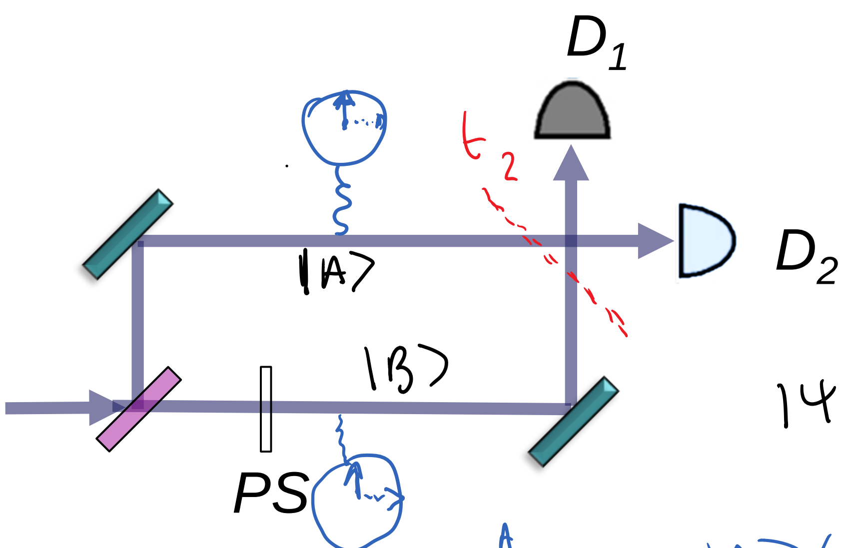
Von Neumann model

$$\langle x|\,\Psi(t)\rangle = \sum_k \langle a_k|\,\psi(t_i)\rangle\,|a_k\rangle\,\langle x|\exp\left(-\frac{i}{\hbar}Ga_kP\right)|\varphi(t_i)\rangle$$
$$= \sum_k \langle a_k|\,\psi(t_i)\rangle\,|a_k\rangle\,\varphi(x+Ga_k,t_i)$$

Entangled state

- Each pointer state $\varphi_k(x,t) \equiv \varphi(x+Ga_k,t_i)$ is correlated with an eigenstate $|a_k\rangle$ (CAVEAT: orthogonality)
- Each pointer state $\varphi_k(x,t)$ is shifted proportionally to the eigenvalue a_k.
- Entangled state: superposition of different configurations
- Definite outcome: random projection to $\varphi_{k_0}(x,t)$ correlated with $|a_{k_0}\rangle$ (MEASUREMENT PROBLEM)
- Premeasurement state radically modified (i) by H_{int}; (ii) by the projection

- interferometer (without BS2)



which path took a particle detected at D_1 or D_2 ?

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|A\rangle + |B\rangle\right) \otimes |\varphi_A\rangle \otimes |\varphi_B\rangle$$

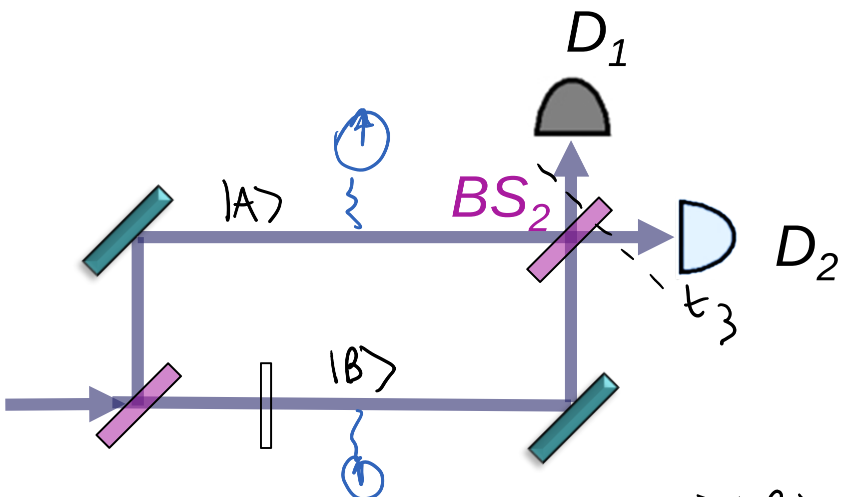
$$H_{int}^{A} = g(t)\,|A\rangle\langle A|\, p_A \qquad H_{int}^{B} = g(t)\,\Pi_B\, p_B$$

$$|\psi(t_2)\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\int H_{int}^{A}\,dt'}\,|A\rangle + e^{-i\int H_{int}^{B}\,dt'}\,|B\rangle \right)|\varphi_A\rangle|\varphi_B\rangle$$

$$= \frac{1}{\sqrt{2}}\left(|A\rangle\,|\varphi_A(q+\lambda_0)\rangle\,|\varphi_B\rangle + |B\rangle\,|\varphi_A\rangle\,|\varphi_B(q+\lambda_0)\rangle \right)$$

EITHER $D_2 \to$ Pointer A clicks OR $D_1 \to$ Pointer B clicks

- interferometer (with BS2)



which path took a particle detected at D_2?

$$|\Psi(t_3)\rangle = |A\rangle|\varphi_A(g+x_0)\rangle|\varphi_B\rangle + |B\rangle|\varphi_A\rangle|\varphi_B(g+x_0)\rangle \longrightarrow D_1$$
$$+ \; |A\rangle|\varphi_A(g+x_0)\rangle|\varphi_B\rangle - |B\rangle|\varphi_A\rangle|\varphi_B(g+x_0)\rangle \longrightarrow D_2$$

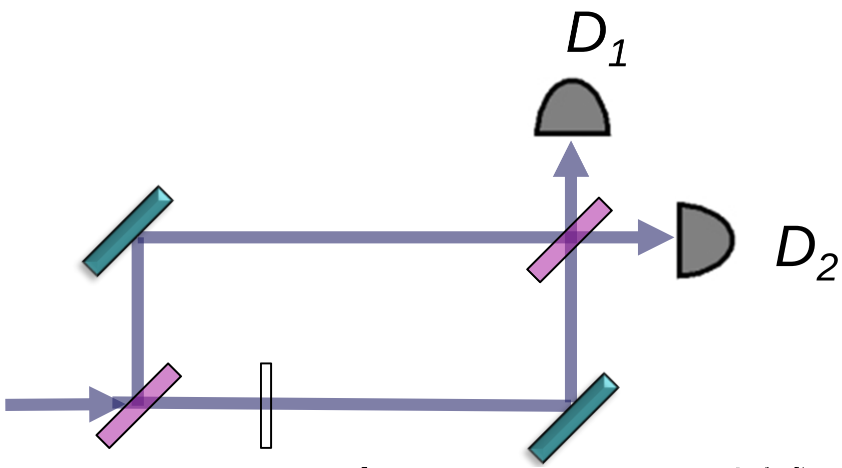
EITHER A CLIKS or B clicks

D_1 or D_2

Delayed choice (Wheeler 1983)

- "paradox": delay insertion of BS2 after the particle has passed the path detectors

wave or particle aspect depend on our choice just before the final detection



we made our decision. This is the sense in which, in a loose way of speaking, we decide what the photon *shall have done* after it has *already* done it. In actuality it is wrong to talk of the "route" of the photon. For a proper way of speaking we recall once more that it makes no sense to talk of the phenomenon until it has been brought to a close by an irreversible act of amplification: "No elementary phenomenon is a phenomenon until it is a registered (observed) phenomenon."

J. A. WHEELER

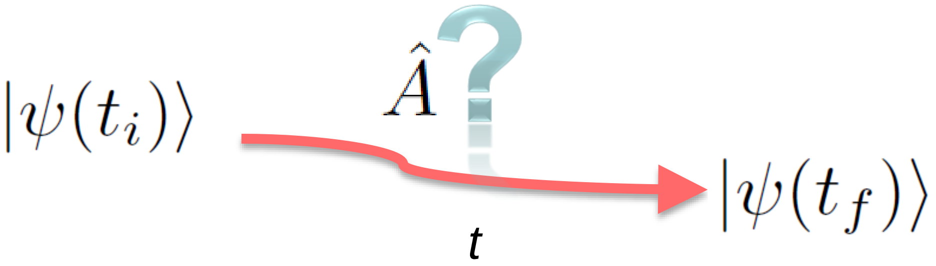
Delayed choice - Context dependence (Wheeler 1983)

- With pointers: particle aspect
- Without pointers : with BS2 wave aspect is inferred but cannot be detected

> we made our decision. This is the sense in which, in a loose way of speaking, we decide what the photon *shall have done* after it has *already* done it. In actuality it is wrong to talk of the "route" of the photon. For a proper way of speaking we recall once more that it makes no sense to talk of the phenomenon until it has been brought to a close by an irreversible act of amplification: "No elementary phenomenon is a phenomenon until it is a registered (observed) phenomenon."

J. A. WHEELER

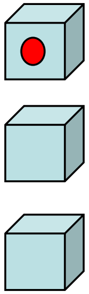
The question we will be interested in throughout :
what is the value of a physical property between two measurements?



Standard answer from standard quantum mechanics:
the question is meaningless, since a measurement would be needed, and that measurement would break the original system evolution (state projection).
Counterfactual reasoning leads to paradoxes (Bohr, Wheeler...)

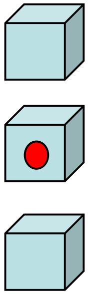
Three Box Paradox (Aharonov and Vaidman JPA 1991)

- 1 particle and 3 boxes



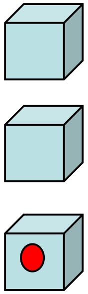
$|A\rangle$

particle in box A



$|B\rangle$

particle in box B



$|C\rangle$

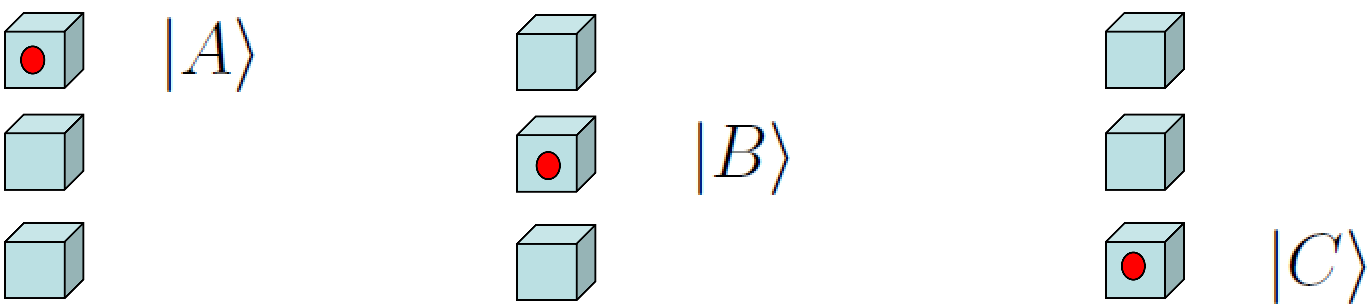
particle in box C

General quantum state:

$$|\psi\rangle = \alpha\,|A\rangle + \beta\,|B\rangle + \gamma\,|C\rangle$$

Three Box Paradox

- Open a box : projectors



$$|\psi\rangle = \alpha\,|A\rangle + \beta\,|B\rangle + \gamma\,|C\rangle$$

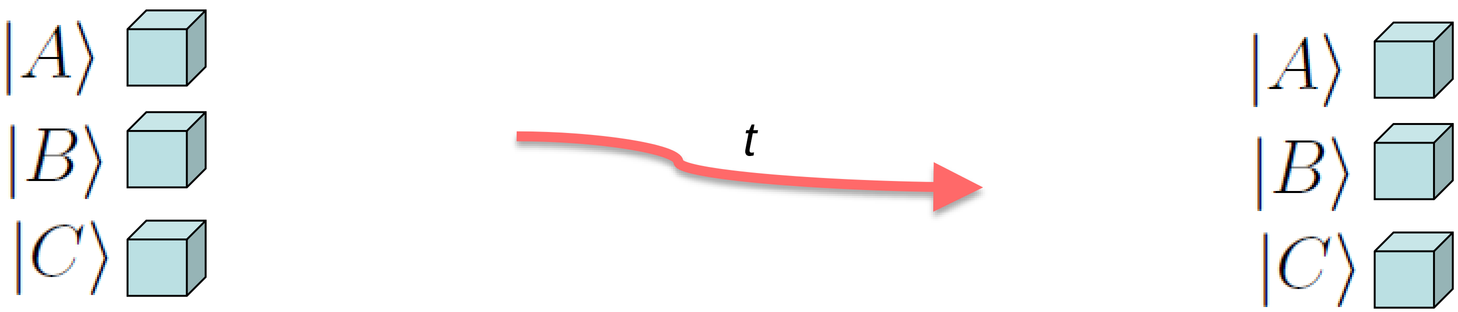
$$\Pi_B \equiv |B\rangle\,\langle B|$$

$$\Pi_B\,|B\rangle = +1\,|B\rangle$$

$$\Pi_B\,(\alpha\,|A\rangle + \gamma\,|C\rangle) = 0\,(\alpha\,|A\rangle + \gamma\,|C\rangle)$$

Three Box Paradox

- Where can we find the particle between $t=t_i$ and $t=t_f$?



Initial state:

$$|\psi_i\rangle = \frac{1}{\sqrt{3}}(|A\rangle + |B\rangle + |C\rangle)$$

pre-selection

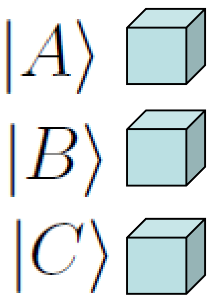
Final state (eigenstate of some observable):

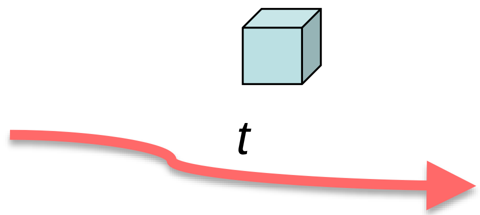
$$|\psi_f\rangle = \frac{1}{\sqrt{3}}(|A\rangle + |B\rangle - |C\rangle)$$

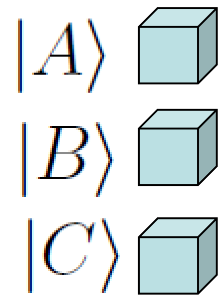
post-selection

Three Box Paradox

- At some intermediate t, is the particle in box A ?





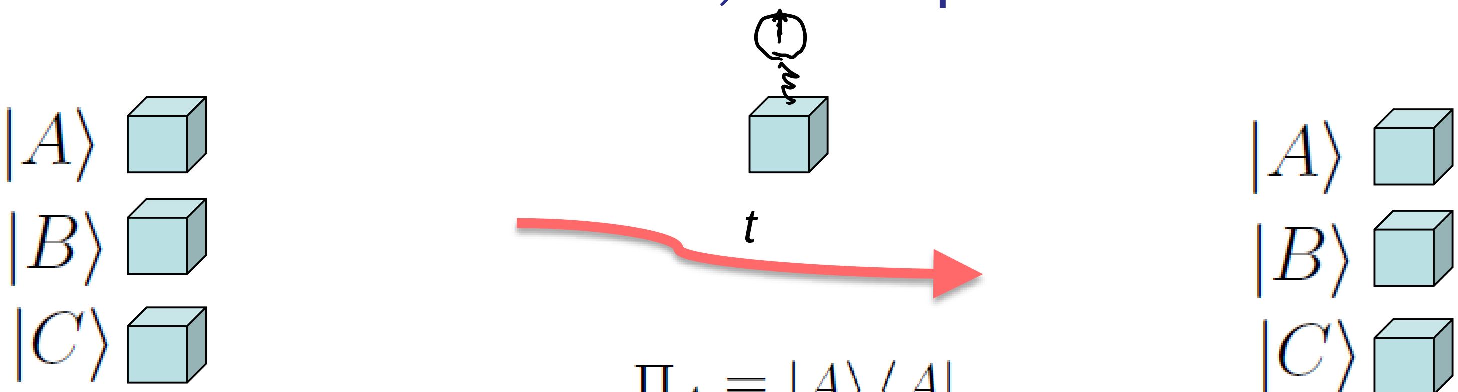


$$\Pi_A \equiv |A\rangle \langle A|$$

Initial state: $$|\psi_i\rangle = \frac{1}{\sqrt{3}} (\underbrace{|A\rangle}_{\textit{eigenvalue 1}} + \underbrace{|B\rangle + |C\rangle}_{\textit{eigenvalue 0}})$$

Three Box Paradox

- At some intermediate t, is the particle in box A ?



$$\Pi_A \equiv |A\rangle\langle A|$$

Initial state: $$|\psi_i\rangle = \frac{1}{\sqrt{3}}\left(|A\rangle + |B\rangle + |C\rangle\right)$$

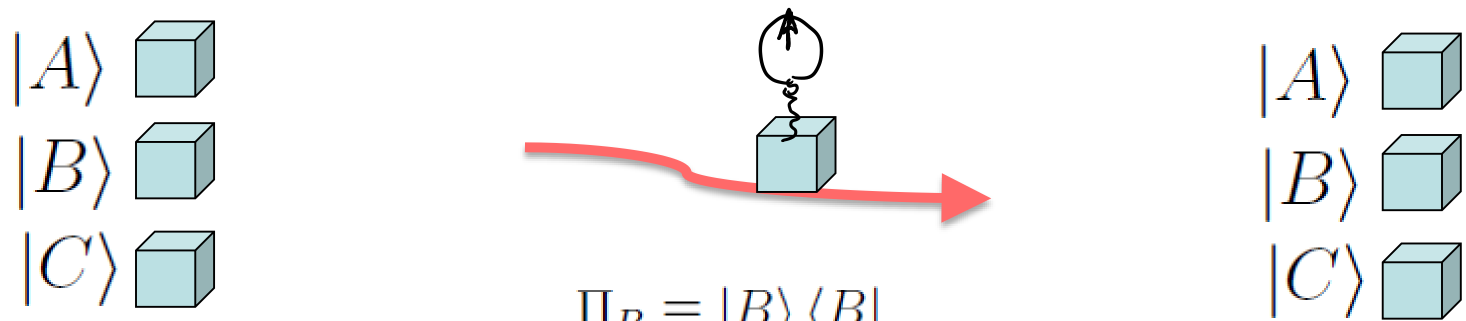
eigenvalue 1 *eigenvalue 0*

Constraint: final state $$|\psi_f\rangle = \frac{1}{\sqrt{3}}\left(|A\rangle + |B\rangle - |C\rangle\right)$$

⇨ ***The particle MUST have been in box A*** $|B\rangle + |C\rangle \perp |\psi_f\rangle$

Three Box Paradox

- At some intermediate t, is the particle in box B ?



$$\Pi_B \equiv |B\rangle\langle B|$$

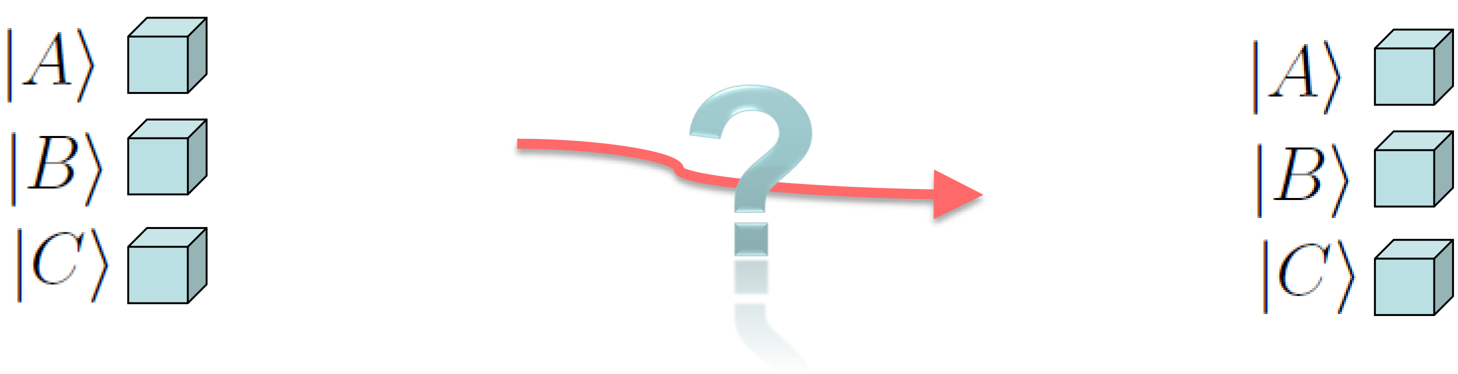
Initial state: $$|\psi_i\rangle = \frac{1}{\sqrt{3}}\left(\underbrace{|B\rangle}_{\text{eigenvalue 1}} + \underbrace{|A\rangle + |C\rangle}_{\text{eigenvalue 0}}\right)$$

Constraint: final state $$|\psi_f\rangle = \frac{1}{\sqrt{3}}\left(|A\rangle + |B\rangle - |C\rangle\right)$$

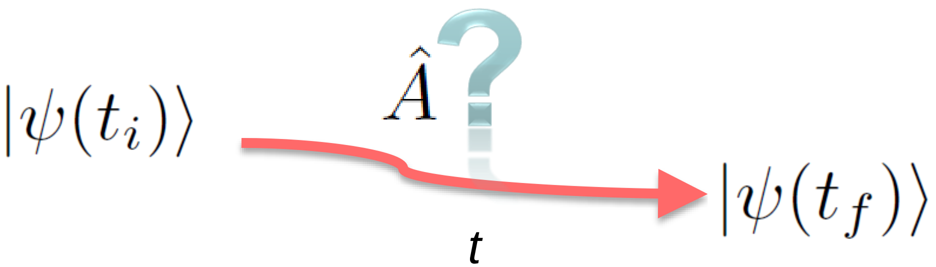
*The particle **MUST have been in box B*** $\quad |A\rangle + |C\rangle \perp |\psi_f\rangle$

The paradox

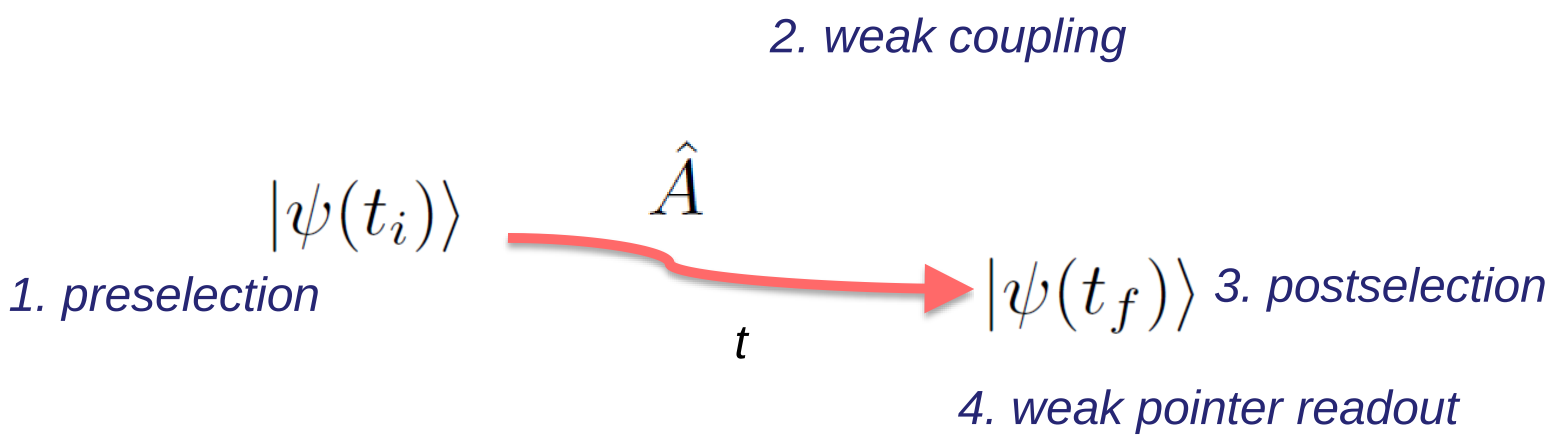
- At some intermediate t, we are sure to find the particle in box A but also in box B (though there is only one particle)
- Paradox ←→ projective measurements and counterfactual reasoning
- But this is forbidden: the experimental setting is modified, postselection may not happen… The paradox is dissolved… but what happens at an intermediate time ?



The question we will be interested in throughout :
what is the value of a physical property between two measurements?



Non-standard answer from standard quantum mechanics: **weak measurements**

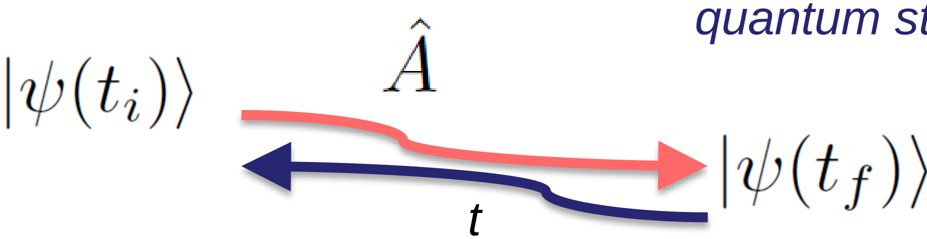


Non-standard answer from standard quantum mechanics:

Weak Measurements

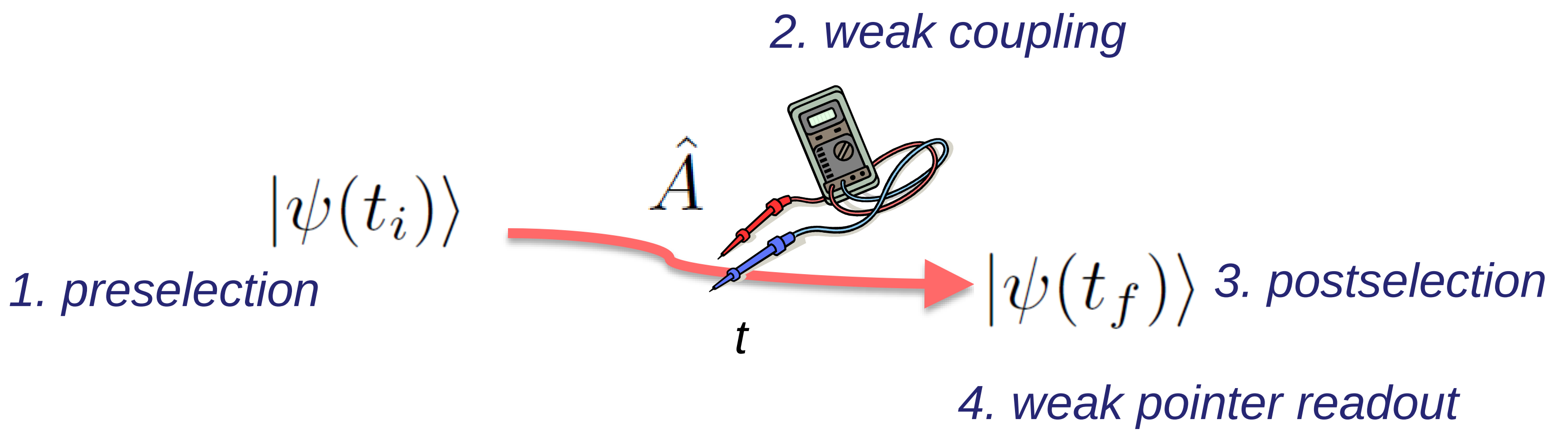
- at some intermediate time t a weak unitary interaction couples the system to a weak pointer
- the system largely unperturbed reaches the same final state (as in the case of no weak interaction)
- the projective measurement on the system at t_f also projects the weak meter wavefunction to a final state, revealing information on the weakly measured observable

weakly coupled meter (ancilla dynamical variable) whose quantum state acts as a pointer



Weak Measurements

- Introduced by Aharonov et al over a number of years (ABL Phys Rev 1964 – time symmetric quantum mechanics, Aharonov Albert & Vaidman PRL 1988 weak values, Phys Today 2011 quantum properties)
- Recent increase in the number of works dealing with weak measurements, incl notable experiments
- Useful as a tool to amplify small signals and estimate unknown parameters



Weak Measurements

1. Preselection (state preparation)
2. Weak coupling (between A and a dynamical variable of the weak pointer)
3. Postselection (projective measurement of a different system observable B, selecting a given outcome)
4. Weak pointer readout: <u>**weak value**</u> of A, given the preselected and postselected states.

Weak Values

- Operationally, value indicated by the weak pointer readout
- It is different from the eigenvalues and can lie outside the eigenvalue range

Weak Values

- Operationally, value indicated by the weak pointer readout (complex number)
- It is different from the eigenvalues and can lie outside the eigenvalue range
- Universal in the weak coupling limit: the derivation in the asymptotic limit gives the expression

$$A^{w}_{\langle\chi_f|,|\psi\rangle} \equiv \frac{\langle\chi(t_w)|\,A\,|\psi(t_w)\rangle}{\langle\chi(t_w)|\,\psi(t_w)\rangle}$$

preselected state (evolved forward in time)

weak value

postselected state (evolved backward in time)

Weak Values: properties

Expectation value

$$
\begin{aligned}
\langle A \rangle_{|\psi\rangle} &= \langle \psi(t_w)| A |\psi(t_w)\rangle \\
&= \sum_k \underbrace{|\langle a_k | \psi(t_w)\rangle|^2}_{\text{prob. eigenvalue } a_k} a_k
\end{aligned}
$$

$$
= \sum_k \underbrace{|\langle \psi(t_f)| \chi_k(t_f)\rangle|^2}_{\text{prob. of postselecting to } |\chi_k\rangle} \cancel{\mathrm{Re}}\, A^w_{\langle \chi_k|,|\psi\rangle}
$$

DERIVATION OF WV EXPRESSION

Same as Von Neumann measurement except that coupling is weak (asymptotic expansion) and that final projection (postselection) due to the measurement of another observable B.

Total Hamiltonian

$$H = H_0 + H_{int}$$

Interaction Hamiltonian:

$$H_{int} = g(t - t_w)AP$$

$g(t - t_w)$ is a smooth function non-vanishing only in the interval $t_w + \tau/2 < t < t_w + \tau/2$ and such that $G \equiv \int_{t_w-\tau/2}^{t_w+\tau/2} g(t)dt$ appears as the effective coupling constant.

$U(t_2, t_1)$ will denote the “free” system evolution generated by H_0. No self-Hamiltonian assumed for pointer.

STEP 1: PRESELECTION $|\psi(t_i)\rangle$ and total initial quantum state is the uncoupled state

$$|\Psi(t_i)\rangle = |\psi(t_i)\rangle\,|\varphi(t_i)\rangle\,.$$

Then the system evolves up to time $t_w - \tau/2$ when the interaction takes place.

STEP 2: WEAK COUPLING. At $t = t_w + \tau/2$ the system and pointer have interacted and the total state $|\Psi(t)\rangle$ becomes

$$\begin{aligned}
|\Psi(t)\rangle &= \exp\left(-\frac{i}{\hbar}\int_{t_w-\tau/2}^{t=t_w+\tau/2} g(t'-t_0)APdt\right) U(t_w-\tau/2, t_i)\,|\psi(t_i)\rangle\,|\varphi(t_i)\rangle \\
&= \exp\left(-\frac{i}{\hbar}GAP\right)|\psi(t_w)\rangle\,|\varphi(t_i)\rangle \qquad \text{(midpoint)} \\
&\simeq \left(I - \frac{i}{\hbar}GAP\right)|\psi(t_w)\rangle\,|\varphi(t_i)\rangle \qquad \text{(“}G\text{ small”)} \\
&\left(= \sum_k \exp\left(-\frac{i}{\hbar}Ga_kP\right)\langle a_k|\,\psi(t_w)\rangle\,|a_k\rangle\,|\varphi(t_i)\rangle\right)
\end{aligned}$$

STEP 3: POSTSELECTION. At $t = t_f$

$$|\Psi(t_f)\rangle \simeq U(t_f, t_w)\left(I - \frac{i}{\hbar} GAP\right)|\psi(t_w)\rangle\,|\varphi(t_i)\rangle$$

and a projective measurement is made for an observable B with eigenstates $|b_k\rangle$. Let us only keep the results corresponding to a chosen eigenvalue b_{k_0} and label the postselected state by

$$|\chi(t_f)\rangle \equiv |b_{k_0}\rangle$$

Then the pointer state correlated with postselection is $|\varphi(t_f)\rangle \equiv \langle\chi(t_f)|\,\Psi(t_f)\rangle$

$$\begin{aligned}
\langle\chi(t_f)|\,\Psi(t_f)\rangle &= \langle\chi(t_f)|\,U(t_f, t_w)\left(I - \frac{i}{\hbar} GAP\right)|\psi(t_w)\rangle\,|\varphi(t_i)\rangle \\
&= \langle\chi(t_w)|\left(I - \frac{i}{\hbar} GAP\right)|\psi(t_w)\rangle\,|\varphi(t_i)\rangle \qquad \text{(backward evolution)} \\
&= \langle\chi(t_w)|\,\psi(t_w)\rangle\left(1 - \frac{i}{\hbar} G \frac{\langle\chi(t_w)|\,A\,|\psi(t_w)\rangle}{\langle\chi(t_w)|\,\psi(t_w)\rangle} P\right)|\varphi(t_i)\rangle \\
&= \langle\chi(t_w)|\,\psi(t_w)\rangle \exp\left(-\frac{i}{\hbar} G \frac{\langle\chi(t_w)|\,A\,|\psi(t_w)\rangle}{\langle\chi(t_w)|\,\psi(t_w)\rangle} P\right)|\varphi(t_i)\rangle \\
|\varphi(t_f)\rangle &= \langle\chi(t_w)|\,\psi(t_w)\rangle \exp\left(-\frac{i}{\hbar} G A^w_{\langle\chi_f|,|\psi\rangle} P\right)|\varphi(t_i)\rangle
\end{aligned}$$

FINAL STATE OF THE WEAKLY COUPLED POINTER

where

$$A^w \equiv A^w_{\langle\chi_f|,|\psi\rangle} \equiv \frac{\langle\chi(t_w)|\,A\,|\psi(t_w)\rangle}{\langle\chi(t_w)|\,\psi(t_w)\rangle}$$

is the **weak value** of the observable A.

STEP 3: POSTSELECTION. At $t = t_f$

$$|\Psi(t_f)\rangle \simeq U(t_f, t_w)\left(I - \frac{i}{\hbar}GAP\right)|\psi(t_w)\rangle\,|\varphi(t_i)\rangle$$

and a projective measurement is made for an observable B with eigenstates $|b_k\rangle$. Let us only keep the results corresponding to a chosen eigenvalue b_{k_0} and label the postselected state by

$$|\chi(t_f)\rangle \equiv |b_{k_0}\rangle$$

Then the pointer state correlated with postselection is $|\varphi(t_f)\rangle \equiv \langle\chi(t_f)|\,\Psi(t_f)\rangle$

$$\langle\chi(t_f)|\,\Psi(t_f)\rangle = \langle\chi(t_f)|\,U(t_f,t_w)\left(I - \frac{i}{\hbar}GAP\right)|\psi(t_w)\rangle\,|\varphi(t_i)\rangle$$

$$= \langle\chi(t_w)|\left(I - \frac{i}{\hbar}GAP\right)|\psi(t_w)\rangle\,|\varphi(t_i)\rangle \qquad \text{(backward evolution)}$$

$$= \langle\chi(t_w)|\,\psi(t_w)\rangle\left(1 - \frac{i}{\hbar}G\frac{\langle\chi(t_w)|\,A\,|\psi(t_w)\rangle}{\langle\chi(t_w)|\,\psi(t_w)\rangle}P\right)|\varphi(t_i)\rangle$$

$$= \langle\chi(t_w)|\,\psi(t_w)\rangle\exp\left(-\frac{i}{\hbar}G\frac{\langle\chi(t_w)|\,A\,|\psi(t_w)\rangle}{\langle\chi(t_w)|\,\psi(t_w)\rangle}P\right)|\varphi(t_i)\rangle$$

$$|\varphi(t_f)\rangle = \langle\chi(t_w)|\,\psi(t_w)\rangle\exp\left(-\frac{i}{\hbar}GA^w_{\langle\chi_f|,|\psi\rangle}P\right)|\varphi(t_i)\rangle$$

where

$$A^w \equiv A^w_{\langle\chi_f|,|\psi\rangle} \equiv \frac{\langle\chi(t_w)|\,A\,|\psi(t_w)\rangle}{\langle\chi(t_w)|\,\psi(t_w)\rangle}$$

is the **weak value** of the observable A.

STEP 4: WEAK POINTER READOUT The quantum state of the pointer is

$$|\varphi(t_f)\rangle \propto \exp\left(-\frac{i}{\hbar}GA^wP\right)|\varphi(t_i)\rangle\,.$$

A^w extraction depends on form of pointer state. For a Gaussian pointer $\varphi(x, t_i)$

$$\begin{aligned}\varphi(x, t_f) &\propto \langle x|\exp\left(-\frac{i}{\hbar}GA^wP\right)|\varphi(t_i)\rangle \\ &\propto \varphi(x + GA^w, t_i)\end{aligned}$$

but validity of the asymptotic expansion implies $\varphi(x, t_i)$ broad.

Expansion

$$\begin{aligned}\langle x|\langle\chi(t_f)|\,\Psi(t_f)\rangle &= \langle x|\langle\chi(t_w)|\exp\left(-\frac{i}{\hbar}GAP\right)|\psi(t_w)\rangle|\varphi(t_i)\rangle \\ &= \Bigg\{\langle\chi(t_w)|\,\psi(t_w)\rangle - \frac{i}{\hbar}G\langle\chi(t_w)|\,A\,|\psi(t_w)\rangle(-i\hbar\partial_x) \\ &+\frac{1}{2}\left(\frac{i}{\hbar}G\right)^2\langle\chi(t_w)|\,A^2\,|\psi(t_w)\rangle(-i\hbar\partial_x)^2\Bigg\}\exp\left(-\frac{x^2}{2\Delta^2}\right) \\ &= \Bigg\{\langle\chi(t_w)|\,\psi(t_w)\rangle - \frac{i}{\hbar}G\langle\chi(t_w)|\,A\,|\psi(t_w)\rangle\left(i\hbar\frac{x}{\Delta^2}\right) \\ &+\frac{1}{2}\left(\frac{i}{\hbar}G\right)^2\langle\chi(t_w)|\,A^2\,|\psi(t_w)\rangle(-i\hbar)^2\left(\frac{x^2-\Delta^2}{\Delta^4}\right)+\ldots\Bigg\}\exp\left(-\frac{x^2}{2\Delta^2}\right)\end{aligned}$$

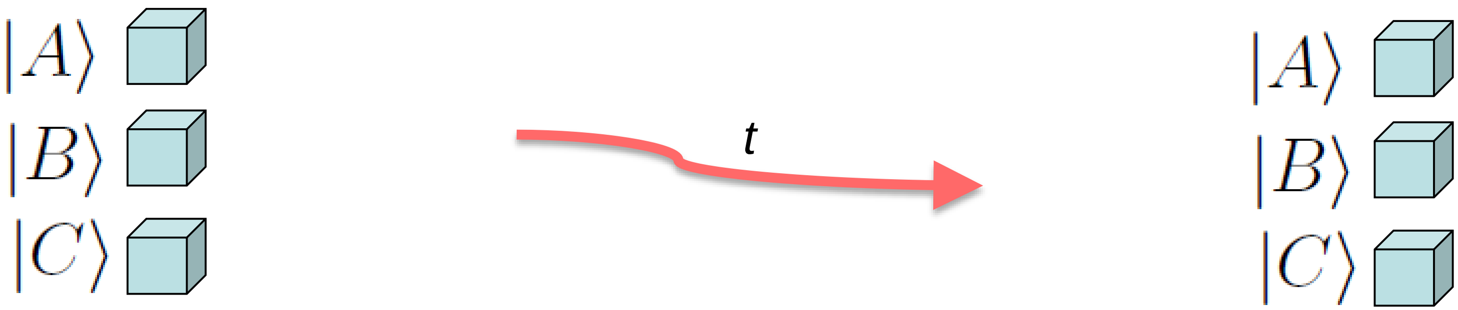
$$
\begin{aligned}
\langle x|\,\langle\chi(t_f)|\,\Psi(t_f)\rangle &= \langle x|\,\langle\chi(t_w)|\exp\left(-\frac{i}{\hbar}GAP\right)|\psi(t_w)\rangle\,|\varphi(t_i)\rangle \\
&= \Bigg\{\langle\chi(t_w)|\,\psi(t_w)\rangle - \frac{i}{\hbar}G\,\langle\chi(t_w)|\,A\,|\psi(t_w)\rangle\,(-i\hbar\partial_x) \\
&+\frac{1}{2}\left(\frac{i}{\hbar}G\right)^2\langle\chi(t_w)|\,A^2\,|\psi(t_w)\rangle\,(-i\hbar\partial_x)^2\Bigg\}\exp\left(-\frac{x^2}{2\Delta^2}\right) \\
&= \Bigg\{\langle\chi(t_w)|\,\psi(t_w)\rangle - \frac{i}{\hbar}G\,\langle\chi(t_w)|\,A\,|\psi(t_w)\rangle\left(i\hbar\frac{x}{\Delta^2}\right) \\
&+\frac{1}{2}\left(\frac{i}{\hbar}G\right)^2\langle\chi(t_w)|\,A^2\,|\psi(t_w)\rangle\,(-i\hbar)^2\left(\frac{x^2-\Delta^2}{\Delta^4}\right)+\ldots\Bigg\}\exp\left(-\frac{x^2}{2\Delta^2}\right)
\end{aligned}
$$

- If $\Delta \to 0$, $x \to x_0$: cannot work.
- If Δ large, $x \sim \kappa\Delta$ with $\kappa \sim 1$ and

$$\Delta \gg G\left|\frac{\langle\chi(t_w)|\,A^2\,|\psi(t_w)\rangle}{\langle\chi(t_w)|\,A\,|\psi(t_w)\rangle}\right|$$

Example : Three Box Paradox

- Where can we find the particle between $t=t_i$ and $t=t_f$?



Initial state:

$$|\psi_i\rangle = \frac{1}{\sqrt{3}}\left(|A\rangle + |B\rangle + |C\rangle\right)$$

pre-selection

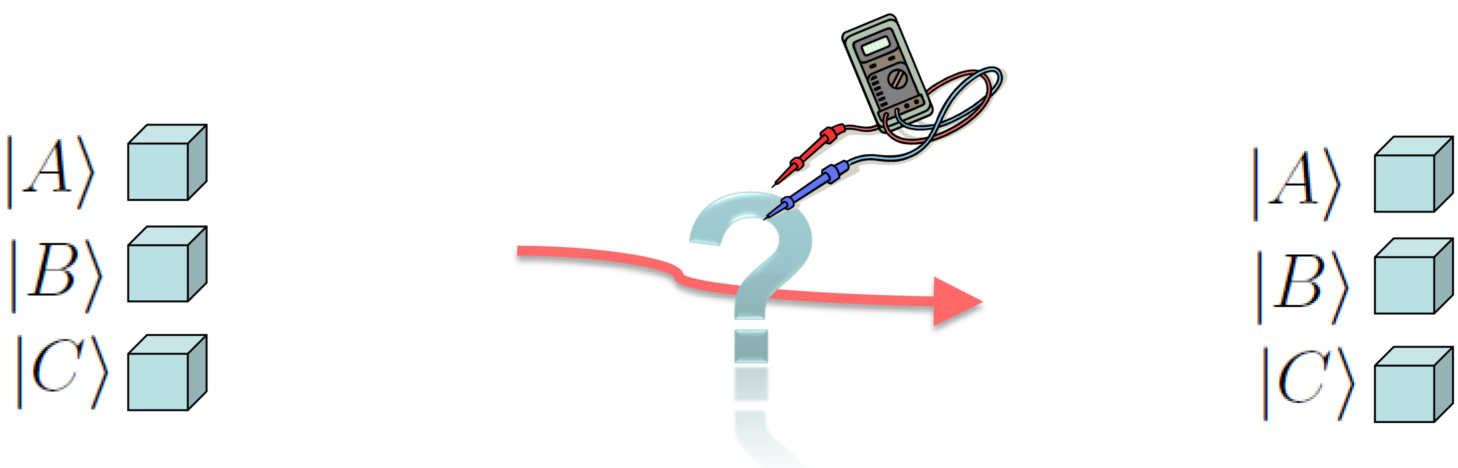
Final state (eigenstate of some observable):

$$|\psi_f\rangle = \frac{1}{\sqrt{3}}\left(|A\rangle + |B\rangle - |C\rangle\right)$$

post-selection

Weak values

$$\Pi_A^w = \frac{\langle \psi_f | \Pi_A | \psi_i \rangle}{\langle \psi_f | \psi_i \rangle} = 1 \qquad\qquad \Pi_B^w = 1$$



Weak values of **spatial projectors**

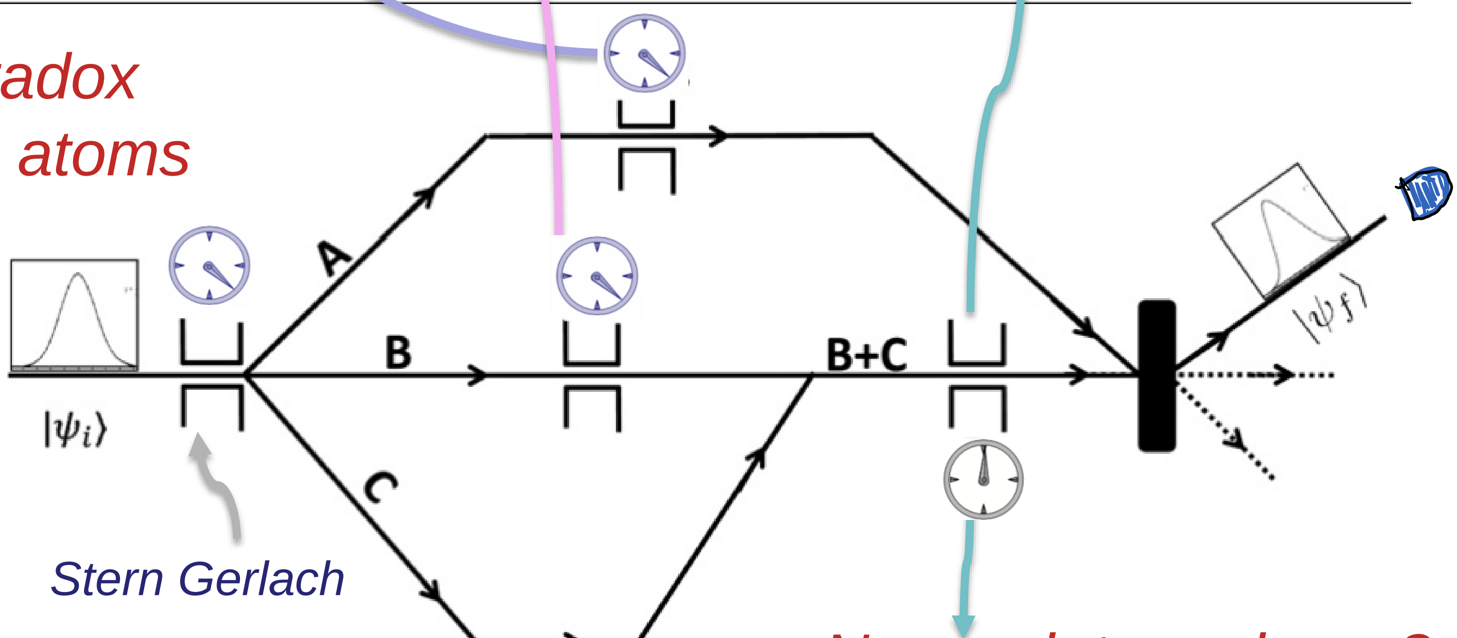
$$\langle \Pi_A \rangle_w = 1 \qquad \langle \Pi_B \rangle_w = 1$$

$$\Pi_{\bar{A}} \equiv 1 - \Pi_A = |B\rangle\langle B| + |C\rangle\langle C|$$

$$\langle \Pi_{\bar{A}} \rangle_w = 0$$

J. Phys. A: Math. Theor. **46** (2013) 315307 A Matzkin and A K Pan

3 box paradox
for spin-1 atoms



Stern Gerlach

No weak trace here ?

Discontinuous weak trajectories

Nested Mach-Zehnder interferometers: couple several weak meters along the paths *(Vaidman et al PRA 87, 052104 2013; PRL 111, 240402 2013)*

Weakly coupled pointers measure the particle's presence at their location (weak value of spatial projectors)

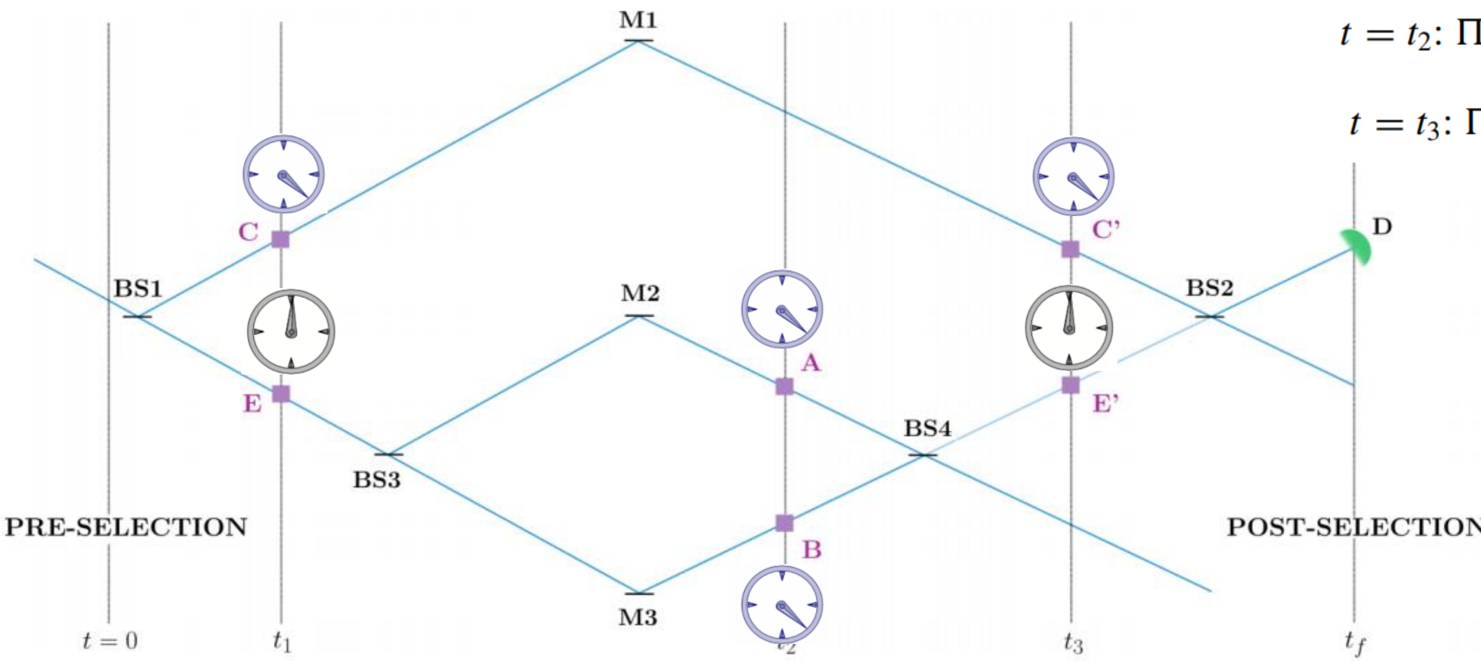
Initial state of each meter: 

$$\Pi_X = |\Gamma_X\rangle\langle\Gamma_X|$$

$$t = t_1\colon \Pi_C^w = 1, \qquad \Pi_E^w = 0,$$

$$t = t_2\colon \Pi_A^w = 1, \qquad \Pi_B^w = -1,$$

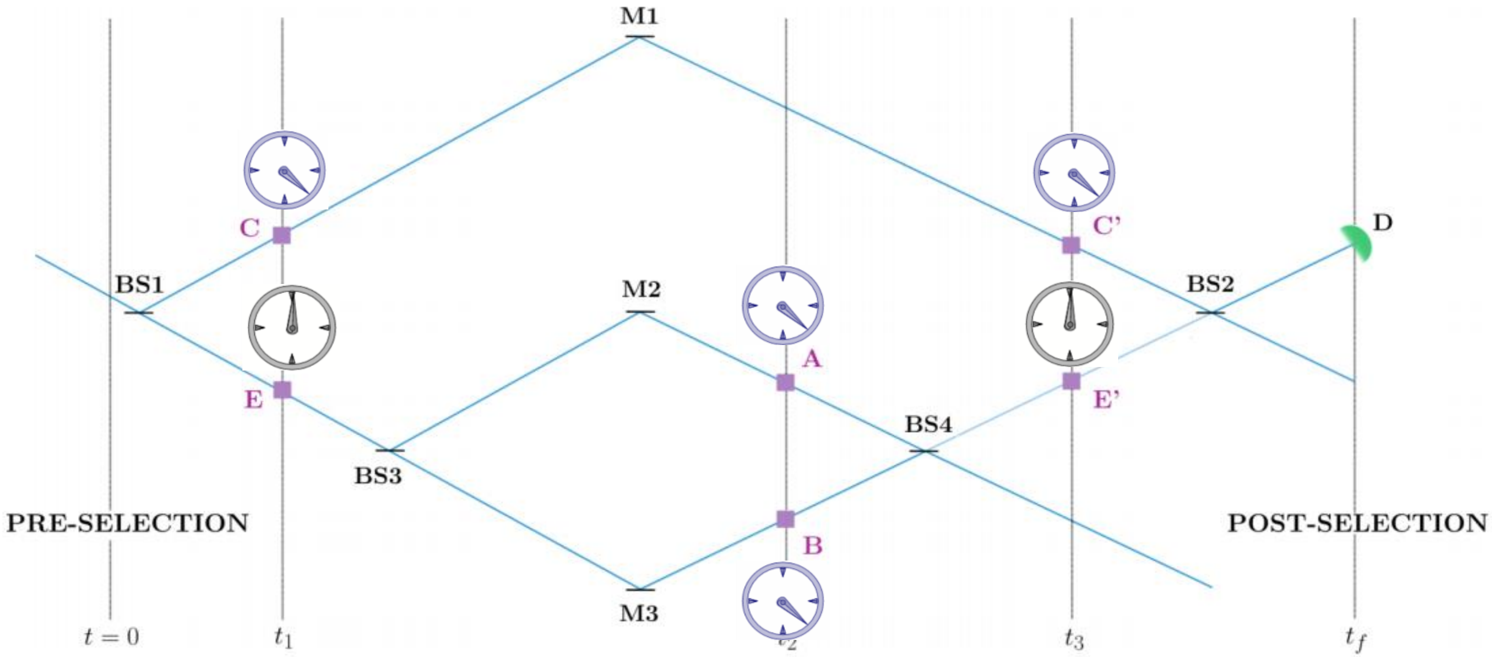
$$t = t_3\colon \Pi_{C'}^w = 1, \qquad \Pi_{E'}^w = 0.$$



Duprey & Matzkin PRA 95, 032110 2017

Discontinuous weak trajectories

Weak Trace criterion (Vaidman): the particle was not present in regions where the projector weak values vanish (the particle's spatial presence cannot be detected) ➔ paradox

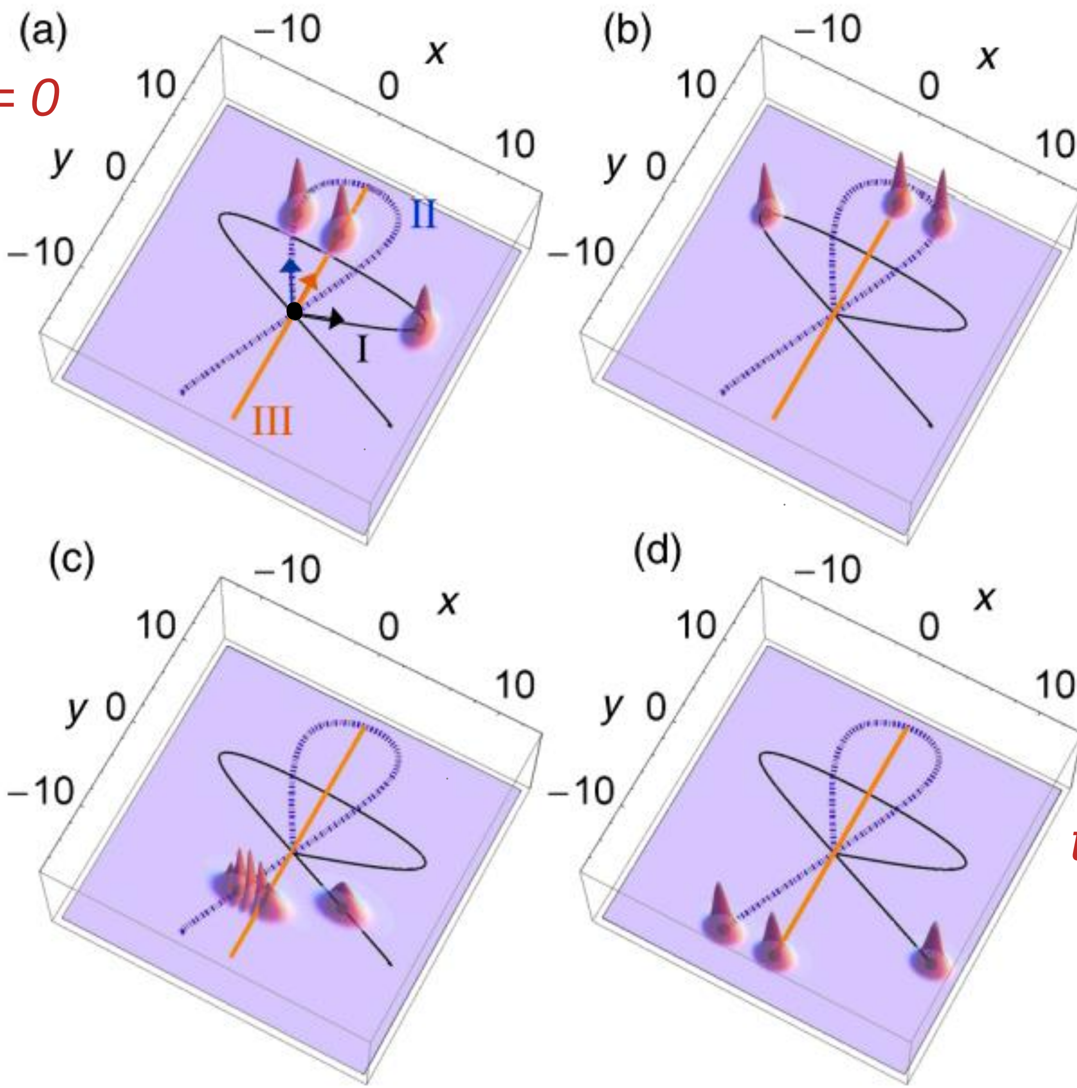


Weak trajectories

TDLO

PROPAGATION OF COHERENT STATES along classical trajectories (semi-classical propagator)

$t = 0$

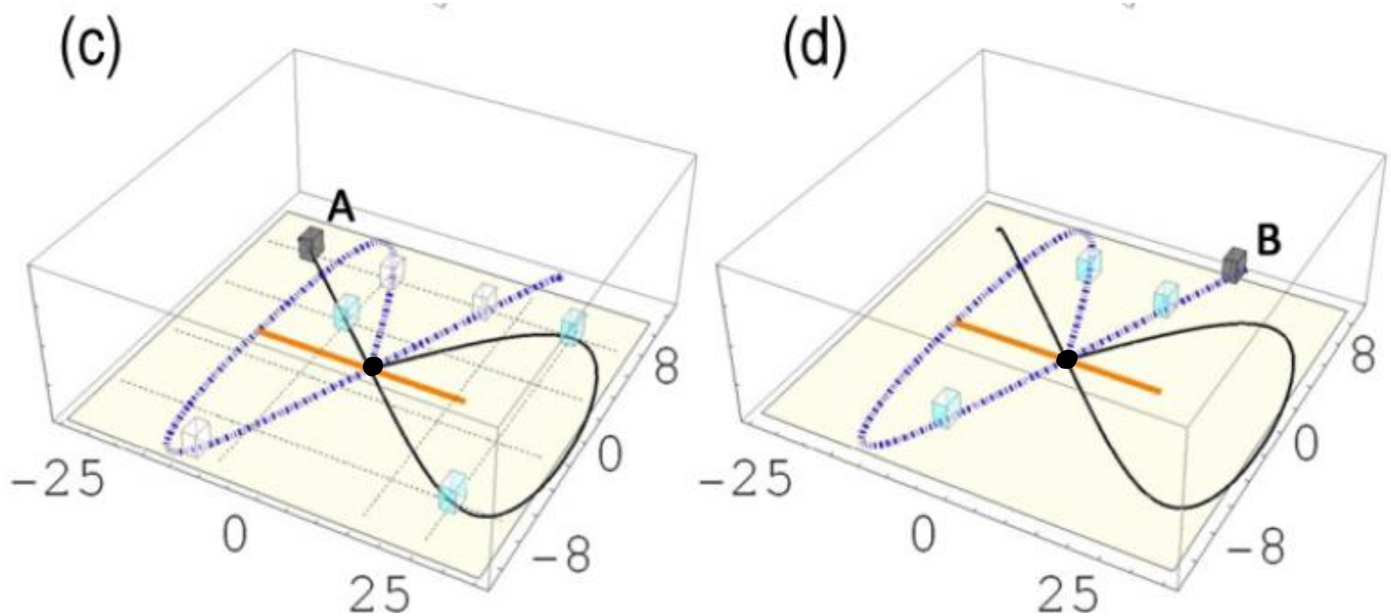


$t = t_f$

POST SELECTION

Weak measurements of trajectories

Weakly coupled pointers on a grid, Postselection at t = tf



$$[\Pi(\mathbf{r})]^w = \frac{\langle \mathbf{r}_f | U(t_f, t_w) | \mathbf{r} \rangle \langle \mathbf{r} | U(t_w, t_i) | \psi_i \rangle}{\langle \mathbf{r}_f | U(t_f, t_i) | \psi_i \rangle}$$

The quantum pointer "fires" if

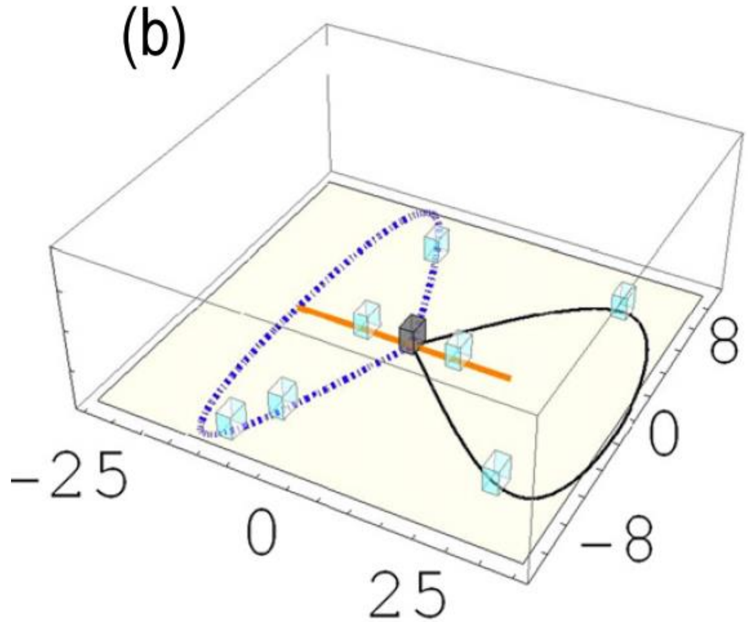
- $\mathbf{r}_f$ lies on a classical trajectory emanating from the initial quantum state centered at $\mathbf{r}_0$
- the pointer is placed along that particular trajectory

Weak measurements of trajectories

Weakly coupled pointers on a grid, Postselection at t = tf

$$[\Pi(\mathbf{r})]^w = \frac{\langle \mathbf{r}_f | U(t_f, t_w) | \mathbf{r} \rangle \langle \mathbf{r} | U(t_w, t_i) | \psi_i \rangle}{\langle \mathbf{r}_f | U(t_f, t_i) | \psi_i \rangle}$$

Sum over paths



Matzkin PRL 2012, JPA 2015, Mori & Tsutsui, PTEP 2015
Sokolovski PLA 2016, Georgiev & Cohen PRA 2018

Weak measurement of momentum field

Momentum weak value

$$\langle \mathbf{p}(t)\rangle_W = \frac{\left\langle \mathbf{r}_f \middle| \hat{\mathbf{p}} \mid \psi(t)\right\rangle}{\left\langle \mathbf{r}_f \middle| \psi(t)\right\rangle},$$

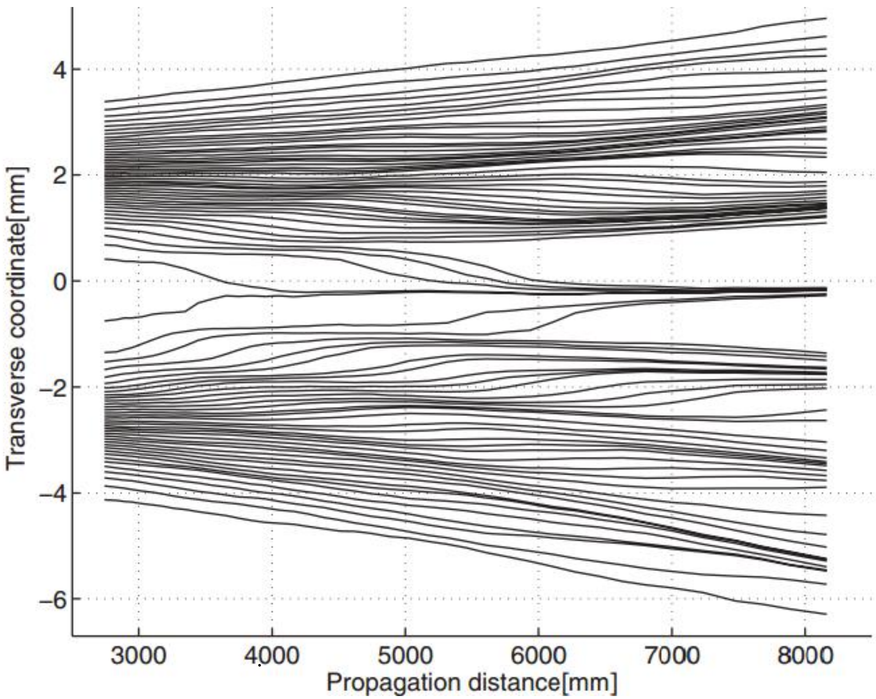
$$= m\mathbf{v}\left(\mathbf{r}_f, t\right) - i\hbar \frac{\nabla\rho\left(\mathbf{r}_f, t\right)}{2\rho\left(\mathbf{r}_f, t\right)}.$$

velocity of the particle
in the de Broglie-Bohm model

$$v(\vec{r},t) = \frac{\vec{j}(\vec{r},t)}{|\psi(\vec{r},t)|^2}$$

Schrodinger
current density

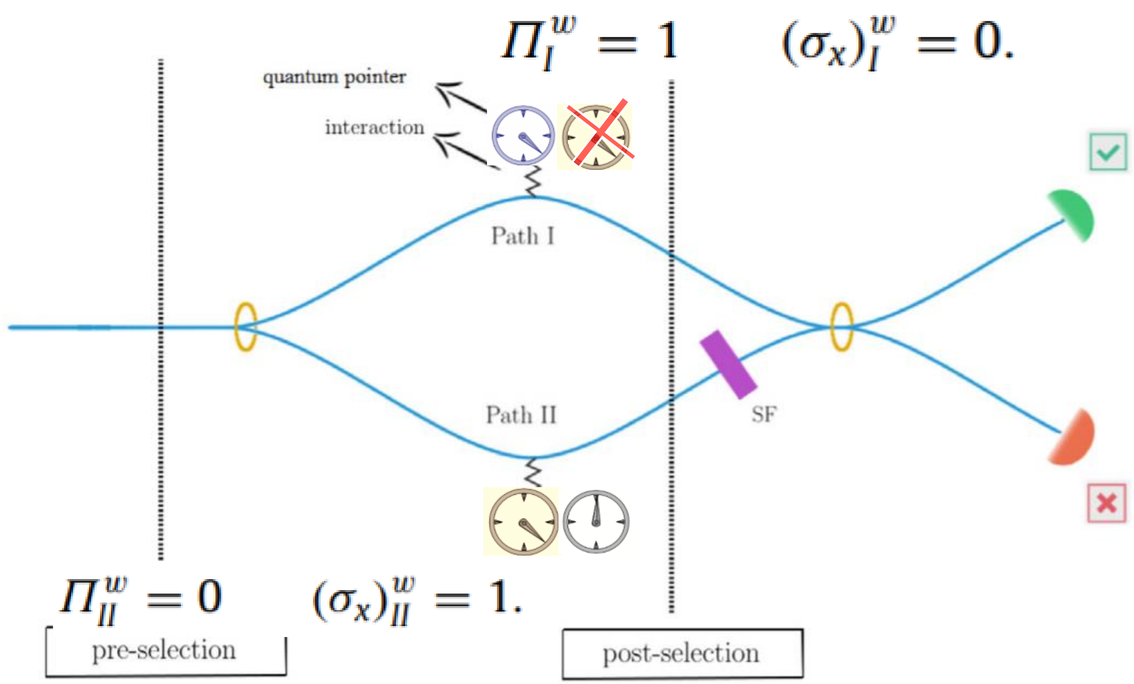
"Bohmian" trajectories Double slit experiment



Steinberg et al, Science, 2011.

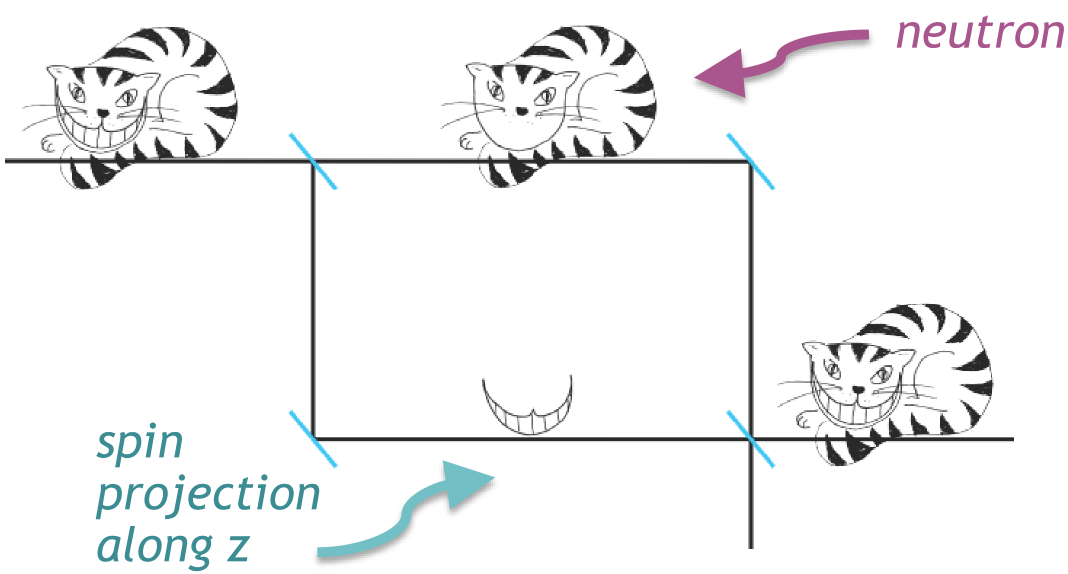
Weak measurements of different observables
Quantum Cheshire Cat effectS

Aharonov et al (NJP 2013, book Quantum Paradoxes 2005)



- The spin left a trace where the spatial wavefunction didn't ➔ spatial separation between the particle and one of its properties

Neutron Mach-Zehnder interferometric experiment



Denkmayr et al,
Nat. Comm. 2014

Inference of weak values without making weak measurements – fit weak values from observed intensities
➔ **not a demonstration of the quantum Cheshire cat effect**

Similar single photon expmt *Ashby & al, PRA 2016*
Experiments are hard to do!

Correa et al NJP 2015, Atherton et al Opt. Lett. 2015, Stuckey et al IJQF 2016, Duprey et al Ann Phys 2018

The quantum Cheshire cat: neutron interferometric experiment

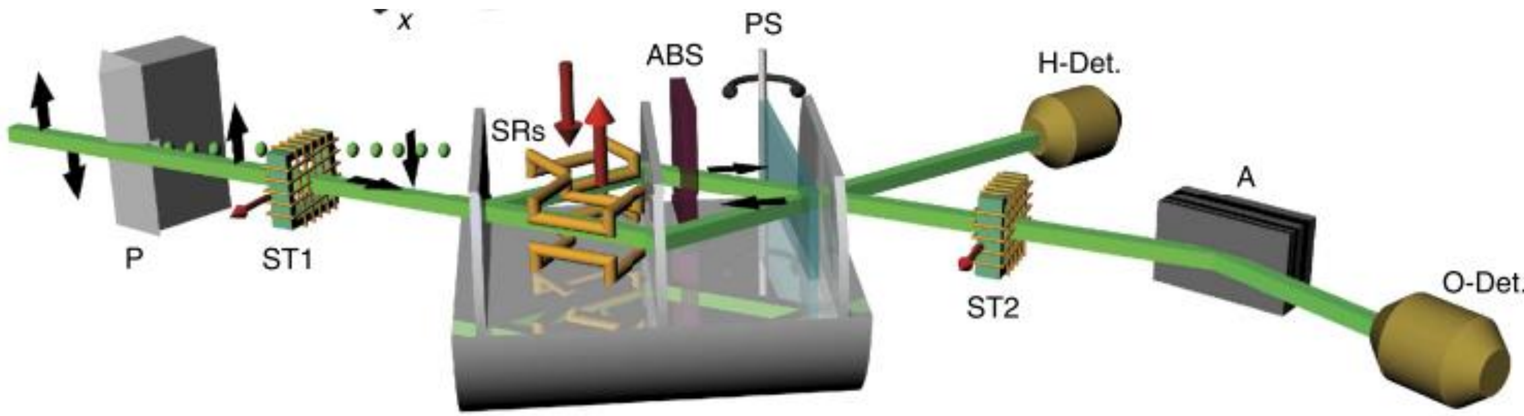
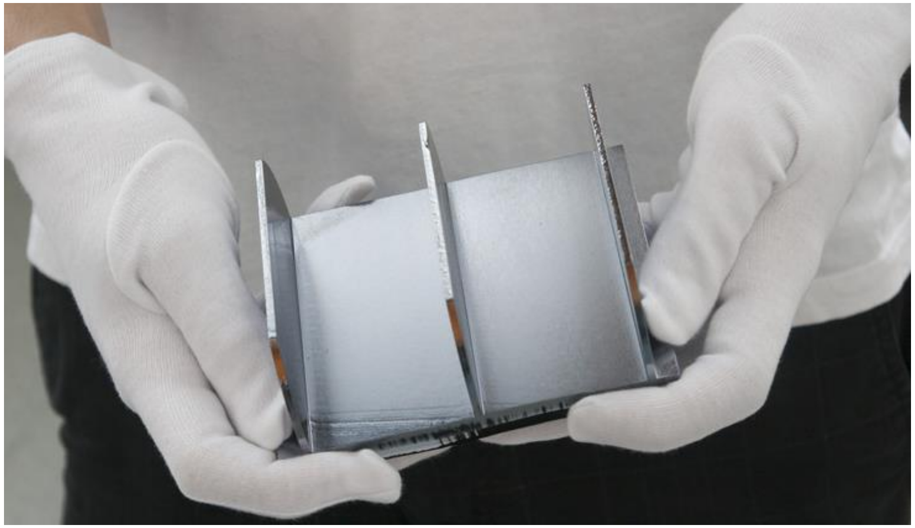


Figure 2 | Illustration of the experimental setup. The neutron beam is polarized by passing through magnetic birefringent prisms (P). To prevent depolarization, a magnetic guide field (GF) is applied around the whole setup. A spin turner (ST1) rotates the neutron spin by $\pi/2$. Preselection of the system's wavefunction $|\psi_i\rangle$ is completed by two spin rotators (SRs) inside the neutron interferometer. These SRs are also used to perform the weak measurement of $\langle\hat{\sigma}_z\hat{\Pi}_I\rangle_w$ and $\langle\hat{\sigma}_z\hat{\Pi}_{II}\rangle_w$. The absorbers (ABS) are inserted in the beam paths when $\langle\hat{\Pi}_I\rangle_w$ and $\langle\hat{\Pi}_{II}\rangle_w$ are determined. The phase shifter (PS) makes it possible to tune the relative phase χ between the beams in path *I* and path *II*. The two outgoing beams of the interferometer are monitored by the H and O detector in reflected and forward directions, respectively. Only the neutrons reaching the O detector are affected by postselection using a spin turner (ST2) and a spin analyzer (A).

The quantum Cheshire cat: neutron interferometric experiment



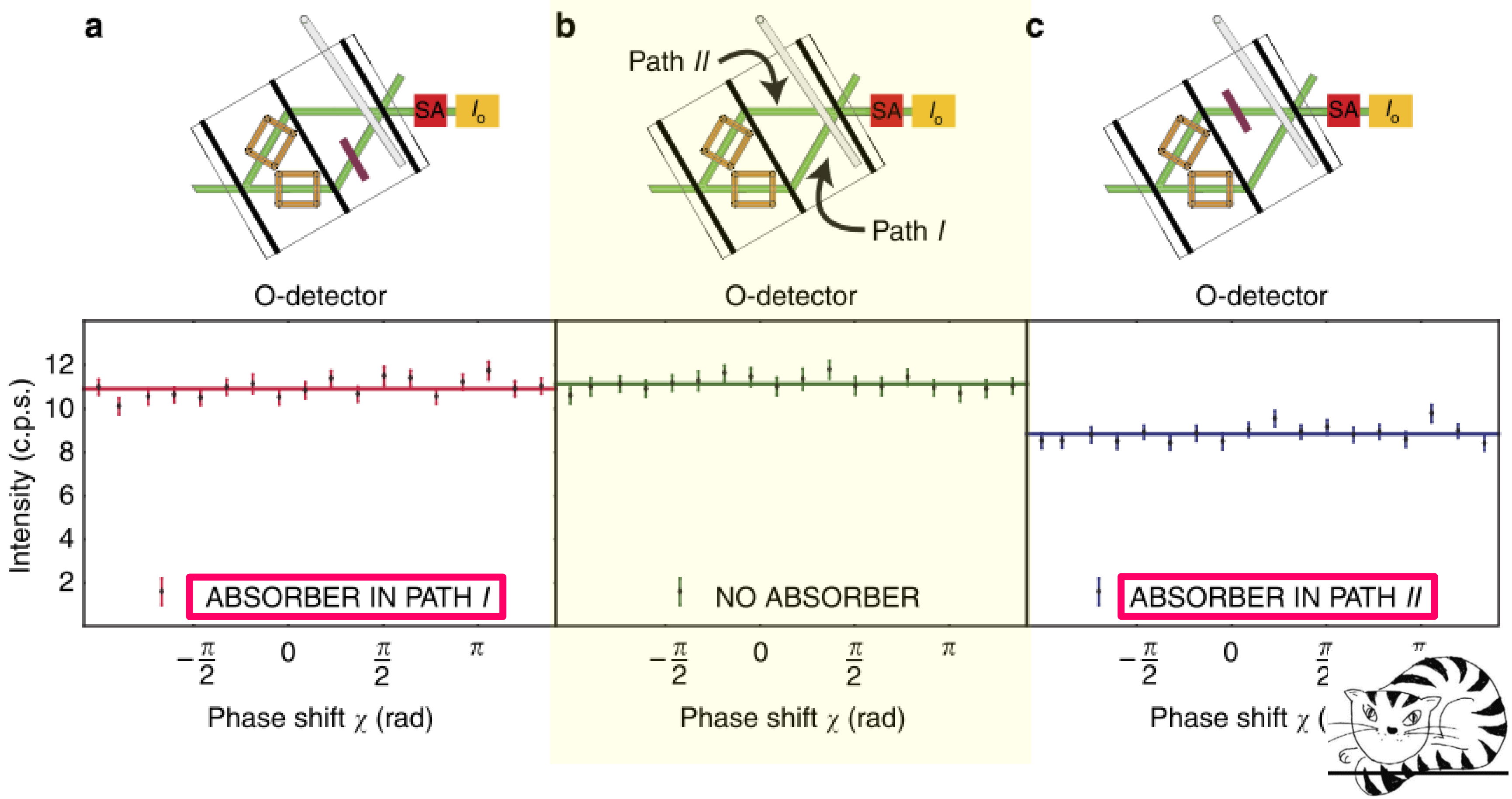
Where is the neutron located? Absorber-Path coupling

preselection

$$|\psi_{\rm i}\rangle = \frac{1}{\sqrt{2}}|S_x;\,+\rangle|I\rangle + \frac{1}{\sqrt{2}}|S_x;-\rangle|II\rangle$$

postselection

$$|\psi_{\rm f}\rangle = \frac{1}{\sqrt{2}}|S_x;\,-\rangle[|I\rangle + |II\rangle]\,.$$



Weak values and quantum properties

Weak measurements controversial (on the significance)

- Eigenstate-eigenvalue link ←→ measurement, property
- Pointer motion, consistent with standard measurements
- Ensemble average or single shot ?
- “Generalized form of eigenvalue (AAV)”?
  - No corresponding “element of reality” for the system (some eigenstate of B)
  - Retrodictive effect on the pointer state

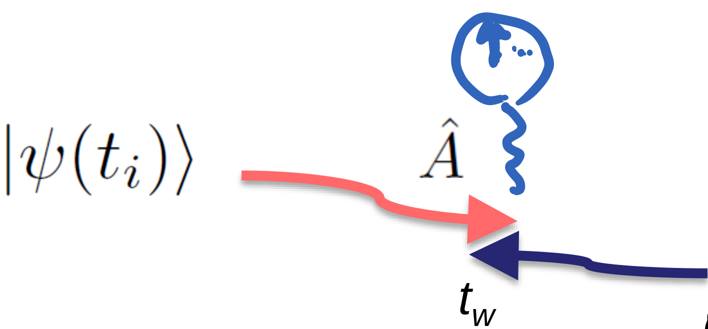
Svensson, Found Phys 2013, Alonso & Jordan, Quant Stud Math Found 2015; Griffiths PRA 2016; Duprey & Matzkin PRA 2017; Cohen, Found Phys 2017; Sokolovski Phys Lett A 2017, Matzkin Found Phys 2019, Vaidman et al PNAS 2019

Weak values and quantum properties

$$\mathrm{Re}\, A^w = \frac{\langle\psi(t_w)|\, \frac{1}{2}\left(\rho_{b_f(t_w)} A + A\rho_{b_f(t_w)}\right) |\psi(t_w)\rangle}{\langle\psi(t_w)|\, \rho_{b_f(t_w)}\, |\psi(t_w)\rangle}$$

$$\mathrm{Im}\, A^w = \frac{\langle\psi(t_w)|\, \frac{1}{2i}\left(\rho_{b_f(t_w)} A - A\rho_{b_f(t_w)}\right) |\psi(t_w)\rangle}{\langle\psi(t_w)|\, \rho_{b_f(t_w)}\, |\psi(t_w)\rangle}.$$

$$A_w = \frac{\langle b_f(t_w) | A | \psi(t_w)\rangle}{\langle b_f(t_w) | \psi(t_w)\rangle}$$



$\langle\, b_f \,|$

$\hat{B} = \sum_k b_k |b_k\rangle\langle b_k|$

Conclusion

- Weak measurements: protocol for non-destructive and non-disturbing measurements

- Weak value (as read from pointers): value related to a partial, local, conditioned property of a system.

- Property ascription:
  eigenvalue ⇔ *weak value*
  global prop./local Measurmt ⇔ *delocalized prop./local value*
  "particle-like" aspect ⇔ *"wave-like" aspect*

- Interesting for quantum foundations (experimentally play with the formalism), and also applications (quantum state measurement, particle tagging, weak signal amplification for parameter estimation)

Thanks!

Grazie!

Work in collaboration:
Q. Duprey, A. Pan (Univ. Cergy-Pontoise)
D. Home (Bose Inst.), U. Sinha (Raman Research Inst.)