How To Win Friends and Influence Functionals

The Influence Functional approach to quantum and classical systems

Gerard McCaul

Tulane University

gmccaul@tulane.edu

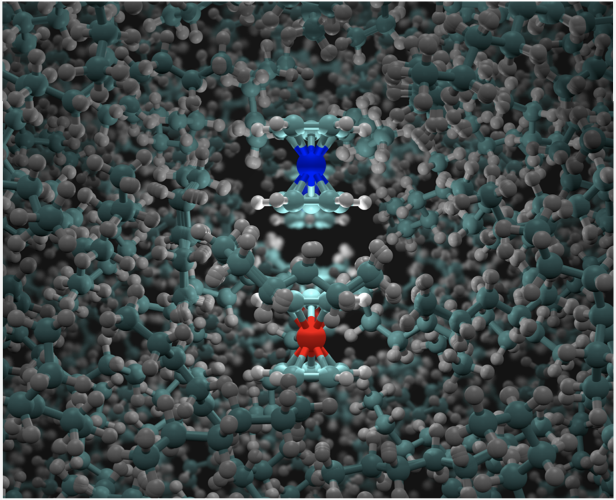
October 4, 2019

This Talk

This talk is about a powerful technique for describing the dynamics of an open system known as the influence functional. Here we will discuss:

- The applications of influence functionals
- How influence functionals are defined and used
- The exact stochastic equations that can be derived from them.
- Their utility in establishing rigorous classical limits.

What are Influence functionals?



- Influence functionals are a path integral technique, where the environmental part of an open system is expressed as single functional.
- They have numerous applications, both analytical and numerical.

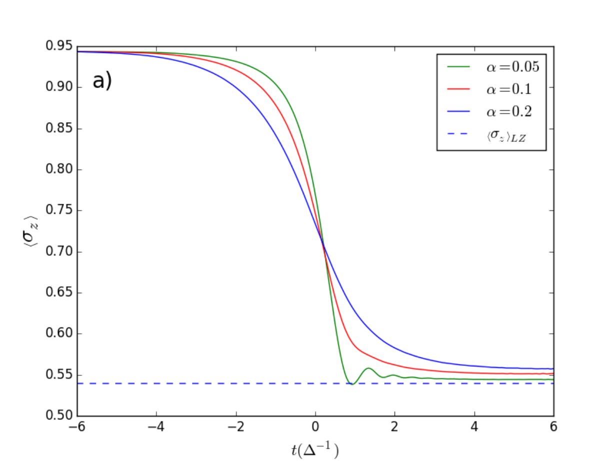
Spin Boson Model

Mappings

- Qubit in environment
- Kondo model
- Bio-molecular friction transfer

Applications

- Quantum computing
- Josephson junctions
- Exotic matter



Equations Derived With Influence Functionals

- One particularly useful application of influence functionals is their use in deriving exact equations.
- One example is the Stochastic Liouville Equation

Stochastic Liouville Equation

$$i\hbar\partial_t\tilde{\rho}\left(t\right) = \left[\hat{H}_Q\left(t\right), \tilde{\rho}\left(t\right)\right] - \eta\left(t\right)\left[\hat{q}\left(t\right), \tilde{\rho}\left(t\right)\right] - \frac{\hbar}{2}\nu\left(t\right)\left\{\hat{q}\left(t\right), \tilde{\rho}\left(t\right)\right\}$$

Generalised Langevin Equation

$$m\ddot{q}\left(t\right) = -V'\left(q,t\right) - 2\int_0^t \mathrm{d}t'\,\dot{q}(t')\gamma\left(t-t'\right) + \eta_{\mathrm{cl}}\left(t\right)$$

Equations Derived With Influence Functionals

- One particularly useful application of influence functionals is their use in deriving exact equations.
- One example is the Stochastic Liouville Equation

Classical Limit?

- The SLE and GLE are clearly closely related to each other.
- If we take the classical limit in a heuristic manner, we find that $\lim_{\hbar\to 0}$ SLE leads to:

$$m\ddot{q}\left(t\right) = -V'\left(q,t\right) + \eta_{\mathrm{cl}}\left(t\right)$$

Generalised Langevin Equation

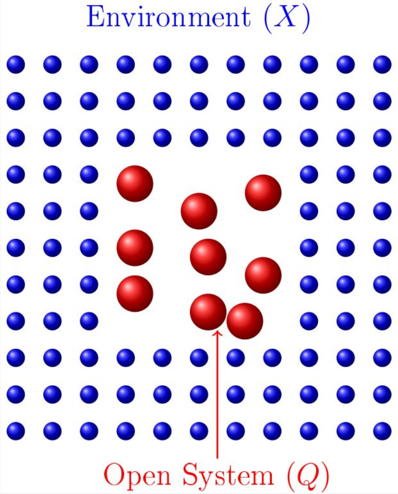
$$m\ddot{q}\left(t\right) = -V'\left(q,t\right) - 2\int_0^t \mathrm{d}t'\,\dot{q}(t')\gamma\left(t-t'\right) + \eta_{\mathrm{cl}}\left(t\right)$$

Evolving The Density Matrix

- For a global system described with canonical coordinates q, p and x, k, the total Hamiltonian may be characterised as

$$H_{\text{tot}} = H_Q + H_X + V_{QX}$$

- To find $\hat{\rho}_Q(t)$, we evolve $\hat{\rho}_0^{\text{tot}}$ before tracing over the environment.



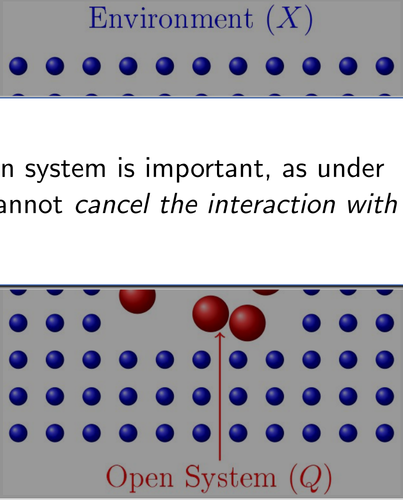
Evolving The Density Matrix

- For a global system described

The Influence of the Environment

Understanding the effect of an environment on an open system is important, as under ordinary circumstances a system's internal dynamics cannot *cancel the interaction with its surroundings...*

- To find $\hat{\rho}_Q(t)$, we evolve $\hat{\rho}_0^{\text{tot}}$ before tracing over the environment.



Brexit Means Brexit

The Goal of Influence Functionals

$$
\begin{tikzcd}
\rho(t_0) \arrow[r, "U_{\mathrm{cl}}(t_1 - t_0)"'] \arrow[d, "\mathrm{Tr}_X"] & \rho(t_1) \arrow[d, "\mathrm{Tr}_X"'] \\
\rho_Q(t_0) \arrow[r, "U_Q[\rho_X(t_0)]"] & \rho_Q(t_1)
\end{tikzcd}
$$

- We would like to describe the evolution of the open system without referring to its environment.
- To do so, we require an effective propagator U_Q.
- This is achieved with the influence functional.

The Goal of Influence Functionals

$$\rho_Q(t_0) \xrightarrow{U_Q[\rho_X(t_0)]} \rho_Q(t_1)$$

- We would like to describe the evolution

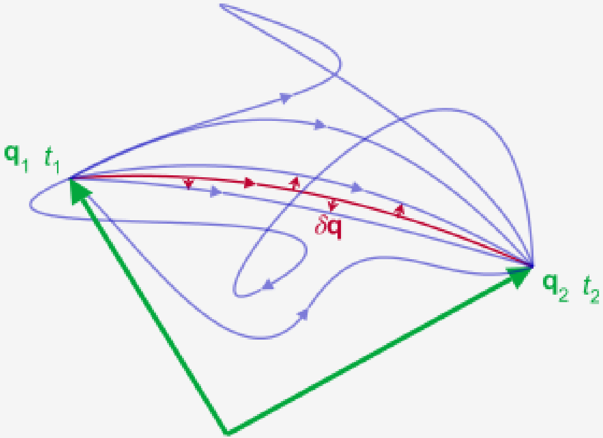
Reduced Density matrix

$$\rho_Q(q,q') = \int \mathrm{d}\bar{x}\mathrm{d}\bar{x}'\mathrm{d}x \; \mathrm{d}\bar{q}\mathrm{d}\bar{q}' \, U\left(q,x,t;\bar{q},\bar{x},0\right)$$

$$\times \rho_0^{\mathrm{tot}}\left(\bar{q},\bar{x};\bar{q}',\bar{x}'\right) U\left(\bar{q}',\bar{x}',0;q',x,t\right)$$

- This is achieved with the influence functional.

Path Integrals



- Influence functionals are a path integral technique
- Expressed as a path integral, the propagator of a state is

$$U(q_2, t_2; q_1, t_1) = \int_{q_1}^{q_2} \mathcal{D}q(t)\, \mathrm{e}^{\frac{iS}{\hbar}}$$

Influence Functionals

- We have an expression for the reduced density matrix, but can we evaluate it?
- Feynman & Vernon (1963) developed a powerful formalism for dealing with systems of this type, using path integrals to describe the system of interest without reference to the environment.

$$\rho_{t_f}\left(q;q'\right)=\frac{1}{Z}\int \mathrm{d}\bar{q}\mathrm{d}\bar{q}'\mathcal{D}q\left(t\right)\mathcal{D}q'\left(t\right)\left\{\mathcal{F}\left[q\left(t\right),q'\left(t\right)\right]\right.$$

$$\left.\times\,\rho_0\left(\bar{q};\bar{q}'\right)\exp\left[\frac{i}{\hbar}S_q\left[q\left(t\right)\right]-\frac{i}{\hbar}S_q\left[q'\left(t\right)\right]\right]\right\}$$

Influence Functionals

- We have an expression for the reduced density matrix, but can we evaluate it?
- Feynman & Vernon (1963) developed a powerful formalism for dealing with systems of this type, using path integrals to describe the system of interest without reference to the

System Action

$S_q\left[q\left(t\right)\right]$ is the action of the isolated system evolving under the Hamiltonian $H_{\mathrm{tot}} = H_Q$

$$\times\, \rho_0\left(\bar{q};\bar{q}'\right)\exp\left[\frac{i}{\hbar}S_q\left[q\left(t\right)\right]-\frac{i}{\hbar}S_q\left[q'\left(t\right)\right]\right]\Bigg\}$$

Influence Functionals

Influence Functional

$\mathcal{F}\left[q\left(t\right),q'\left(t\right)\right]$ is the *Influence Functional*. It characterises the environment's effect on the system.

$$\mathcal{F}\left[q\left(t\right),q'\left(t'\right)\right]=\int\mathcal{D}\bar{X}(t)\rho_{X}(x_{0};x_{0}')\exp\left(iS_{\mathcal{F}}/\hbar\right)$$

$$\mathcal{D}\bar{X}(t)=\mathrm{d}x_{0}\mathrm{d}x_{0}'\mathrm{d}x\mathcal{D}x(t)\mathcal{D}x'(t)$$

$$S_{\mathcal{F}}=S_{X}\left[x\left(t\right)\right]-S_{X}\left[x'\left(t\right)\right]+S_{QX}\left[q\left(t\right),x\left(t\right)\right]-S_{QX}\left[q'\left(t\right),x'\left(t\right)\right]$$

Using the Influence Functional

- Influence functionals can be evaluated numerically, but in some circumstances it is possible to derive analytic expressions.
- In this case they may be used to derive effective equations of motion for the reduced system.

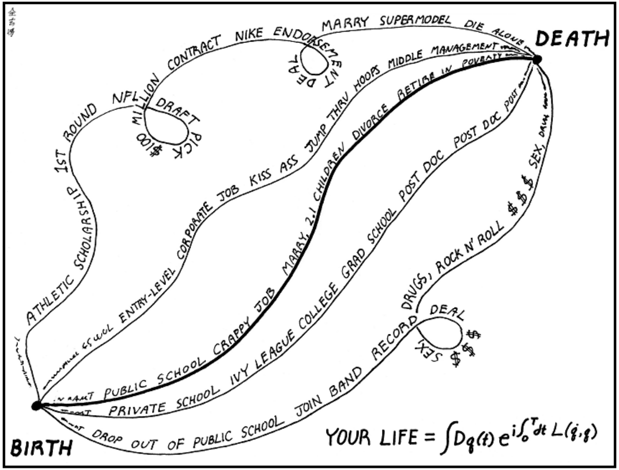
- if $\mathcal{F}\left[q\left(t\right), q'\left(t'\right)\right]$ can be decomposed into a product of the form

$$F\left[q\left(t\right), q'\left(t'\right)\right] = \exp\left(\frac{i}{\hbar}\Phi_1\left[q\left(t\right)\right]\right)\exp\left(\frac{i}{\hbar}\Phi_2\left[q'\left(t'\right)\right]\right)$$

- Then the effective propagator U_Q into a product of the form

$$U_{\mathrm{Q}}\left[q\left(t\right), q'\left(t'\right)\right] = \tilde{U}_{\mathrm{Q}}\left[q\left(t\right)\right]\tilde{U}_{\mathrm{Q}}^{\dagger}\left[q'\left(t'\right)\right]$$

- from which an effective equation of motion may be found.



The Path Integral Formulation of Your Life

- Influence functionals do not have to be restricted to quantum mechanics!
- A connection can be made to the classical limit of quantum results using the Classical Influence Functional
- Doing so requires a Hilbert space formulation of classical mechanics, in order to construct a classical path integral

What is Koopman von-Neumann mechanics?

- Koopman von-Neumann (KvN) mechanics was proposed in 1932 as a Hilbert space formulation of classical mechanics, providing an operational framework to assess classical problems.
- In both KvN and quantum mechanics, the propagator is a unitary transformation of the form:

$$\hat{U}(t) = \mathrm{e}^{-i\hat{A}t}$$

Classical

- $[\hat{q}, \hat{p}] = 0$
- $\dot{\phi} = \{H, \phi\}$
- $\hat{A} = \hat{K} = \frac{\hat{p}}{m}\hat{\lambda} - \hat{V}'(x)\,\hat{\theta}$

Quantum

- $[\hat{q}, \hat{p}] = i$
- $\dot{\phi} = -i\hat{H}\phi$
- $\hat{A} = \hat{H}$

What is Koopman von-Neumann mechanics?

- Koopman von-Neumann (KvN) mechanics was proposed in 1932 as a Hilbert space formulation of classical mechanics, providing an operational framework to assess classical problems

Bopp Operators

$\hat{\lambda}$ and $\hat{\theta}$ are *Bopp operators*, defined by their commutation relations:

$$\left[\hat{x}, \hat{\lambda}\right] = \left[\hat{p}, \hat{\theta}\right] = i$$

$$\left[\hat{\lambda}, \hat{\theta}\right] = \left[\hat{\lambda}, \hat{p}\right] = \left[\hat{\theta}, \hat{x}\right] = 0$$

- $\phi = \{H, \phi\}$
- $\hat{A} = \hat{K} = \frac{\hat{p}}{m}\hat{\lambda} - \hat{V}'(x)\,\hat{\theta}$

- $\phi = -iH\phi$
- $\hat{A} = \hat{H}$

Constructing a Classical Path Integral

- In order to derive the Classical influence functional, we require a Classical Path Integral.
- Given the KvN propagator

$$\hat{U}_{\mathrm{cl}} = \mathrm{e}^{-it\hat{K}}$$

- We can find a representation for it in phase space

$$U_{\mathrm{cl}}(q_f, p_f, t_f; q_i, p_i) = \left\langle q_f, p_f \left| \mathrm{e}^{-it_f \hat{K}} \right| q_i, p_i \right\rangle .$$

- Just like in the quantum case, this propagator can be expressed as a path integral:

$$U_{\mathrm{cl}}\left(q_f, p_f, t_f; q_i, p_i\right) = \int_{x_i, p_i}^{x_f, p_f} \underbrace{\mathcal{D}q\mathcal{D}p\mathcal{D}\lambda\mathcal{D}\theta}_{\mathcal{D}Q} \; \mathrm{e}^{iR}$$

- Since the classical wavefunction and probability density are evolved by the same equation, this propagator can be used to evolve *both*.

Constructing a Classical Path Integral

- In order to derive the Classical influence functional, we require a Classical Path Integral.
- Given the KvN propagator

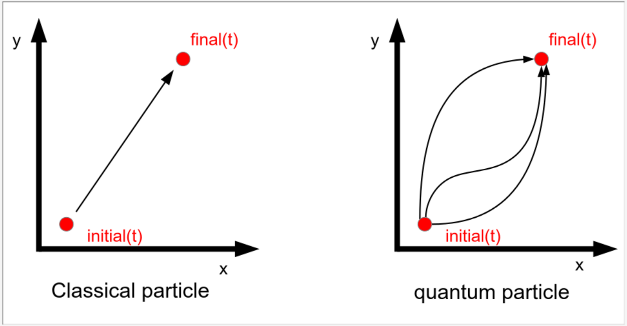
$$\hat{U}_{\mathrm{cl}} = \mathrm{e}^{-it\hat{K}}$$

The Classical "Quantum Action"

$$R = \int_0^{t_f} \mathrm{d}t \left[\lambda(t)\left(\dot{x}(t) - \frac{p(t)}{m}\right) + \theta(t)\left(\dot{p}(t) + V'(x,t)\right)\right]$$

$$U_{\mathrm{cl}}(q_f, p_f, t_f; q_i, p_i) = \int_{x_i,p_i}^{x_f,p_f} \underbrace{\mathcal{D}q\mathcal{D}p\mathcal{D}\lambda\mathcal{D}\theta}_{\mathcal{D}Q} \ \mathrm{e}^{iR}$$

- Since the classical wavefunction and probability density are evolved by the same equation, this propagator can be used to evolve *both*.



- Classical path integral for total system is as expected (a delta functional).
- The path integral describing a *reduced* system is much more interesting.
- The process of tracing out the environment results in a non-trivial sum of trajectories.

Classical Influence Functional

- Evolving the composite system with the path integral propagator, and tracing out the environment gives:

$$\rho_Q(q_f, p_f, t_f) = \int \mathrm{d}x_f \mathrm{d}k_f \left(\int \mathcal{D}Q\mathcal{D}X \mathrm{d}q_0 \mathrm{d}p_0 \mathrm{d}x_0 \mathrm{d}k_0 \ \mathrm{e}^{iR} \rho_0 \right)$$

- Using the influence functional it is possible to describe an *effective* propagator for the reduced system $U_Q \left[\rho_X(t_0)\right]$.

$$U_Q(q_f, p_f, t_f) = \int \mathcal{D}q(t)\mathcal{D}\theta_Q(t)\, \mathcal{F}\left[q(t), p(t), \theta_Q(t)\right] \exp\left(i \int_0^{t_f} \mathrm{d}t \ \left[\theta_Q(t) \left(m\ddot{q} + \frac{\partial V_Q}{\partial q} \right) \right] \right)$$

Classical Influence Functional

- Evolving the composite system with the path integral propagator, and tracing out the

The form of this propagator depends on the influence functional, which is in turn a functional of the initial environment state.

$$\mathcal{F}\left[q\left(t\right),p\left(t\right),\theta_Q\left(t\right)\right]=\int \mathrm{d}x_0\mathrm{d}k_0\mathrm{d}x_f\mathrm{d}k_f\mathcal{D}x(t)\mathcal{D}\theta_X\left(t\right)\;\rho_X(x_0,k_0,q_0,p_0)$$
$$\times\exp\left(i\int_0^{t_f}\mathrm{d}t\;\left[\theta_X\left(t\right)\left(m_k\ddot{x}+\frac{\partial V_{QX}}{\partial x}+\frac{\partial V_X}{\partial x}\right)+\theta_Q\left(t\right)\frac{\partial V_{QX}}{\partial q}\right]\right)$$

$$U_Q(q_f,p_f,t_f)=\int \mathcal{D}q(t)\mathcal{D}\theta_Q\left(t\right)\mathcal{F}\left[q\left(t\right),p\left(t\right),\theta_Q\left(t\right)\right]\exp\left(i\int_0 \mathrm{d}t\;\left[\theta_Q\left(t\right)\left(m\ddot{q}+\frac{\partial}{\partial q}\right)\right]\right)$$

Under certain circumstances an effective equation of motion may be defined from the influence functional. This is the case when it is possible to express $\mathcal{F}$ as

$$\mathcal{F}\left[q\left(t\right),p\left(t\right),\theta_{Q}\left(t\right)\right]=\exp\left(i\int_{0}^{t_{f}}\mathrm{d}t\;\theta_{Q}\left(t\right)\chi\left[q(t),p(t)\right]\right) \tag{1}$$

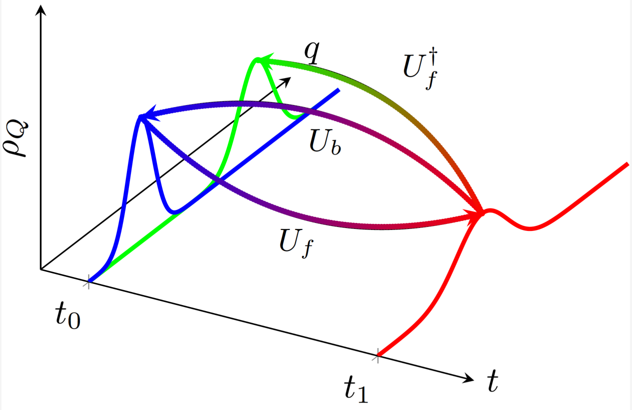
where $\chi\left[q(t),p(t)\right]$ is an arbitrary functional of the phase space coordinates only. In the case, the effective equation of motion will be:

$$m\ddot{q}=-\frac{\partial V_{Q}}{\partial q}-\chi\left[q,\dot{q}\right] \tag{2}$$

The Effective Propagator

In both the classical and quantum cases, the effective propagator U_Q is now a functional of the environment state at the initial time. The "true" effective propagator for a system depends on the point in time it is being evolved from.

- $U_f = U_Q\left[\rho_X(t_0)\right]$
- $U_b = U_Q^{\dagger}\left[\rho_X(t_1)\right]$



Effective Propagators

Applying Influence functionals

- Using influence functionals, it is possible to derive both the Stochastic Liouville equation and Generalised Langevin Equation.
- To do so, we must specify an environment and interaction Hamiltonian.
- We will use the Caldeira-Leggett (CL) model

$$
\begin{aligned}
H_{\text{tot}} = H_Q + \frac{1}{2}\sum_n \left(m_n \dot{x}_n^2 + m_n \omega_n^2 x_n^2\right) \\
- q\sum_n c_n x_n + \frac{q^2}{2}\sum_n \frac{c_n^2}{m_n \omega_n^2}.
\end{aligned}
$$

- Furthermore, we will specify that the environment is initially in thermal equilibrium.

In both cases, the influence functional can be expressed as

$$\mathcal{F} = \mathrm{e}^{\Phi}$$

Quantum Influence Functional

Quantum

$$\Phi\left[q(t),q'(t)\right] = \frac{i}{2\hbar}\int_0^{t_f}\mathrm{d}t\int_0^{t_f}\mathrm{d}t'\,K^R\left(t-t'\right)\epsilon\left(t\right)\epsilon\left(t'\right)$$
$$-\frac{1}{\hbar}\int_0^{t_f}\mathrm{d}t\int_0^{t_f}\mathrm{d}t'\,\left[\theta\left(t-t'\right)K^I\left(t-t'\right)\right]\epsilon\left(t\right)y\left(t'\right)$$

$$\epsilon\left(t\right) = q\left(t\right) - q'\left(t\right) \qquad y\left(t\right) = \frac{1}{2}\left(q\left(t\right) + q'\left(t\right)\right)$$

$$K^R\left(t-t'\right) = \int_0^{\infty}\mathrm{d}\omega\;I\left(\omega\right)\coth\left(\frac{1}{2}\hbar\beta\omega\right)\cos\left(\omega t\right)$$

$$K^I\left(t\right) = -\int_0^{\infty}\mathrm{d}\omega\;I\left(\omega\right)\sin\left(\omega t\right)$$

Classical Influence Functional

Classical

$$\Phi\left[q\left(t\right),p\left(t\right),\theta_{Q}\left(t\right)\right] = 2i\int_{0}^{t_f}\mathrm{d}t\ \theta_{Q}\left(t\right)\int_{0}^{t}\mathrm{d}t'\,q(t')\frac{\mathrm{d}\gamma\left(t-t'\right)}{\mathrm{d}t'} - \int^{t_f}\mathrm{d}t\int^{t_f}\mathrm{d}t'\ \theta_{Q}\left(t\right)k_B T\gamma\left(t-t'\right)\theta_{Q}\left(t'\right)$$

$$\gamma\left(t-t'\right) = \int_{0}^{\infty}\frac{\mathrm{d}\omega}{\omega\pi}\ I\left(\omega\right)\cos\left(t-t'\right)$$

- In order to use the influence functional as part of an effective equation of motion, it must conform to the following functional form:

Classical

$$\mathcal{F}\left[q\left(t\right),p\left(t\right),\theta_{Q}\left(t\right)\right]=\exp\left(i\int_{0}^{t_f}\mathrm{d}t\;\theta_{Q}\left(t\right)\chi\left[q(t),p(t)\right]\right)$$

Quantum

$$\mathcal{F}\left[q\left(t\right),q'\left(t'\right)\right]=\exp\left(\frac{i}{\hbar}\Phi_{1}\left[q\left(t\right)\right]\right)\exp\left(\frac{i}{\hbar}\Phi_{2}\left[q'\left(t'\right)\right]\right)$$

- In order to use the influence functional as part of an effective equation of motion, it must conform to the following functional form:

Classical

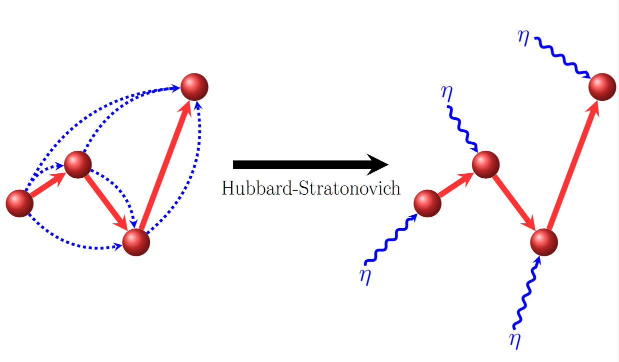
Problem

There is a problem here. Neither the quantum or classical influence functional is of a form from which an equation of motion can be derived. Is there a solution?

$$\mathcal{F}\left[q\left(t\right),q'\left(t'\right)\right]=\exp\left(\frac{}{\hbar}\Phi_1\left[q\left(t\right)\right]\right)\exp\left(\frac{}{\hbar}\Phi_2\left[q'\left(t'\right)\right]\right)$$

The Hubbard-Stratonovich Transformation

Use the Hubbard-Stratonovich Transformation!



- The HS transformation can be considered as converting a system of two body potentials into a set of independent particles in a fluctuating field.

The Hubbard-Stratonovich Transformation

Fourier-Transforming a Gaussian

$$\begin{tikzcd}
 & \langle \mathrm{e}^{ikx} \rangle_r \arrow[d, leftrightarrow] \\
\mathrm{e}^{-\frac{1}{2\sigma}x^2} \arrow[r, "\text{Fourier}"'] & \int \mathrm{d}x \; \mathrm{e}^{-\frac{1}{2\sigma}x^2} \mathrm{e}^{ikx} \arrow[d, leftrightarrow] \\
 & \mathrm{e}^{-\frac{1}{2}\sigma k^2}
\end{tikzcd}$$

into a set of independent particles in a fluctuating field.

Trajectories vs Distribution

Transformed Influence Functionals

Classical Propagator

$$\tilde{U}_{\text{cl}} = \int_{q_0,\dot{q}_0}^{q_f,\dot{q}_f} \mathcal{D}q(t)\delta\left[m\ddot{q}\,(t) + V'\,(q,t) + 2\int_0^t \mathrm{d}t'\,\dot{q}(t')\gamma\left(t-t'\right) - \eta_{\text{cl}}\,(t)\right]$$

Quantum Propagator

$$\hat{U}^{\pm}\,(t_f) = \widehat{T}\exp\left(-\frac{i}{\hbar}\int_0^{t_f}\hat{L}_Q\,(t) - \left[\eta\,(t) \pm \frac{\hbar}{2}\nu\,(t)\right]\hat{q}\,(t)\,\mathsf{d}t\right)$$

Transformed Influence Functionals

Classical Propagator

Making Some Noise

$$\left\langle \eta\left(t\right)\eta\left(t'\right)\right\rangle_r = \hbar\int_0^{\infty}\frac{\mathrm{d}\omega}{\pi}I\left(\omega\right)\coth\left(\frac{1}{2}\omega\hbar\beta\right)\cos\left(\omega\left(t-t'\right)\right)$$

$$\left\langle \eta(t)\nu\left(t'\right)\right\rangle_r = -2i\Theta\left(t-t'\right)\int_0^{\infty}\frac{\mathrm{d}\omega}{\pi}I\left(\omega\right)\sin\left(\omega\left(t-t'\right)\right)$$

$$\left\langle \eta_{\mathrm{cl}}\left(t\right)\eta_{\mathrm{cl}}\left(t'\right)\right\rangle = \lim_{\hbar\to 0}\left\langle \eta\left(t\right)\eta\left(t'\right)\right\rangle = \gamma(t-t')$$

Equations of Motions

Stochastic Liouville Equation

$$i\hbar\partial_t\tilde{\rho}\left(t\right) = \left[\hat{H}_Q\left(t\right), \tilde{\rho}\left(t\right)\right] - \eta\left(t\right)\left[\hat{q}\left(t\right), \tilde{\rho}\left(t\right)\right] - \frac{\hbar}{2}\nu\left(t\right)\left\{\hat{q}\left(t\right), \tilde{\rho}\left(t\right)\right\}$$

Generalised Langevin Equation

$$m\ddot{q}\left(t\right) = -V'\left(q,t\right) - 2\int_0^t \mathrm{d}t'\,\dot{q}(t')\gamma\left(t-t'\right) + \eta_{\mathrm{cl}}\left(t\right)$$

Classical Limit

- To show the SLE is equivalent in the classical limit to the GLE, we take the *effective* propagator of the reduced quantum system

$$\tilde{\rho}_{t_f}(q;q') = \int \mathrm{d}\bar{q}\mathrm{d}\bar{q}'\ \tilde{U}_{\text{eff}}\left(q,q',t_f;\bar{q},\bar{q}',0\right)\tilde{\rho}_0\left(\bar{q};\bar{q}'\right)$$

- Here $\tilde{U}_{\text{eff}}\left(q,q',t_f;\bar{q},\bar{q}',0\right)$ is $\hat{U}^{+}\left(t_f\right)\hat{U}^{-}\left(t_f\right)$ in path integral form.
- It is then possible to show:

$$\lim_{\hbar\to 0}\tilde{U}_{\text{eff}}\left(q,q',t_f;\bar{q},\bar{q}',0\right) = \tilde{U}_{\text{cl}}$$

Classical Limit

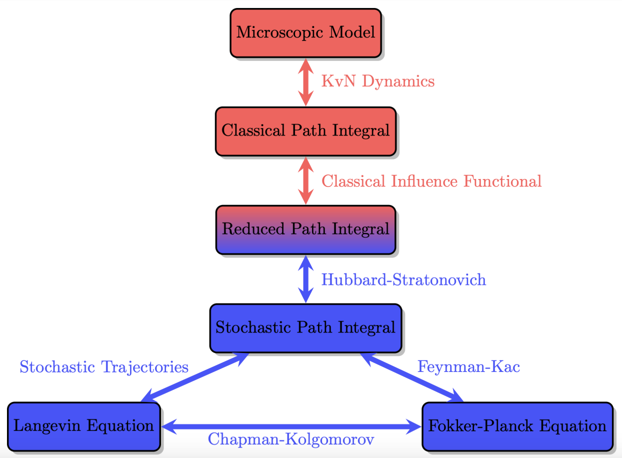
- To show the SLE is equivalent in the classical limit to the GLE, we take the *effective* propagator of the reduced quantum system

Propagator classical limit

$$\lim_{\hbar\to 0}\tilde{U}_{\text{eff}} = \int_{q_0,\dot{q}_0}^{q_f,\dot{q}_f}\mathcal{D}q(t)\;\delta\left[m\ddot{q}\left(t\right)+V'\left(q,t\right)-\eta_{\text{cl}}\left(t\right)\right]\exp\left(i\int_0^{t_f}\mathrm{d}t\;q\left(t\right)\nu_{\text{cl}}\left(t\right)\right)$$

$$\left\langle\eta_{\text{cl}}\left(t\right)\nu_{\text{cl}}\left(t'\right)\right\rangle_r = -2i\Theta\left(t-t'\right)\frac{\mathrm{d}\gamma\left(t-t'\right)}{\mathrm{d}t}$$

Summing up



- Influence functionals are a useful technique in the analysis of physical systems.
- One application is in the derivation of exact classical and quantum stochastic equations.
- The formalism also helps to link classical results as the limit of their quantum equivalents.

References

R.P. Feynman, F.L. Vernon Jr. (1963)
Ann. Phys. **24**, 118-173

H. Grabert, P. Schramm. G. Ingold (1988)
Physics Reports **3**, 115-207

H. Kleinert, S.V. Shabanov (1995)
Phys. Lett. A. **200**, 224-232

J. Stockburger (2004)
J. Chem. Phys. **296**, 159-169

P. Walters, N. Makri (2015)
Chem. Phys. Lett. **24**, 4959-4965

G.M.G McCaul, C.D. Lorenz, L. Kantorovitch (2018)
Phys. Rev. B **97**, 224310

G. McCaul, A. Pechen, D. Bondar (2019)
Phys. Rev. E **99**, 062121