Feynman Integrals & Intersection Theory

Pierpaolo Mastrolia

LFC 19

ECT*, Villa Tambosi, Trento 10.9.2019

Based on:

- **PM, Mizera**,
Feynman Integrals and Intersection Theory
JHEP 1902 (2019) 139 [arXiv: 1810.03818]

- **Frellesvig, Gasparotto, Laporta, Mandal, PM, Mattiazzi, Mizera**,
Decomposition of Feynman Integrals on the Maximal Cut by Intersection Numbers
JHEP 1095 (2019) 153 [arXiv: 1901.1151]

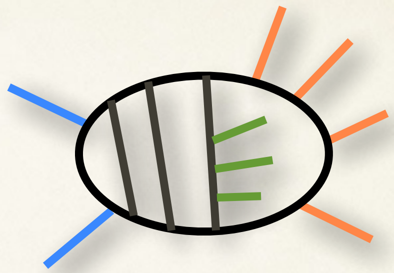
- **Frellesvig, Gasparotto, Mandal, PM, Mattiazzi, Mizera**,
Vector Space of Feynman Integrals & Multivariate Intersection Numbers
arXiv: 1907.02000

Physics and Astronomy Department
Galileo Galilei
University of Padova - Italy

Outline

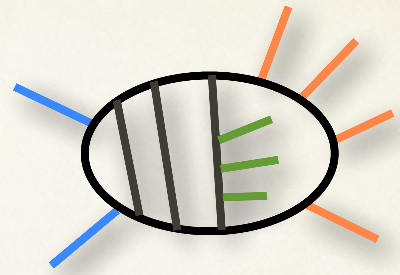
- **Feynman Integrals in Dim Reg**
  - Integration-by-parts Identities
- **Basics of Intersection Theory**
- **Intersection Numbers for 1-forms**
- **Integral Relations by Intersection Numbers**
  - Special Functions
  - Feynman Integrals
- **Intersection Numbers for n-forms**
- **Conclusions**

Scattering Amplitudes



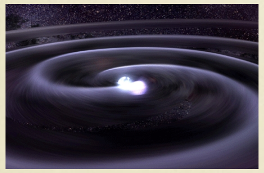
- **Very healthy status**
  - **Progress @ High Loops**
  - **Progress @ High Legs**
  - **New Ideas in the multi-loop integral evaluation**
    - Differential Equations and Path ordered exponential
    - Iterated integrals and special/pure Functions
  - **New Ideas exploiting the (hidden symmetries) of the integrands**
    - Unitarity and on-shell methods beyond one-loop
    - Double-copy relations
  - **New Ideas and tool to boost the Automated Algorithms**
    - Exploiting Finite Field Arithmetic
    - Advanced linear system resolution algorithm

Scattering Amplitudes



- **Very healthy status**
  - **Progress @ High Loops**
  - **Progress @ High Legs**

- **a couple of interesting directions I have been involved in:**
  - **The proposal of a new CERN experiments for the muon (g-2)**
    - NNLO QED corrections required
    - Calculation relevant for di-muon in e+e- collision and t-tbar production

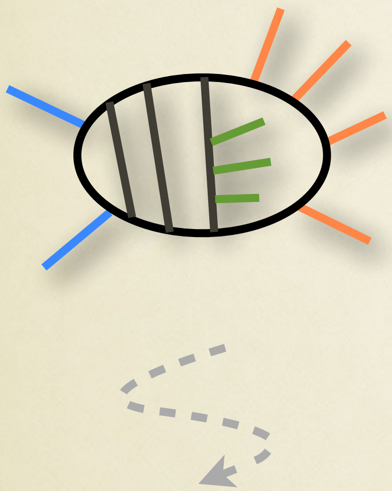


  - **Effective Field Theory approach to General Relativity**
    - New applications of Feynman Calculus to Gravitational Wave Physics

for the investigation of physical problems that admit a field-theoretic perturbative approach: computation of multi-loop Feynman integrals cannot be considered as optional.

Feynman Integrals

- **Momentum-space Representation**



$$= \quad I_{a_1,a_2,\ldots,a_N} \equiv \int \prod_{i=1}^{L} \frac{d^d k_i}{\pi^{d/2}} \prod_{j=1}^{N} \frac{1}{D_j^{a_j}}$$

N-denominator generic Integral

L loops, $E{+}1$ external momenta,

$N = LE + \frac{1}{2}L(L{+}1)$ (generalised) denominators

total number of *reducible* and *irreducible* scalar products

't Hooft & Veltman
Passarino & Veltman

- **Integration-by-parts Identites** Tkachov; Chetyrkin & Tkachov

Laporta, Remiddi, Baikov, Smirnov, vanRitbergen, Melnikov, Gehrmann, Weinzierl… …many of us here…

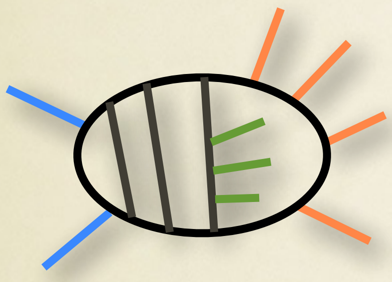
$$\int \prod_{i=1}^{L} \frac{d^d k_i}{\pi^{d/2}} \frac{\partial}{\partial k_j^{\mu}} \left(v_\mu \prod_{n=1}^{N} \frac{1}{D_n^{a_n}} \right) = 0 \qquad v_\mu = v_\mu(p_i, k_j) \quad \text{arbitrary}$$

The role of the Integration Domain is hidden

Feynman Integrals :: *Baikov Representation*

- **Denominators as integration variables** Baikov

$$\{D_1, \ldots, D_N\} \to \{z_1, \ldots, z_N\} \equiv \mathbf{z}$$



$$= \quad I_{a_1,\ldots,a_N} \equiv K(d, s_{ij}) \int_{\mathcal{C}} d\mathbf{z}\ B(\mathbf{z})^\gamma \prod_{i=1}^{N} \frac{1}{z_i^{a_n}}$$

N-denominator
generic Integral

Volume

$$B(\mathbf{z}) = \det(q_i \cdot q_j)$$

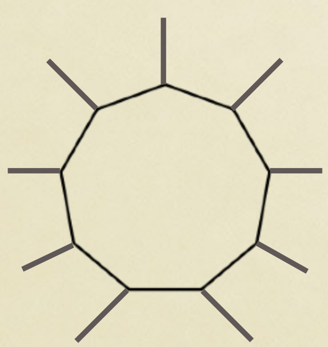
$$\gamma \equiv (d - E - L - 1)/2$$

$$q = \{p_i, k_j\} \qquad s_{ij} = p_i \cdot p_j$$

$$B(\partial\mathcal{C} = 0)$$

Fundamental property

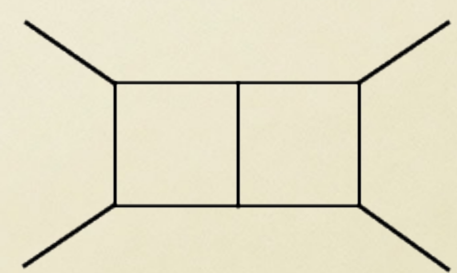
- **1-loop Nonagon**



$$N = LE + \frac{1}{2}L(L+1)$$

$$\int_{\mathcal{C}} dz_1 \wedge \cdots \wedge dz_9 \ \frac{B(\mathbf{z})^\gamma}{z_1^{n_1} \cdots z_9^{n_9}}$$

- **2-loop Box**



$B(\mathbf{z})$, $\mathcal{C}$, γ depend on the graph.

Feynman Integrals :: *Baikov Representation*

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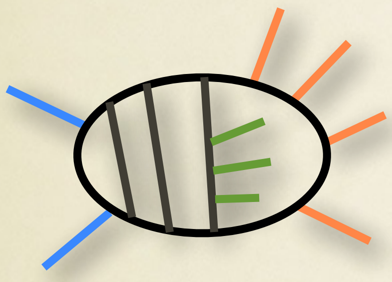
Volume



$$= \quad I_{a_1,\ldots,a_N} \equiv K(d, s_{ij}) \int_{\mathcal{C}} d\mathbf{z}\; B(\mathbf{z})^\gamma \prod_{i=1}^{N} \frac{1}{z_i^{a_n}}$$

$B(\mathbf{z}) = \det(q_i \cdot q_j)$

$\gamma \equiv (d - E - L - 1)/2$

$q = \{p_i, k_j\} \qquad s_{ij} = p_i \cdot p_j$

$B(\partial\mathcal{C} = 0)$

Fundamental property

N-denominator
generic Integral

- **Integration-by-parts Identites** Zhang, Larsen; Lee;

$$\int_{\mathcal{C}} d \left(h(\mathbf{z})\; B(\mathbf{z})^\gamma \prod_{i=1}^{N} \frac{1}{z_i^{a_n}} \right) = 0$$

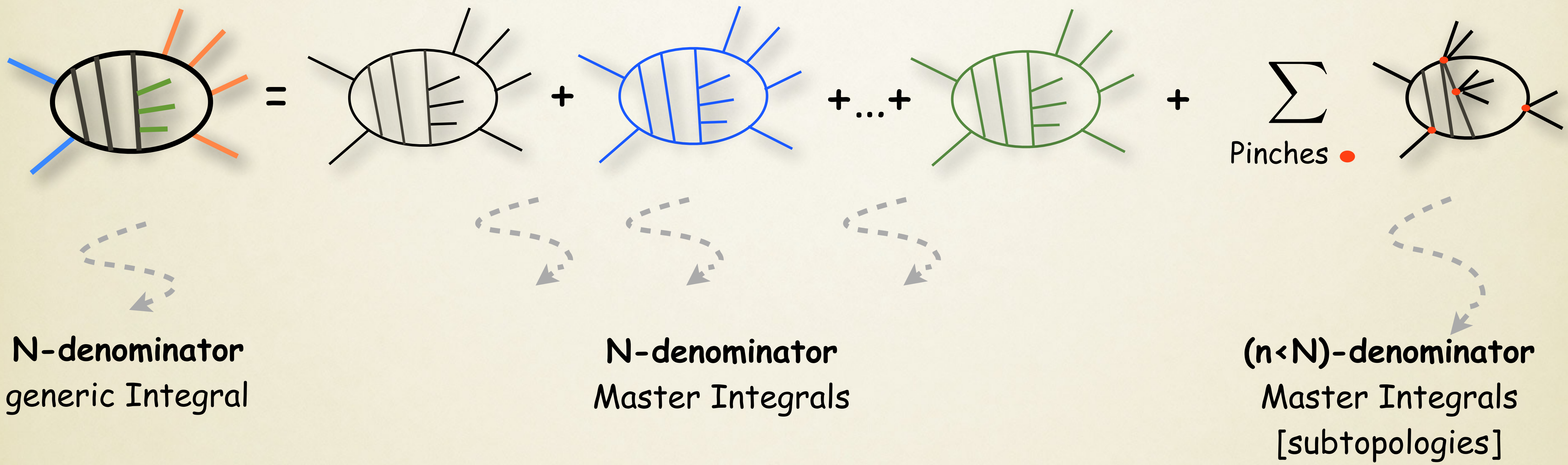
$h(\mathbf{z})$ arbitrary rational function

$$B(\partial\mathcal{C}) = 0$$

Fundamental property

Integration-by-parts identities

- Relations among Integrals in dim. reg.



Integration-by-parts identities

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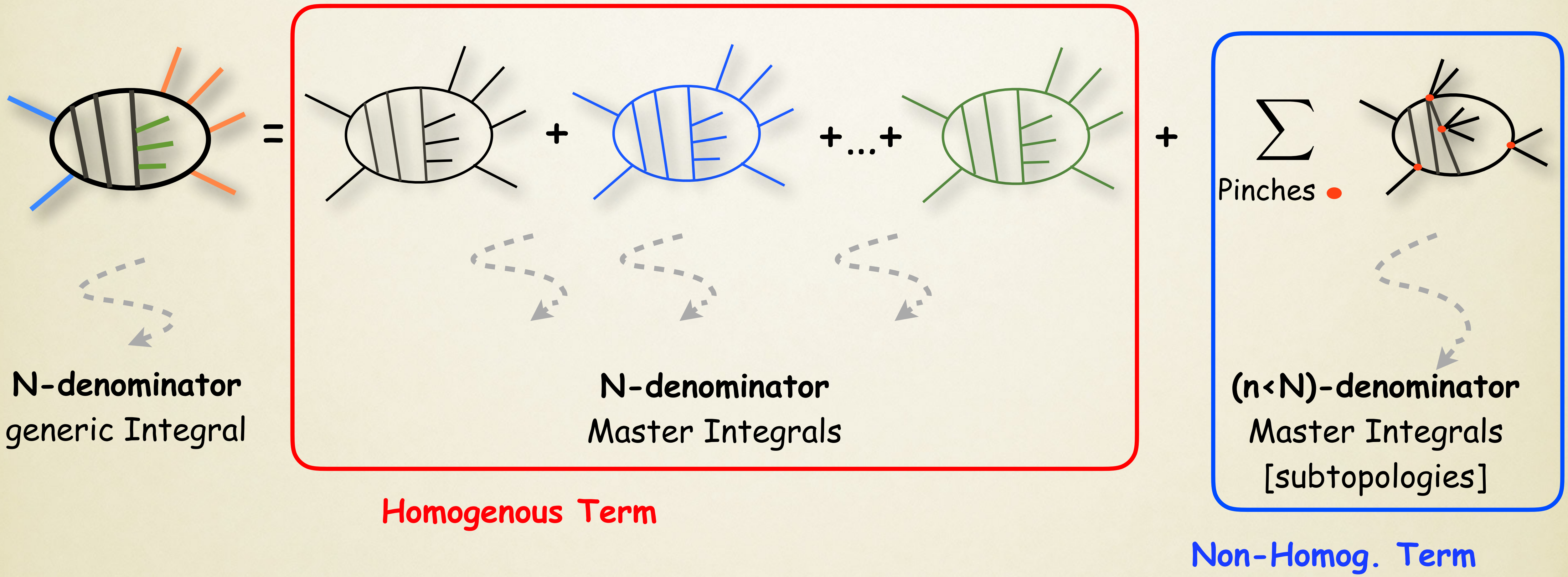
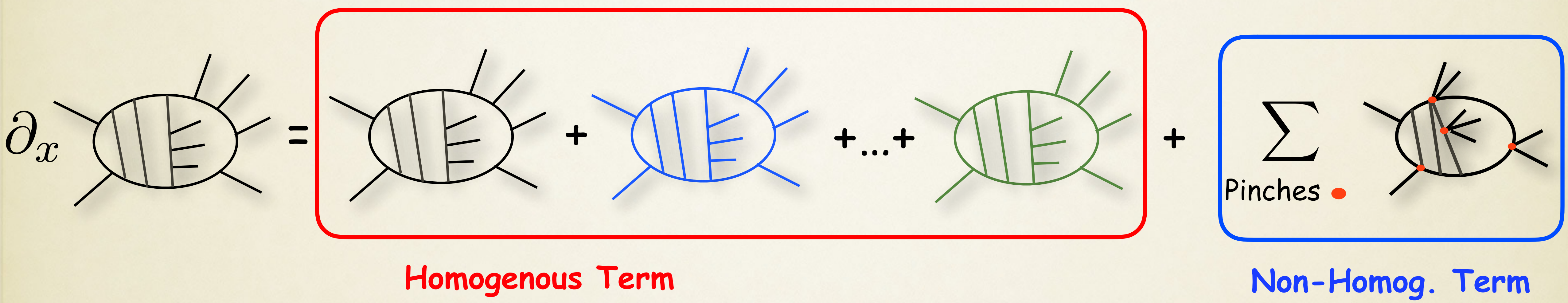


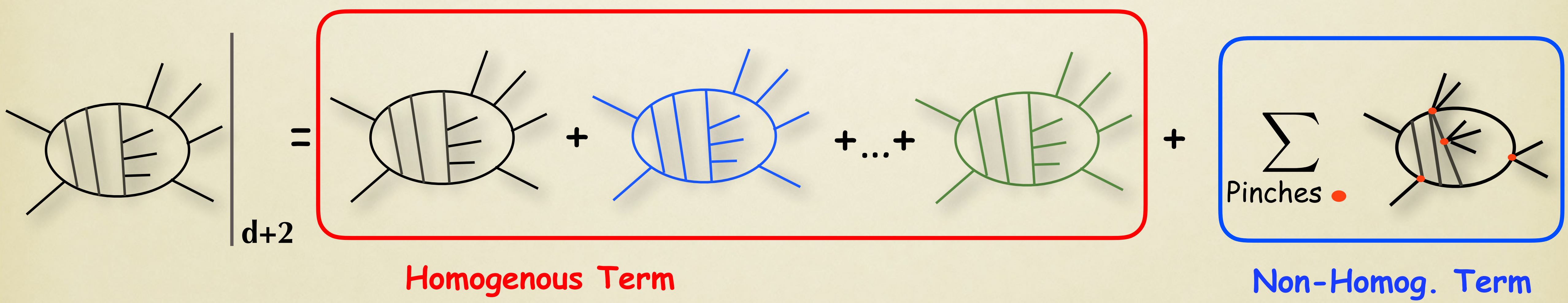
Integration-by-parts identities :: *byproducts*

- **1st order Differential Equations for MIs** Barucchi, Ponzano; Kotikov; Remiddi, & Gerhmann; …Weinzierl, Adams, Bogner … Henn; Lee; Argeri, diVita, Mirabella, Schubert, Tancredi, Schlenck **& P.M.**; …



- **Dimension-Shift relations** Bern Dixon Kosower; Tarasov; Lee; Bernstein-Tkachov;

  \+ related work by Gluza, Kajda, Kosower; Remiddi, Tancredi



Basics of Intersection Theory

Aomoto, Cho, Kita , Mazumoto, Mimachi, Mizera, Yoshida,…

Consider an integral I over the variables $\mathbf{z} = (z_1, z_2, \ldots, z_m)$

$$I \;=\; \int_{\mathcal{C}} u(\mathbf{z}) \;\; \varphi(\mathbf{z})$$

$u(\mathbf{z})$ is a multi-valued function $\qquad u(\partial\mathcal{C}) = 0$

$\varphi(\mathbf{z}) = \hat{\varphi}(\mathbf{z}) d^m \mathbf{z}$ is a differential m-form.

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$$I = \underbrace{\int_{\mathcal{C}} u(\mathbf{z})}_{\text{twisted cycle}} \underbrace{\varphi(\mathbf{z})}_{\text{twisted cocycle}}$$

$u(\mathbf{z})$ is a multi-valued function $\qquad u(\partial\mathcal{C}) = 0$

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Equivalence classes, Integration-by-parts Identities, and Covariant Derivative

there could exist many forms φ that integrate to give the result I.

$(m-1)$-differential form ξ

$$0 = \int_{\mathcal{C}} d\,(u\,\xi) = \int_{\mathcal{C}} (du \wedge \xi + u\,d\xi) = \int_{\mathcal{C}} u \left(\frac{du}{u} \wedge + d \right) \xi \equiv \int_{\mathcal{C}} u\,\nabla_{\omega}\xi$$

$$\omega \equiv d \log u \qquad\qquad \nabla_{\omega} \equiv d + \omega\wedge$$

$${}_{\omega}\langle\varphi| : \;\; \varphi \sim \varphi + \nabla_{\omega}\xi. \qquad\qquad \int_{\mathcal{C}} u\,\varphi = \int_{\mathcal{C}} u\,(\varphi + \nabla_{\omega}\xi)$$

Equivalence classes, Integration-by-parts Identities, and Covariant Derivative

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Space of m-forms :: *Twisted cohomology Group*

$$H^m_\omega \equiv \{m\text{–forms } \varphi_{\mathrm{m}} \,|\, \nabla_\omega \varphi_{\mathrm{m}} = 0\} / \{\nabla_\omega \varphi_{\mathrm{m}-1}\},$$

Dual space

$$H^m_{-\omega} \;, \qquad\qquad \nabla_{-\omega} = d - \omega \wedge$$

Pairings of Cycles and Co-cycles

Aomoto, Cho, Kita , Mazumoto, Mimachi, Mizera, Yoshida,…

- **Basic building blocks**

$$\langle\varphi_L| \equiv \varphi_L(\mathbf{z}) \in H^m_\omega \qquad |\varphi_R\rangle \equiv \varphi_R(\mathbf{z}) \in H^m_{-\omega} \qquad |\mathcal{C}_L] \equiv \int_{\mathcal{C}_L} u(\mathbf{z}) \qquad [\mathcal{C}_R| \equiv \int_{\mathcal{C}_R} u(\mathbf{z})^{-1}$$

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- **Integrals ::** pairings of cycles and co-cycles

$$\langle \, \varphi_L \, | \, \mathcal{C}_L \,] \; \equiv \; \int_{\mathcal{C}_L} \; u(\mathbf{z}) \; \varphi_L(\mathbf{z}) \; = \; I$$

- **Dual Integrals ::** pairings of cycles and co-cycles

$$[\, \mathcal{C}_R \, | \, \varphi_R \, \rangle \; \equiv \; \int_{\mathcal{C}_R} \; u(\mathbf{z})^{-1} \; \varphi_R(\mathbf{z}) \; = \; \tilde{I}$$

- **Intersection numbers for cycles ::** pairings of cycles

$$[\, \mathcal{C}_\mathrm{L} \, | \, \mathcal{C}_\mathrm{R} \,] \equiv \text{intersection number}$$

- **Intersection numbers for co-cycles ::** pairings of co-cycles

$$\langle \, \varphi_\mathrm{L} \, | \, \varphi_\mathrm{R} \, \rangle \equiv \int_{\mathcal{C}} \iota(\varphi_\mathrm{L}) \wedge \varphi_\mathrm{R}$$

- **Riemann Twisted Period Relations**

$$\langle \, \varphi_\mathrm{L} \, | \, \varphi_\mathrm{R} \, \rangle = \langle \, \varphi_\mathrm{L} \, | \, \mathcal{C}_\mathrm{L} \,] \; [\, \mathcal{C}_\mathrm{L} \, | \, \mathcal{C}_\mathrm{R} \,]^{-1} \; [\, \mathcal{C}_\mathrm{R} \, | \, \varphi_\mathrm{R} \, \rangle$$

Cho & Mazumoto (1994)

Pairings of Cycles and Co-cycles

Aomoto, Cho, Kita , Mazumoto, Mimachi, Mizera, Yoshida,…

- **Basic building blocks**

$$\langle \varphi_L| \equiv \varphi_L(\mathbf{z}) \in H^m_\omega \qquad |\varphi_R\rangle \equiv \varphi_R(\mathbf{z}) \in H^m_{-\omega} \qquad |\mathcal{C}_L] \equiv \int_{\mathcal{C}_L} u(\mathbf{z}) \qquad [\mathcal{C}_R| \equiv \int_{\mathcal{C}_R} u(\mathbf{z})^{-1}$$

- **Integrals ::** pairings of cycles and co-cycles $\langle \varphi_L | \mathcal{C}_L] \equiv \int_{\mathcal{C}_L} u(\mathbf{z})\, \varphi_L(\mathbf{z}) = I$

- **Dual Integrals ::** pairings of cycles and co-cycles $[\mathcal{C}_R | \varphi_R \rangle \equiv \int_{\mathcal{C}_R} u(\mathbf{z})^{-1}\, \varphi_R(\mathbf{z}) = \tilde{I}$

- **Intersection numbers for cycles ::** pairings of cycles $[\mathcal{C}_\mathrm{L} | \mathcal{C}_\mathrm{R}] \equiv \text{intersection number}$

- **Intersection numbers for co-cycles ::** pairings of co-cycles $\langle \varphi_\mathrm{L} | \varphi_\mathrm{R} \rangle \equiv \int_{\mathcal{C}} \iota(\varphi_\mathrm{L}) \wedge \varphi_\mathrm{R}$

- **Riemann Twisted Period Relations** $\langle \varphi_\mathrm{L} | \varphi_\mathrm{R} \rangle = \langle \varphi_\mathrm{L} | \mathcal{C}_\mathrm{L}] \, [\mathcal{C}_\mathrm{L} | \mathcal{C}_\mathrm{R}]^{-1} \, [\mathcal{C}_\mathrm{R} | \varphi_\mathrm{R} \rangle$

Cho & Mazumoto (1994)

Integral Decomposition from Differential Forms

$$I = \langle \varphi | \mathcal{C}]$$

Consider a set of ν MIs,

$$J_i = \int_{\mathcal{C}} u(\mathbf{z})\, e_i(\mathbf{z}) = \langle e_i | \mathcal{C}] \;, \qquad i = 1, \ldots, \nu \,,$$

$$I = \sum_{i=1}^{\nu} c_i \, J_i$$

$$\langle \varphi | = \sum_{i=1}^{\nu} c_i \, \langle e_i |$$

Vector spaces of differential forms

- **Space Dimensions**

$$\nu \equiv \dim H^n_{\pm\omega}$$

$$\nu = \text{number of independent forms (twisted cocycles)}$$

$$= \{\text{the number of solutions of } \omega = 0$$

Lee, Pomeransky (2013)

- **Basis :: bra** $\langle e_i|$ $\quad i = 1, 2, \ldots, \nu$

- **dual-Basis :: ket** $|h_j\rangle$ $\quad j = 1, 2, \ldots, \nu$

- **Metric Matrix** $\mathbf{C}_{ij} = \langle e_i | h_j \rangle$

intersection number

Master Decomposition Formula

Mizera **& P.M.** (2018)

+ Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi (2019)

- **Decomposition of differential forms**

projecting $\langle\varphi|$ onto a basis of $\langle e_i|$

$$\langle\varphi| = \sum_{i,j=1}^{\nu} \langle\varphi|h_j\rangle \left(\mathbf{C}^{-1}\right)_{ji} \langle e_i|$$

- ***Proof***

for an arbitrary $|\psi\rangle$

$$\mathbf{M} = \begin{pmatrix} \langle\varphi|\psi\rangle & \langle\varphi|h_1\rangle & \langle\varphi|h_2\rangle & \dots & \langle\varphi|h_\nu\rangle \\ \langle e_1|\psi\rangle & \langle e_1|h_1\rangle & \langle e_1|h_2\rangle & \dots & \langle e_1|h_\nu\rangle \\ \langle e_2|\psi\rangle & \langle e_2|h_1\rangle & \langle e_2|h_2\rangle & \dots & \langle e_2|h_\nu\rangle \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \langle e_\nu|\psi\rangle & \langle e_\nu|h_1\rangle & \langle e_\nu|h_2\rangle & \dots & \langle e_\nu|h_\nu\rangle \end{pmatrix} \equiv \begin{pmatrix} \langle\varphi|\psi\rangle & \mathbf{A}^\intercal \\ \mathbf{B} & \mathbf{C} \end{pmatrix}$$

$(\nu+1)\times(\nu+1)$ matrix $\mathbf{M}$

$$\det\mathbf{M} = \det\mathbf{C}\left(\langle\varphi|\psi\rangle - \mathbf{A}^\intercal\,\mathbf{C}^{-1}\,\mathbf{B}\right) = 0$$

$$\langle\varphi|\psi\rangle = \mathbf{A}^\intercal\,\mathbf{C}^{-1}\,\mathbf{B} = \sum_{i,j=1}^{\nu} \langle\varphi|h_j\rangle\,(\mathbf{C}^{-1})_{ji}\,\langle e_i|\psi\rangle$$

Intersection Numbers :: *1-forms*

- **1-form** $\langle \varphi | \equiv \hat{\varphi}(z)\; dz$ $\hat{\varphi}(z)$ rational function

- **Zeroes and Poles of** ω

$$\omega \equiv d \log u$$

$$\nu = \{\text{the number of solutions of } \omega = 0\}$$

$$\mathcal{P} \equiv \{\, z \mid z \text{ is a pole of } \omega \,\}$$

$\mathcal{P}$ can also include the pole at infinity if $\text{Res}_{z=\infty}(\omega) \neq 0$.

- **Intersection Numbers (for cocycles)**

Matsumoto (1996, 1998)

1-forms φ_L and φ_R

$$\langle \varphi_L | \varphi_R \rangle_\omega = \sum_{p \in \mathcal{P}} \text{Res}_{z=p} \Big(\psi_p \, \varphi_R \Big)$$

ψ_p is a function (0-form), solution to the differential equation $\nabla_\omega \psi = \varphi_L$, around p

Intersection Numbers :: *1-forms*

Solving a 1st ODE

$$\nabla_\omega \psi = \varphi_L \qquad \frac{d}{dz}\psi + \omega\psi = \varphi_L$$

- **Way-1 :: Laurent expansions**

$\tau \equiv z - p$

known: $\varphi_{L,p}$ and ω_p

ansatz: $$\psi_p = \sum_{j=\min}^{\max} \psi_p^{(j)} \tau^j + \mathcal{O}\left(\tau^{\max+1}\right)$$

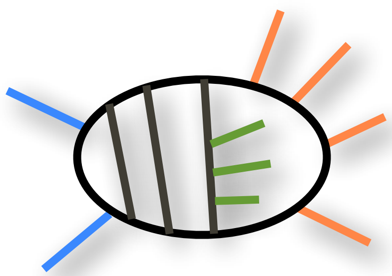
- **Fixing the coefficients ==>**
  ==> solving a simple, triangular system

- **Way-2 :: Variation of parameters** NEW

$$\psi = \frac{1}{u} \int u \, \varphi_L \qquad \langle \varphi_L | \varphi_R \rangle = \sum_{p \in \mathcal{P}} \text{Res}_{z=p} \left\{ \left(\int u \, \varphi_L \right) \left(u^{-1} \, \varphi_R \right) \right\}$$

- **left term ::** series expansion + integration
- **right term ::** series expansion
- **Residue extraction ::** no system-solving required!

Feynman Integrals & Intersection Theory



$$= \; I_{a_1,a_2,\ldots,a_N} \equiv \int \prod_{i=1}^{L} \frac{d^d k_i}{\pi^{d/2}} \prod_{j=1}^{N} \frac{1}{D_j^{a_j}}$$

$$\equiv K \int_{\mathcal{C}} u\,\varphi \quad \equiv K \, \langle \varphi | \mathcal{C}]_\omega$$

Mizera **& P.M.** (2018)

- **Baikov representation**

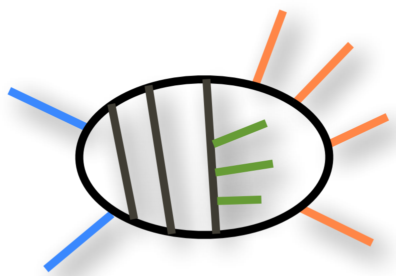
$$u = B^\gamma \,, \qquad \gamma \equiv (d{-}E{-}L{-}1)/2$$

$$\omega \equiv d\log(u) = \gamma d\log(B)$$

$$\varphi \equiv \hat{\varphi}\, d^N \mathbf{z} \,, \qquad \hat{\varphi} \equiv \frac{1}{z_1^{a_1} z_2^{a_2} \cdots z_N^{a_N}} \,,$$

$$d^N \mathbf{z} \equiv dz_1 \wedge dz_2 \wedge \cdots \wedge dz_N$$

Feynman Integrals & Intersection Theory



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$$\equiv K \int_{\mathcal{C}} u\,\varphi \quad \equiv K \,\langle \varphi | \mathcal{C}]_\omega$$

Mizera **& P.M.** (2018)

- **Loop-by-Loop (LBL) Baikov repr'n**

  Frellesvig, Papadopoulos (2017)

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera **& P.M.** (2019)

$$u = B_1^{\gamma_1} B_2^{\gamma_2} \cdots B_m^{\gamma_m},$$

$$\omega \equiv d\log(u) = \sum_{i=1}^{m} \gamma_i \, d\log(B_i)$$

$$\varphi \equiv \hat{\varphi}\, d^M \mathbf{z}, \qquad \hat{\varphi} \equiv \frac{f(z_1,\ldots,z_M)}{z_1^{a_1} z_2^{a_2} \cdots z_M^{a_M}},$$

$$d^M \mathbf{z} \equiv dz_1 \wedge dz_2 \wedge \cdots \wedge dz_M$$

$(N - M)$ ISPs
integrated out

f rational function

Integrals reduction and Master Integrals

Mizera & P.M. (2018)

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera & P.M. (2019)

$\nu = \{\text{the number of solutions of } \omega = 0\}$

- Basis of Master Forms $\qquad \langle e_i| \quad |h_j\rangle \qquad i = 1, 2, \ldots, \nu$

- **Master Integrals** $\qquad J_i \equiv K\, E_i\ , \quad \text{with} \quad E_i \equiv \langle e_i|\mathcal{C}]$

- **Integral Decomposition**

$$I = K\langle\varphi|\mathcal{C}] = \sum_{i=1}^{\nu} c_i\, J_i$$

$$\langle\varphi| = \sum_{i,j=1}^{\nu} \langle\varphi|h_j\rangle\ \left(\mathbf{C}^{-1}\right)_{ji}\ \langle e_i|$$

$$c_i \equiv \sum_{j=1}^{\nu} \langle\varphi|h_j\rangle\ \left(\mathbf{C}^{-1}\right)_{ji}$$

Basis choices

for $i = 1, 2, \ldots, \nu$

- **dLog Basis** $\qquad \langle e_i| = \langle \varphi_i| \equiv \frac{dz}{z - z_i}$ $\qquad z_i$ are *poles* of ω

- **Monomial Basis** $\qquad \langle e_i| = \langle \phi_i| \equiv z^{i-1} dz$

- **Orthonormal Basis**

$\mathcal{P} = \{z_1, z_2, \ldots, z_{\nu+1}, z_{\nu+2}\}$ $\qquad$ pick two special ones, say $z_{\nu+1}$ and $z_{\nu+2}$

$$\langle e_i| \equiv d\log \frac{z - z_i}{z - z_{\nu+1}}, \qquad |h_i\rangle \equiv \mathrm{Res}_{z=z_i}(\omega)\, d\log \frac{z - z_i}{z - z_{\nu+2}}$$

$$\mathbf{C}_{ij} = \delta_{ij} \qquad \langle \varphi| = \sum_{i=1}^{\nu} \langle \varphi | h_i \rangle \langle e_i|$$

- Gram-Schmidt method

- **...or any arbitrary rational basis...**

Dimensional Recurrence Relation

- **MIs in (d+2n) dimensions**

$$J_i^{(d+2n)} \equiv K(d+2n)\, E_i^{(d+2n)} \qquad E_i^{(d+2n)} \equiv \langle B^n e_i | \mathcal{C}] = \int_{\mathcal{C}} u\,(B^n\, e_i)\ , \qquad i = 1, 2, \ldots, \nu$$

- **Master Decomposition Formula**

$$\langle B^\nu e_i | = \sum_{n=0}^{\nu-1} c_n\, \langle B^n e_i | \qquad n = 0, 1, \ldots, \nu - 1$$

- **Recurrence Relations for Master Forms**

$$\sum_{n=0}^{\nu} c_n\, \langle B^n e_i | = 0\ , \qquad c_\nu \equiv -1$$

- **Recurrence Relations for Master Integrals**

$$\sum_{n=0}^{\nu} \alpha_n\, J_i^{(d+2n)} = 0 \qquad \alpha_n \equiv c_n / K(d+2n)$$

System of Differential Equations

- **External Derivative**

$$\partial_x I = \partial_x \langle \varphi | \mathcal{C}] = \partial_x \int_{\mathcal{C}} u\varphi = \int_{\mathcal{C}} u \left(\frac{\partial_x u}{u} \wedge + \partial_x \right) \varphi = \langle (\partial_x + \sigma)\varphi | \mathcal{C}] \qquad \sigma = \partial_x \log u$$

$$\partial_x \langle e_i | = \langle (\partial_x + \sigma \wedge) e_i | \equiv \langle \Phi_i |$$

- **Master Decomposition Formula**

$$\langle \Phi_i | = \langle \Phi_i | h_k \rangle \left(\mathbf{C}^{-1} \right)_{kj} \langle e_j | = \mathbf{\Omega}_{ij} \langle e_j |$$

$$\mathbf{\Omega} \equiv \mathbf{F}\mathbf{C}^{-1} \qquad \mathbf{F}_{ik} \equiv \langle \Phi_i | h_k \rangle$$

The C-matrix is important!

- **System of DEQ for Master Forms**

$$\partial_x \langle e_i | = \mathbf{\Omega}_{ij} \langle e_j | \ , \qquad \mathbf{\Omega} = \mathbf{\Omega}(d, x)$$

System of Differential Equations

- **System of DEQ for Master Integrals**

$J_i \equiv K\, E_i \;, \quad \text{with} \quad E_i \equiv \langle e_i | \mathcal{C}],$

$$\partial_x J_i = \mathbf{A}_{ij}\, J_j \qquad \mathbf{A} \equiv \mathbf{\Omega} + \mathbf{K} \qquad \mathbf{K} = \partial_x \log(K)\, \mathbb{I}$$

- **(Homogenous) Solutions**

For each i, the ν independent solutions

$$\mathbf{P}_{ij} = \langle e_i | \mathcal{C}_j] = \int_{\mathcal{C}_j} u\, e_i \;, \qquad i, j = 1, 2, \ldots, \nu \;,$$

$\nu \times \nu$ matrix $\mathbf{P}$

- Basic math :: Resolvent matrix
- De Rahm int. th. :: (Riemann) Twisted Period matrix

- Example :: Derivative basis

ν-dimensional basis formed by $\langle e_i|$ and its derivatives up the $(\nu - 1)^{\text{th}}$-order

$$\mathbf{P} \;=\; \text{Wronski matrix}$$

Contiguity relations for Special Functions

Euler Beta Integrals

$$I_n \equiv \int_{\mathcal{C}} u\, z^n dz\ , \qquad u \equiv B^\gamma\ , \qquad B \equiv z(1-z)\ , \qquad \mathcal{C} \equiv [0,1]$$

- **Direct Integration**

$$I_n = \frac{\Gamma(1+\gamma)\Gamma(1+\gamma+n)}{\Gamma(2+2\gamma+n)}$$

  - **Integral relation**

    a relation between I_n and I_0

$$I_n = \frac{\Gamma(1+\gamma+n)\Gamma(2+2\gamma)}{\Gamma(1+\gamma)\Gamma(2+2\gamma+n)}\, I_0$$

  - **Special case** $n = 1$

$$I_1 = \frac{1}{2} I_0$$

Euler Beta Integrals

$$I_n \equiv \int_{\mathcal{C}} u\ z^n dz\ , \qquad u \equiv B^\gamma\ , \qquad B \equiv z(1-z)\ , \qquad \mathcal{C} \equiv [0,1]$$

- **IBP identities**

$$\int_{\mathcal{C}} d(B^{\gamma+1} z^{n-1}) = 0$$

$$(\gamma+n) I_{n-1} - (1+2\gamma+n) I_n = 0$$

$$I_n = \frac{(\gamma+n)}{(1+2\gamma+n)}\, I_{n-1}$$

  - Special case $n = 1$ $\qquad I_1 = \frac{1}{2} I_0$

Euler Beta Integrals

- **Intersection Theory**

$$I_n \equiv \int_{\mathcal{C}} u\,\phi_{n+1} \equiv {}_\omega\langle \phi_{n+1}|\mathcal{C}] \ , \qquad \phi_{n+1} \equiv z^n dz$$

$$u = B^\gamma \qquad B = z(1-z) \ , \qquad \omega = d\log u = \gamma\left(\frac{1}{z} + \frac{1}{z-1}\right)dz \qquad \nu = 1, \qquad \mathcal{P} = \{0, 1, \infty\}$$

- **Monomial Basis**

1 master integral $\qquad I_0 = {}_\omega\langle \phi_1|\mathcal{C}]$

  - **Integral relation**

$$I_1 = c_1\ I_0 \qquad \Longleftrightarrow \qquad \langle\phi_2| = c_1\langle\phi_1|$$

$$c_1 = \langle\phi_2|\phi_1\rangle\langle\phi_1|\phi_1\rangle^{-1}$$

  - **Master Decomposition Formula**

$\mathbf{C}_{ij}$ has just one element $\mathbf{C}_{11} = \langle\phi_1|\phi_1\rangle$

$$\langle\phi_1|\phi_1\rangle = \mathrm{Res}_{z=\infty}(\psi_\infty\phi_1) = \frac{\gamma}{2(2\gamma-1)(2\gamma+1)}$$

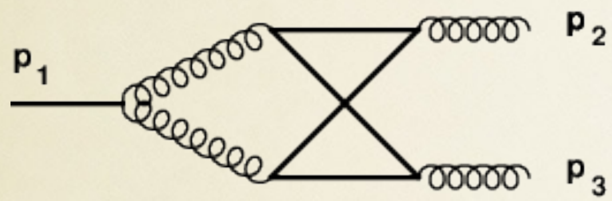
$$\langle\phi_2|\phi_1\rangle = \mathrm{Res}_{z=\infty}(\psi_\infty\phi_1) = \frac{\gamma}{4(2\gamma-1)(2\gamma+1)}$$

$$c_1 = \frac{1}{2}$$

Feynman Integrals Decomposition
:: on the maximal cut :: 1-forms

- On the maximal cut :: simpler integrals
- 1-forms :: univariate integral representations
- Operation required :: Intersection Numbers for 1-forms

Two-Loop Non-Planar Triangle



$D_1 = k_1^2\,,\; D_2 = k_2^2 - m^2\,,\; D_3 = (p_1 - k_1)^2\,,\; D_4 = (p_3 - k_1 + k_2)^2 - m^2,$
$D_5 = (k_1 - k_2)^2 - m^2\,,\; D_6 = (p_2 - k_2)^2 - m^2\,.$

$$z = D_7 = 2(p_2 + k_1)^2 - p_1^2$$

$$u = B^\gamma\,, \qquad B = \left(z^2 - \tau_1^2\right)\left(z^2 - \tau_2^2\right)\,, \qquad \tau_1 = s\sqrt{1 + (4m)^2/s}\,, \qquad \tau_2 = s\,,$$

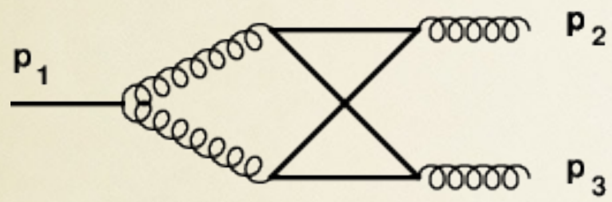
$$\gamma = \frac{d-5}{2} \qquad \omega = \frac{2\gamma z\left(2z^2 - \tau_1^2 - \tau_2^2\right)}{\left(z^2 - \tau_1^2\right)\left(z^2 - \tau_2^2\right)} dz\,, \qquad \nu = 3\,, \qquad \mathcal{P} = \{-\tau_1, -\tau_2, \tau_2, \tau_1, \infty\}$$

dlog-basis. $\varphi_1 = \left(\frac{1}{\tau_1 + z} - \frac{1}{\tau_2 + z}\right) dz\,, \qquad \varphi_2 = \left(\frac{1}{\tau_2 + z} - \frac{1}{z - \tau_2}\right) dz\,, \qquad \varphi_3 = \left(\frac{1}{z - \tau_2} - \frac{1}{z - \tau_1}\right) dz\,,$

$$\mathbf{C} = \begin{pmatrix} \langle\varphi_1|\varphi_1\rangle & \langle\varphi_1|\varphi_2\rangle & \langle\varphi_1|\varphi_3\rangle \\ \langle\varphi_2|\varphi_1\rangle & \langle\varphi_2|\varphi_2\rangle & \langle\varphi_2|\varphi_3\rangle \\ \langle\varphi_3|\varphi_1\rangle & \langle\varphi_3|\varphi_2\rangle & \langle\varphi_3|\varphi_3\rangle \end{pmatrix} = \frac{1}{\gamma}\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \qquad \mathbf{C}^{-1} = \gamma \begin{pmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{pmatrix}$$

the projection of $\phi_1 = dz$ is $\langle\phi_1| = \frac{\gamma\tau_1}{4\gamma + 1}\langle\varphi_1| + \frac{\gamma(\tau_1 + \tau_2)}{4\gamma + 1}\langle\varphi_2| + \frac{\gamma\tau_1}{4\gamma + 1}\langle\varphi_3|$ verified with REDUZE.

Two-Loop Non-Planar Triangle



System of Differential Equations $\quad x \equiv \frac{\tau_1}{\tau_2} \qquad \sigma(x) = \partial_x \log\left(B(z,x)^\gamma\right) = -\frac{2\gamma\tau_2^2 x}{z^2 - \tau_2^2 x^2}.$

$$\langle \Phi_i(x)| \equiv \langle (\partial_x + \sigma(x))\varphi_i|$$

$$\langle \Phi_1(x)| = -\frac{\tau_2\left(2\gamma\tau_2^2x^2 - 2\gamma\tau_2^2x + \tau_2^2x + \tau_2 xz - z^2 - \tau_2 z\right)}{(\tau_2+z)(\tau_2x-z)(\tau_2x+z)^2}dz,$$

$$\langle \Phi_2(x)| = \frac{4\gamma\tau_2^3x}{(\tau_2-z)(\tau_2+z)(\tau_2x-z)(\tau_2x+z)}dz,$$

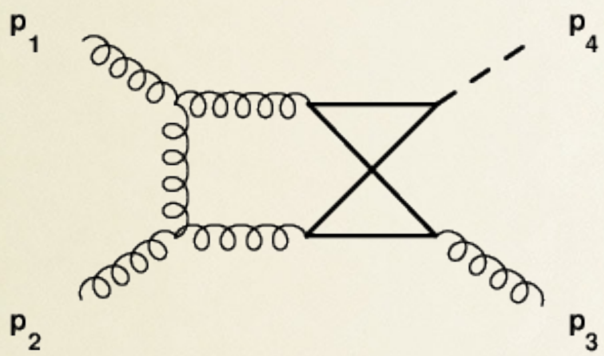
$$\langle \Phi_3(x)| = -\frac{\tau_2\left(2\gamma\tau_2^2x^2 - 2\gamma\tau_2^2x + \tau_2^2x - \tau_2 xz - z^2 + \tau_2 z\right)}{(\tau_2-z)(\tau_2x-z)^2(\tau_2x+z)}dz.$$

$$\mathbf{F}_{ij} = \langle \Phi_i|\varphi_j\rangle \qquad \mathbf{F} = \begin{pmatrix} \frac{7x^2+2x-1}{(x-1)x(x+1)} & -\frac{2}{x-1} & -\frac{x-1}{x(x+1)} \\ -\frac{2}{x-1} & \frac{4x}{(x-1)(x+1)} & -\frac{2}{x-1} \\ -\frac{x-1}{x(x+1)} & -\frac{2}{x-1} & \frac{7x^2+2x-1}{(x-1)x(x+1)} \end{pmatrix}$$

$$\mathbf{\Omega} = \mathbf{F}\mathbf{C}^{-1} = \gamma\begin{pmatrix} \frac{4x^2+x-1}{(x-1)x(x+1)} & \frac{1}{x} & \frac{1}{x(x+1)} \\ -\frac{2}{(x-1)(x+1)} & \frac{2}{x+1} & -\frac{2}{(x-1)(x+1)} \\ \frac{1}{x(x+1)} & \frac{1}{x} & \frac{4x^2+x-1}{(x-1)x(x+1)} \end{pmatrix}$$

- Canonical

Non-Planar Contribution to H+j Production



Loop-by-Loop form of the Baikov representation

$$D_1 = k_1^2\,,\quad D_2 = (k_1+p_1)^2\,,\quad D_3 = (k_1-p_3-p_4)^2\,,$$
$$D_4 = (k_2-p_3)^2 - m_t^2 \quad D_5 = k_2^2 - m_t^2\,,\quad D_6 = (k_1-k_2)^2 - m_t^2\,,$$
$$D_7 = (k_1-k_2-p_4)^2 - m_t^2\,.$$

$$z = D_8 = (k_1-p_3)^2$$

$$D_9 = (k_2+p_1)^2$$

$$u = \frac{\left(-m_H^2+s+t+z\right)^{d-5}\left(z\left(m_H^2-s-z\right)+4sm_t^2\right)^{\frac{d-5}{2}}}{\sqrt{z\left(-m_H^2+s+z\right)}}\,,$$

$$\omega = \frac{q_0+q_1\,z+q_2\,z^2+q_3\,z^3+q_4\,z^4}{2z\left(-m_H^2+s+z\right)\left(-m_H^2+s+t+z\right)\left(z\left(-m_H^2+s+z\right)-4sm_t^2\right)}\,dz\,,$$

$$\nu = 4\,,$$

$$\mathcal{P} = \{0,\, m_H^2-s,\, \tfrac{1}{2}(m_H^2-s-\rho),\, \tfrac{1}{2}(m_H^2-s+\rho),\, m_H^2-s-t,\, \infty\}\,,\qquad \rho = \sqrt{m_H^4-2sm_H^2+16sm_t^2+s^2}\,.$$

Mixed Bases $J_1 = I_{1,1,1,1,1,1,1;0} = \langle e_1|\mathcal{C}]$, $J_2 = I_{1,2,1,1,1,1,1:0} = \langle e_2|\mathcal{C}]$, $J_3 = I_{1,1,1,2,1,1,1;0} = \langle e_3|\mathcal{C}]$ and $J_4 = I_{1,1,1,1,2,1,1;0} = \langle e_4|\mathcal{C}]$,

$$\hat{e}_1 = 1\,,$$
$$\hat{e}_2 = \frac{(d-5)\left(m_H^4 - m_H^2(2s+t+z)+s^2+s(t+z)+2tz\right)}{s(-m_H^2+s+t+z)^2}\,,$$
$$\hat{e}_3 = \frac{(d-5)(s+z)}{z(m_H^2-s-z)+4sm_t^2}\,,$$
$$\hat{e}_4 = \frac{(d-5)(m_H^2-z)}{z(m_H^2-s-z)+4sm_t^2}\,.$$

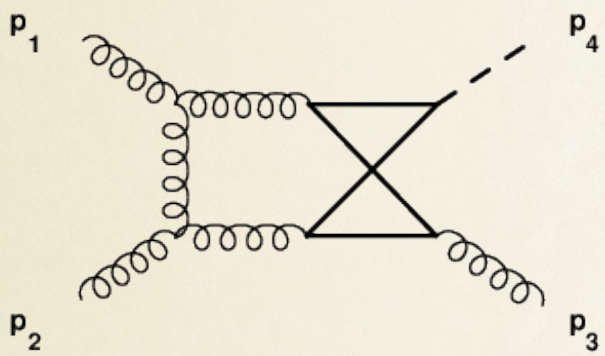
$$\hat{\varphi}_1 = \frac{1}{z} - \frac{1}{-m_H^2+s+z}\,,$$
$$\hat{\varphi}_2 = \frac{1}{-m_H^2+s+z} - \frac{1}{\frac{1}{2}\left(-m_H^2+\rho+s\right)+z}\,,$$
$$\hat{\varphi}_3 = \frac{1}{\frac{1}{2}\left(-m_H^2+\rho+s\right)+z} - \frac{1}{\frac{1}{2}\left(-m_H^2-\rho+s\right)+z}\,,$$
$$\hat{\varphi}_4 = \frac{1}{\frac{1}{2}\left(-m_H^2-\rho+s\right)+z} - \frac{1}{-m_H^2+s+t+z}\,.$$

$$\mathbf{C}_{ij} = \langle e_i|\varphi_j\rangle\;,\quad 1\le i,j\le 4,$$

$$\langle\varphi| = \sum_{i,j=1}^{\nu}\langle\varphi|h_j\rangle\left(\mathbf{C}^{-1}\right)_{ji}\langle e_i|$$

$$I_{1,1,1,1,1,1,1;-1} = c_1\,J_1 + c_2\,J_2 + c_3\,J_3 + c_4\,J_4$$

Non-Planar Contribution to $H+j$ Production



Loop-by-Loop form of the Baikov representation

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$$u = \frac{\left(-m_H^2+s+t+z\right)^{d-5}\left(z\left(m_H^2-s-z\right)+4sm_t^2\right)^{\frac{d-5}{2}}}{\sqrt{z\left(-m_H^2+s+z\right)}}\,,$$

$$\omega = \frac{q_0+q_1\,z+q_2\,z^2+q_3\,z^3+q_4\,z^4}{2z\left(-m_H^2+s+z\right)\left(-m_H^2+s+t+z\right)\left(z\left(-m_H^2+s+z\right)-4sm_t^2\right)}\,dz\,, \qquad \nu = 4\,,$$

$$\mathcal{P} = \{0,\, m_H^2-s,\, \tfrac{1}{2}(m_H^2-s-\rho),\, \tfrac{1}{2}(m_H^2-s+\rho),\, m_H^2-s-t,\, \infty\}\,, \qquad \rho = \sqrt{m_H^4-2sm_H^2+16sm_t^2+s^2}\,.$$

Mixed Bases $J_1 = I_{1,1,1,1,1,1,1;0} = \langle e_1|\mathcal{C}]$, $J_2 = I_{1,2,1,1,1,1,1:0} = \langle e_2|\mathcal{C}]$, $J_3 = I_{1,1,1,2,1,1,1;0} = \langle e_3|\mathcal{C}]$ and $J_4 = I_{1,1,1,1,2,1,1;0} = \langle e_4|\mathcal{C}]$,

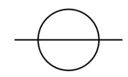
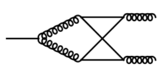
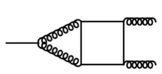
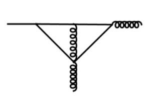
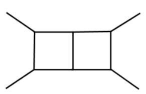
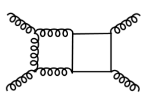
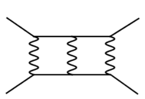
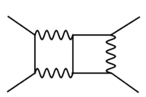
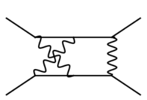
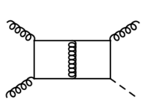
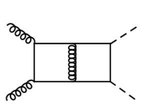
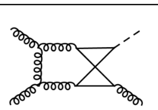
Checks. KIRA, leaves us with 6 MIs, 2 more: $J_5 = I_{1,1,1,1,1,2,1;0,0}\,,\qquad J_6 = I_{1,1,2,1,1,1,1;0,0}\,.$

- **Higher sectors IBPs** $J_6 = \dfrac{10-2d}{s}J_1 + \dfrac{(2m_t^2-m_H^2)s+m_H^4}{m_H^2 s}J_3 + \dfrac{2m_t^2}{s}J_4 + \dfrac{s(m_H^2-2m_t^2)+2m_H^2m_t^2}{m_H^2 s}J_5\,.$ (on the cut)
- **Self similarity** $k_1 \to -k_1-p_1-p_2\,,\quad k_2 \to -k_2+p_3\,,\quad p_1 \leftrightarrows p_2\,,$ $J_5 = \dfrac{s}{m_H^2+s}J_3 - \dfrac{m_H^2}{m_H^2+s}J_4$ (on the cut)

after using these 2 extra relations KIRA is in perfect agreement

$\nu = 4$ verified with a numerical evaluation of the integrals on the maximal cut + PSLQ [80 digits]

Other Applications :: *proof of concepts*

Integral family	Sec.	ν_{LBL}	ν_{std}
	7	1	1
	8	3	3
	9	1	1
	10	2	1
	11	2	2
	12	3	4
	13.1	2	2
	13.2	3	4
	13.3	3	4
	14.1	4	4
	14.1	4	4
	14.2	4	6

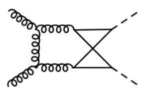
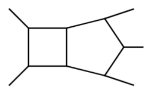
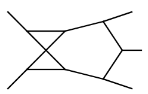
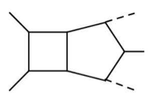
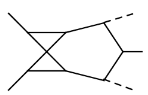
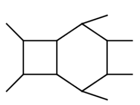
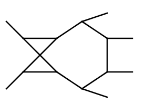
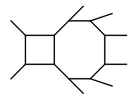
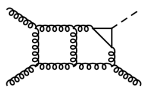
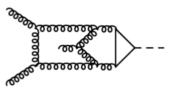
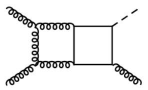
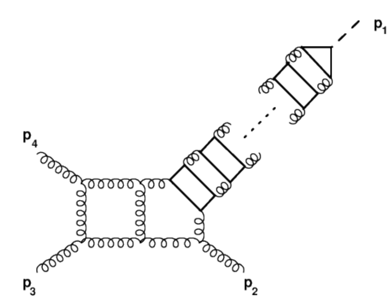
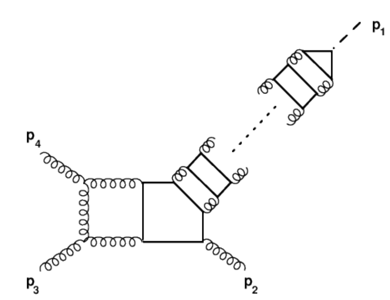
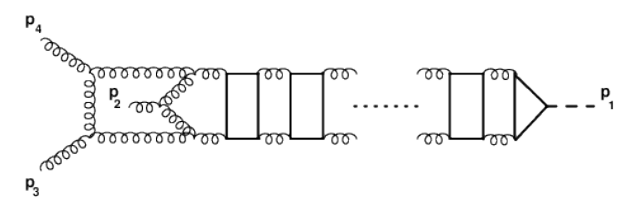
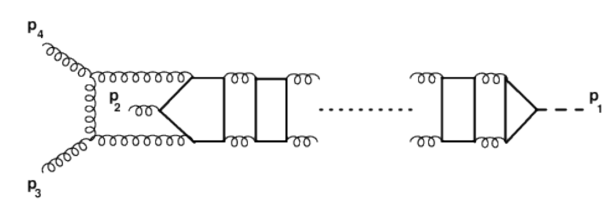
Integral family	Sec.	ν_{LBL}	ν_{std}
	14.3	4	6
	15.1	3	3
	15.2	3	3
	16	3	3
	16	3	3
	16	3	3
	16	3	3
	16.1	3	3
	17.1	2	2
	17.2	3	3
	17.3	3	4

Table 1: Comparisons of the number of masters obtained by the LP criterion, from Loop-by-Loop (ν_{LBL}) and standard Baikov parametrization (ν_{std}).







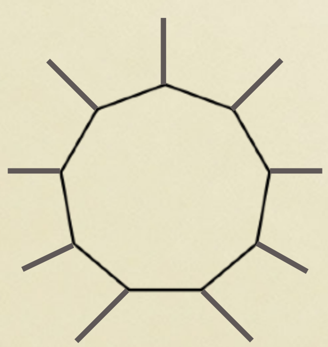


Feynman Integrals Decomposition
:: n-forms ::

Frellesvig, Gasparotto, Mandal, Mattiazzi, Mizera **& P.M.**
arXiv:1907.02000

- **n-forms :: n-variable integral representations**
- Operation required :: **Intersection Numbers for n-forms**
- **n** ***steps*** **down in the decomposition**

- **1-loop Nonagon**

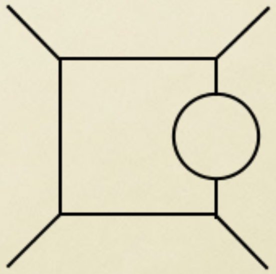


- **Int. Num. for 8-forms**

$$N = LE + \frac{1}{2}L(L+1)$$

$$\int_{\mathcal{C}} dz_1 \wedge \cdots \wedge dz_9 \; \frac{B(\mathbf{z})^\gamma}{z_1^{n_1} \cdots z_9^{n_9}}$$

- **2-loop Box**



- **Int. Num. for 7-forms**

n-Forms

$$I = \int_{\mathcal{C}} u(\mathbf{z})\, \varphi(\mathbf{z}) \qquad \varphi(\mathbf{z}) = \hat{\varphi}(\mathbf{z})\, d^n\mathbf{z}\ , \qquad d^n\mathbf{z} \equiv dz_1 \wedge \ldots \wedge dz_n$$

- **Number of Master Integrals** $\nu \equiv \dim H^n_{\pm\omega}$

1) Counting Critical Points

Lee, Pomeransky (2013)

$$\omega \equiv d\log u(\mathbf{z}) = \sum_{i=1}^{n} \hat{\omega}_i\, dz_i$$

$\nu \equiv$ number of solutions of the system of equations

$$\hat{\omega}_i \equiv \partial_{z_i} \log u(\mathbf{z}) = 0\,, \qquad i = 1, \ldots, n$$

2) Euler Characteristics

Aluffi, Marcolli (2008)

Bitoun, Bogner, Klausen, Panzer (2017)

$$\nu \ = (-1)^n\, (n{+}1 - \chi(\mathcal{P}_\omega))$$

in terms of the Euler characteristic $\chi(\mathcal{P}_\omega)$ of the projective variety $\mathcal{P}_\omega$ defined as the set of poles of ω.

Frellesvig, Gasparotto, Mandal, Mattiazzi, Mizera **& P.M.** (2019)

Multivariate Intersection Numbers

Mizera (2019)

- **(n-1)-form Vector Space: known!**

Frellesvig, Gasparotto, Mandal, Mattiazzi, Mizera **& P.M.** (2019)

$$\nu_{\mathbf{n-1}} \qquad \langle e_i^{(\mathbf{n-1})}| \qquad |h_i^{(\mathbf{n-1})}\rangle \qquad \left(\mathbf{C}_{(\mathbf{n-1})}\right)_{ij} \equiv {}_{\mathbf{n-1}}\langle e_i^{(\mathbf{n-1})}|h_j^{(\mathbf{n-1})}\rangle$$

- **n-form decomposition: n = (n-1) + (n)**

$$\langle \varphi_L^{(\mathbf{n})}| = \sum_{i=1}^{\nu_{\mathbf{n-1}}} \langle e_i^{(\mathbf{n-1})}| \wedge \langle \varphi_{L,i}^{(n)}| \, , \qquad |\varphi_R^{(\mathbf{n})}\rangle = \sum_{i=1}^{\nu_{\mathbf{n-1}}} |h_i^{(\mathbf{n-1})}\rangle \wedge |\varphi_{R,i}^{(n)}\rangle \, ,$$

Intersection Numbers for n-forms :: Recursive Formula (I)

$${}_{\mathbf{n}}\langle \varphi_L^{(\mathbf{n})}|\varphi_R^{(\mathbf{n})}\rangle = -\sum_{p\in\mathcal{P}_n} \underset{z_n=p}{\mathrm{Res}} \left({}_{\mathbf{n-1}}\langle \varphi_L^{(\mathbf{n})}|h_i^{(\mathbf{n-1})}\rangle \, \psi_i^{(n)} \right)$$

Mizera (2019)

$$\nabla_{-\Omega^{(n)}} \vec{\psi}^{(n)} = \vec{\varphi}_R^{(n)}$$

$$\partial_{z_n} \langle e_i{}^{(\mathbf{n-1})}| = \Omega_{ij}^{(n)} \langle e_i{}^{(\mathbf{n-1})}|$$

Multivariate Intersection Numbers

Mizera (2019)

- **(n-1)-form Vector Space: known!**

Frellesvig, Gasparotto, Mandal, Mattiazzi, Mizera **& P.M.** (2019)

$$\nu_{\mathbf{n-1}} \qquad \langle e_i^{(\mathbf{n-1})}| \qquad |h_i^{(\mathbf{n-1})}\rangle \qquad \left(\mathbf{C}_{(\mathbf{n-1})}\right)_{ij} \equiv {}_{\mathbf{n-1}}\langle e_i^{(\mathbf{n-1})}|h_j^{(\mathbf{n-1})}\rangle$$

- **n-form decomposition: n = (n-1) + (n)**

$$\langle \varphi_L^{(\mathbf{n})}| = \sum_{i=1}^{\nu_{\mathbf{n-1}}} \langle e_i^{(\mathbf{n-1})}| \wedge \langle \varphi_{L,i}^{(n)}| \,, \qquad |\varphi_R^{(\mathbf{n})}\rangle = \sum_{i=1}^{\nu_{\mathbf{n-1}}} |h_i^{(\mathbf{n-1})}\rangle \wedge |\varphi_{R,i}^{(n)}\rangle \,,$$

Intersection Numbers for n-forms :: Recursive Formula (II)

$${}_{\mathbf{n}}\langle \varphi_L^{(\mathbf{n})}|\varphi_R^{(\mathbf{n})}\rangle = (-1)^n \sum_{p_n \in \mathcal{P}_n} \cdots \sum_{p_1 \in \mathcal{P}_1} \operatorname*{Res}_{z_n = p_n} \cdots \operatorname*{Res}_{z_1 = p_1} \left(\varphi_L^{(\mathbf{n})} \psi_{1 i_1}^{(1)} \psi_{i_1 i_2}^{(2)} \cdots \psi_{i_{\mathbf{n-2}} i_{\mathbf{n-1}}}^{(n-1)} \psi_{i_{\mathbf{n-1}}}^{(n)} \right)$$

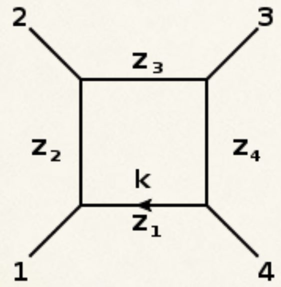
Frellesvig, Gasparotto, Mandal, Mattiazzi, Mizera **& P.M.** (2019)

$$\nabla_{-\Omega^{(n)}} \psi_{i_{\mathbf{m}} i_{\mathbf{m-1}}}^{(n)} = \hat{h}_{i_{\mathbf{m}} i_{\mathbf{m-1}}}^{(n)} \qquad |h_{i_{\mathbf{m}}}^{(\mathbf{m})}\rangle = |h_{i_{\mathbf{m-1}}}^{(\mathbf{m-1})}\rangle \wedge |h_{i_{\mathbf{m-1}} i_{\mathbf{m}}}^{(m)}\rangle$$

$$\partial_{z_n} \langle e_i{}^{(\mathbf{n-1})}| = \Omega_{ij}^{(n)} \langle e_i{}^{(\mathbf{n-1})}|$$

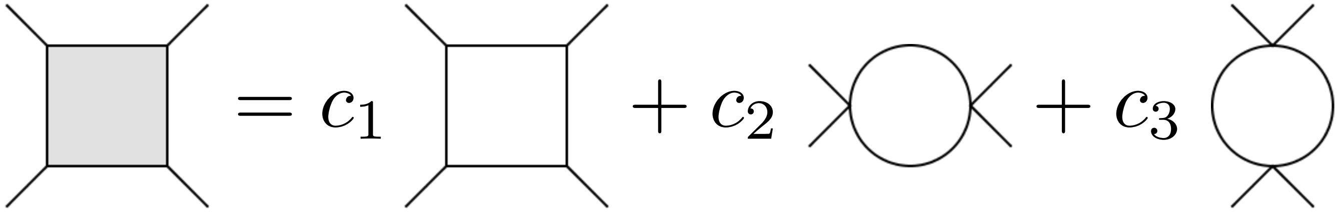
Feynman Integrals Reduction

Massless Box

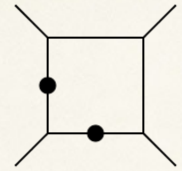


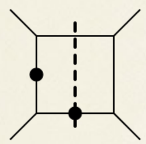
$$u(\mathbf{z}) = \left((st - sz_4 - tz_3)^2 - 2tz_1(s(t+2z_3-z_2-z_4)+tz_3) + s^2z_2^2 + t^2z_1^2 - 2sz_2(t(s-z_3)+z_4(s+2t))\right)^{\frac{d-5}{2}}$$

- **Integral Decomposition**



Example.

 $= \int_{\mathcal{C}} \frac{u \, d^4\mathbf{z}}{z_1^2 \, z_2^2 \, z_3 \, z_4}$

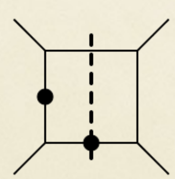
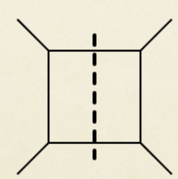
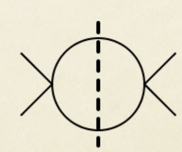
- **Cut$_{\{1,3\}}$:**  $= \int_{\mathcal{C}} u_{1,3} \, \varphi_{1,3} \,, \qquad \varphi_{1,3} = \hat{\varphi}_{1,3} \, dz_2 \wedge dz_4 \,, \qquad \hat{\varphi}_{1,3} = \frac{\hat{\omega}_1}{z_2^2 z_4}$

$$u_{1,3} = z_2^{\rho_2} z_4^{\rho_4} \, u(0, z_2, 0, z_4)$$

$$\nu_{(24)} = 2 \qquad \hat{e}_1^{(24)} = \hat{h}_1^{(24)} = \frac{1}{z_2 z_4}, \qquad \hat{e}_2^{(24)} = \hat{h}_2^{(24)} = 1 \,,$$

$$\nu_{(4)} = 2 \qquad \hat{e}_1^{(4)} = \hat{h}_1^{(4)} = \frac{1}{z_4}, \qquad \hat{e}_2^{(4)} = \hat{h}_2^{(4)} = 1$$

Integral Decomposition

 $= c_1$  $+ \, c_2$  ,

$$c_1 = \sum_{j=1}^{2} \langle \varphi_{1,3} | h_j^{(24)} \rangle \left(\mathbf{C}_{(24)}^{-1} \right)_{j1} = \frac{(d-6)(d-5)}{st} \,,$$

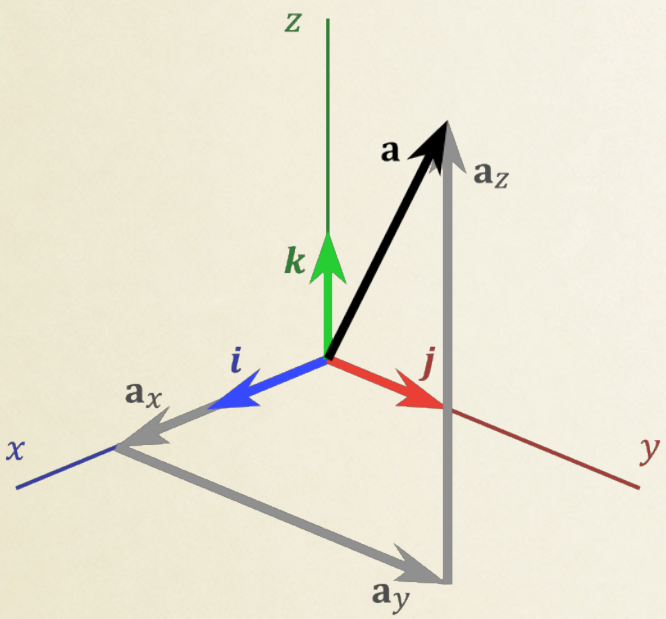
$$c_2 = \sum_{j=1}^{2} \langle \varphi_{1,3} | h_j^{(24)} \rangle \left(\mathbf{C}_{(24)}^{-1} \right)_{j2} = -\frac{4(d-5)(d-3)}{s^3 t} \,.$$

in agreement with the IBP decomposition.

To Conclude:

Amplitudes Decomposition:

the algebraic way



$\mathbf{a} = a_x\,\mathbf{i} + a_y\,\mathbf{j} + a_z\,\mathbf{k}$

- **Basis**: {**i j k**}

- **Scalar product/Projection**: to extract the components

  $a_x = \mathbf{a}.\mathbf{i}$ $a_y = \mathbf{a}.\mathbf{j}$ $a_z = \mathbf{a}.\mathbf{k}$

Summary

- **Novel Math for Quantum Field Theory**
  - De Rahm (co)Homology and Intersection Theory
  - Rich theory :: Differential and Algebraic Geometry, Topology, Number Theory
- **Novel Property Discovered**
  - The **algebra** of Feynman Integrals (and not only) is controlled by **Intersection Numbers**
  - Intersection Numbers ~ Scalar Product/Projection between Feynman Integrals
  - Exploiting the geometric properties of the integrands, dictated by graph polynomials
- **Novel Formulae for Multivariate Intersection number**
  - Useful in Physics and Math
- **(towards a) Novel Decomposition Method**
  - Direct **decomposition** into a Integral Basis
  - Direct construction of system of **differential equations** for the Integral Basis
  - Direct construction of **finite difference equations** for the Integral Basis

The unreasonable effectiveness of mathematics

E. Wigner

Wigner was referring to the mysterious phenomenon in which areas of pure mathematics, originally constructed without regard to application, are suddenly discovered to be exactly what is required to describe the structure of the physical world.

M. Berry

Extra

Gauss ${}_2F_1$ Hypergeometric Functions

$$\beta(b, c-b)\ {}_2F_1(a,b,c;x) = \int_0^1 z^{b-1}(1-z)^{c-b-1}(1-xz)^{-a}\, dz$$

$$= \int_{\mathcal{C}} u\,\varphi = {}_\omega\langle\varphi|\mathcal{C}] \qquad u = z^{b-1}(1-xz)^{-a}(1-z)^{-b+c-1}\,, \qquad \varphi = dz$$

$$\omega = d\log u = \frac{xz^2(c-a-2) + z(ax-c+x+2) - bxz + b - 1}{(z-1)z(xz-1)}\, dz\,, \qquad \nu = 2\,, \qquad \mathcal{P} = \{0,\ 1,\ \tfrac{1}{x},\ \infty\}$$

Gauss ${}_2F_1$ Hypergeometric Functions

- **Monomial Basis** $\{\langle\phi_i|\}_{i=1,2} \qquad \phi_{n+1} \equiv z^n dz$

  - **Metric** $$\mathbf{C} = \begin{pmatrix} \langle\phi_1|\phi_1\rangle & \langle\phi_1|\phi_2\rangle \\ \langle\phi_2|\phi_1\rangle & \langle\phi_1|\phi_2\rangle \end{pmatrix}$$

$$\langle\phi_1|\phi_1\rangle = \Big(x^2(-(a-b+1))(b-c+1) - 2ax(-b+c-1) + a(c-2)\Big) \,/\, \Big(x^2(a-c+1)(a-c+2)(a-c+3)\Big),$$

$$\langle\phi_1|\phi_2\rangle = \Big(x^3(-(a-b+1))(a-b+2)(b-c+1) - ax^2(-b+c-1)(2a-3b+c+2) + ax(a+2c-5)(-b+c-1) - a(c-3)(c-2)\Big) \,/\, \Big(x^3(a-c+1)(a-c+2)(a-c+3)(a-c+4)\Big),$$

$$\langle\phi_2|\phi_1\rangle = \Big(x^3(-(a-b))(a-b+1)(b-c+1) - ax^2(-b+c-1)(2a-3b+c) + ax(a+2c-3)(-b+c-1) - a(c-2)(c-1)\Big) \,/\, \Big(x^3(a-c)(a-c+1)(a-c+2)(a-c+3)\Big),$$

$$\langle\phi_2|\phi_2\rangle = \Big(-ax^2(a^2b - a^2c + a^2 - 3ab^2 + 7abc - 8ab - 4ac^2 + 9ac - 5a - 3b^2c + 6b^2 + 4bc^2 - 10bc + 6b - c^3 + 2c^2 - c) + x^4(-(a^3 - 3a^2b + 3a^2 + 3ab^2 - 6ab + 2a - b^3 + 3b^2 - 2b))(b-c+1)$$
$$+ 2ax^3(a-b+1)(ab - ac + a - 2b^2 + 3bc - 2b - c^2 + c) + 2a(c-2)x(a+c-2)(b-c+1) + a(c^3 - 6c^2 + 11c - 6)\Big) \,/\, \Big(x^4(a-c)(a-c+1)(a-c+2)(a-c+3)(a-c+4)\Big).$$

- **Master Decomposition Formula** $$\langle\phi_n| = \sum_{i,j=1}^{2} \langle\phi_n|\phi_j\rangle \left(\mathbf{C}^{-1}\right)_{ji} \langle\phi_i|$$

- **Gauss' contiguity relation**

$$\langle\phi_3|\mathcal{C}] \equiv \beta(b+2, c-b)\,{}_2F_1(a, b+2, c+2; x)$$

$$= \left(\frac{b}{x(a-c-1)}\right)\beta(b, c-b)\,{}_2F_1(a, b, c; x) + \left(\frac{(b-a+1)x + c}{x(c-a+1)}\right)\beta(b+1, c-b)\,{}_2F_1(a, b+1, c+1; x)$$

Gauss ${}_2F_1$ Hypergeometric Functions

- **dLog Basis** $\varphi_1 = \left(\frac{1}{z} - \frac{1}{z-1}\right) dz \qquad \varphi_2 = \left(\frac{1}{z-1} - \frac{x}{xz-1}\right) dz.$

$$I_1 = \langle \varphi_1 | \mathcal{C}] = {}_2F_1(a, b-1, c-2; x)\,, \qquad I_2 = \langle \varphi_2 | \mathcal{C}] = \frac{(b-1)(x-1)}{c-2} {}_2F_1(a+1, b, c-1; x)$$

$$\mathbf{C}_{ij} = \langle \varphi_i | \varphi_j \rangle \qquad \mathbf{C} = \frac{1}{c-b-1} \begin{pmatrix} \frac{c-2}{b-1} & -1 \\ -1 & \frac{a+b-c+1}{a} \end{pmatrix}$$

- **Canonical System of Differential Equations**

$a = -\gamma,\, b = \gamma{+}1,\, c = 2(\gamma{+}1)$

$$\partial_x I_i = \mathbf{A}_{ij} I_j \qquad \mathbf{A} = \gamma \begin{pmatrix} 0 & \frac{-1}{x-1} \\ \frac{-1}{x} & \frac{2}{x-1} - \frac{2}{x} \end{pmatrix}$$

Appell F_1 Functions

$$\beta(a, c-a)\ F_1(a, b_1, b_2, c; x, y) = \int_{\mathcal{C}} z^{a-1}(1-z)^{-a+c-1}(1-xz)^{-b_1}(1-yz)^{-b_2}\, dz$$

$$= \int_{\mathcal{C}} u\,\varphi = \ {}_{\omega}\langle\varphi|\mathcal{C}] \qquad\qquad \mathcal{C} = [0,1]$$

$$u = z^{a-1}(1-z)^{-a+c-1}(1-xz)^{-b_1}(1-yz)^{-b_2},$$

$$\omega = \left(\frac{-a+c-1}{z-1} + \frac{a-1}{z} - \frac{b_1 x}{xz-1} - \frac{b_2 y}{yz-1}\right) dz, \qquad \nu = 3, \qquad \mathcal{P} = \left\{0,\ 1,\ \frac{1}{x},\ \frac{1}{y},\ \infty\right\}$$

- **dLog Basis** $\quad \varphi_1 = \left(\frac{1}{z} - \frac{1}{z-1}\right) dz, \qquad \varphi_2 = \left(\frac{1}{z-1} - \frac{x}{xz-1}\right) dz, \qquad \varphi_3 = \left(\frac{x}{xz-1} - \frac{y}{yz-1}\right) dz$

$$\mathbf{C} = \frac{1}{c-a-1}\begin{pmatrix} \frac{c-2}{a-1} & -1 & 0 \\ -1 & \frac{a-c+b_1+1}{b_1} & \frac{-a+c-1}{b_1} \\ 0 & \frac{-a+c-1}{b_1} & \frac{(a-c+1)(b_1+b_2)}{b_1 b_2} \end{pmatrix}$$

Lauricella F_D Functions

$$\beta(a, c-a)\, F_D(a, b_1, b_2, \ldots, b_m, c; x_1, \ldots, x_m) = \int_{\mathcal{C}} u\, \varphi = \ {}_\omega\langle \varphi | \mathcal{C}]$$

$$u = z^{a-1}\,(1-z)^{-a+c-1} \prod_{i=1}^{m} (1 - x_i z)^{-b_i}\,,$$

$$\mathcal{C} = [0,1], \qquad \varphi = dz\,, \qquad \omega = d\log(u),$$

$$\nu = m+1, \qquad \mathcal{P} = \left\{0, \frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_m}, 1, \infty\right\}$$

$\nu = \dim H^1_{\pm\omega}$ = [number of P-poles - 2] = [number of P-poles - (1+1)]

Is this relation accidental?

(other) **Parametric Representations:**

- **Schwinger Parameterization**
- **Lee-Pomeransky Parameterization**

Frellesvig, Gasparotto, Mandal, Mattiazzi,
Mizera, Ossola, Sameshima **& P.M.**
(in progress)

Gamma Function :: 1-variate InterX

$$\Gamma(s) = \int_{x=0}^{\infty} x^{s-1} e^{-x} dx.$$

$$u(x) := x^{s-1} e^{-x} \qquad C := [0, \infty]$$

$$\omega := d \ln u \ = \left(\frac{s-1}{x} - 1 \right) dx \qquad \nu = 1 \qquad P = \{0, \infty\}$$

$$I(n) := \int_C u\, \phi_n := \langle \phi_n | C] \ , \qquad \phi_n := x^n \, dx$$

$$\phi_0 = 1dx \qquad I(0) := \langle \phi_0 | \, C] \qquad \langle \phi_0 | \, \phi_0 \rangle = s - 1$$

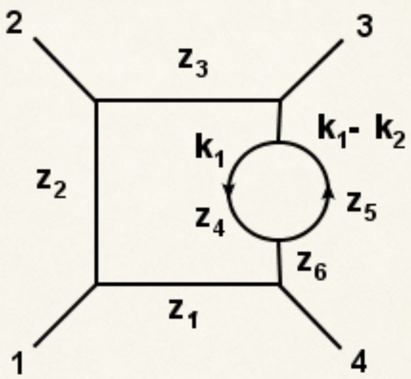
$$\phi_1 = xdx \qquad I(1) := \langle \phi_1 | \, C] \qquad \langle \phi_1 | \, \phi_0 \rangle = s(s-1).$$

- Master Decomposition Formula

$$\langle \phi_1 | = \langle \phi_1 | \, \phi_0 \rangle \, \langle \phi_0 | \, \phi_0 \rangle^{-1} \, \langle \phi_0 | \ = s \, \langle \phi_0 | \qquad \Longleftrightarrow \qquad I(1) = sI(0)$$

$$\Gamma(s+1) = s\Gamma(s).$$

Box w/ self-energy

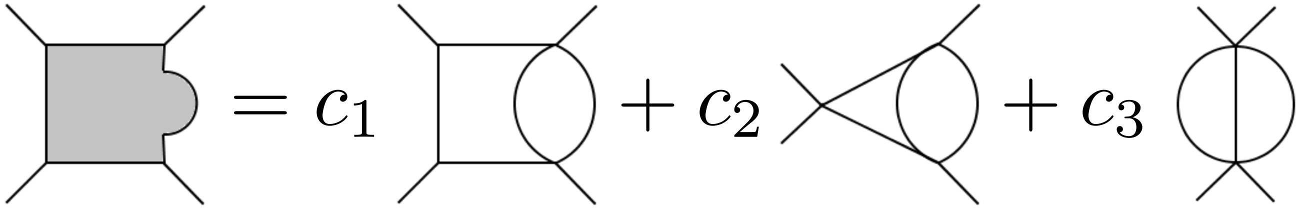


$$u(\mathbf{z}) = \mathcal{B}_1^{\frac{2-d}{2}} \mathcal{B}_2^{\frac{d-3}{2}} \mathcal{B}_3^{\frac{d-5}{2}} \,,$$

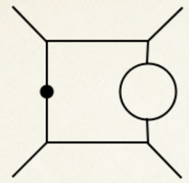
$$\mathcal{B}_1 = z_6 \,, \quad \mathcal{B}_2 = 2(z_5+z_6)z_4 - z_4^2 - (z_5-z_6)^2 \,,$$

$$\mathcal{B}_3 = t^2 z_1^2 + s^2 z_2^2 - 2tz_1((2s+t)z_3+s(t-z_2-z_6)) - 2sz_2(st-tz_3+(s+2t)z_6) + (tz_3+s(z_6-t))^2$$

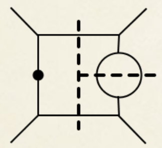
- **Integral Decomposition**



Example.

 $$= \int_{\mathcal{C}} \frac{u\, d^6\mathbf{z}}{z_1 z_2^2 z_3 z_4 z_5 z_6^2}$$

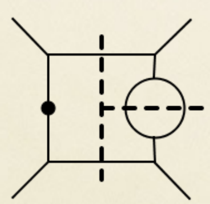
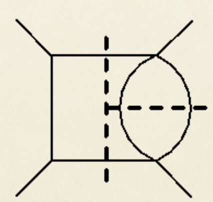
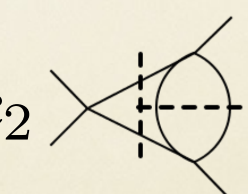
- **Cut**$_{\{\mathbf{1,3,4,5}\}}$:

 $$= \int_{\mathcal{C}} u_{1,3,4,5}\, \varphi_{1,3,4,5}\,, \qquad \varphi_{1,3,4,5} = \hat{\varphi}_{1,3,4,5}\, dz_2 \wedge dz_6 \qquad \hat{\varphi}_{1,3,4,5} = \frac{\hat{\omega}_2}{z_2 z_6^2}$$

$$u_{1,3,4,5} = z_2^{\rho_2}\, u(0, z_2, 0, 0, 0, z_6)$$

$$\nu_{(62)} = 2 \qquad \hat{e}_1^{(62)} = \hat{h}_1^{(62)} = \frac{1}{z_2}\,, \qquad \hat{e}_2^{(62)} = \hat{h}_2^{(62)} = 1\,,$$

$$\nu_{(2)} = 2 \qquad \hat{e}_1^{(2)} = \hat{h}_1^{(2)} = \frac{1}{z_2}\,, \qquad \hat{e}_2^{(2)} = \hat{h}_2^{(2)} = 1\,.$$

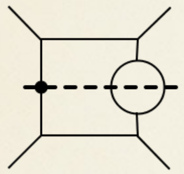
 $= c_1$  $+\, c_2$  ,

$$c_1 = \sum_{j=1}^{2} \langle \varphi_{1,3,4,5} | h_j^{(62)} \rangle \big(\mathbf{C}^{-1}_{(62)}\big)_{j1} = \frac{-3(3d-16)(3d-14)(2s+t)}{2(d-6)st^3}\,,$$

$$c_2 = \sum_{j=1}^{2} \langle \varphi_{1,3,4,5} | h_j^{(62)} \rangle \big(\mathbf{C}^{-1}_{(62)}\big)_{j2} = \frac{-3(3d-16)(3d-14)(3d-10)(2ds-10s-t)}{4(d-6)(d-5)(d-4)s^2t^3}\,.$$

in agreement with the IBP decomposition.

- **Cut**$_{\{\mathbf{2,4,5}\}}$:

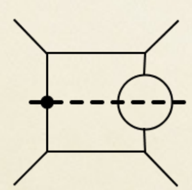
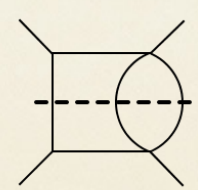
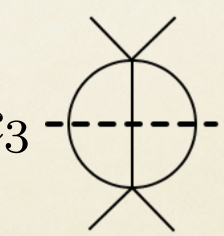
 $= \int_{\mathcal{C}} u_{2,4,5}\, \varphi_{2,4,5}\,, \qquad \varphi_{2,4,5} = \hat{\varphi}_{2,4,5}\, dz_1 \wedge dz_3 \wedge dz_6\,, \qquad \hat{\varphi}_{2,4,5} = \frac{\hat{\omega}_2}{z_1 z_3 z_6^2}.$

$$u_{2,4,5} = z_1^{\rho_1} z_3^{\rho_3} u(z_1, 0, z_3, 0, 0, z_6)$$

$$\nu_{(631)} = 2 \qquad \hat{e}_1^{(631)} = \hat{h}_1^{(631)} = \frac{1}{z_1 z_3}, \qquad \hat{e}_2^{(631)} = \hat{h}_2^{(631)} = 1\,,$$

$$\nu_{(31)} = 2 \qquad \hat{e}_1^{(31)} = \hat{h}_1^{(31)} = z_1, \qquad \hat{e}_2^{(31)} = \hat{h}_2^{(31)} = 1,$$

$$\nu_{(1)} = 2. \qquad \hat{e}_1^{(1)} = \hat{h}_1^{(1)} = z_1, \quad \hat{e}_2^{(1)} = \hat{h}_2^{(1)} = 1.$$

 $= c_1$  $+ c_3$  ,

c_1 is the same as found in $\text{Cut}_{1,3,4,5}$

$$c_3 = \sum_{j=1}^{2} \langle \varphi_{2,4,5} | h_j^{(631)} \rangle \left(\mathbf{C}_{(631)}^{-1} \right)_{j2} = \frac{3(3d-16)(3d-14)(3d-10)(3d-8)}{2(d-6)^2(d-4)st^4}\,.$$

in agreement with the IBP decomposition.

Contiguity relations for Special Functions

Hypergeometric ${}_3F_2$

$$\beta(a_1, b_1-a_1)\beta(a_2, b_2-a_2)\,{}_3F_2\left({a_1 a_2 a_3 \atop b_1 b_2}; x\right) = \int_{\mathcal{C}} u\, d^2\mathbf{z} = \langle 1^{(12)}|\mathcal{C}]$$

$$u = (1-z_1 z_2 x)^{-a_3} \prod_{i=1}^{2} z_i^{a_i-1}(1-z_i)^{b_i-a_i-1}\,, \qquad d^2\mathbf{z} = dz_1 \wedge dz_2,$$

$\mathcal{C}$ is the square with $z_i \in [0,1]$

(z_1, z_2)-space

$\hat{\omega}_1 = \hat{\omega}_2 = 0$ $\qquad \nu_{(12)} = 3$

$$\hat{e}_1^{(12)} = \frac{1}{z_1}\,, \quad \hat{e}_2^{(12)} = \frac{1}{z_2}\,, \quad \hat{e}_3^{(12)} = \frac{1}{1-z_2}\,,$$

$$\hat{h}_i^{(12)} = \hat{e}_i^{(12)} \;\; (i = 1, 2, 3)$$

z_2-subspace

$\hat{\omega}_2 = 0$ (w.r.t. z_2), $\qquad \nu_{(2)} = 2.$

$$\hat{e}_1^{(2)} = \frac{1}{z_2}\,, \quad \hat{e}_2^{(2)} = \frac{1}{1-z_2}$$

$$\hat{h}_i^{(2)} = \hat{e}_i^{(2)} \;\; (i = 1, 2)$$

- **Integral Decomposition**

$$\langle 1^{(12)}| = \sum_{i=1}^{3} c_i\, \langle e_i^{(12)}|$$

$${}_3F_2\left({a_1 a_2 a_3 \atop b_1 b_2}; x\right) = \alpha_1\; {}_3F_2\left({a_1-1, a_2, a_3 \atop b_1-1, b_2}; x\right), \; +\alpha_2\; {}_3F_2\left({a_1, a_2-1, a_3 \atop b_1, b_2-1}; x\right), \; +\alpha_3\; {}_3F_2\left({a_1, a_2, a_3 \atop b_1, b_2-1}; x\right)$$

Mathematica: ok