Renormalon effects in top-mass sensitive observables

Silvia Ferrario Ravasio*
IPPP Durham

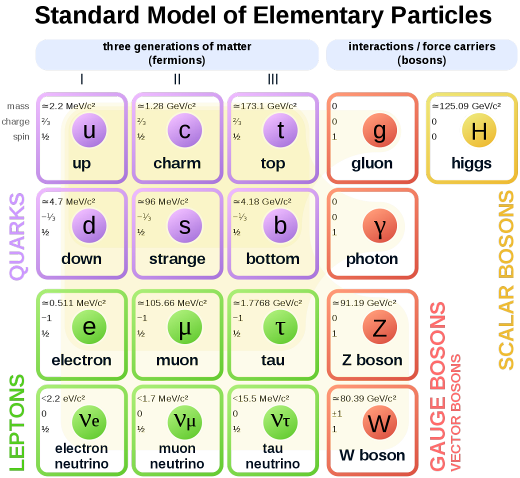
ECT*, Trento

LFC19: Strong dynamics for physics within and beyond the Standard Model at LHC and Future Colliders

Based on S.F.R., P. Nason and C. Oleari [arxiv:1810.10931]

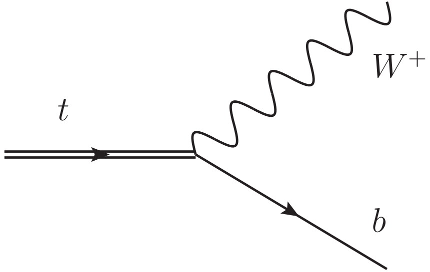
Top quark phenomenology

- Top: last quark to be observed and **heaviest** elementary particle in the SM



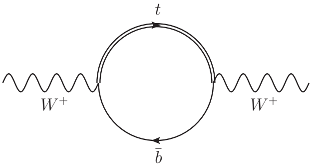
Top quark phenomenology

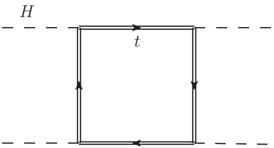
- Top: last quark to be observed and **heaviest** elementary particle in the SM
  - only quark that **decays** instead of hadronizing

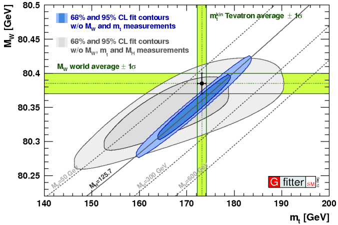


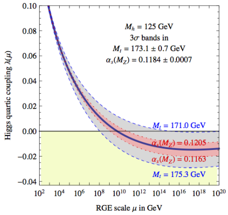
Top quark phenomenology

- Top: last quark to be observed and **heaviest** elementary particle in the SM
  - only quark that **decays** instead of hadronizing
  - $\mathbf{m_t}$ affects significantly many parameters of the SM, e.g. the mass of the W boson and the Higgs self-coupling λ





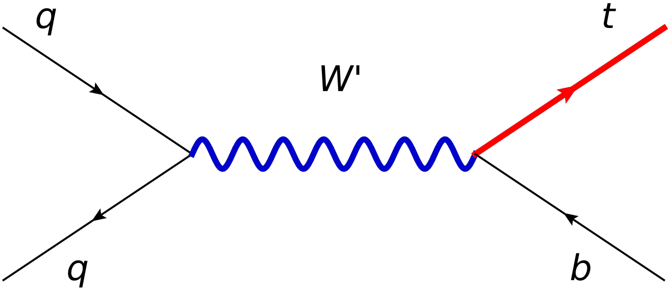




[arXiv:1512.01222, Espinosa] [arXiv:1407.3792, Baak et al.]

Top quark phenomenology

- Top: last quark to be observed and **heaviest** elementary particle in the SM
  - only quark that **decays** instead of hadronizing
  - $\mathbf{m_t}$ affects significantly many parameters of the SM, e.g. the mass of the W boson and the Higgs self-coupling λ
  - it may couple with exotic particles we hope to see with the new colliders (**BSM**)



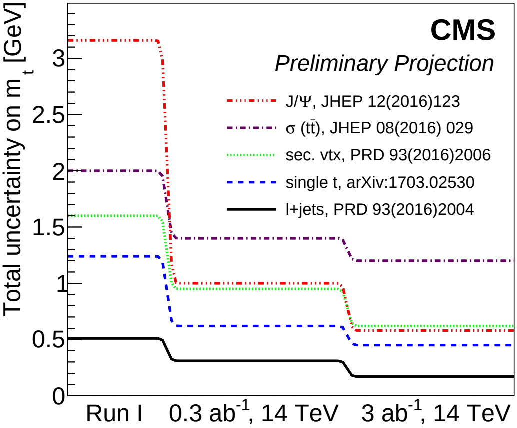
Top quark phenomenology

- Top: last quark to be observed and **heaviest** elementary particle in the SM
  - only quark that **decays** instead of hadronizing
  - $\mathbf{m_t}$ affects significantly many parameters of the SM, e.g. the mass of the W boson and the Higgs self-coupling λ
  - it may couple with exotic particles we hope to see with the new colliders (**BSM**)

We want a precise determination of $\mathbf{m_t}$ in a given **renormalization scheme**

Top-quark mass measurements

- Direct measurements give us the most precise determination, provided that the theoretical errors are small and under control.
  - CMS: $m_t = 172.44 \pm 0.13\ \text{(stat)} \pm 0.47\ \text{(syst)}$ GeV
  - ATLAS: $m_t = 172.51 \pm 0.27\ \text{(stat)} \pm 0.42\ \text{(syst)}$ GeV
  - The projected future experimental uncertainty is 200 MeV



From 1902.04070, **SM Physics at the HL-LHC and HE-LHC**

Top-quark mass measurements

- Direct measurements give us the most precise determination, provided that the theoretical errors are small and under control.
  - CMS: $m_t = 172.44 \pm 0.13\ (\text{stat}) \pm 0.47\ (\text{syst})$ GeV
  - ATLAS: $m_t = 172.51 \pm 0.27\ (\text{stat}) \pm 0.42\ (\text{syst})$ GeV
  - The projected future experimental uncertainty is 200 MeV
- High precision $\Rightarrow$ high level of **scrutiny** of extracted m_t.
- Direct mesurements are based on Monte Carlo (MC) generators. Experimental collaborations do not identify the **"MC" mass** with a renormalization scheme.
- Full consensus not reached by theorists, but some more precise statements can be done when looking predictions regarding the mass of the t-decay products (m_{rec}).

Top-quark mass measurements

- Direct measurements give us the most precise determination, provided that the theoretical errors are small and under control.
  - CMS: $m_t = 172.44 \pm 0.13\ \text{(stat)} \pm 0.47\ \text{(syst)}$ GeV
  - ATLAS: $m_t = 172.51 \pm 0.27\ \text{(stat)} \pm 0.42\ \text{(syst)}$ GeV
  - The projected future experimental uncertainty is 200 MeV
- High precision $\Rightarrow$ high level of **scrutiny** of extracted m_t.
- Direct mesurements are based on Monte Carlo (MC) generators. Experimental collaborations do not identify the **"MC" mass** with a renormalization scheme.
- Full consensus not reached by theorists, but some more precise statements can be done when looking predictions regarding the mass of the t-decay products (m_{rec}).
  - ✓ in NWA, $m_{\text{rec}} = m_{\text{pole}}$ at all-orders. SMC preserves t virtuality: $m_{\text{MC}} = m_{\text{pole}}$. Finite-width effects lead to $\alpha_s \Gamma_t$ difference.
  - ✓ Several works suggest the MC mass being close to the **pole mass**, with a theoretical error of few hundred MeV.
  - ✓ E.g Hoang, Platzer and Samitz find that the shower cutoff $Q_0 \approx 1$ GeV implies that m_t^{MC} s a short-distance mass and $m_t^{\text{MC}}(Q_0) - m_t^{\text{pole}} = -\frac{2}{3}\alpha_s(Q_0)Q_0 + \mathcal{O}(\alpha_s(Q_0)^2 Q_0)$

Top-quark mass measurements

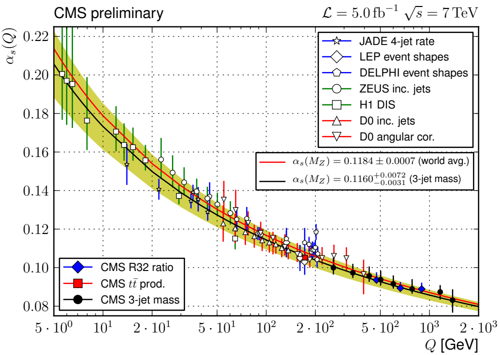
- Direct measurements give us the most precise determination, provided that the theoretical errors are small and under control.
  - CMS: $m_t = 172.44 \pm 0.13\ (\text{stat}) \pm 0.47\ (\text{syst})$ GeV
  - ATLAS: $m_t = 172.51 \pm 0.27\ (\text{stat}) \pm 0.42\ (\text{syst})$ GeV
  - The projected future experimental uncertainty is 200 MeV
- High precision $\Rightarrow$ high level of **scrutiny** of extracted m_t.
- Direct mesurements are based on Monte Carlo (MC) generators. Experimental collaborations do not identify the **"MC" mass** with a renormalization scheme.
- Full consensus not reached by theorists, but some more precise statements can be done when looking predictions regarding the mass of the t-decay products (m_{rec}).
  - ✓ in NWA, $m_{\text{rec}} = m_{\text{pole}}$ at all-orders. SMC preserves t virtuality: $m_{\text{MC}} = m_{\text{pole}}$. Finite-width effects lead to $\alpha_s \Gamma_t$ difference.
  - ✓ Several works suggest the MC mass being close to the **pole mass**, with a theoretical error of few hundred MeV.
  - ✓ E.g Hoang, Platzer and Samitz find that the shower cutoff $Q_0 \approx 1$ GeV implies that m_t^{MC} s a short-distance mass and $m_t^{\text{MC}}(Q_0) - m_t^{\text{pole}} = -\frac{2}{3}\alpha_s(Q_0)Q_0 + \mathcal{O}(\alpha_s(Q_0)^2 Q_0)$

The pole mass has infrared renormalons!!!

IR Renormalons

- QCD is affected by **infrared slavery**:

$$\alpha_s(k) = \frac{\alpha_s(Q)}{1 + 2b_0\alpha_s(Q)\log\left(\frac{k}{Q}\right)} = \frac{1}{2b_0\log\left(\frac{k}{\Lambda_{QCD}}\right)}; \quad b_0 = \frac{11C_A}{12\pi} - \frac{n_l T_R}{3\pi} > 0$$



IR Renormalons

- QCD is affected by **infrared slavery**:

$$\alpha_s(k) = \frac{\alpha_s(Q)}{1 + 2b_0\alpha_s(Q)\log\left(\frac{k}{Q}\right)} = \frac{1}{2b_0 \log\left(\frac{k}{\Lambda_{\mathrm{QCD}}}\right)}; \quad b_0 = \frac{11\mathrm{C_A}}{12\pi} - \frac{n_l \mathrm{T_R}}{3\pi} > 0$$

- All orders contribution coming from low-energy region

$$\underbrace{\int_0^Q \mathrm{d}k\, k^{p-1}\alpha_s(Q)}_{\text{NLO}} \Longrightarrow \underbrace{\int_0^Q \mathrm{d}k\, k^{p-1}\alpha_s(k)}_{\text{all orders}} = \boxed{Q^p \times \alpha_s(Q) \sum_{n=0}^{\infty} \left(\frac{2\, b_0}{p}\, \alpha_s(Q)\right)^n n!}$$

IR Renormalons

- QCD is affected by **infrared slavery**:

$$\alpha_s(k) = \frac{\alpha_s(Q)}{1 + 2b_0\alpha_s(Q)\log\left(\frac{k}{Q}\right)} = \frac{1}{2b_0\log\left(\frac{k}{\Lambda_{\mathrm{QCD}}}\right)}; \quad b_0 = \frac{11C_A}{12\pi} - \frac{n_l T_R}{3\pi} > 0$$

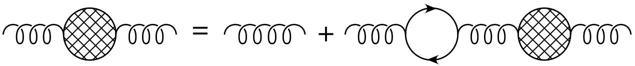
- All orders contribution coming from low-energy region

$$\underbrace{\int_0^Q dk\, k^{p-1}\alpha_s(Q)}_{\text{NLO}} \Longrightarrow \underbrace{\int_0^Q dk\, k^{p-1}\alpha_s(k)}_{\text{all orders}} = \boxed{Q^p \times \alpha_s(Q)\sum_{n=0}^{\infty}\left(\frac{2\,b_0}{p}\,\alpha_s(Q)\right)^n n!}$$

- **Asymptotic series**
  $\Rightarrow$ Minimum for $n_{\min} \approx \frac{p}{2b_0\alpha_s(Q)}$
  $\Rightarrow$ Size $Q^p \times \alpha_s(Q)\sqrt{2\pi n_{\min}}e^{-n_{\min}} \approx \boxed{\Lambda^p_{\mathrm{QCD}}}$
  We are interested in $p = 1$, i.e. in **linear renormalons**

Large n_f limit

- All-orders computation can be carried out exactly in the large number of flavour n_f limit

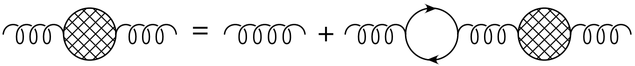


$$\frac{-ig^{\mu\nu}}{k^2+i\eta} \to \frac{-ig^{\mu\nu}}{k^2+i\eta} \times \frac{1}{1+\Pi(k^2+i\eta,\mu^2)-\Pi_{\rm ct}}$$

$$\Pi(k^2+i\eta,\mu^2)-\Pi_{\rm ct} = \alpha_s(\mu)\left(-\frac{n_f \mathrm{T_R}}{3\pi}\right)\left[\log\left(\frac{|k^2|}{\mu^2}\right) - i\pi\theta(k^2) - \frac{5}{3}\right]$$

Large n_f limit

- All-orders computation can be carried out exactly in the large number of flavour n_f limit



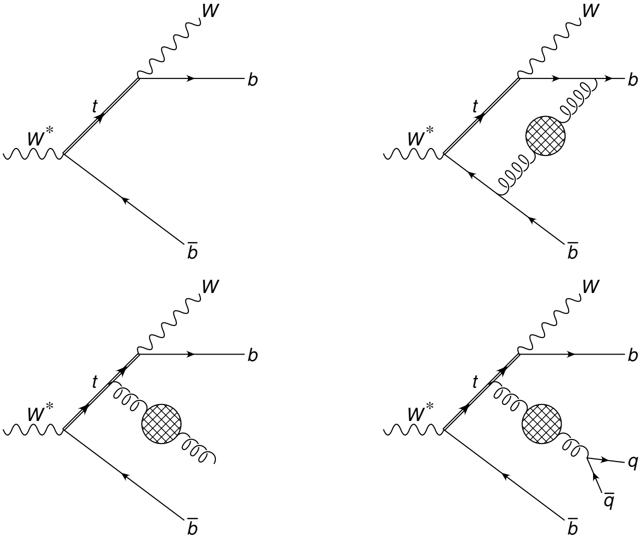
$$\frac{-ig^{\mu\nu}}{k^2+i\eta} \to \frac{-ig^{\mu\nu}}{k^2+i\eta} \times \frac{1}{1+\Pi(k^2+i\eta,\mu^2)-\Pi_{\rm ct}}$$

$$\Pi(k^2+i\eta,\mu^2)-\Pi_{\rm ct} = \alpha_s(\mu)\left(-\frac{n_f \mathrm{T_R}}{3\pi}\right)\left[\log\left(\frac{|k^2|}{\mu^2}\right) - i\pi\theta(k^2) - \frac{5}{3}\right]$$

- **naive non-abelianization** at the end of the computation

$$\Pi(k^2+i\eta,\mu^2)-\Pi_{\rm ct} \to \alpha_s(\mu)\underbrace{\left(\frac{11\mathrm{C_A}}{12\pi} - \frac{n_l \mathrm{T_R}}{3\pi}\right)}_{b_0}\left[\log\left(\frac{|k^2|}{\mu^2}\right) - i\pi\theta(k^2) - C\right]$$

$W^* \to t\bar{b} \to W b\bar{b}$ at all orders using the (complex) pole scheme



Integrated cross section

Integrated cross section (with cuts $\Theta(\Phi)$):

$$\sigma = \int \mathrm{d}\Phi \, \frac{\mathrm{d}\sigma(\Phi)}{\mathrm{d}\Phi} \, \Theta(\Phi)$$

$$= \sigma_{\text{LO}} - \frac{1}{\pi b_0} \int_0^\infty \mathrm{d}\lambda \, \frac{\mathrm{d}}{\mathrm{d}\lambda} \left[\frac{T(\lambda)}{\alpha_s(\mu)} \right] \arctan \left[\pi \, b_0 \, \alpha_s \left(\lambda e^{-C/2} \right) \right]$$

$\lambda =$ gluon mass

- $T(0) = \sigma_{\text{NLO}}$
- $T(\lambda) = \boxed{\sigma_{\text{NLO}}(\lambda)} + \frac{3\lambda^2}{2 \mathrm{T_R} \alpha_s} \int \mathrm{d}\Phi_{g^*} \mathrm{d}\Phi_{\text{dec}} \frac{\mathrm{d}\sigma^{(2)}_{q\bar{q}}(\lambda, \Phi)}{\mathrm{d}\Phi} \left[\Theta(\Phi) - \underbrace{\Theta(\Phi_{g^*})}_{q\bar{q} \to g^*} \right]$
- $T(\lambda) \xrightarrow{\lambda \to \infty} \frac{1}{\lambda^2}$
- $\alpha_s\left(\lambda e^{-C/2}\right) \approx \alpha_s(\lambda) \left[1 + \frac{K_g}{2\pi} \alpha_s(\lambda) \right] + \mathcal{O}(\alpha_s^3) = \alpha_s^{\text{MC}}(\lambda)$

Integrated cross section

Integrated cross section (with cuts $\Theta(\Phi)$):

$$\sigma = \int d\Phi \, \frac{d\sigma(\Phi)}{d\Phi} \, \Theta(\Phi)$$

$$= \sigma_{\rm LO} \quad - \frac{1}{\pi b_0} \int_0^\infty d\lambda \, \frac{d}{d\lambda} \left[\frac{T(\lambda)}{\alpha_s(\mu)} \right] \arctan \left[\pi \, b_0 \, \alpha_s \left(\lambda e^{-C/2} \right) \right]$$

So, if

$$\left. \frac{dT(\lambda)}{d\lambda} \right|_{\lambda=0} = A \neq 0$$

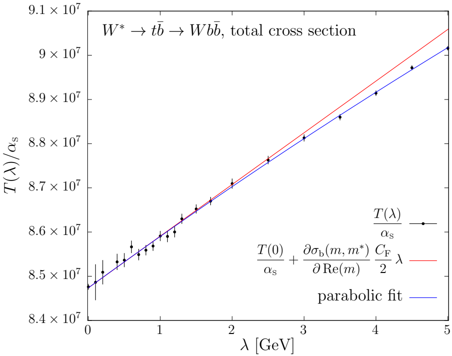
the low-λ contribution takes the form

$$\langle O \rangle \sim -A \sum_{n=0}^{\infty} \int_0^m d\lambda \left[-2b_0 \, \alpha_s(m) \log \left(\frac{\lambda^2}{m^2} \right) \right]^n = -Am \sum_{n=0}^{\infty} \left(2 \, b_0 \, \alpha_s(m) \right)^n n!$$

Linear λ term $\leftrightarrow$ Linear renormalons

Total cross section

$\sigma^{\text{tot}}(\overline{m}(\mu))$ is renormalon free: $\underbrace{\frac{T(\lambda)}{\alpha_s}}_{\text{pole}} \to \underbrace{\frac{T(\lambda)}{\alpha_s} - \frac{\partial \sigma_b}{\partial \text{Re}(m)} \frac{C_F}{2} \lambda}_{\overline{\text{MS}}}$

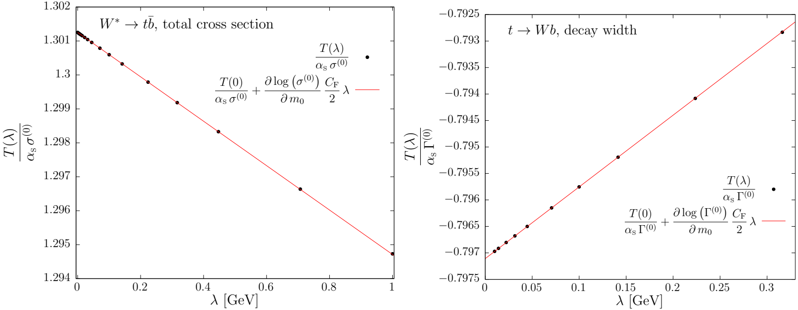


$\Rightarrow$ If a complex mass is used, the top can never be on-shell and the only term that can develop a linear λ sensitivity is the mass counterterm.

Total cross section in NWA

For $\Gamma_t \to 0$ the cross section factorizes

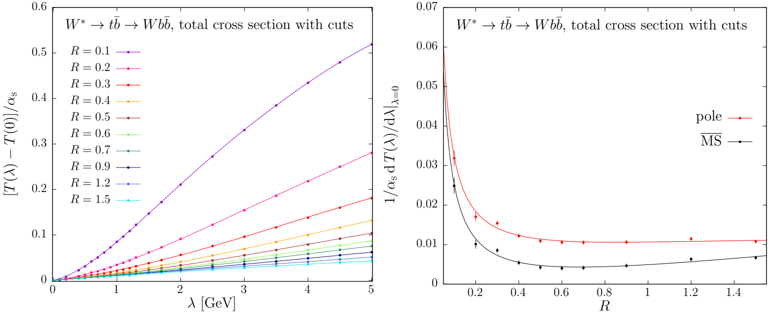
$$\sigma(W^* \to W\, b\, \bar{b}) = \sigma(W^* \to t\bar{b}) \times \frac{\Gamma(t \to W\, b)}{\Gamma_t}$$



Since both terms are free from linear renormalons, also $\sigma(W^* \to W\, b\, \bar{b})$ is free from linear renormalons.

Total cross section with cuts

Cuts: a b jet and a separate $\bar{b}$ jet with $k_\perp > 25$ GeV (anti-$k_\perp$ jets).



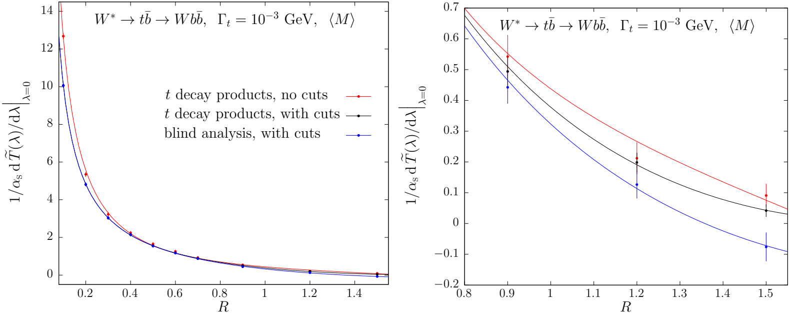
$$\text{Small } R\text{: } \left.\frac{dT(\lambda)}{d\lambda}\right|_{\lambda=0} \propto \frac{1}{R} \Rightarrow \textbf{jet renormalon};$$

Large R: small slope for $\overline{\text{MS}}$.

$\sigma(e^+e^- \to t\bar{t})$, calculated using a short-distance mass, is free from linear renormalons (in absence of cuts). For this kind of measurements, a statistical uncertainty of 20 MeV is predicted in 1611.03399 , by F. Simon.

Reconstructed-top mass in NWA

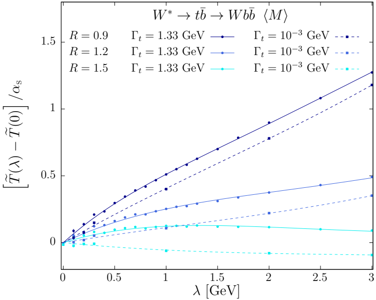
$$O = M = \sqrt{(p_W + p_{b_j})^2}$$

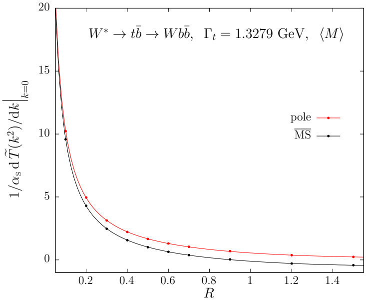


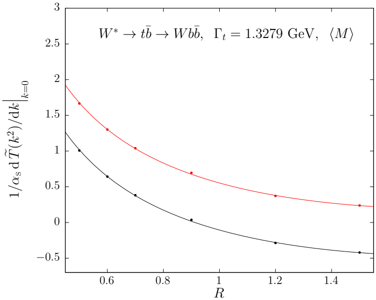
- For $\Gamma_t \to 0$, we can define the "top-decay products"
- For large R, $\langle M \rangle \approx m_{\text{pole}}$ and $T'(0) = 0$: no linear renormalon
- If we move to $\overline{\text{MS}}$ we add $-\frac{C_F}{2}\frac{\partial \langle M \rangle}{\text{Re}(m)} \approx -0.67$: physical linear renormalon

Reconstructed-top mass

For the blind analysis, restoring $\Gamma_t = 1.3279$ GeV only slightly changes this picture







Reconstructed-top mass: some numbers

$$M = \sum_{i=0}^{\infty} c_i \alpha_s^i$$

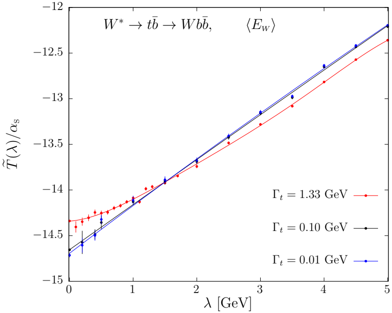
	$c_i\alpha_s^i$ [MeV]		
i	$\mathrm{Re}(m_{\rm pole} - \overline{m}(\mu))$	$\langle M \rangle_{\rm pole}$, $R = 1.5$	$\langle M \rangle_{\overline{\rm MS}}$, $R = 1.5$
5	+89	−10(1)	+79(1)
6	+60	−11(1)	+49(1)
7	+47	−11(1)	+35(1)
8	+44	−12(1)	+31(1)
9	+46	−15(1)	+31(1)
10	+55	−19(1)	+36(1)

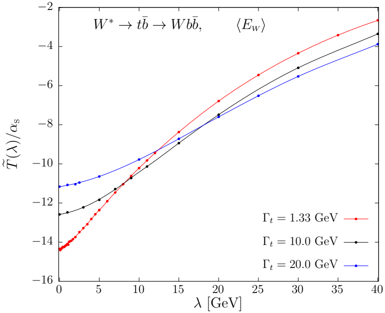
More accurate estimates of $m_{\rm pole} - \overline{m}(\mu)$ (e.g. inclusion of b and c mass effects) can be found in

- [Beneke, Marquad, Nason, Steinhauser, arXiv:1605.03609]: Δm =110 MeV
- [Hoang, Lepenik, Preisser, arXiv:1802.04334]: $\Delta m = 250$ MeV

Energy of the W boson, pole scheme (lab frame)

$\mathbf{E_W}$ = simplified **leptonic observable**. In absence of cuts, is this observable free from renormalons?





When the pole scheme is used we always have renormalons

- Vanishing Γ_t (left): slope ≈ 0.5 near 0;
- Large Γ_t (right): slope ≈ 0.06 near 0;

Energy of the W boson, $\overline{\text{MS}}$ scheme (lab frame)

$\mathbf{E_W}$ = simplified **leptonic observable**. In absence of cuts, is this observable free from physical renormalons?

Γ_t	slope (pole)	$\frac{\partial \langle E_W \rangle_b}{\partial\,\text{Re}(m)}$	$-\frac{C_F}{2}\frac{\partial \langle E_W \rangle_b}{\partial\,\text{Re}(m)}$	slope ($\overline{\text{MS}}$)
NWA	0.53 (2)	0.10 (3)	−0.066 (4)	0.46 (2)
10 GeV	0.058 (8)	0.0936 (4)	−0.0624 (3)	0.004 (8)
20 GeV	0.061 (2)	0.0901 (2)	−0.0601 (1)	0.001 (2)

Yes, if a **finite width** is used, but …

Energy of the W boson (lab frame)

But $\mathcal{O}(\alpha_s^n)$ corrections are dominated by scales of the order $\mu = m_t e^{1-n}$:
we can see the presence of Γ_t only for $\mathbf{n \geq 1 + \log(m_t/\Gamma_t) \approx 6}$

	$\langle E_W \rangle$ [GeV]			
	pole scheme		$\overline{\text{MS}}$ scheme	
i	c_i	$c_i\ \alpha_S^i$	c_i	$c_i\ \alpha_S^i$
0	121.5818	121.5818	120.8654	120.8654
1	$-1.435\,(0) \times 10^{1}$	$-1.552\,(0) \times 10^{0}$	$-7.192\,(0) \times 10^{0}$	$-7.779\,(0) \times 10^{-1}$
2	$-4.97\,(4) \times 10^{1}$	$-5.82\,(4) \times 10^{-1}$	$-3.88\,(4) \times 10^{1}$	$-4.54\,(4) \times 10^{-1}$
3	$-1.79\,(5) \times 10^{2}$	$-2.26\,(6) \times 10^{-1}$	$-1.45\,(5) \times 10^{2}$	$-1.84\,(6) \times 10^{-1}$
4	$-6.9\,(4) \times 10^{2}$	$-9.4\,(6) \times 10^{-2}$	$-5.7\,(4) \times 10^{2}$	$-7.8\,(6) \times 10^{-2}$
5	$-2.9\,(3) \times 10^{3}$	$-4.4\,(5) \times 10^{-2}$	$-2.4\,(3) \times 10^{3}$	$-3.5\,(5) \times 10^{-2}$
6	$-1.4\,(3) \times 10^{4}$	$-2.2\,(4) \times 10^{-2}$	$-1.0\,(3) \times 10^{4}$	$-1.7\,(4) \times 10^{-2}$
7	$-8\,(2) \times 10^{4}$	$-1.3\,(4) \times 10^{-2}$	$-5\,(2) \times 10^{4}$	$-8\,(4) \times 10^{-3}$
8	$-5\,(2) \times 10^{5}$	$-9\,(4) \times 10^{-3}$	$-2\,(2) \times 10^{5}$	$-4\,(4) \times 10^{-3}$
9	$-3\,(2) \times 10^{6}$	$-7\,(4) \times 10^{-3}$	$-1\,(2) \times 10^{6}$	$-2\,(4) \times 10^{-3}$
10	$-3\,(2) \times 10^{7}$	$-6\,(5) \times 10^{-3}$	$0\,(2) \times 10^{6}$	$-1\,(5) \times 10^{-4}$
11	$-3\,(3) \times 10^{8}$	$-7\,(6) \times 10^{-3}$	$0\,(3) \times 10^{6}$	$0\,(6) \times 10^{-5}$
12	$-4\,(3) \times 10^{9}$	$-9\,(9) \times 10^{-3}$	$0\,(3) \times 10^{8}$	$1\,(9) \times 10^{-3}$

Warning!

Despite the fact the energy of the W boson is not affected by linear renormalons, an accurate determination of the top mass is limited by the reduced sensitivity on the top-mass value:

$$2\mathrm{Re}\left[\frac{\partial\langle E_W\rangle_{\mathrm{LO}}}{\partial m}\right] = 0.1$$

$$2\mathrm{Re}\left[\frac{\partial\langle M\rangle_{\mathrm{LO}}}{\partial m}\right] = 1$$

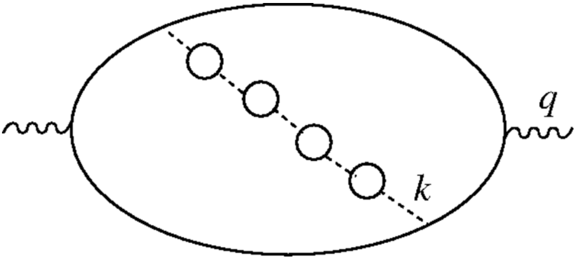
for $E = 300$ GeV, $m_W = 80.4$ GeV, $m_t = 172.5$ GeV ($\beta = 0.5$).
At the threshold, (think about e^+e^- future colliders)

$$\langle E_W\rangle_{\mathrm{LO}} \approx \frac{m_t^2 + m_W^2}{2m_t} \to 2\mathrm{Re}\left[\frac{\partial\langle E_W\rangle_{\mathrm{LO}}}{\partial m}\right] \approx 0.4.$$

Furthermore, the top will be produced in its rest frame, so E_W is renormalon free in NWA ...

Conclusions

- We devised a simple method that enables us to investigate the presence of linear infrared renormalons in **any infrared safe observable**.
- The **inclusive cross section** and $\mathbf{E_W}$ are free from physical renormalons if $\Gamma_t > 0$ (for σ also in NWA).
- Once jets requirements are introduced, the **jet renormalon** leads to an unavoidable ambiguity.
- For large R, $\langle \mathbf{M} \rangle \approx \mathbf{m}_{\text{pole}}$. This observable has a **physical renormalon**.



THANK YOU FOR THE ATTENTION!

IR-safe observables

Average value of an observable O (e.g. reconstructed-top mass, W-boson energy, ...)

$$\langle O \rangle = \frac{1}{\sigma} \int d\Phi \, \frac{d\sigma(\Phi)}{d\Phi} \, O(\Phi)$$

$$= \langle O \rangle_{\text{LO}} - \frac{1}{\pi b_0} \int_0^\infty d\lambda \, \frac{d}{d\lambda} \left[\frac{\widetilde{T}(\lambda)}{\alpha_s(\mu)} \right] \arctan \left[\pi \, b_0 \, \alpha_s \left(\lambda e^{-C/2} \right) \right]$$

- $\widetilde{T}(0) = \langle O \rangle_{\text{NLO}}$
- $\widetilde{T}(\lambda) = \boxed{\langle O(\lambda) \rangle_{\text{NLO}}} + \frac{3\lambda^2}{2 n_f \mathrm{T_R} \alpha_s} \int d\Phi_{g^*} \, d\Phi_{\text{dec}} \, \frac{d\sigma^{(2)}_{q\bar{q}}(\lambda, \Phi)}{d\Phi} \left[\overline{O}(\Phi) - \underbrace{\overline{O}(\Phi_{g^*})}_{q\bar{q} \to g^*} \right]$

  with $\lambda =$ gluon mass, $\overline{O}(\Phi) = [O(\Phi) - O_{\text{LO}}] \, \Theta(\Phi) / \sigma_{\text{LO}}$
- $\widetilde{T}(\lambda) \xrightarrow{\lambda \to \infty} \frac{1}{\lambda^2}$

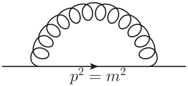
pole-$\overline{\text{MS}}$ mass relation

$\overline{m}(\mu)$ ⇒UV-divergent contribution of self-energy corrections

m_{pole} ⇒UV-divergent + $\underbrace{\text{IR (finite)}}_{\alpha_s^{n+1} n!}$ contributions

- At $\mathcal{O}(\alpha_s)$:

$$m_{\text{pole}} - \overline{m}(\mu) = \text{Fin}\Bigg[i \times$$



$$\Bigg] = \text{Fin}\left[i\Sigma^{(1)}(\epsilon) \right]$$

$$i\Sigma^{(1)}(\epsilon) = -i\, g^2\, \text{C}_\text{F} \left(\frac{\mu^2}{4\pi} \text{e}^{\Gamma_E} \right)^\epsilon \int \frac{\text{d}^d k}{(2\pi)^d} \frac{\gamma^\alpha (\not{p} + \not{k} + m) \gamma_\alpha}{[k^2 + i\eta] \, [(k+p)^2 - m^2 + i\eta]} \Bigg|_{\not{p}=m}$$

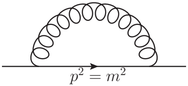
pole-$\overline{\text{MS}}$ mass relation

$\overline{m}(\mu)$ ⇒UV-divergent contribution of self-energy corrections

m_{pole} ⇒UV-divergent + $\underbrace{\text{IR (finite)}}_{\alpha_s^{n+1} n!}$ contributions

- At $\mathcal{O}(\alpha_s)$:

$$m_{\text{pole}} - \overline{m}(\mu) = \text{Fin}\Big[i \times \text{(diagram)} \Big] = \text{Fin}\left[i\Sigma^{(1)}(\epsilon) \right]$$



$$i\Sigma^{(1)}(\epsilon) = -i\, g^2\, \text{C}_\text{F} \left(\frac{\mu^2}{4\pi} \text{e}^{\Gamma_E} \right)^{\epsilon} \int \frac{\text{d}^d k}{(2\pi)^d} \frac{\gamma^\alpha (\not{p} + \not{k} + m)\gamma_\alpha}{[k^2 + i\eta]\,[(k+p)^2 - m^2 + i\eta]} \Bigg|_{\not{p}=m}$$

- At all-orders:

$$i\Sigma(\epsilon) = -\, i\, g^2\, \text{C}_\text{F} \left(\frac{\mu^2}{4\pi} \text{e}^{\Gamma_E} \right)^{\epsilon} \int \frac{\text{d}^d k}{(2\pi)^d} \frac{\gamma^\alpha (\not{p} + \not{k} + m)\gamma_\alpha}{[k^2 + i\eta]\,[(k+p)^2 - m^2 + i\eta]} \Bigg|_{\not{p}=m}$$

$$\times \frac{1}{1 + \Pi(k^2 + i\eta, \mu^2, \epsilon) - \Pi_{\text{ct}}}$$

- At all-orders:

$$i\Sigma(\epsilon) = -\frac{1}{\pi}\int_{0^-}^{+\infty}\frac{\mathrm{d}\lambda^2}{2\pi}\left[i\underbrace{\Sigma^{(1)}(\epsilon,\lambda)}_{\lambda=\text{gluon mass}}\right]\mathrm{Im}\left[\frac{1}{\lambda^2+i\eta}\frac{1}{1+\Pi(\lambda^2+i\eta,\mu^2,\epsilon)-\Pi_{\mathrm{ct}}}\right]$$

$$\mathrm{Fin}\left[i\Sigma(\epsilon)\right] = -\frac{1}{\pi b_0}\int_0^\infty \lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{r_{\mathrm{fin}}(\lambda)}{\alpha_s(\mu)}\right]\arctan\left[\pi b_0\alpha_s(\lambda e^{-C/2})\right]+\ldots$$

where $\boxed{r_{\mathrm{fin}}(\lambda)\xrightarrow{\lambda\ll 1} -\alpha_s(\mu)\frac{\mathrm{C_F}}{2}\lambda}$, $\quad r_{\mathrm{fin}}(\lambda)\xrightarrow{\lambda\to\infty}\mathcal{O}\left(\frac{m^2}{\lambda^2}\right)$

- At all-orders:

$$i\Sigma(\epsilon) = -\frac{1}{\pi}\int_{0^-}^{+\infty}\frac{d\lambda^2}{2\pi}\left[i\underbrace{\Sigma^{(1)}(\epsilon,\lambda)}_{\lambda=\text{gluon mass}}\right]\text{Im}\left[\frac{1}{\lambda^2+i\eta}\frac{1}{1+\Pi(\lambda^2+i\eta,\mu^2,\epsilon)-\Pi_{\text{ct}}}\right]$$

$$\text{Fin}\left[i\Sigma(\epsilon)\right] = -\frac{1}{\pi b_0}\int_0^{\infty}\lambda\frac{d}{d\lambda}\left[\frac{r_{\text{fin}}(\lambda)}{\alpha_s(\mu)}\right]\arctan\left[\pi b_0\alpha_s(\lambda e^{-C/2})\right]+\ldots$$

where $\boxed{r_{\text{fin}}(\lambda)\xrightarrow{\lambda\ll 1}-\alpha_s(\mu)\frac{\text{C}_\text{F}}{2}\lambda}$, $\quad r_{\text{fin}}(\lambda)\xrightarrow{\lambda\to\infty}\mathcal{O}\left(\frac{m^2}{\lambda^2}\right)$

- Small λ contribution (independent from C):

$$\frac{\text{C}_\text{F}}{2}\sum_{n=0}^{\infty}\int_0^m d\lambda\left[-2b_0\,\alpha_s(m)\log\left(\frac{\lambda^2}{m^2}\right)\right]^n = \frac{\text{C}_\text{F}}{2}m\sum_{n=0}^{\infty}(2\,b_0\,\alpha_s(m))^n\,n!$$

The resummed series has an ambiguity proportional to $\mathbf{\Lambda_{QCD}}$:

Linear k term $\leftrightarrow$ Linear renormalons

pole-$\overline{\text{MS}}$ mass relation

- In the pure n_f limit: arxiV:hep-ph/9502300, Ball et all

$$b_0 = -\frac{n_f T_R}{3\pi}\,, C = \frac{5}{3}\,, \qquad \frac{m - \overline{m}(\overline{m})}{m} = \frac{4}{3}\alpha_s(\overline{m})\left[1 + \sum_{i=1}^{\infty} d_i\left(b_0\,\alpha_s(\overline{m})\right)^i\right]$$

i	1	2	3	4	5	6	7	8
d_i	5×10^0	2×10^1	1×10^2	9×10^2	9×10^3	1×10^5	1×10^6	2×10^7

- "Realistic" large b_0 approximation:

$$\alpha_s(\lambda\, e^{-C/2}) = \frac{\alpha_s(\lambda)}{1 - b_0 C\alpha_s(\lambda)} \approx \underbrace{\alpha_s(\lambda)\left[1 + b_0 C\alpha_s(\lambda)\right] = \alpha_s^{\text{CMW}}(\lambda)}_{b_0\,C = \frac{1}{2\pi}\left[\left(\frac{67}{18} - \frac{\pi^2}{6}\right)C_A - \frac{10}{9}n_l T_R\right]}$$

Pole-$\overline{\text{MS}}$ mass relation

$$m_0 = 172.5 \text{ GeV}\,, \qquad \Gamma = 1.3279 \text{ GeV}\,, \quad m^2 = m_0^2 - i m_0 \Gamma\,, \qquad \mu = m_0$$

$$m - \overline{m}(\mu) = m \sum_{i=1}^{n} c_i \alpha_s^i(\mu)$$

$m - \overline{m}(\mu)$				
i	$\mathrm{Re}\,(c_i)$	$\mathrm{Im}\,(c_i)$	$\mathrm{Re}\,(m\, c_i\, \alpha_s^i)$	$\mathrm{Im}\,(m\, c_i\, \alpha_s^i)$
1	4.244×10^{-1}	2.450×10^{-3}	$7.919 \times 10^{+0}$	$+1.524 \times 10^{-2}$
2	6.437×10^{-1}	2.094×10^{-3}	$1.299 \times 10^{+0}$	-7.729×10^{-4}
3	$1.968 \times 10^{+0}$	8.019×10^{-3}	4.297×10^{-1}	$+9.665 \times 10^{-5}$
4	$7.231 \times 10^{+0}$	2.567×10^{-2}	1.707×10^{-1}	-5.110×10^{-5}
5	$3.497 \times 10^{+1}$	1.394×10^{-1}	8.930×10^{-2}	$+1.240 \times 10^{-5}$
6	$2.174 \times 10^{+2}$	8.164×10^{-1}	6.005×10^{-2}	-5.616×10^{-6}
7	$1.576 \times 10^{+3}$	$6.133 \times 10^{+0}$	4.709×10^{-2}	$+2.009 \times 10^{-6}$
8	$1.354 \times 10^{+4}$	$5.180 \times 10^{+1}$	4.376×10^{-2}	-1.031×10^{-6}
9	$1.318 \times 10^{+5}$	$5.087 \times 10^{+2}$	4.608×10^{-2}	$+4.961 \times 10^{-7}$
10	$1.450 \times 10^{+6}$	$5.572 \times 10^{+3}$	5.481×10^{-2}	-2.909×10^{-7}

Pole-$\overline{\text{MS}}$ mass relation

$$m_0 = 172.5 \text{ GeV}\,, \qquad \Gamma = 1.3279 \text{ GeV}\,, \quad m^2 = m_0^2 - i m_0 \Gamma\,, \qquad \mu = m_0$$

$$m - \overline{m}(\mu) = m \sum_{i=1}^{n} c_i \alpha_s^i(\mu)$$

$m - \overline{m}(\mu)$				
i	$\text{Re}\,(c_i)$	$\text{Im}\,(c_i)$	$\text{Re}\left(m\,c_i\,\alpha_s^i\right)$	$\text{Im}\left(m\,c_i\,\alpha_s^i\right)$
5	$3.497 \times 10^{+1}$	1.394×10^{-1}	8.930×10^{-2}	$+1.240 \times 10^{-5}$
6	$2.174 \times 10^{+2}$	8.164×10^{-1}	6.005×10^{-2}	-5.616×10^{-6}
7	$1.576 \times 10^{+3}$	$6.133 \times 10^{+0}$	4.709×10^{-2}	$+2.009 \times 10^{-6}$
8	$1.354 \times 10^{+4}$	$5.180 \times 10^{+1}$	4.376×10^{-2}	-1.031×10^{-6}
9	$1.318 \times 10^{+5}$	$5.087 \times 10^{+2}$	4.608×10^{-2}	$+4.961 \times 10^{-7}$
10	$1.450 \times 10^{+6}$	$5.572 \times 10^{+3}$	5.481×10^{-2}	-2.909×10^{-7}

More accurate estimates of $m_{\text{pole}} - \overline{m}(\mu)$ (e.g. inclusion of b and c mass effects) can be found in

- [Beneke, Marquad, Nason, Steinhauser, arXiv:1605.03609]: Δm =110 MeV
- [Hoang, Lepenik, Preisser, arXiv:1802.04334]: Δm = 250 MeV

NB: Actual systematic uncertainty is 500 MeV!