Scattering amplitudes and contour deformations

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Diquark correlations in hadron physics
ECT*, Trento, Italy

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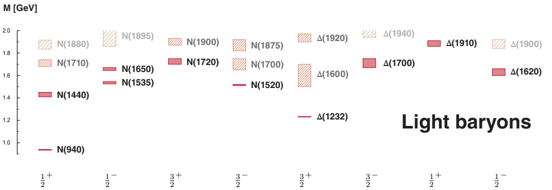
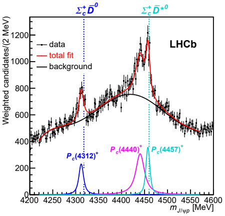
Motivation

Spectroscopy:

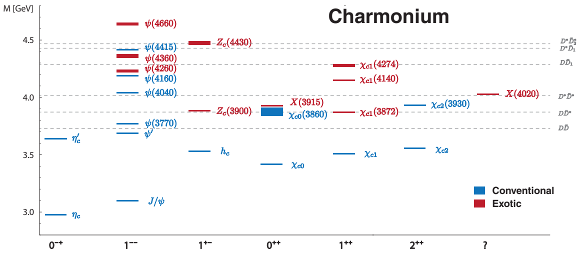
- Light hadrons are relativistic, chiral symmetry important
- Multiquarks?
- Resonances!





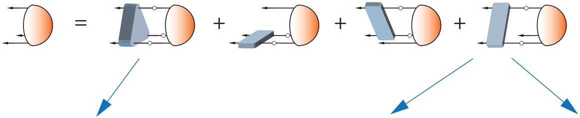
LHCb pentaquarks

Aaij, PRL 112 (2019) 222001

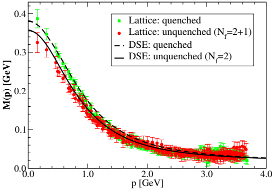


Bound-state equations

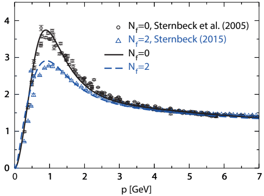
Bethe-Salpeter equation for baryons: GE, Sanchis-Alepuz, Williams, Alkofer, Fischer, PPNP 91 (2016), 1606.09602



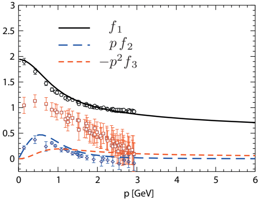
Quark:



Gluon:



Quark-gluon vertex:



- Many **running quantities** go into calculation of observables
- Relativistic BS amplitudes carry **rich tensor structure**
  (pion: **4**, ρ-meson: **8**, nucleon: **64**, Δ-baryon: **128**, ...)

Bound-state equations

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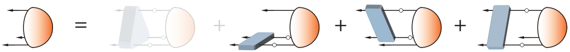
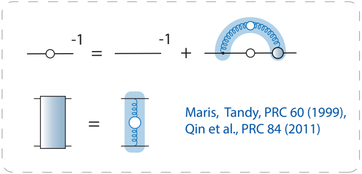


Rainbow-ladder:



Maris, Tandy, PRC 60 (1999),
Qin et al., PRC 84 (2011)

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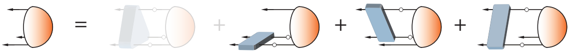
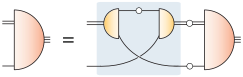
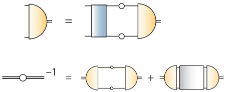


Quark-diquark approximation:





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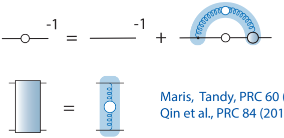

Maris, Tandy, PRC 60 (1999),
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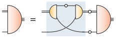
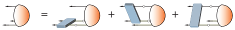
DSE / BSE / Faddeev landscape



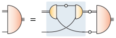


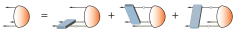
		NJL / contact	q-dq model	DSE (RL)		DSE (bRL)
u/d	N, Δ masses	√	√	√	√	*√*
	N, Δ em. FFs	√	√	√	√	
	$N \to \Delta\gamma$	√	√	√	*√*	
	N^*, Δ^* masses (+)	√	√	*√*	*√*	
	$N \to N^*\gamma$	√	√			
	N^*, Δ^* masses (-)	*√*	*√*	*√*	*√*	
	$N \to N^*\gamma$					
s	ground states	*√*	*√*	*√*	*√*	
	excited states	*√*	*√*	*√*	*√*	
	em. FFs & TFFs				*√*	
c, b	ground states	*√*	*√*		*√*	
	excited states		*√*		*√*	
		Cloet, Thomas, Roberts, Bashir, Segovia, Chen, Wilson, Lu, . . .	Oettel, Alkofer, Roberts, Cloet, Segovia, Chen, El-Bennich, . . .	GE, Alkofer, Nicmorus, Sanchis-Alepuz, Fischer	GE, Sanchis-Alepuz, Fischer, Alkofer, Qin, Roberts	Sanchis-Alepuz, Williams, Fischer

√ . . . before 2015

√ . . . after 2015

DSE / BSE / Faddeev landscape





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	excited states	√	√	√	√	
	em. FFs & TFFs				√	
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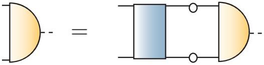
see talks by
Chen Chen,
Marco Bedolla,
Jorge Segovia,
...

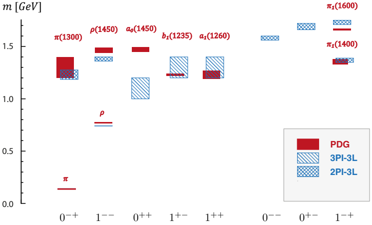
√ . . . before 2015
√ . . . after 2015

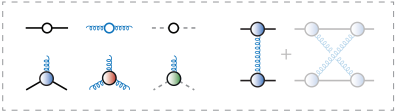
Bound-state equations

Bethe-Salpeter equation for mesons: GE, Sanchis-Alepuz, Williams, Alkofer, Fischer, PPNP 91 (2016), 1606.09602



Light meson spectrum beyond rainbow-ladder:
All two & three-point functions calculated (3PI)
Williams, Fischer, Heupel, PRD 93 (2016)





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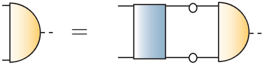

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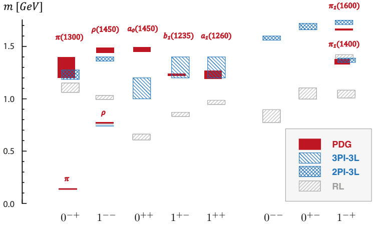
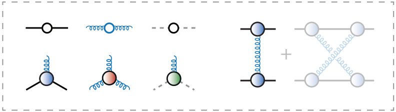
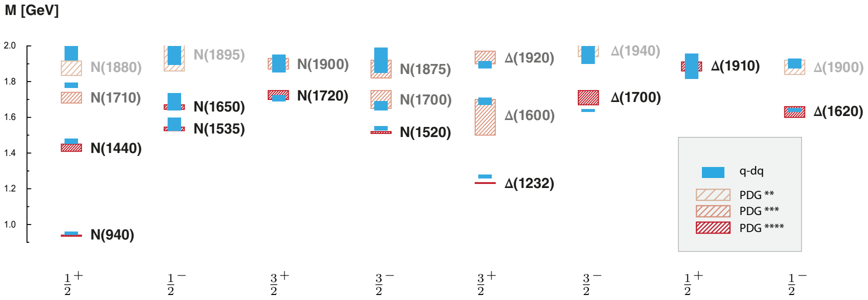




Light baryons

Light baryon spectrum (DSE-RL'): GE, Fischer, Sanchis-Alepuz, PRD 94 (2016)



- Spectrum in 1:1 agreement with experiment
- Correct level ordering (without coupled-channel effects...)

2 parameters,
1 scale, $m_{u,d,s}$

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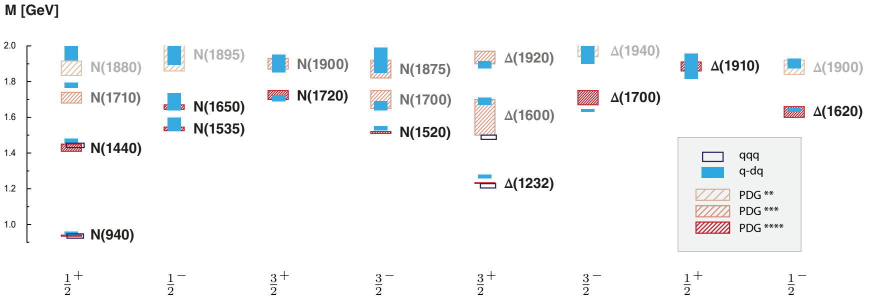

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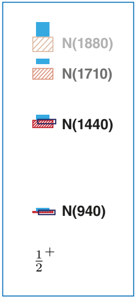
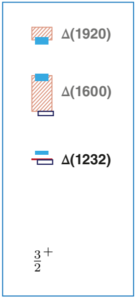


here
scalar and
axialvector
diquarks are
sufficient



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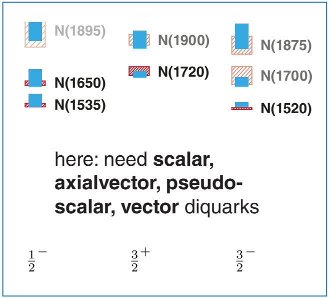

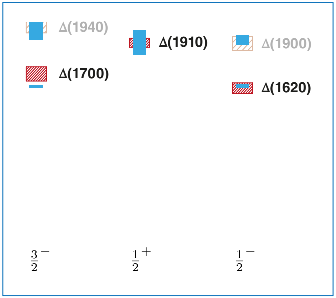

- Spectrum in 1:1 agreement with experiment
- Correct level ordering (without coupled-channel effects...)
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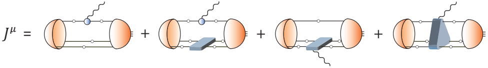
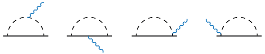
similar: Chen et al., PRD 97 (2018)
see **Chen Chen**’s talk

Form factors

Nucleon em. form factors
from three-quark calculation
GE, PRD 84 (2011)

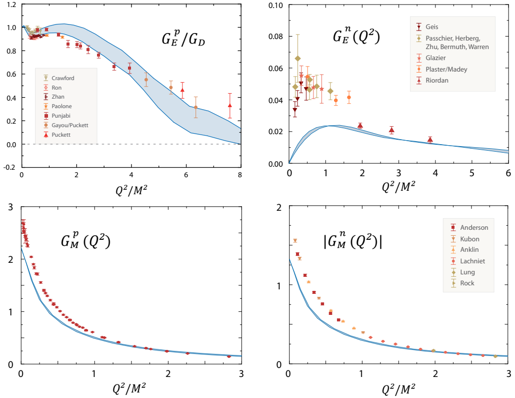


- **"Quark core without pion cloud"**



- **similar:** $N \to \Delta\gamma$ transition, axial & pseudoscalar FFs, octet & decuplet em. FFs

  Review: GE, Sanchis-Alepuz, Williams, Fischer, Alkofer, PPNP 91 (2016), 1606.09602



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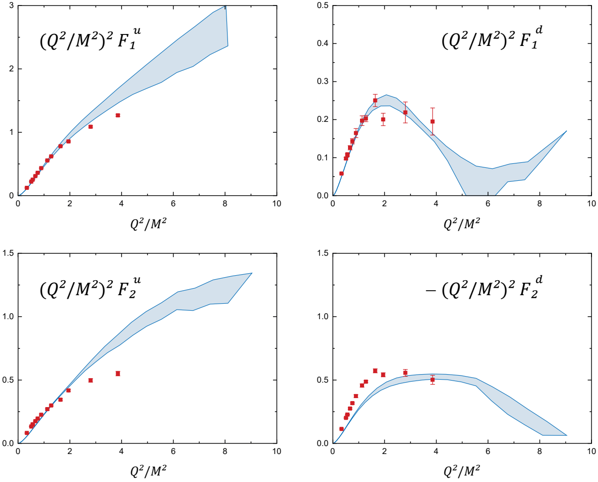
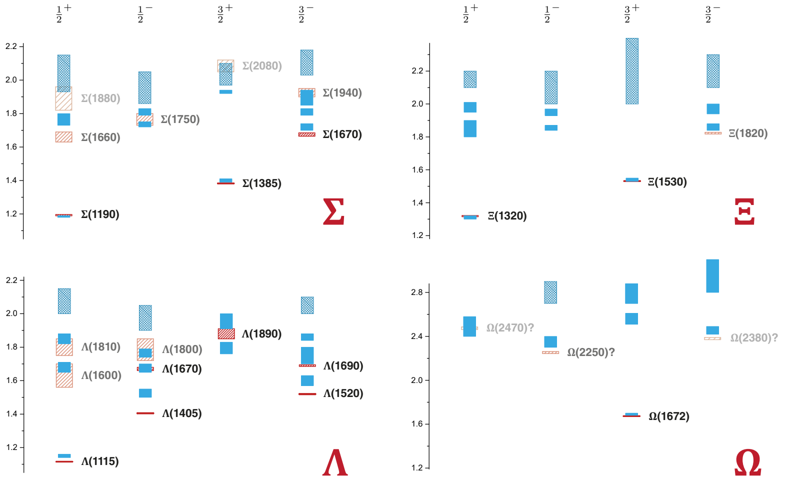


u/d quark contributions
from three-quark calculation
GE, PRD 84 (2011)

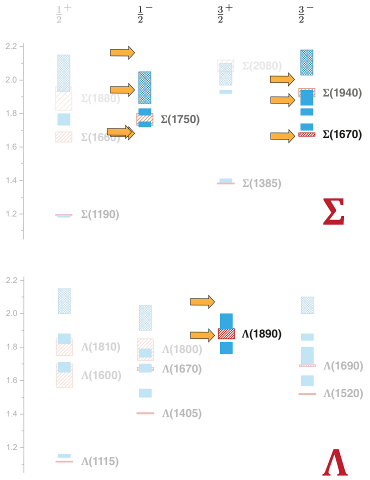
Data:
Cates, de Jager, Riordan, Wojtsekhowski, PRL 106 (2011)

Strange baryons



GE, Fischer, FBS 60 (2019), Fischer, GE, PoS Hadron 2017

Strange baryons

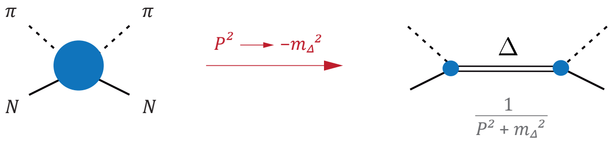


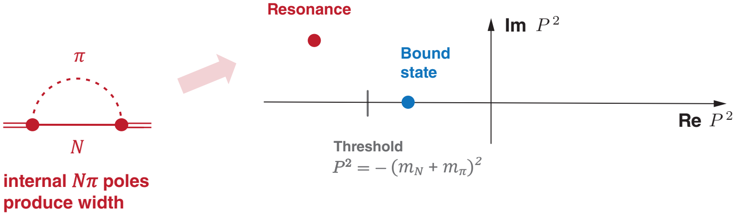
New states from Bonn-Gatchina

Sarantsev et al., 1907.13387 [nucl-ex]

GE, Fischer, FBS 60 (2019), Fischer, GE, PoS Hadron 2017

Resonances?

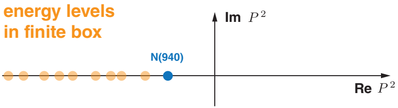

π
π
N
N
$P^2 \longrightarrow -m_\Delta{}^2$
Δ
$\frac{1}{P^2 + m_\Delta{}^2}$


π
N
internal $N\pi$ poles
produce width
Resonance
Bound
state
Im P^2
Re P^2
Threshold
$P^2 = -(m_N + m_\pi)^2$

Resonances?

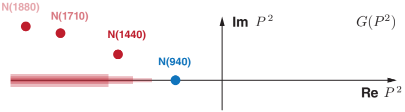
Lattice QCD:

$$\langle \dots \rangle = \int \mathcal{D}[\psi, \bar{\psi}, A]\, e^{-S[\psi,\bar{\psi},A]} \; (\dots)$$



- **Finite volume:**
  bound states & scattering states

vary volume,
Luescher method

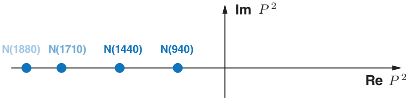


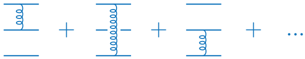
- **Infinite volume:**
  Bound states, resonances,
  branch cuts

Resonances?

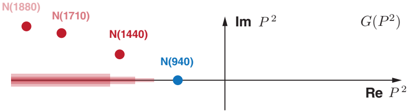
In terms of quarks and gluons?

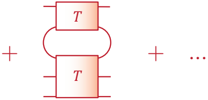
Bound states:





Resonances by **meson-baryon interactions:**

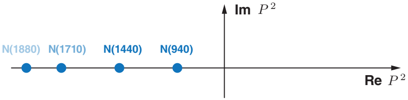




Resonances?

In terms of quarks and gluons?

Bound states:



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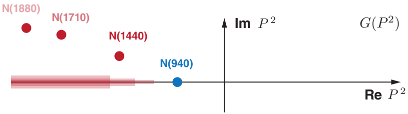
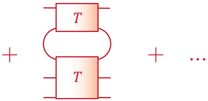
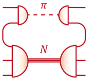




Both **bound states** and **resonances** must be generated from quark-gluon structure!

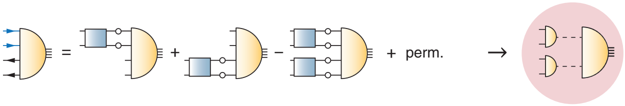


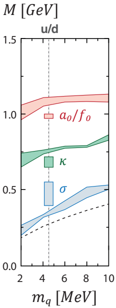
Analogue for $\rho \to \pi\pi$:

Williams, 1804.11161 [hep-ph], Miramontes, Sanchis-Alepuz, 1906.06227 [hep-ph]

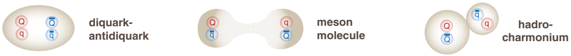
Tetraquarks *are* resonances

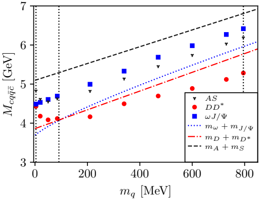
Light scalar mesons σ, κ, a_0, f_0 as **tetraquarks:**
BSE dynamically generates **meson poles** in BS amplitude
GE, Fischer, Heupel, PLB 753 (2016)

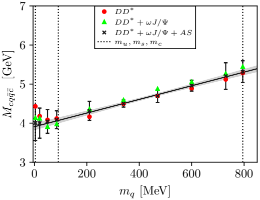




X(3872): Can we distinguish different **tetraquark subclusters?**

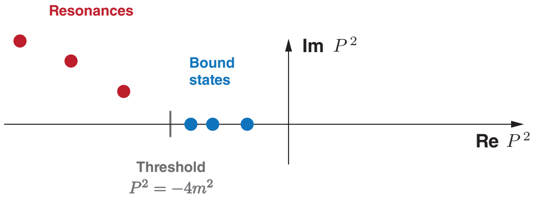






X(3872) dominated by DD*
Wallbott, GE, Fischer, PRD 100 (2019)

How to extract resonance information?



Instead of extracting resonance information from below threshold,
can we calculate them directly in **complex plane,**
on the **second Riemann sheet?**

→ Need to take care of **singularities** in integrals & integral equations

→ Much progress using **Nakanishi representation** → LFWFs, PDFs, GPDs, . . .
see talk by **Gianni Salmè**

→ Alternative: use numerical **contour deformations**

Euclidean vs. Minkowski

- *"We live in Minkowski space and not Euclidean space!"*

  Choice of **metric** cannot affect physics: $P_M^\mu = \begin{pmatrix} P_0 \\ \boldsymbol{P} \end{pmatrix} \Leftrightarrow P_E^\mu = \begin{pmatrix} \boldsymbol{P} \\ iP_0 \end{pmatrix}$

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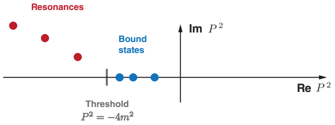
- **Spacelike** (“Euclidean”) vs. **timelike** (“Minkowski”)?



What about $P^2 \in \mathbb{C}$?

What if phase space is multi-dimensional?

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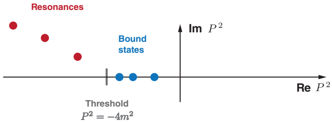


What about $P^2 \in \mathbb{C}$?

What if phase space is multi-dimensional?

- It's about the **integration path**... but

E = M

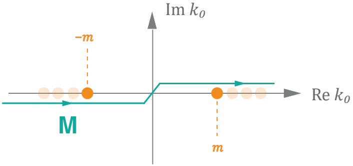
≠ **E'** ... "naive Euclidean"

"We need XY in Minkowski space"

"We calculate XY directly in Minkowski space"

"The Euclidean calculation is wrong"

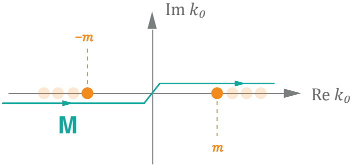
Textbook example



$$i\int d^4k\,\frac{1}{k^2-m^2+i\epsilon}\;\cdots = \;i\int d^3k\int_{-\infty(1+i\epsilon)}^{\infty(1+i\epsilon)} dk_0\,\frac{1}{k_0^2-\omega^2}\;\cdots$$

- Do k_0 integration first, pick up $\boldsymbol{k}$-dependent residues, integrate over $\boldsymbol{k}$

Textbook example



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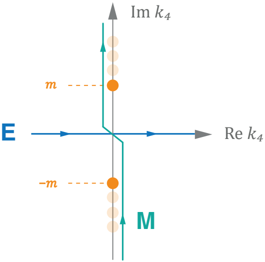
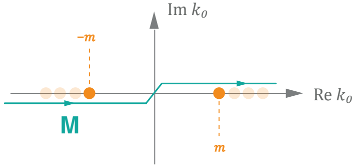


Euclidean: $k_4 = ik_0$, but $d^4k_E = -id^4k_M$

$$\int d^3k\int_{-\infty}^{\infty} dk_4\,\frac{1}{k_4^2+\omega^2}\,\cdots$$

Textbook example



$$i \int d^4k \, \frac{1}{k^2 - m^2 + i\epsilon} \cdots = i \int d^3k \int_{-\infty(1+i\epsilon)}^{\infty(1+i\epsilon)} dk_0 \, \frac{1}{k_0^2 - \omega^2} \cdots$$

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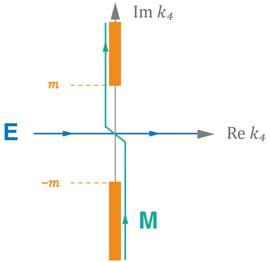

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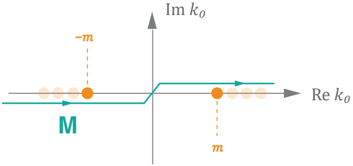
$$\int d^3k \int_{-\infty}^{\infty} dk_4 \, \frac{1}{k_4^2 + \omega^2} \cdots$$

- Now exchange $d^3k \leftrightarrow dk_4$ integration:

$$\int_{-\infty}^{\infty} dk_4 \int d^3k \, \frac{1}{k_4^2 + \omega^2} \cdots$$

has **cuts** instead of poles
→ avoid cuts in k_4 integration

Textbook example



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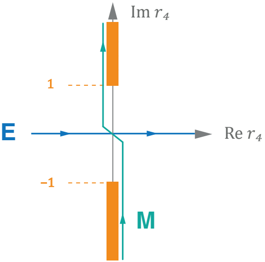


Euclidean:

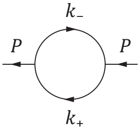
- Make everything dimensionless: $r^\mu = k_E^\mu / m$
- For manifest Lorentz invariance: k_E^2, $d\Omega$ instead of k_4, d^3k:

$$\int_{-\infty}^{\infty} dk_E^2 \int d\Omega\,\frac{1}{k_E^2+m^2}\,\cdots \qquad\qquad k_E^2 = -k^2$$

→ avoid cuts in k_E^2 integration

Two poles

Consider two-point function (current correlator, self energy, vacuum polarization, ...)



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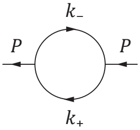

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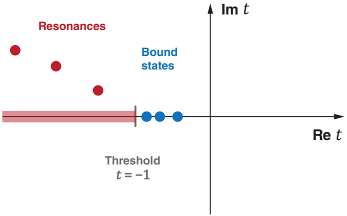
Define $P^2 = 4m^2 t$:



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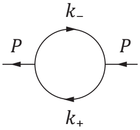

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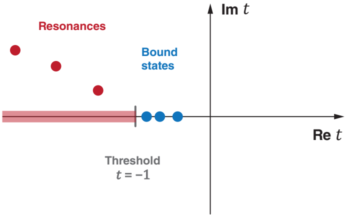
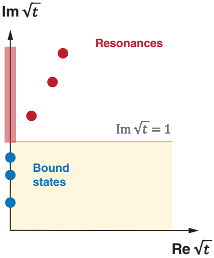


Simpler in $\sqrt{t}$:



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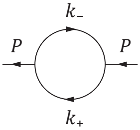
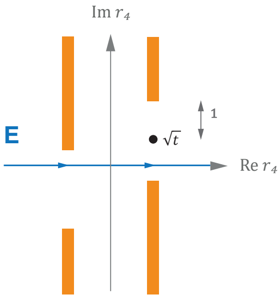


$$\int d^4k \, \frac{1}{k_+^2 + m^2} \, \frac{1}{k_-^2 + m^2}$$

Then:



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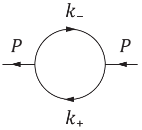

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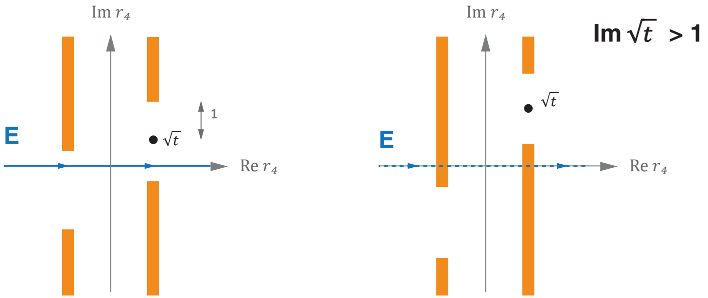

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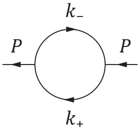

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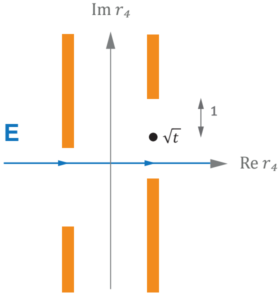

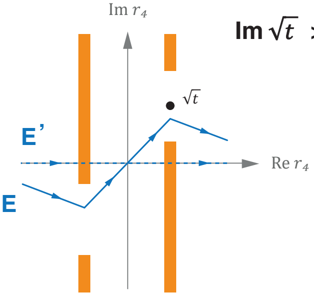

Im $\sqrt{t}$ > 1

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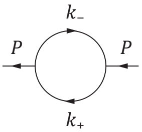

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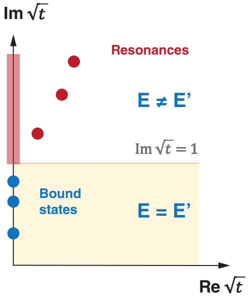

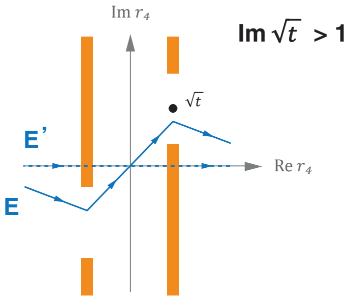

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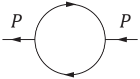

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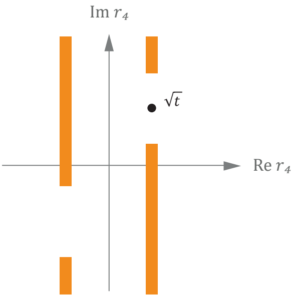

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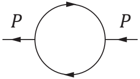

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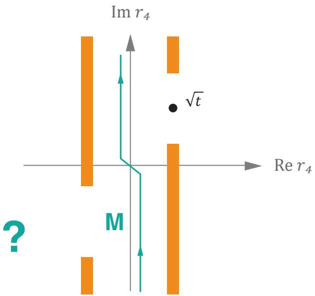

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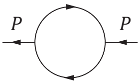

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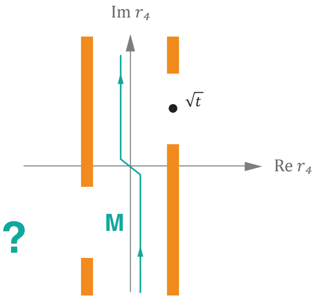
Where does the $i\epsilon$ come from?

$$\sum_{n=0}^{\infty} e^{-iE_nT} |n\rangle\langle n|\Omega\rangle \xrightarrow{T\to\infty(1-i\epsilon)} e^{-iE_0T}|0\rangle\langle 0|\Omega\rangle$$

$$\int_{-\infty(1+i\epsilon)}^{\infty(1+i\epsilon)} dk_0 \quad \Leftrightarrow \quad \int_{-\infty(i-\epsilon)}^{\infty(i-\epsilon)} dr_4$$



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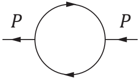

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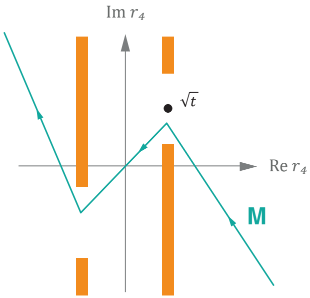

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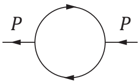

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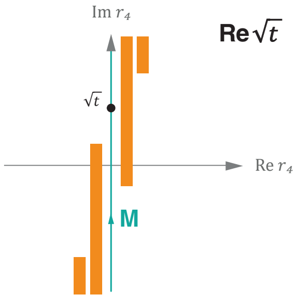

Re $\sqrt{t}$ = 0

Two poles

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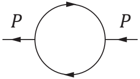

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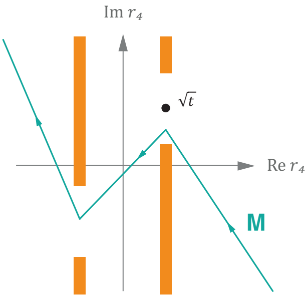

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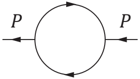

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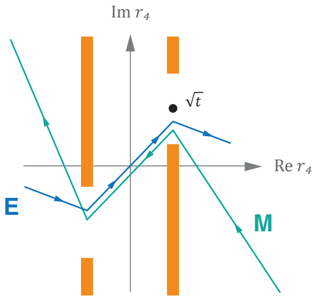


So:

E = M

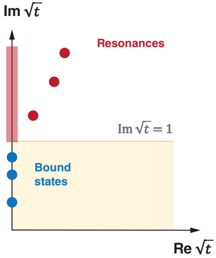
$$\int d^3k \int_{-\infty}^{\infty} dk_4$$ ... close contours analytically, pick up **residues**

$$\int_{-\infty}^{\infty} dk_4 \int d^3k$$ ... avoid cuts by numerical **contour deformation**

Suggestions for better wording:

"We need XY ~~in Minkowski space"~~
"We calculate XY ~~directly in Minkowski space"~~
... in the full kinematical domain
... above threshold
... using residue calculus

The naive Euclidean calculation ~~is~~ would be wrong
in certain kinematical regions (if anyone actually did that)



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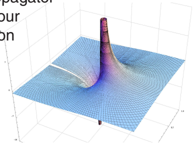
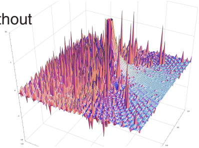
Quark propagator **with** contour deformation



... and without



Contour deformations

2-point functions:

- Fermion propagator in QED3
  Maris, PRD 52 (1995)
- Quark propagator in QCD
  GE, PhD thesis (2009)
- Gluon and ghost propagators in QCD
  Strauss, Fischer, Kellermann, PRL 109 (2012)
- Glueball correlator in YM
  Windisch, Alkofer, Haase, Liebmann, CPC 184 (2013),
  Windisch, Huber, Alkofer, PRD 87 (2013)
- Finite-T spectral functions from FRG
  Pawlowski, Strodthoff, Wink, PRD 98 (2018)

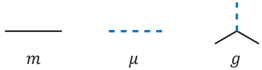
3-point functions:

- Rare pion decay $\pi^0 \to e^+e^-$
  Weil, GE, Fischer, Williams, PRD 96 (2017)
- Rho-meson decay
  Williams, 1804.11161
- Quark-photon vertex
  Miramontes, Sanchis-Alepuz, 1906.06227

4-point functions:

- Scalar scattering amplitude
  GE, Duarte, Pena, Stadler, 1907.05402 [hep-ph]

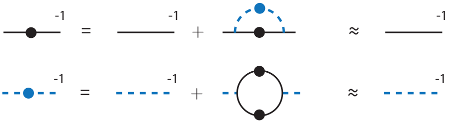
Scalar system



2 parameters:

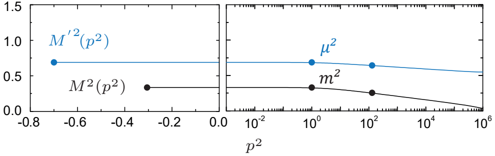
$$c = \frac{g^2}{(4\pi m)^2}\,, \quad \beta = \frac{\mu}{m}$$

Dressed propagators do not change much:



Tree-level propagators ok –
at least for small coupling

Ahlig, Alkofer, Ann. Phys. 275 (1999)

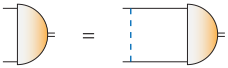


$$D(p^2) = \frac{1}{Z}\,\frac{1}{p^2+M^2(p^2)}$$

$$D'(p^2) = \frac{1}{Z'}\,\frac{1}{p^2+M'^2(p^2)}$$

Bound states & resonances

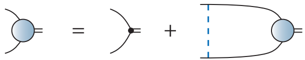
- **Homogeneous BSE:** $\psi = KG_0\,\psi$



→ **BS amplitude:**
eigenvalue spectrum of KG_0
for given J^{PC} channel

Wick 1954,
Cutkosky 1954,
Nakanishi 1969, ...

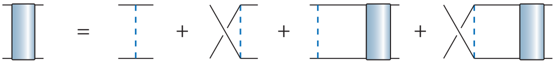
- **Inhomogeneous BSE:** $\Gamma = \Gamma_0 + KG_0\,\Gamma$



→ **Vertex:** bound-state
and resonance poles
for given J^{PC} channel

$$\Gamma = \frac{\Gamma_0}{1 - KG_0}$$

- **Scattering equation:** $T = K + KG_0\,T$

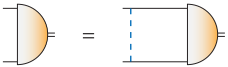


→ **Scattering amplitude,**
all singularities

$$T = \frac{K}{1 - KG_0}$$

Bound states & resonances

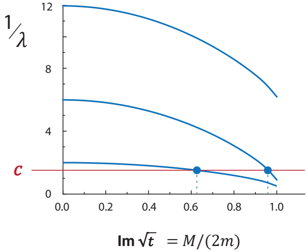
- **Homogeneous BSE:**

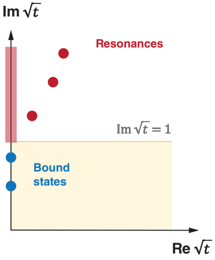


$$\Rightarrow \quad \psi(t) = c\, K G_0(t)\, \psi(t)$$

$$\psi(X,Z,t) = c \int dx \int dz\; K(X,x,Z,z,t)\; G_0(x,z,t)\; \psi(x,z,t)$$

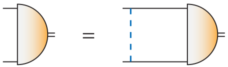
$$\Rightarrow \quad \frac{1}{\lambda(t)} \stackrel{!}{=} c$$





Bound states & resonances

- **Homogeneous BSE:**



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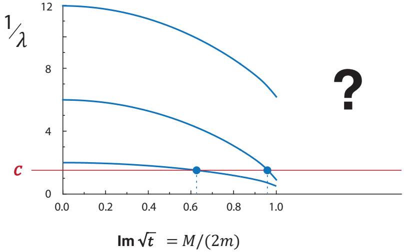
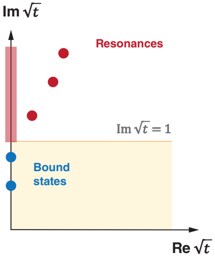
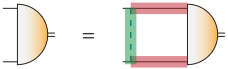


?



Contour deformation

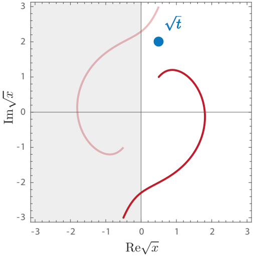
- **Homogeneous BSE:**



$$\psi(X,Z,t) = \int_0^\infty dx \int_{-1}^{1} dz \; K(X,x,Z,z)\; \underbrace{G_0(x,z,t)}_{\frac{1}{(x+t+1)^2 - 4\,x\,t\,z^2}}\; \psi(x,z,t)$$

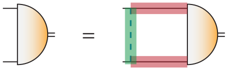
→ cuts from G_0 in complex x plane for given t

→ cuts from K in complex x plane for given X



Contour deformation

- **Homogeneous BSE:**



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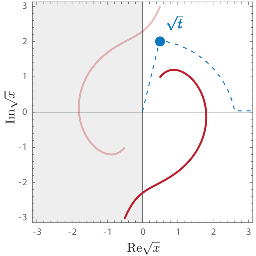
→ cuts from G_0 in complex x plane for given t

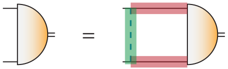
→ cuts from K in complex x plane for given X



- Find path in x that avoids G_0 cuts
- Paths in X and x must match →
  each point on path creates another cut from K

Contour deformation

- **Homogeneous BSE:**



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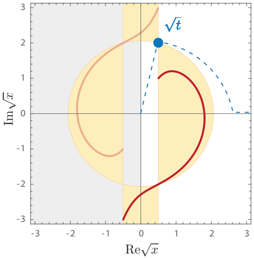
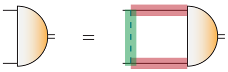
→ cuts from G_0 in complex x plane for given t
→ cuts from K in complex x plane for given X



- Find path in x that avoids G_0 cuts
- Paths in X and x must match → each point on path creates another cut from K
- All cuts in yellow area
- $\mathrm{Re}\sqrt{x}$ and $\mathrm{Abs}\sqrt{x}$ must never decrease
- Can cover **entire complex t plane!**

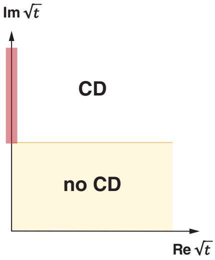
Contour deformation

- **Homogeneous BSE:**



$$\psi(X,Z,t) = \int_0^\infty dx \int_{-1}^{1} dz \; K(X,x,Z,z) \; \underbrace{G_0(x,z,t)}_{\frac{1}{(x+t+1)^2 - 4\,x\,t\,z^2}} \; \psi(x,z,t)$$

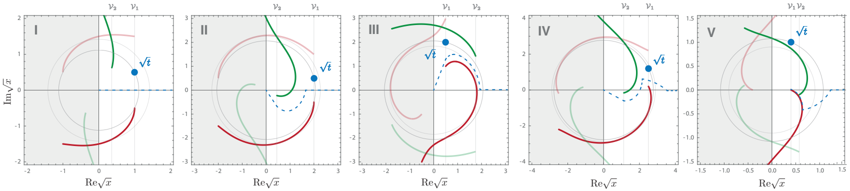
→ cuts from G_0 in complex x plane for given t
→ cuts from K in complex x plane for given X



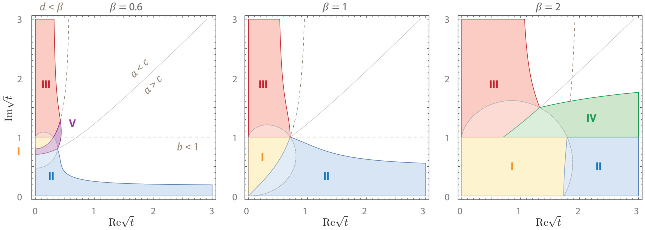
- Find path in x that avoids G_0 cuts
- Paths in X and x must match → each point on path creates another cut from K
- All cuts in yellow area
- $\mathrm{Re}\sqrt{x}$ and $\mathrm{Abs}\sqrt{x}$ must never decrease
- Can cover **entire complex t plane!**

Contour deformation

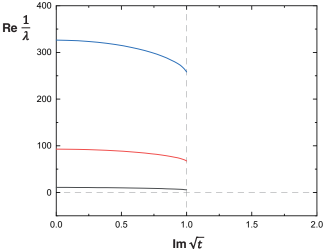
For **onshell scattering amplitude** more complicated:

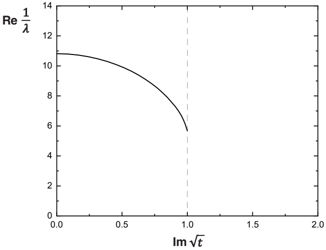


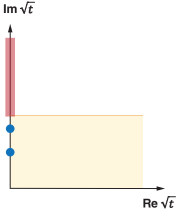
Can still cover **parts** of complex t plane:



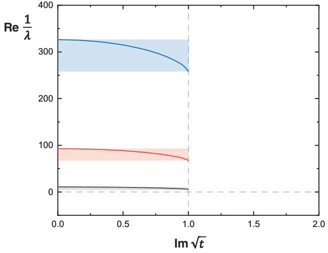
BSE Eigenvalues

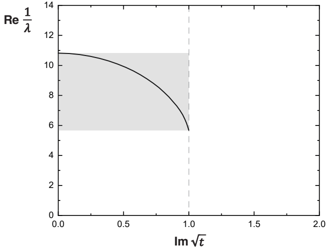


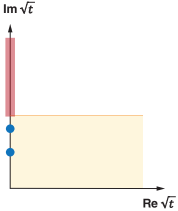




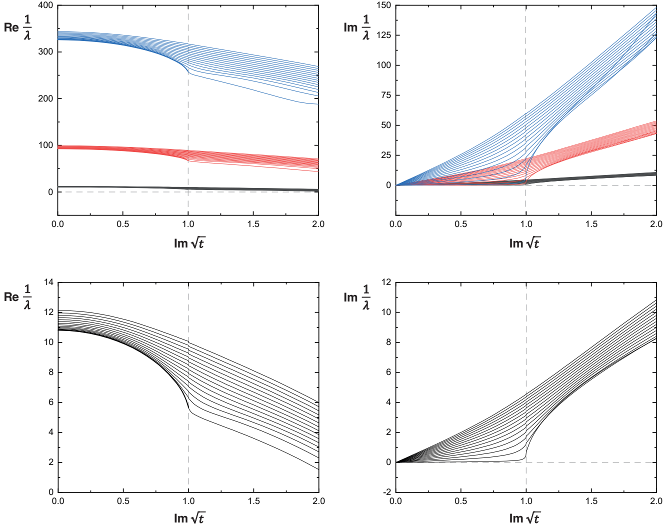
BSE Eigenvalues

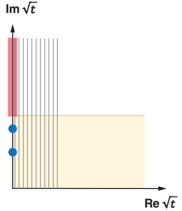




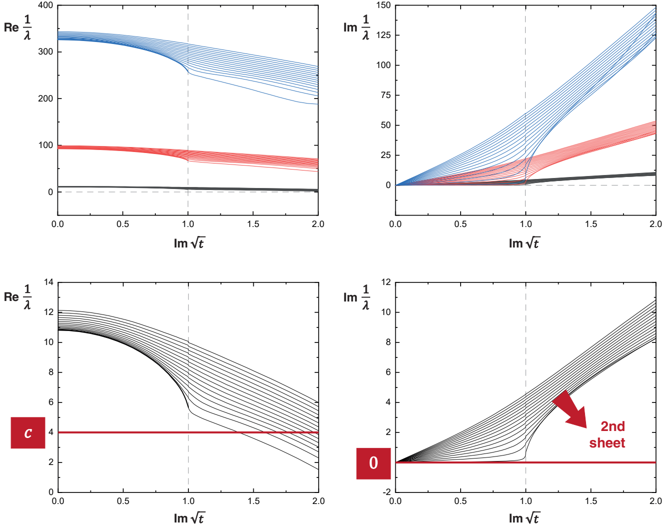


BSE Eigenvalues





BSE Eigenvalues

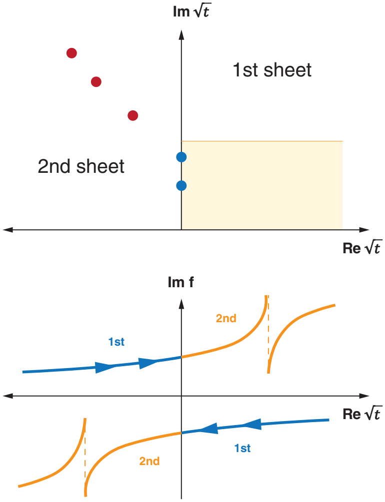


$$\frac{1}{\lambda(t)} \stackrel{!}{=} c + 0 \cdot i$$

still valid for
complex poles:

can detect
resonances from
homogeneous BSE

How to access 2nd sheet?

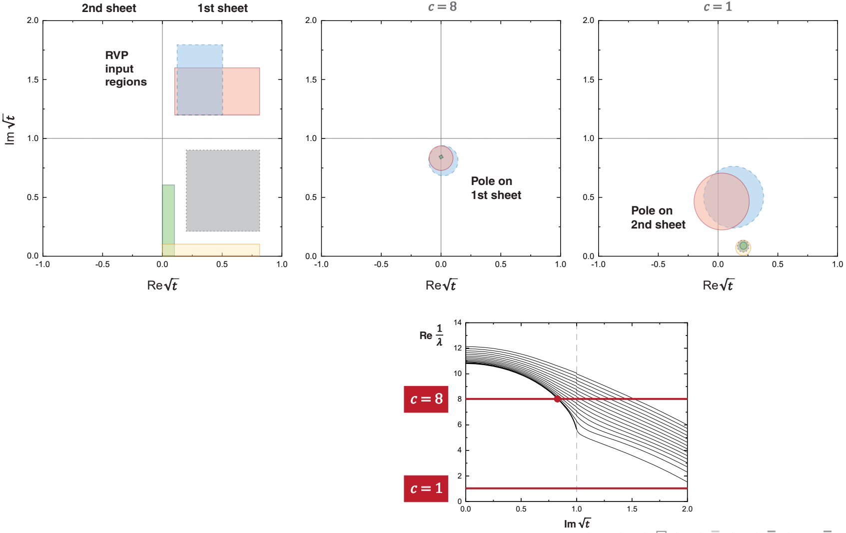


RVP: Resonances via Padé / Schlessinger point method / Continued fraction

Schlessinger, Phys. Rev. 167 (1968)
Tripolt, Haritan, Wambach, Moiseyev, PLB 774 (2017)
Binosi, Tripolt, 1904.08172 [hep-ph]

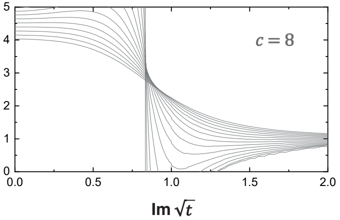
$$f(z) = \cfrac{c_1}{1 + \cfrac{c_2\,(z-z_1)}{1 + \cfrac{c_3\,(z-z_2)}{1 + \cfrac{c_4\,(z-z_3)}{\cdots}}}}$$

Poles on 2nd sheet



Poles on 2nd sheet

Vertex from inhomogeneous BSE:
only threshold cusp, no resonance bump



Poles on 2nd sheet

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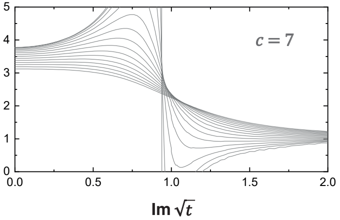

Poles on 2nd sheet

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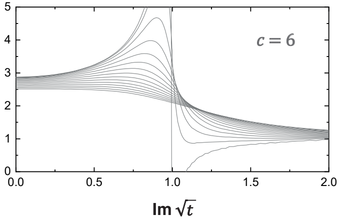

Poles on 2nd sheet

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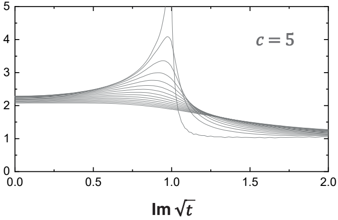

Poles on 2nd sheet

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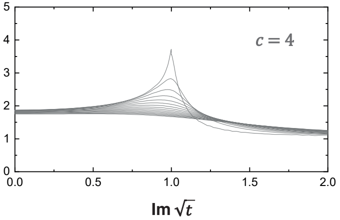

Poles on 2nd sheet

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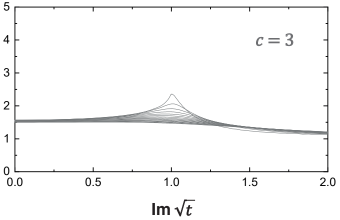

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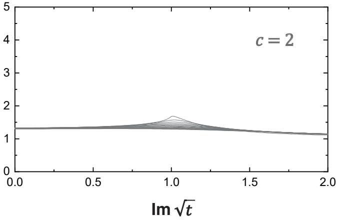

Poles on 2nd sheet

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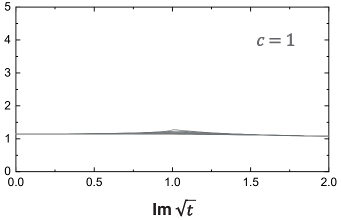
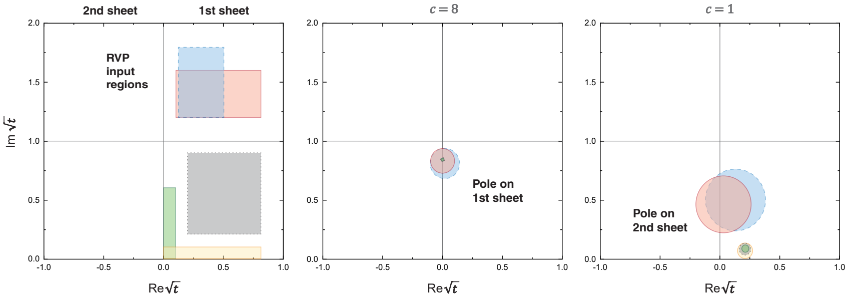


Poles on 2nd sheet



Large couplings: RVP accurately reproduces **bound-state poles** on 1st sheet

Small couplings:
virtual states?
(poles on axis of 2nd sheet)

Glöckle, "The QM Few-Body Problem", 1983
Hanhart, Pelaez, Rios, PLB 739 (2014)

RVP not good enough to determine them precisely

Two-body unitarity

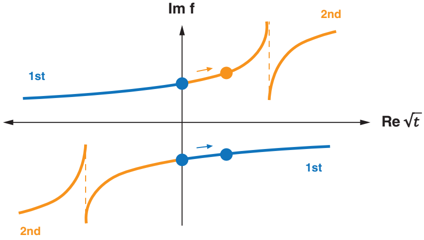
Follows from scattering equation:

$$T = K + K G_0 T \quad \Rightarrow \quad T^{-1} = K^{-1} - G_0$$

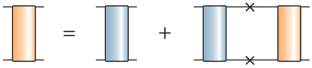
$$\Rightarrow \quad T_+^{-1} - T_-^{-1} = (K_+^{-1} - K_-^{-1}) - (G_{0+} - G_{0-})$$

$$\Rightarrow \quad T_+ - T_- = T_+ (G_{0+} - G_{0-}) \, T_- + (\dots)$$

e.g.: Gribov Lectures on Theoretical Physics, Cambridge 2008



If $T_\pm = T(\, t \pm i\epsilon \,)$:



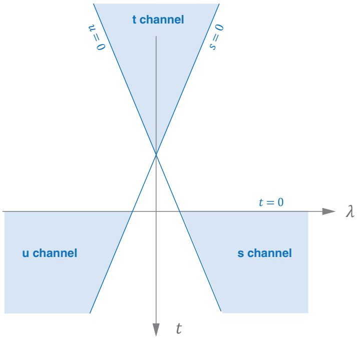
With **partial-wave decomposition:**

$$f_l(t)_{II} = \frac{f_l(t)_I}{1 - 2i\,\tau(t)\, f_l(t)_I}$$

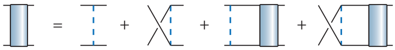
→ But this requires **scattering amplitude**

Scattering amplitude

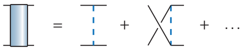
Depends on two variables: t and crossing variable $\lambda = \dfrac{s-u}{4m^2}$



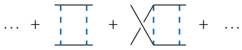
- Bound states, resonances and t-channel cuts at fixed $t \rightarrow$ determined by scattering equation



- Exchange-particle poles from K at fixed $s = \mu^2$ and $u = \mu^2$ (no poles in $T - K$)

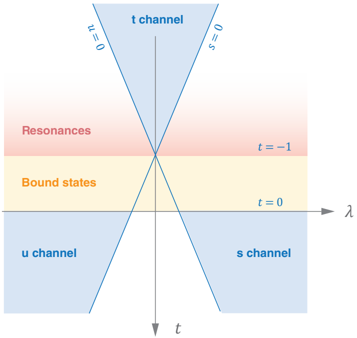


- Perturbative cuts for $s > 4\mu^2$ and $u > 4\mu^2$



Scattering amplitude

Depends on two variables: t and crossing variable $\lambda = \frac{s-u}{4m^2}$



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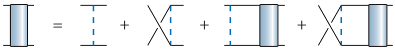

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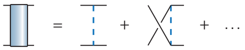

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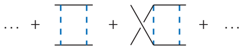
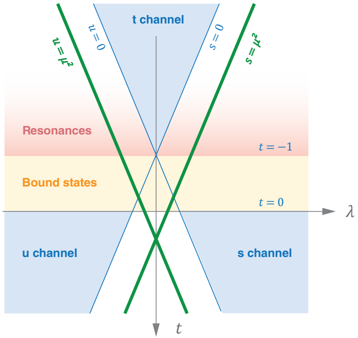
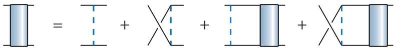


Scattering amplitude

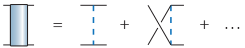
Depends on two variables: t and crossing variable $\lambda = \dfrac{s-u}{4m^2}$



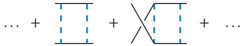
- Bound states, resonances and t-channel cuts at fixed $t \to$ determined by scattering equation



- Exchange-particle poles from K at fixed $s=\mu^2$ and $u=\mu^2$ (no poles in $T-K$)

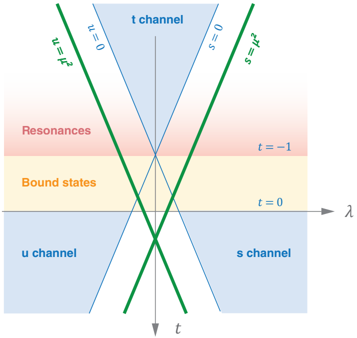


- Perturbative cuts for $s>4\mu^2$ and $u>4\mu^2$

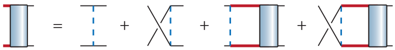


Scattering amplitude

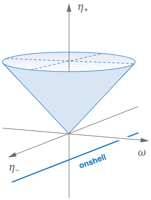
Depends on two variables: t and crossing variable $\lambda = \frac{s-u}{4m^2}$



- To obtain onshell scattering amplitude, must first solve **half-offshell** scattering equation



- Kinematics same as in **Compton scattering**
  GE, Ramalho, PRD 98 (2018)



Scattering amplitude

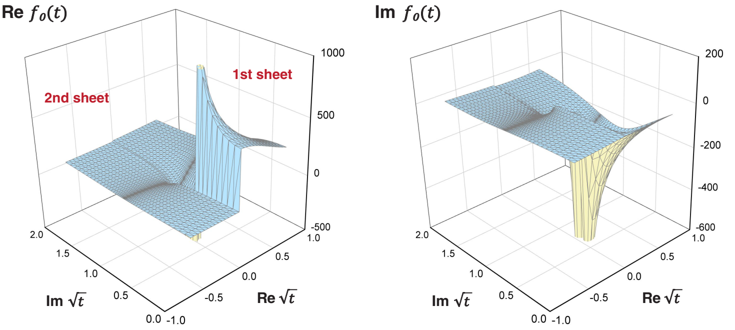
Partial-wave expansion:

$$T(t,\lambda) = \sum_{l=0}^{\infty} (2l+1)\, f_l(t)\, P_l(\cos\Theta) \approx f_0(t)$$

Amplitude on **2nd sheet:**

$$f_l(t)_{II} = \frac{f_l(t)_I}{1 - 2i\,\tau(t)\, f_l(t)_I}$$

$c = 9$



Scattering amplitude

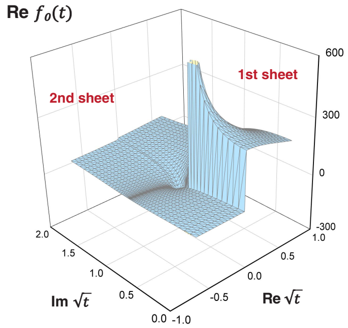
Partial-wave expansion:

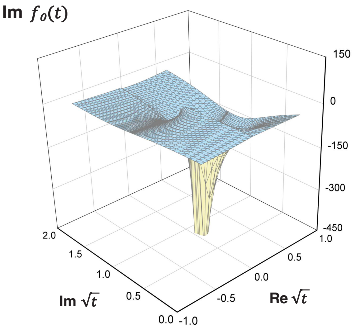
$$T(t,\lambda) = \sum_{l=0}^{\infty} (2l+1)\, f_l(t)\, P_l(\cos\Theta) \approx f_0(t)$$

Amplitude on **2nd sheet:**

$$f_l(t)_{II} = \frac{f_l(t)_I}{1 - 2i\,\tau(t)\, f_l(t)_I}$$

$c = 8$





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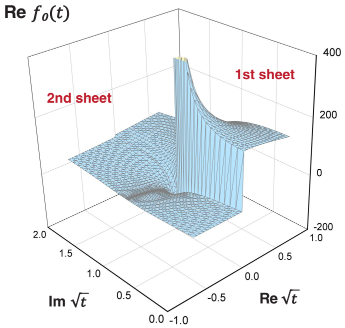
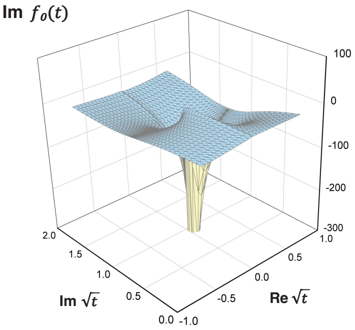
Amplitude on **2nd sheet:**

$$f_l(t)_{II} = \frac{f_l(t)_I}{1 - 2i\,\tau(t)\, f_l(t)_I}$$

$c = 7$





Scattering amplitude

Partial-wave expansion:

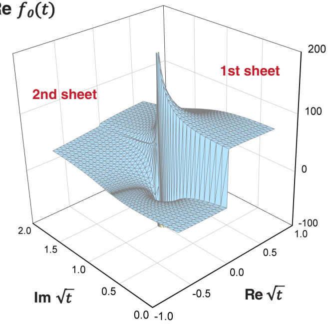
$$T(t,\lambda) = \sum_{l=0}^{\infty} (2l+1)\, f_l(t)\, P_l(\cos\Theta) \approx f_0(t)$$

Amplitude on **2nd sheet:**

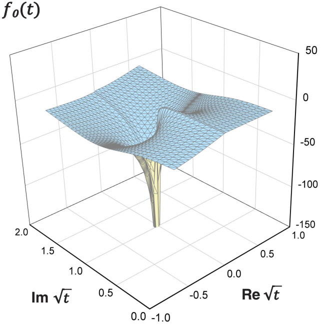
$$f_l(t)_{II} = \frac{f_l(t)_I}{1 - 2i\,\tau(t)\, f_l(t)_I}$$

$c = 5$

Re $f_0(t)$



Im $f_0(t)$



Scattering amplitude

Partial-wave expansion:

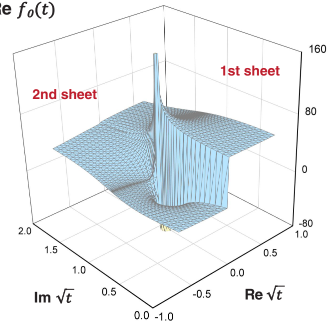
$$T(t,\lambda) = \sum_{l=0}^{\infty} (2l+1)\, f_l(t)\, P_l(\cos\Theta) \approx f_0(t)$$

Amplitude on **2nd sheet:**

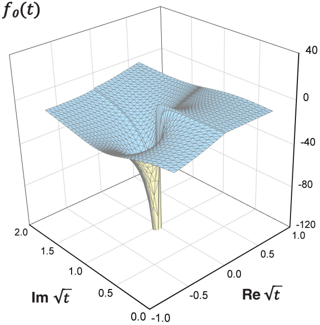
$$f_l(t)_{II} = \frac{f_l(t)_I}{1 - 2i\,\tau(t)\, f_l(t)_I}$$

$c = 4$

Re $f_0(t)$



Im $f_0(t)$



Scattering amplitude

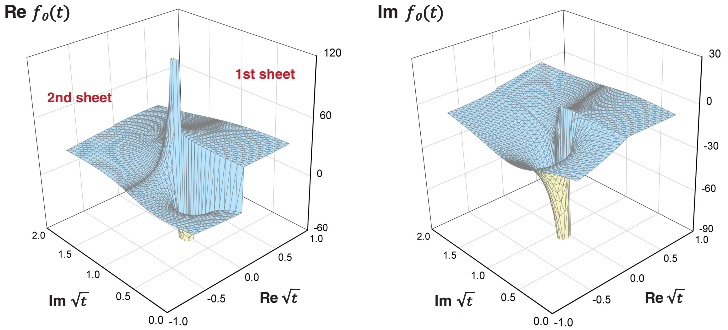
Partial-wave expansion:

$$T(t,\lambda) = \sum_{l=0}^{\infty} (2l+1)\, f_l(t)\, P_l(\cos\Theta) \approx f_0(t)$$

Amplitude on **2nd sheet:**

$$f_l(t)_{II} = \frac{f_l(t)_I}{1 - 2i\,\tau(t)\, f_l(t)_I}$$

$c = 3$



Scattering amplitude

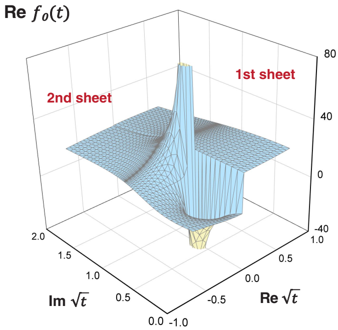
Partial-wave expansion:

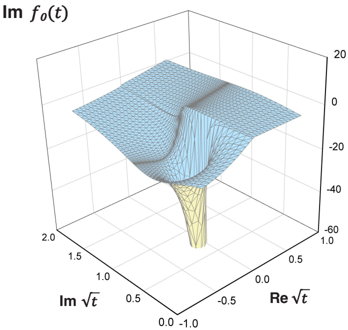
$$T(t,\lambda) = \sum_{l=0}^{\infty} (2l+1)\, f_l(t)\, P_l(\cos\Theta) \approx f_0(t)$$

Amplitude on **2nd sheet:**

$$f_l(t)_{II} = \frac{f_l(t)_I}{1 - 2i\,\tau(t)\, f_l(t)_I}$$

$c = 2$





Scattering amplitude

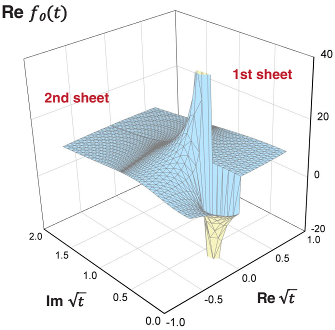
Partial-wave expansion:

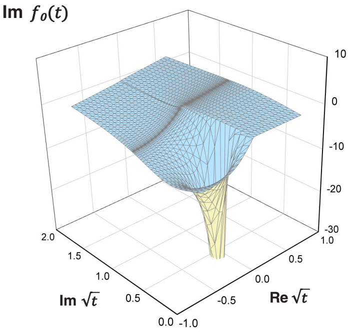
$$T(t,\lambda) = \sum_{l=0}^{\infty} (2l+1)\, f_l(t)\, P_l(\cos\Theta) \approx f_0(t)$$

Amplitude on **2nd sheet:**

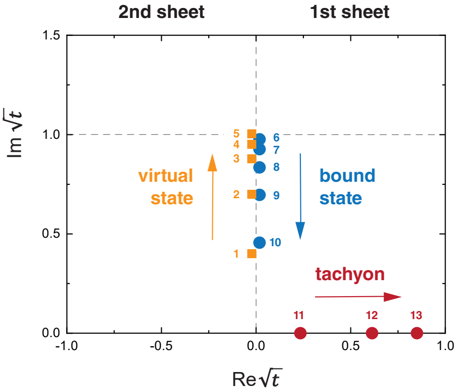
$$f_l(t)_{II} = \frac{f_l(t)_I}{1 - 2i\,\tau(t)\, f_l(t)_I}$$

$c = 1$





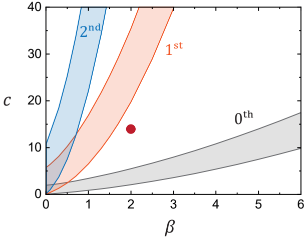
Pole trajectories



- Scalar model doesn't have resonances but only **virtual bound states**
- Need full **scattering equation** to find them (2-body unitarity)
- For **nearby resonances,** (in-)homogeneous BSE + RVP probably sufficient

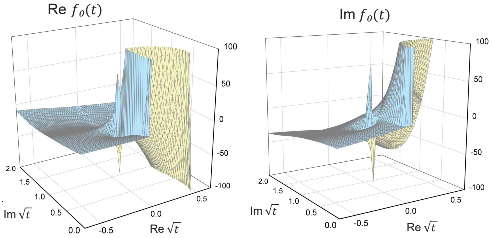
Pole trajectories

Analogous for other values of β
(i.e., exchange-particle masses)



Inside each band a state is bound

At fixed β, when increasing coupling:
virtual states → bound states → tachyons



Here for $\beta = 2, \; c = 12$:

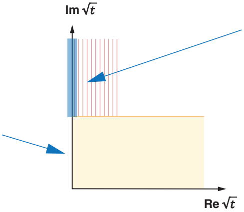
- Ground state has become tachyonic, 1st excited state is not yet bound
- Large structure is exchange particle pole at fixed s (or u), cf. Mandelstam plane

Benchmarks

- **Binding energies**

$c = 1, \beta = 0.5$

$\mathrm{Im}\sqrt{t}$	π/λ_0 this work	π/λ_0 [1, 2]	π/λ_0 [3]
0.999	1.18(3)	1.211	1.216
0.995	1.43(1)	1.440	1.440
0.99	1.623	1.624	1.623
0.95	2.498	2.498	2.498
0.90	3.251	3.251	3.251
0.80	4.416	4.416	4.416
0.75	4.901	4.901	4.901
0.6	6.094	6.096	6.094
0.4	7.205	7.206	7.204
0.2	7.849	7.850	7.849
0	8.061	8.062	8.061

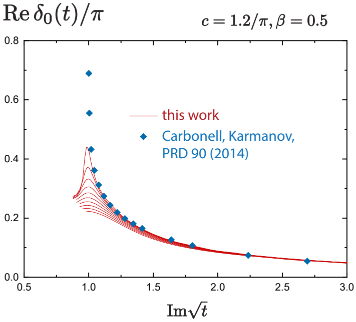


[1, 2] Kusaka, Simpson, Williams, PRD 56 (1997)
Karmanov, Carbonell, EPJ A 28 (2006)

[3] Frederico, Salmè, Viviani, PRD 89 (2014)

- **Phase shifts**

$$f_l(t) = \frac{1}{2i\,\tau(t)}\left[e^{2i\,\delta_l(t)} - 1\right]$$



Summary & Outlook

- **Contour deformations:**
  Toolbox for treating resonances with integral equations

- **Pole positions on 2nd sheet** determined from
  → homogeneous BSE + RVP (nearby resonances, otherwise ballpark estimates);
  → scattering amplitude (2nd sheet directly via two-body unitarity)

- Generally applicable for **circumventing singularities** (e.g. from quark propagator)
  → highly excited states, timelike FFs, FFs at large Q^2, . . .
  → LFWFs, PDFs, GPDs, TMDs, . . .

- Can be taken over without changes to NN, $N\pi$ scattering, etc.
  → amplitude analyses

Backup slides

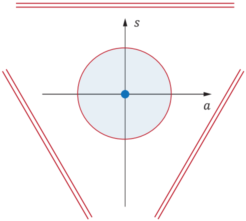
The role of diquarks?

- **Singlet:** symmetric variable, carries overall scale:

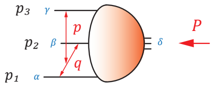
$$\mathcal{S}_0 \sim p_1^2 + p_2^2 + p_3^2 + \frac{M^2}{3}$$

- **Doublet:** $\mathcal{D}_0 \sim \frac{1}{\mathcal{S}_0} \begin{bmatrix} -\sqrt{3}\,(\delta x + 2\delta\omega) \\ x + 2\omega \end{bmatrix}$

  Mandelstam plane, outside: **diquark poles!**



Lorentz invariants can be grouped into **multiplets of the permutation group S3:**
GE, Fischer, Heupel, PRD 92 (2015)

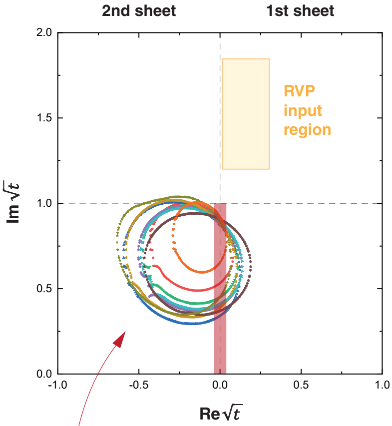


- Second **doublet:** $\mathcal{D}_1 \sim \frac{1}{\sqrt{\mathcal{S}_0}} \begin{bmatrix} -\sqrt{3}\,(\delta x - \delta\omega) \\ x - \omega \end{bmatrix}$

$f_i\,(\mathcal{S}_0\,,\bigcirc\,,\bigcirc\,)$ → full result as before

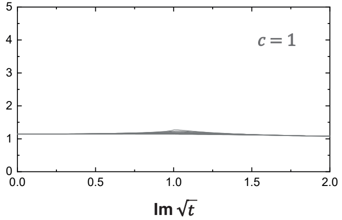
$f_i\,(\mathcal{S}_0\,,\bigcirc\,,\bigcirc\,)$ → **same ground-state spectrum,** but diquark poles switched off!

Pole trajectories



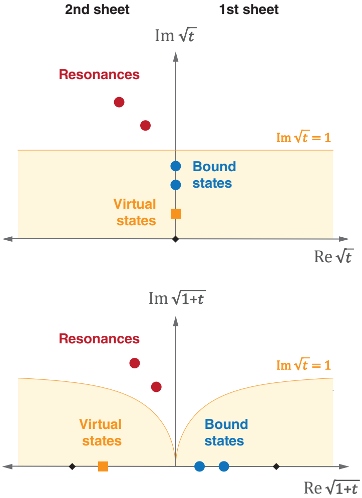
Pole trajectories:
Zeros of **Im** $1/\lambda$

- **No resonances above threshold**
- But RVP sensitive to # input points, also doesn't handle cuts well
- **Vertex from inhomogeneous BSE:** only threshold cusp, no resonance bump



- **Virtual bound states?**
  Glöckle, "The QM Few-Body Problem", 1983
  Hanhart, Pelaez, Rios, PLB 739 (2014)

Poles on 2nd sheet



No cut in $\sqrt{1+t}$ plane

Hanhart, Pelaez, Rios, PLB 739 (2014)

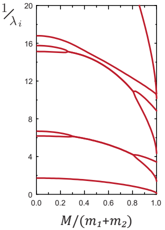
→ can analytically continue eigenvalues of homogeneous BSE!

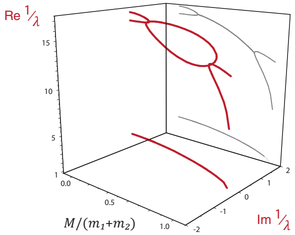
Complex eigenvalues?

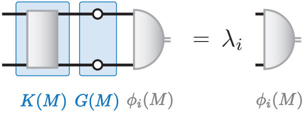
Excited states: some EVs are complex conjugate?

Typical for **unequal-mass** systems, already in Wick-Cutkosky model
Wick 1954, Cutkosky 1954

Connection with "**anomalous**" **states?**
Ahlig, Alkofer, Ann. Phys. 275 (1999)







K and G are Hermitian (even for unequal masses!) but KG is not

If $G = G^\dagger$ and $G > 0$:
Cholesky decomposition $G = L^\dagger L$

$$K\, L^\dagger L\, \phi_i = \lambda_i\, \phi_i$$

$$(L K L^\dagger)\,(L\phi_i) = \lambda_i\,(L\phi_i)$$

$\Rightarrow$ Hermitian problem with same EVs!

Complex eigenvalues?

Excited states: some EVs are complex conjugate?

Typical for **unequal-mass** systems, already in Wick-Cutkosky model
Wick 1954, Cutkosky 1954

Connection with "**anomalous**" **states?**
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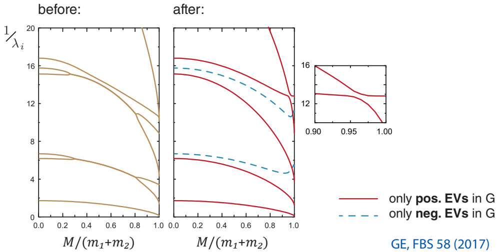
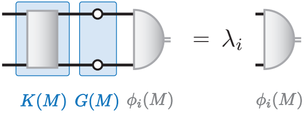


GE, FBS 58 (2017)



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$$K\, L^\dagger L\, \phi_i = \lambda_i\, \phi_i$$
$$(LKL^\dagger)\,(L\phi_i) = \lambda_i\,(L\phi_i)$$

⇒ Hermitian problem with same EVs!

⇒ all EVs strictly **real**
⇒ level repulsion
⇒ "anomalous states" removed?

Complex eigenvalues?

Excited states: some EVs are complex conjugate?

Typical for **unequal-mass** systems, already in Wick-Cutkosky model
Wick 1954, Cutkosky 1954

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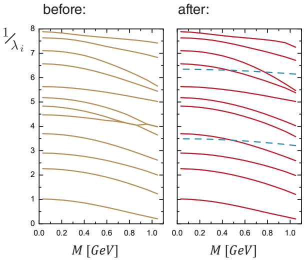
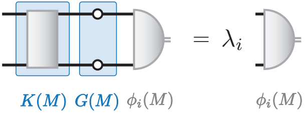


Eigenvalue spectrum for pion channel
GE, FBS 58 (2017)

only **pos. EVs** in G
only **neg. EVs** in G



K and G are Hermitian (even for unequal masses!) but KG is not

If $G = G^\dagger$ and $G > 0$:
Cholesky decomposition $G = L^\dagger L$

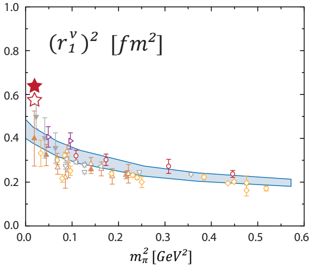
$$K\, L^\dagger L\, \phi_i = \lambda_i\, \phi_i$$
$$(LKL^\dagger)\,(L\phi_i) = \lambda_i\,(L\phi_i)$$

⇒ Hermitian problem with same EVs!

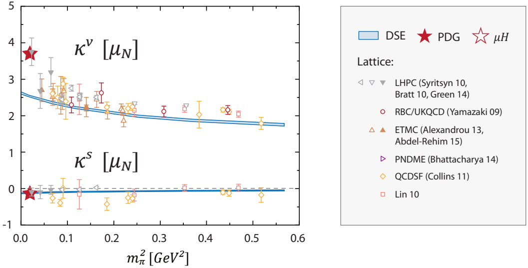
⇒ all EVs strictly **real**
⇒ level repulsion
⇒ "anomalous states" removed?

Nucleon em. form factors

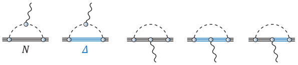
Nucleon charge radii:
isovector (p-n) Dirac (F1) radius



Nucleon magnetic moments:
isovector (p-n), isoscalar (p+n)



- **Pion-cloud effects** missing (⇒ divergence!), agreement with lattice at larger quark masses.

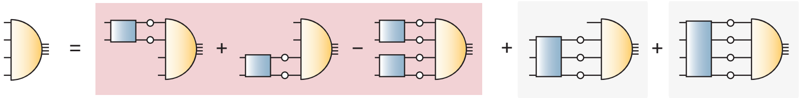


- **But:** pion-cloud **cancels** in κ^s ⟺ **quark core**

Exp: $\kappa^s = -0.12$
Calc: $\kappa^s = -0.12(1)$ **!!**

GE, PRD 84 (2011)

Four-body equation



Two-body interactions
... plus permutations:

$(qq)(\bar{q}\bar{q}),\ (q\bar{q})(q\bar{q}),\ (q\bar{q})(q\bar{q})$
(12) (34)   (23) (14)   (13) (24)

3-body
(+ permutations)

4-body

Bethe-Salpeter amplitude:

$$\Gamma(p,q,k,P) = \sum_i f_i\left(p^2,q^2,k^2,\{\omega_j\},\{\eta_j\}\right)\, \tau_i(p,q,k,P) \quad \otimes \quad \text{Color} \quad \otimes \quad \text{Flavor}$$

9 Lorentz invariants:

$$p^2,\quad q^2,\quad k^2$$

$$\begin{aligned} \omega_1 &= q\cdot k & \eta_1 &= p\cdot P \\ \omega_2 &= p\cdot k & \eta_2 &= q\cdot P \\ \omega_3 &= p\cdot q & \eta_3 &= k\cdot P \end{aligned}$$

$$P^2 = -M^2$$

256 Dirac-Lorentz tensors

2 Color tensors:

$3\otimes\bar{3},\quad 6\otimes\bar{6}$ or
$1\otimes 1,\quad 8\otimes 8$
(Fierz-equivalent)

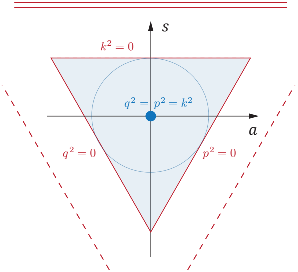
Structure of the amplitude

- **Singlet:** symmetric variable, carries overall scale:

$$\mathcal{S}_0 = \tfrac{1}{4}\,(p^2+q^2+k^2)$$

- **Doublet:** $\mathcal{D}_0 = \frac{1}{4\mathcal{S}_0}\begin{bmatrix}\sqrt{3}\,(q^2-p^2) \\ p^2+q^2-2k^2\end{bmatrix}$

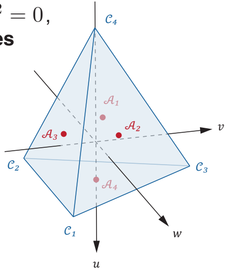
Mandelstam triangle,
outside: **meson and diquark poles!**



Lorentz invariants can be grouped into **multiplets of the permutation group S4:**
GE, Fischer, Heupel, PRD 92 (2015)

- **Triplet:** $\mathcal{T}_0 = \frac{1}{4\mathcal{S}_0}\begin{bmatrix}2\,(\omega_1+\omega_2+\omega_3) \\ \sqrt{2}\,(\omega_1+\omega_2-2\omega_3) \\ \sqrt{6}\,(\omega_2-\omega_1)\end{bmatrix}$

tetrahedron bounded by $p_i^2=0$,
outside: **quark singularities**



- Second **triplet:** 3dim. sphere

$$\mathcal{T}_1 = \frac{1}{4\mathcal{S}_0}\begin{bmatrix}2\,(\eta_1+\eta_2+\eta_3) \\ \sqrt{2}\,(\eta_1+\eta_2-2\eta_3) \\ \sqrt{6}\,(\eta_2-\eta_1)\end{bmatrix}$$

Tetraquark mass