THE ROLE OF SYMMETRY IN THE CLUSTER STRUCTURE OF LIGHT NUCLEI

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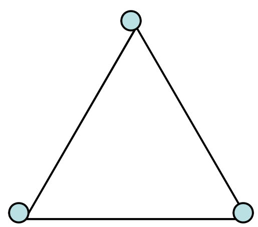
HISTORICAL INTRODUCTION

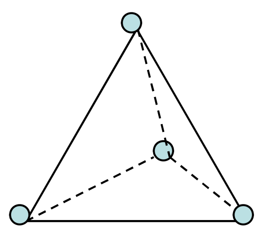
Wheeler (1937)
Dennison (1954)
Kameny (1956)
Brink (1965)

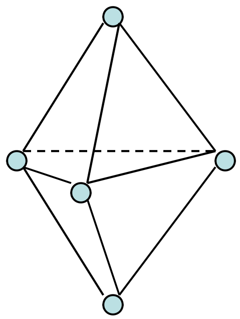
kα NUCLEI

Suggested cluster (quasi-molecular) structure of kα nuclei (Brink-Bloch), based on a microscopic calculation with Volkov forces

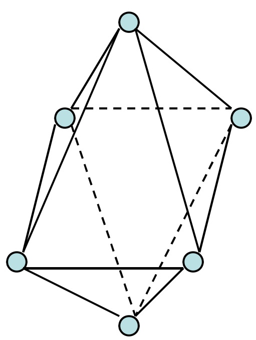
k=2  ^{8}Be Dumbbell Z_2

k=3  ^{12}C Triangle D_{3h}

k=4  ^{16}O Tetrahedron T_d

k=5 

^{20}Ne Trigonal bipyramid D_{3h}

k=6 

^{24}Mg Triaxial bipyramid D_{2h}

Octahedral ? O_h

Open questions 2019

- What are the signatures of symmetry in clustering
- How far cluster extends in excitation energy
- How far it extends in mass number

An answer to the first question can be obtained by a novel theoretical approach: the Algebraic Cluster Model (ACM)

This model provides explicit analytic expressions for all observables: energies, electromagnetic transitions, form factors,…

⇨ Part of the simplicity in complexity program

An answer to the second and third question can be obtained by new experiments.

ALGEBRAIC CLUSTER MODEL (ACM)

Algebraic structure U(3k-2)

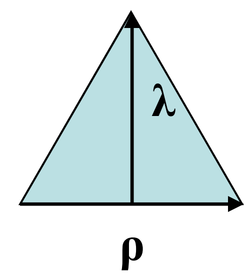
Bosonic quantization of the Jacobi variables according to the general quantization scheme¶ for problems with ν degrees of freedom, $U(\nu+1)$.

This approach bypasses the solution of the Schrödinger equation for k interacting α-particles.

¶ F. Iachello, in *Lie Algebras, Cohomologies and New Applications of Quantum Mechanics*, N. Kamran and P. Olver eds., Contemporary Mathematics, Vol. 160, p.151, Amer. Math. Soc., Providence, RI, 1994.

Explicit construction of the algebra completed for k=2,3,4

k	Nucleus	U(3k-2)	Discrete symmetry	Jacobi variables
2 α	^{8}Be	U(4)$^{\#}$	Z_2	$\boldsymbol{\rho}$
3 α	^{12}C	U(7)¶	D_3	$\boldsymbol{\rho},\boldsymbol{\lambda}$
4 α	^{16}O	U(10)§	T_d	$\boldsymbol{\rho},\boldsymbol{\lambda},\boldsymbol{\eta}$



[#] F. Iachello, Chem. Phys. Lett. 78, 581 (1981); Phys. Rev. C23, 2778 (1981).
[¶] R. Bijker and F. Iachello, Phys. Rev. C61, 067305 (2000).
R. Bijker and F. Iachello, Ann. Phys. (N.Y.) 298, 334 (2002).
[§] R. Bijker, AIP Conf. Proc. 1323, 28 (2010); J. Phys.Conf.Ser.380, 012003(2012).
R. Bijker and F. Iachello, Phys. Rev. Lett. 112, 152501 (2014).
R. Bijker and F. Iachello, Nucl. Phys. A 957, 154 (2017).
[DOI:10.1016/j.nuclphysa.2016.08.008]

CLASSIFICATION OF STATES

The discrete symmetry of cluster states imposes conditions on the allowed quantum states.

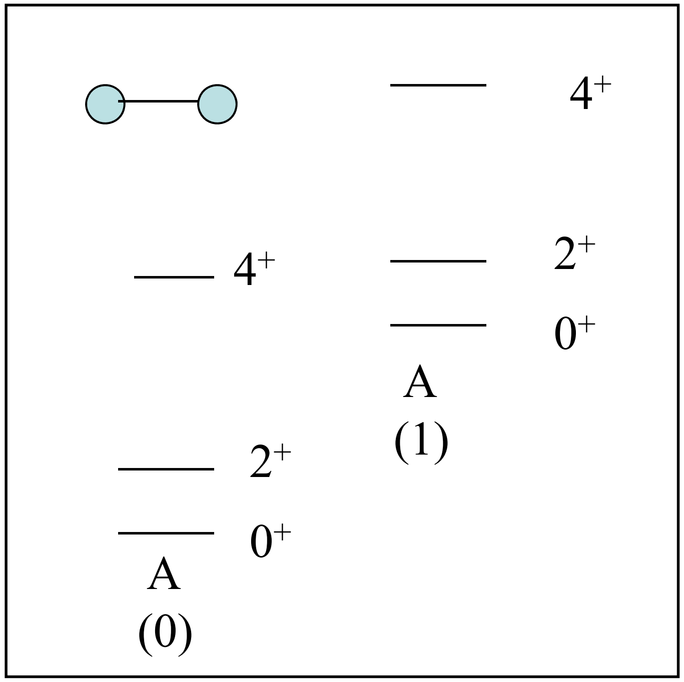
Mathematical method for determining the allowed states (i.e. constructing representations of the discrete group G): Diagonalization of the symmetry adapter operators

⇨ Expected states L^P for $k\alpha$ configurations (k=2,3,4) with Z_2, D_{3h} and T_d symmetry.

^8Be

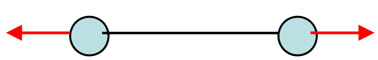
k=2

Z_2



A irreps: $L^P=0^+,2^+,4^+,\ldots$

Vibrations of a 2α configuration

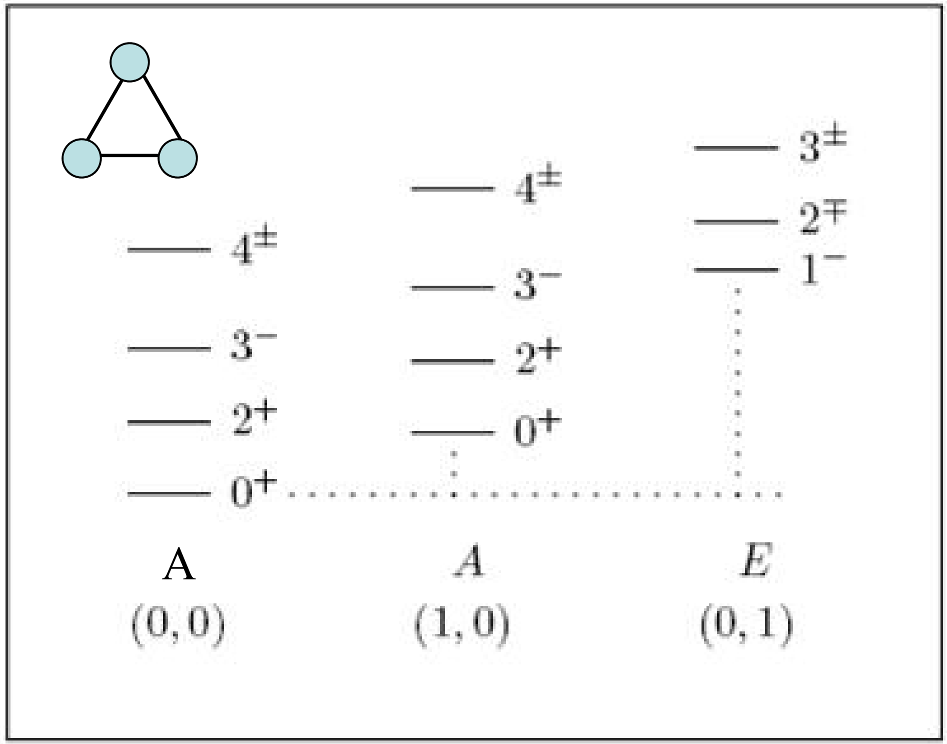


A

^{12}C

k=3

D_{3h}



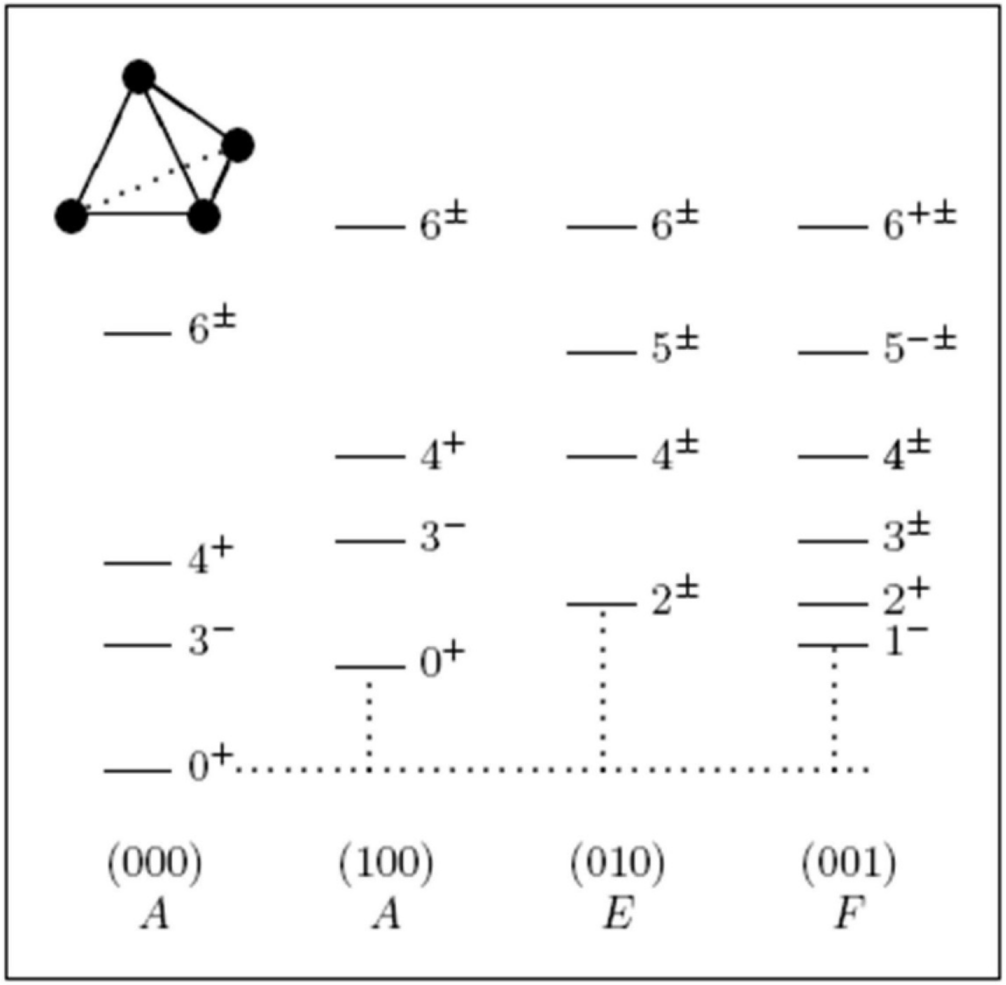
A irreps: $L^P = 0^+ , 2^+ , 3^- , 4^\pm$, … (K=0,3,6,…)

E irreps: $L^P=1^-, 2^\pm, 3^\pm$, … (K=1,2,4,5, …)

Both positive and negative parity states sit in the same irrep because of the lack of reflection symmetry of a D_3 configuration: parity doubling!

^{16}O

k=4



A irreps: $L^P=0^+, 3^-, 4^+, 6^\pm$, …

E irreps: $L^P=2^\pm, 4^\pm, 5^\pm, 6^\pm$, … ; F irreps: $1^-, 2^+, 3^\pm, 4^\pm$, …

Both positive and negative parity states in the same irrep:

Parity doubling!

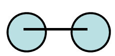
ENERGY FORMULAS

Within U(3k-2), in the large N limit, it is possible to derive energy formulas. These formulas are identical to those used in physical chemistry to study molecular structures. For a *rigid* roto-vibrator these are:

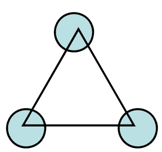
2 α $$E(v,L)=E_0+\omega\left(v+\frac{1}{2}\right)+BL(L+1)$$

3 α $$E(v_1,v_2,L,K)=E_0+\omega_1\left(v_1+\frac{1}{2}\right)+\omega_2\left(v_2+1\right)+BL(L+1)+(A-B)(K\mp 2\ell_2)^2$$

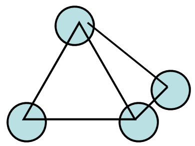
4 α $$E(v_1,v_2,v_3,L)=E_0+\omega_1\left(v_1+\frac{1}{2}\right)+\omega_2\left(v_2+1\right)+\omega_3\left(v_3+\frac{3}{2}\right)+BL(L+1)$$



Linear



Symmetric top



Spherical top

ELECTROMAGNETIC TRANSITIONS

Within U(3k-2) it is also possible to calculate all properties in explicit analytic form

(a) B(EL) values along the rotational band

2 α $$B(EL;0\rightarrow L)=\left(\frac{Ze\beta^{L}}{2}\right)^{2}\frac{2L+1}{4\pi}\left[2+2P_{L}(-1)\right]$$

3 α $$B(EL;0\rightarrow L)=\left(\frac{Ze\beta^{L}}{3}\right)^{2}\frac{2L+1}{4\pi}\left[3+6P_{L}\left(-\frac{1}{2}\right)\right]$$

4 α $$B(EL;0\rightarrow L)=\left(\frac{Ze\beta^{L}}{4}\right)^{2}\frac{2L+1}{4\pi}\left[4+12P_{L}\left(-\frac{1}{3}\right)\right]$$

(b) Form factors in electron scattering

$$F_L\left(0 \to L; q\right) = c_L\, j_L(q\beta)$$

2 α $\quad c_L^2 = \frac{2L+1}{4}\left[2 + 2P_L(-1)\right] \qquad c_0^2 = 1, c_2^2 = 5, c_4^2 = 9$

3 α $\quad c_L^2 = \frac{2L+1}{9}\left[3 + 6P_L(-\frac{1}{2})\right] \qquad c_0^2 = 1, c_2^2 = \frac{5}{4}, c_3^2 = \frac{35}{8}, c_4^2 = \frac{81}{64}$

4 α $\quad c_L^2 = \frac{2L+1}{16}\left[4 + 12P_L(-\frac{1}{3})\right] \qquad c_0^2 = 1, c_3^2 = \frac{35}{9}, c_4^2 = \frac{7}{3}, c_6^2 = \frac{416}{81}$

COMPARISON WITH EXPERIMENT: ENERGIES

The occurrence of Z_2 symmetry in ^{8}Be was emphasized by many authors and confirmed by recent experiments.

The occurrence of D_3 symmetry in ^{12}C has been confirmed by recent experiments (Freer, Gai *et al.*)[¶] .

The occurrence of T_d symmetry in ^{16}O was emphasized by Robson[#] (1978-1982) and more recently revisited by Bijker (2014)[§] .

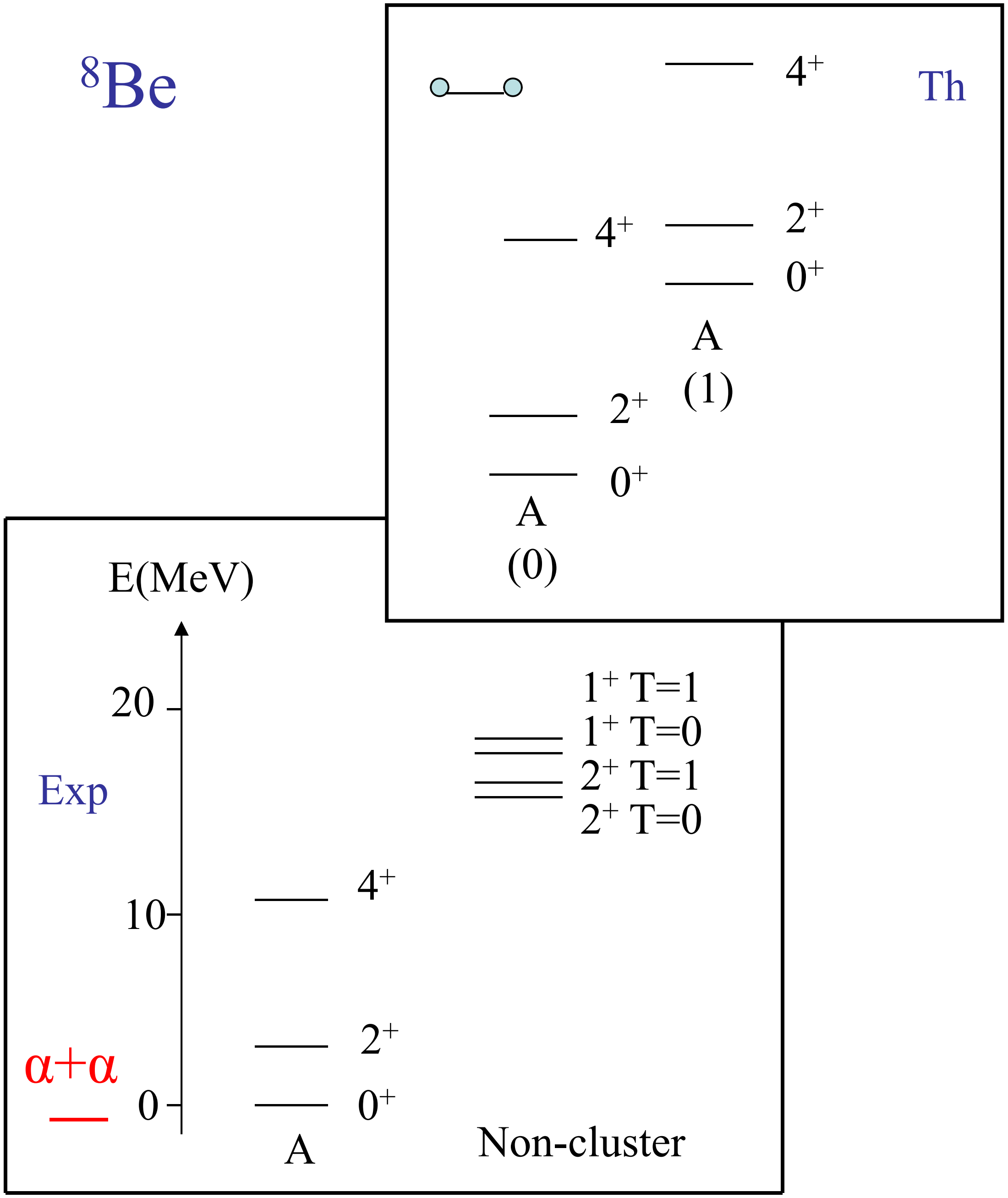
[¶] M. Freer *et al.*, Phys. Rev. C 83, 034314 (2011).
D.J. Marin-Lambarri, R. Bijker, M. Freer, M. Gai, Tz. Kokalova, D.J. Parker, and C. Wheldon, Phys. Rev. Lett. 113, 012502 (2014).

[#] D. Robson, Nucl. Phys. A308, 281 (1978); Phys. Rev. Lett. 42, 876 (1979); Phys. Rev. C25, 1108 (1982).

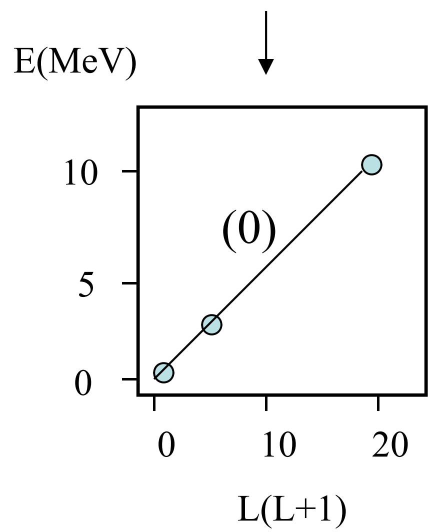
[§] R. Bijker and F. Iachello, Phys. Rev. Lett. 112, 152501 (2014); Nucl. Phys. A957, 154 (2017).

EVIDENCE FOR Z_2 SYMMETRY

^{8}Be



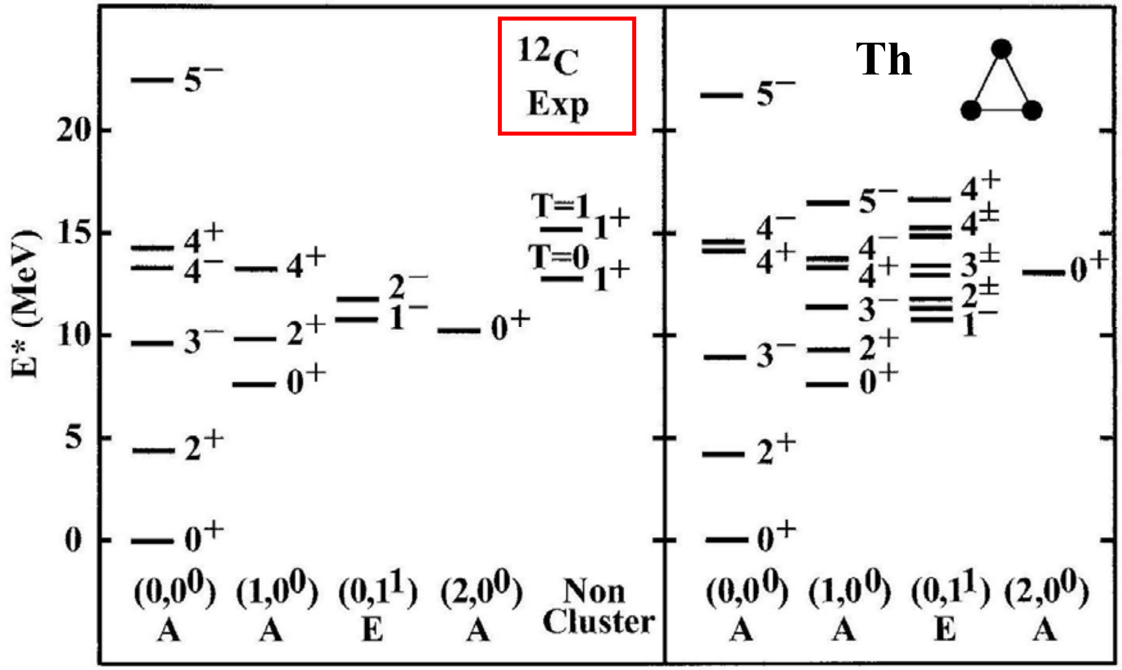
Observed
cluster
rotational
bands



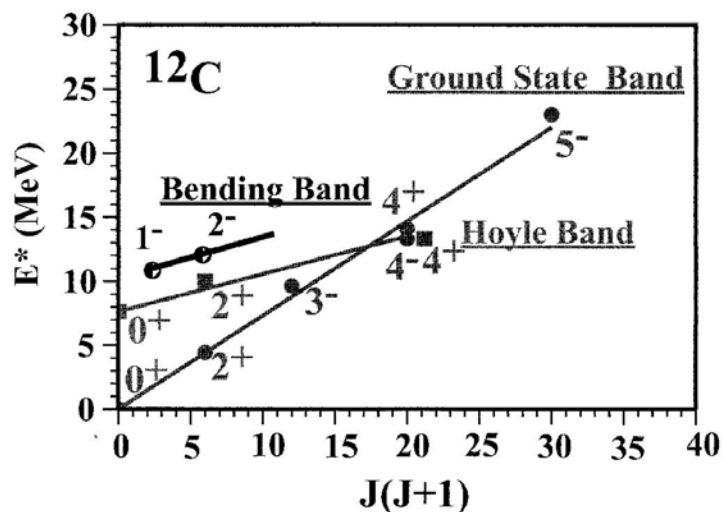
^{12}C

EVIDENCE FOR D_3 SYMMETRY

$^{8}Be+\alpha$



Observed cluster rotational bands

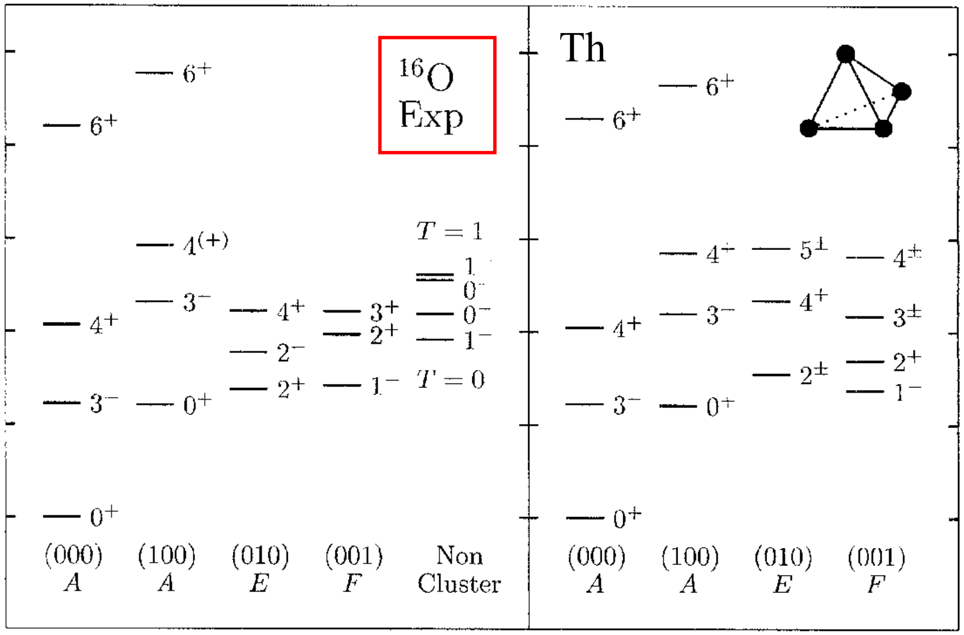


From D.J. Marin-Lambarri *et al.*, *loc. cit.* (2014).

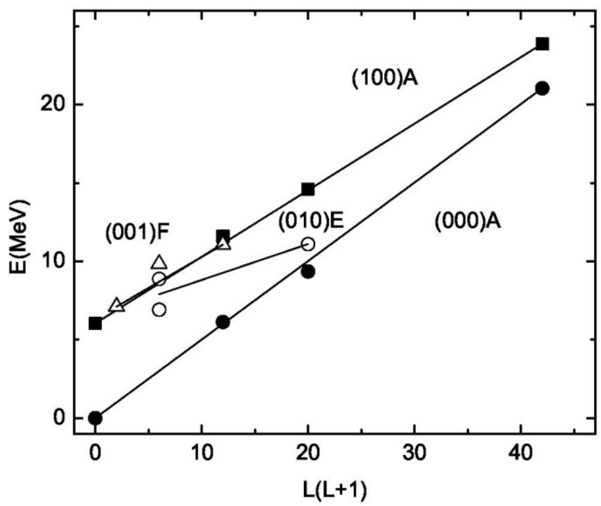
^{16}O

EVIDENCE FOR T_d SYMMETRY

$^{12}C+\alpha$



Observed cluster rotational bands



From R. Bijker and F. Iachello, *loc. cit.* (2014).

COMPARISON WITH EXPERIMENT: B(EL) VALUES

^{8}Be

$B(EL;L^P\rightarrow 0^+)$	Th	Exp	$E(L^P)$	Th*	Exp
$B(E2;2^+\rightarrow 0^+)$	20.4	20.0±0.8¶ (21.9(9) W.u.)	$E(2^+)$	3060	3030
$B(E4;4^+\rightarrow 0^+)$	326.1		$E(4^+)$	10200	11350

B(EL) values in e^2fm^{2L} and E in keV

β=2.0fm

* E(keV)=510L(L+1)

¶ Model dependent estimate from radiative capture and GFMC, D.M. Daker *et al.*, Phys. Rev. Lett. 111, 062502 (2013)

^{12}C

$B(EL;L^P\rightarrow 0^+)$	Th	Exp	$E(L^P)$	Th*	Exp
$B(E2;2^+\rightarrow 0^+)$	9.3	7.6±0.4 (4.65(26) W.u.)	$E(2^+)$	4400	4439
$B(E3;3^-\rightarrow 0^+)$	84	103±17 (12(2) W.u.)	$E(3^-)$	9640	9641
$B(E4;4^+\rightarrow 0^+)$	68		$E(4^+)$	14670	14080

↑

Parameter free: consequence of symmetry alone

B(EL) values in e^2fm^{2L} and E in keV

β=1.9fm estimated from the elastic form factor measured in electron scattering

* E(keV)= 730 L(L+1)

^{16}O

$B(EL; L^P \to 0^+)$	Th	Exp	$E(L^P)$	Th*	Exp
$B(E3; 3^- \to 0^+)$	181	205±10 (13.5(7) W.u.)	$E(3^-)$	6132	6130
$B(E4{:}4^+ \to 0^+)$	338	378±133 (3.7(13) W.u.)	$E(4^+)$	10220	10356
$B(E6{:}6^+ \to 0^+)$	8245		$E(6^+)$	21462	21052

↑

Parameter free: consequence of symmetry alone!

B(EL) values in $e^2 fm^{2L}$ and E in keV

β=2.0fm extracted from the elastic form factor measured in electron scattering

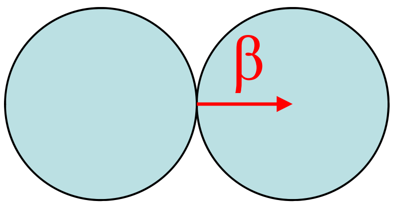
* E(keV)=511 L(L+1)

CHARGE AND MATTER DENSITIES IN ACM

Charge and matter densities can be calculated analytically in ACM. They are obtained by convoluting the charge (and matter) density of the point-like cluster with that of the α-particle, parametrized as a Gaussian

$$\rho_{\alpha}(r) = \left(\frac{\alpha}{\pi}\right)^{3/2} e^{-\alpha r^2} \qquad \alpha = 0.56\, fm^{-2} \longleftarrow \text{From } \langle r^2 \rangle^{1/2} \text{ of } {}^4\text{He}$$

and are given in terms of the distance, β, of the alpha-particles from the center of mass



Charge (and matter) densities in clusters with k=2,3,4 and symmetry Z_2 , D_{3h} and T_d ¶ .

$$\boxed{\begin{aligned}\rho(\vec{r}) &= \sum_{\lambda\mu} \frac{Ze}{k} f_\lambda(r) Y_{\lambda\mu}(\vartheta,\varphi) \sum_{i=1}^{k} Y^*_{\lambda\mu}(\vartheta_i,\varphi_i) \\ f_\lambda(r) &= \left(\frac{\alpha}{\pi}\right)^{3/2} e^{-\alpha(r^2+\beta^2)} 4\pi i_\lambda(2\alpha\beta r)\end{aligned}}$$

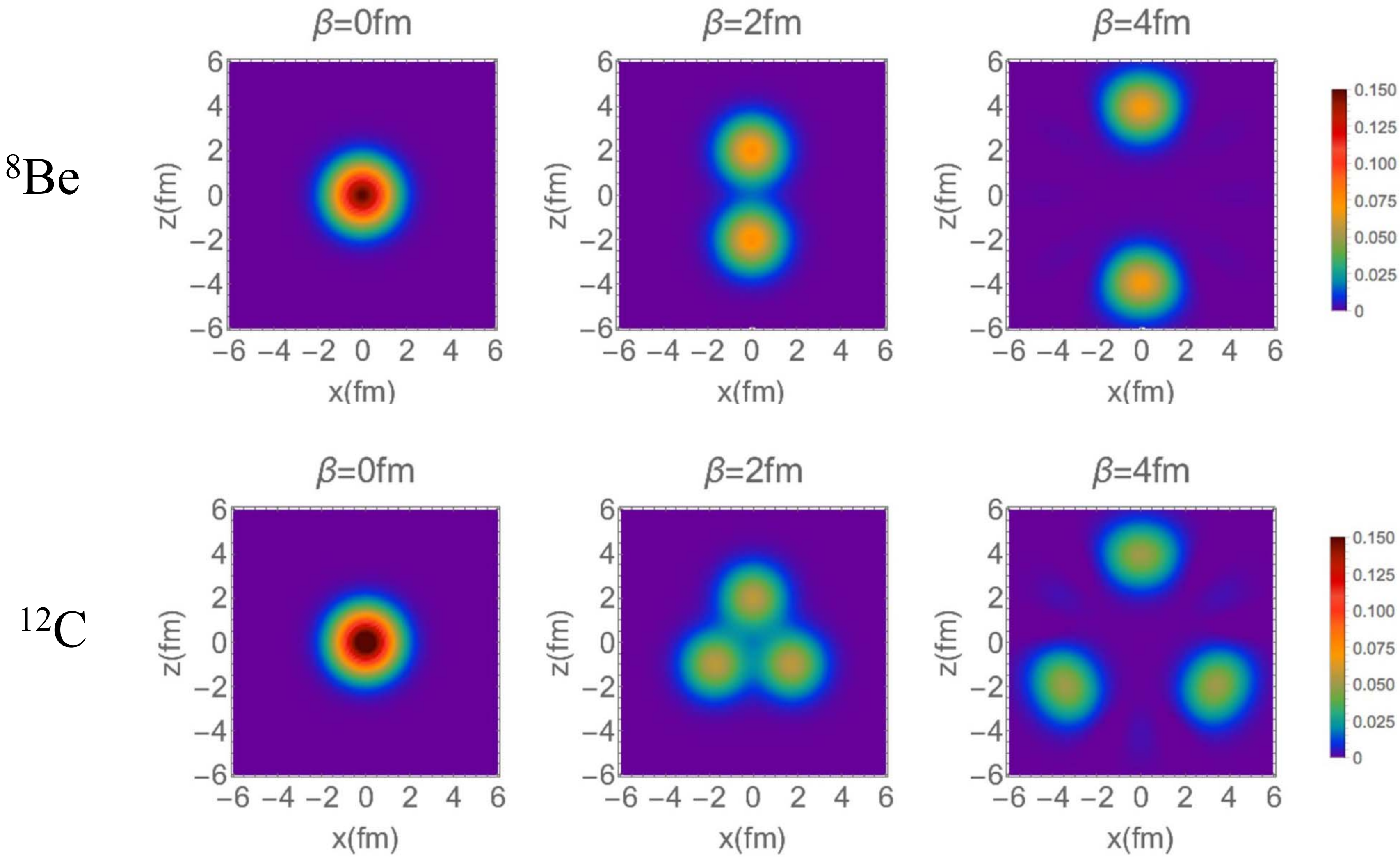
$$\sum_{i=1}^{2} Y^*_{\lambda\mu} = \delta_{\mu,0}\sqrt{\frac{2\lambda+1}{4\pi}}\left[1+P_\lambda(-1)\right]$$

$$\sum_{i=1}^{3} Y^*_{\lambda\mu} = \left\{\begin{array}{l} \sqrt{\dfrac{2\lambda+1}{4\pi}}\left[1+2P_\lambda(-\dfrac{1}{2})\right]; \mu=0 \\ \sqrt{\dfrac{2\lambda+1}{4\pi}}\sqrt{\dfrac{(\lambda+\mu)!}{(\lambda-\mu)!}}2P_\lambda^{-\mu}(-\dfrac{1}{2}); \mu=2k\neq 0 \end{array}\right\}$$

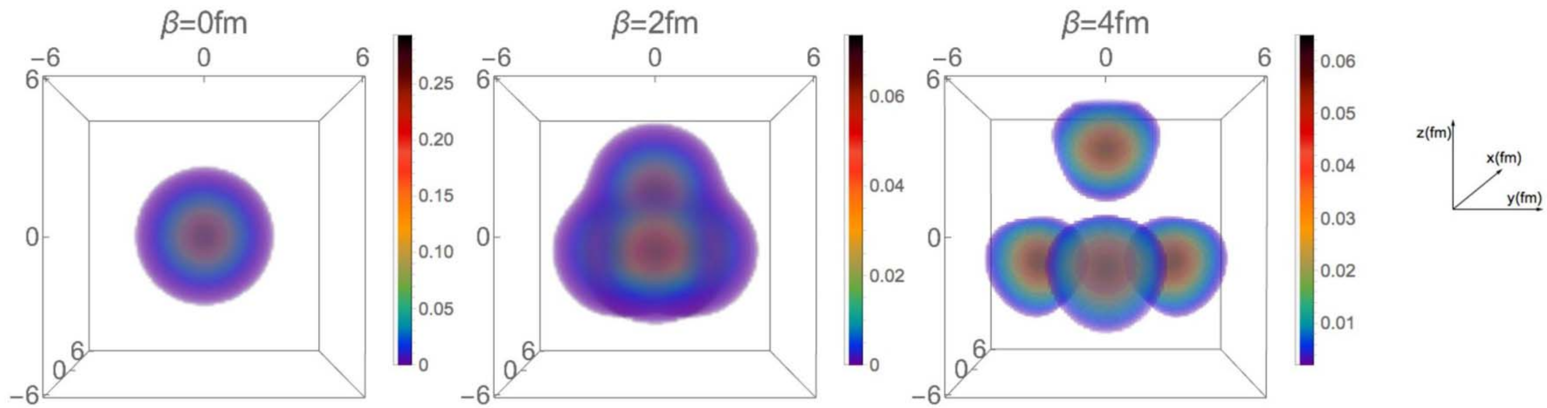
$$\sum_{i=1}^{4} Y^*_{\lambda\mu} = \left\{\begin{array}{l} \sqrt{\dfrac{2\lambda+1}{4\pi}}\left[1+3P_\lambda(-\dfrac{1}{3})\right]; \mu=0 \\ \sqrt{\dfrac{2\lambda+1}{4\pi}}\sqrt{\dfrac{(\lambda+\mu)!}{(\lambda-\mu)!}}3P_\lambda^{-\mu}; \mu=3k\neq 0 \end{array}\right\}$$

¶ V. Della Rocca, R. Bijker and F. Iachello, Nucl. Phys. A966, 158 (2017).

CHARGE DENSITIES IN ACM



^{16}O



Values of beta extracted from the experimental moment of inertia (^{8}Be) and from the first zero in the electron scattering form factor (^{12}C and ^{16}O)

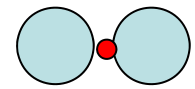
^{8}Be	1.82(4) fm
^{12}C	1.90(6) fm
^{16}O	2.00(6) fm

corresponding to close-packing of the α-particles

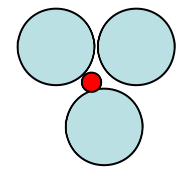
CLUSTERING IN kα STRUCTURES PLUS ADDITIONAL PARTICLES ¶

Clustering in kα structures+ x neutrons (protons) §

^{9}Be,^{9}B 2α+1neutron (proton) →



^{13}C,^{13}N 3α+1neutron (proton) →



^{17}O,^{17}F 4α+1neutron (proton)

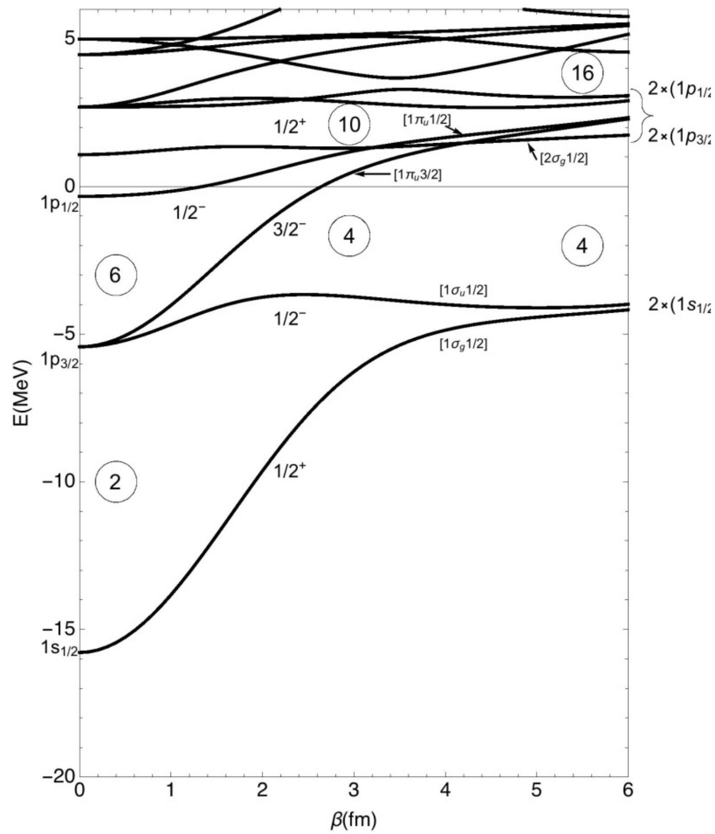
The extra neutron (proton) acts as a glue to further bind the structure as in covalent bonding of molecules.

A novel approach (CSM) ¶ has been introduced recently to study these structures.

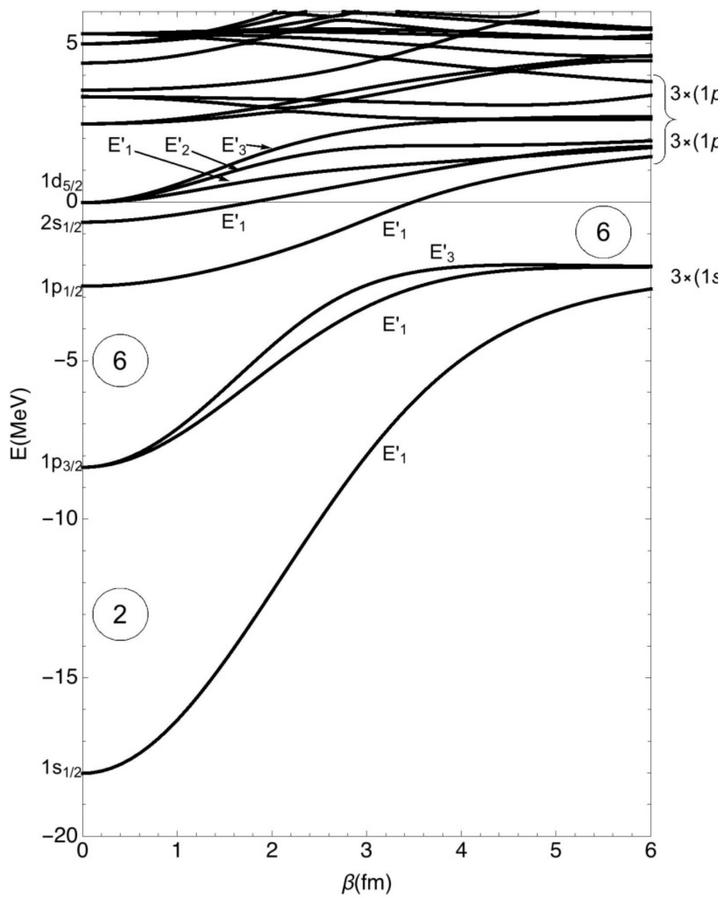
§ These structures were suggested by W. van Oertzen, Z. Phys. A 354, 37 (1996); A357, 355 (1997).

¶ V. Della Rocca, R. Bijker and F. Iachello, Nucl. Phys. A 966, 158 (2017).

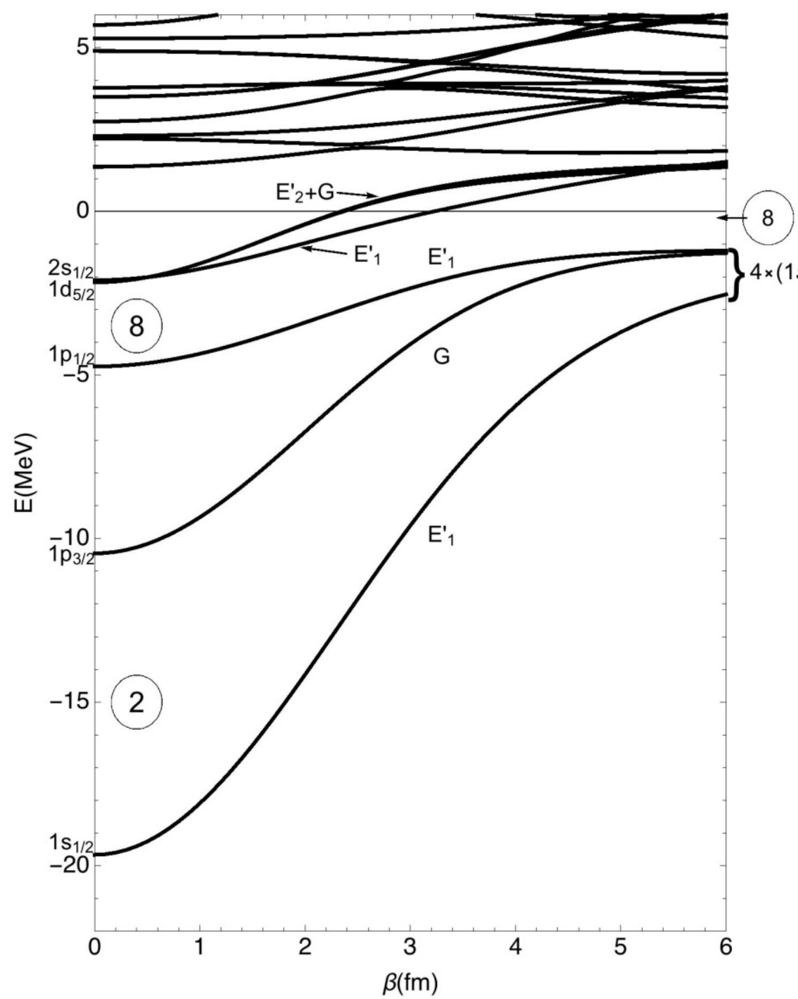
SINGLE PARTICLE LEVELS IN CLUSTER POTENTIALS



Single particle levels in a potential with Z_2 symmetry



Single particle levels in a potential with D_{3h} symmetry



Single particle levels in a potential with T_d symmetry

ROTATION VIBRATION STATES

Rotation-vibration states built on top of the single-particle states can be obtained by a novel approach

$\Longrightarrow$Representations of double groups ¶

(fermionic representations of the point groups)

	Nucleus	Double group
2α+1 particle	^{9}Be-^{9}B	Z_2’
3α+1 particle	^{13}C-^{13}N	D_{3h}’
4α+1 particle	^{17}O-^{17}F	T_d’

¶ R. Bijker and F. Iachello, Phys. Rev. Lett. 122, 162501 (2019).

ENERGY FORMULAS ¶

Z_2'

$$E(J,K)=\varepsilon_K+\frac{(\hbar)^2}{2\mathfrak{I}}\left[J(J+1)-K^2+\delta_{(K,1/2)}a(-)^{J+1/2}(J+1/2)\right]$$

$$a=-\sum_{nlj}(-)^{j+(1/2)}(j+\frac{1}{2})\left|c_{nlj}^{1/2}\right|^2$$ ⟵decoupling parameter

D_{3h}'

$$E_{rot}(\Omega,J,K)=\varepsilon_\Omega+A_\Omega\left[J(J+1)+b_\Omega K^2+a_\Omega g_\Omega(J)\right]$$

$$g_\Omega(J)=\delta_{K,1/2}(-)^{J+1/2}(J+1/2)$$

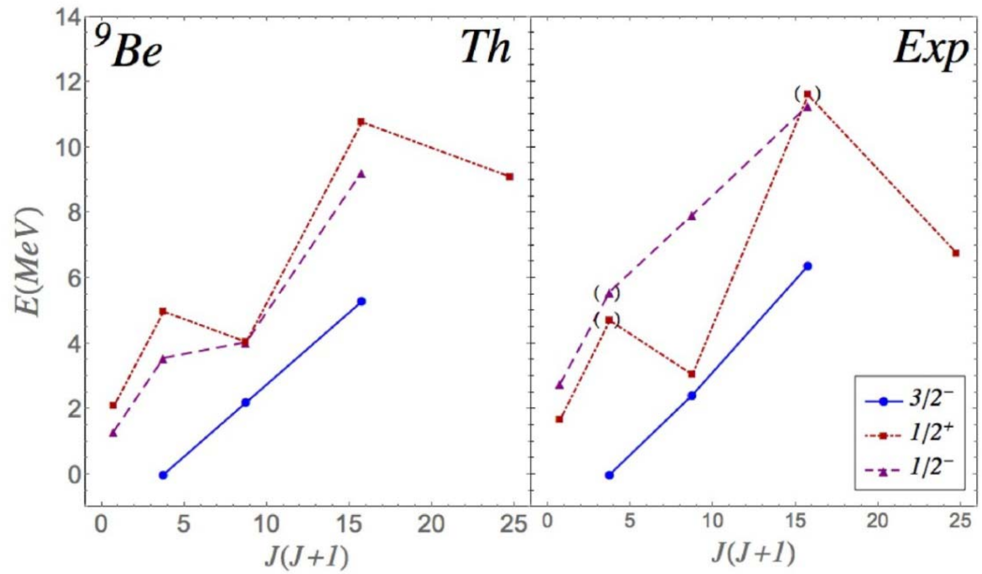
T_d'

$$E(\Omega,J,K)=\varepsilon_\Omega+A_\Omega\left[J(J+1)+a_\Omega g_\Omega(J)\right]$$

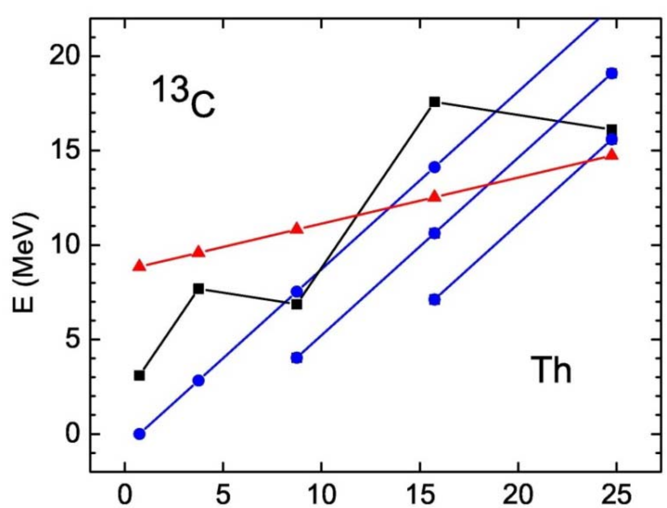
¶ V. Della Rocca and F. Iachello, Nucl. Phys. A 973, 1 (2018).

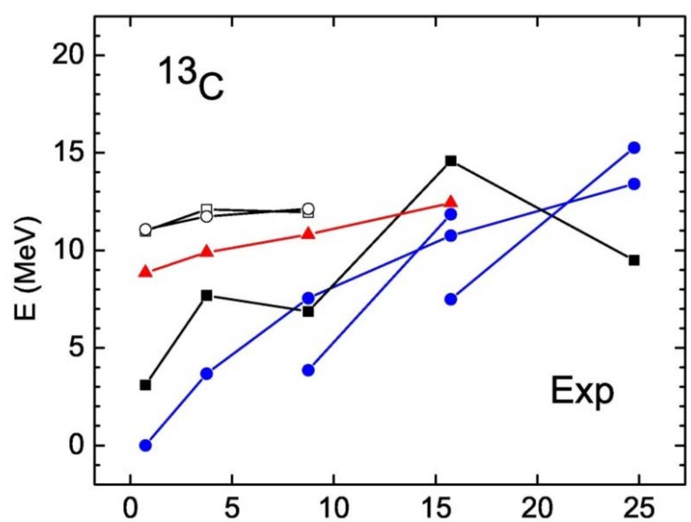
EVIDENCE FOR Z_2 AND D_{3h} SYMMETRY

Z_2'



D_{3h}'





The group D'$_{3h}$ has three spinor representations. The allowed values of K^P for each of them is given by:

$$\Omega = E_{1/2}^{(+)}: \quad K^P = 1/2^+$$

$$K = 3n \pm \frac{1}{2}, P = (-)^n$$

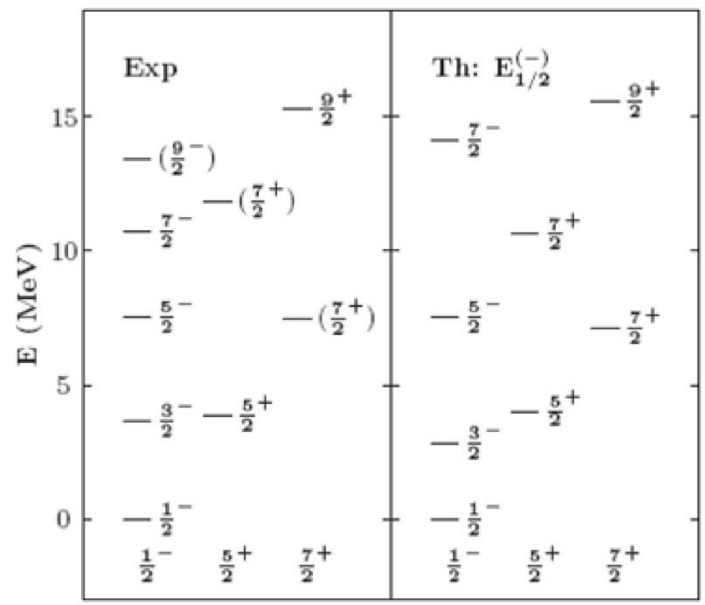
$$\Omega = E_{1/2}^{(-)}: \quad K^P = 1/2^-$$

$$K = 3n \pm \frac{1}{2}, P = (-)^{n+1}$$

$$\Omega = E_{3/2}: \quad K^P = (3n - \frac{3}{2})^{\pm}$$

with n=1,2,3,…, and K>0. The angular momenta of each K are J=K, K+1, K+2, …

^{13}C



Ground state band assigned to the representation $\Omega = E_{1/2}^{(-)}$ of D'$_{3h}$ with A=0.942 MeV, b=-0.62 and a=0.

ELECTROMAGNETIC TRANSITIONS

$$B(\lambda;\Omega',I',M',K' \to \Omega,I,M,K) =$$

$$= \left\| \begin{bmatrix} \langle I',K',\lambda,K-K' | I,K \rangle \left(\delta_{v,v'} G_\lambda(\Omega,\Omega') + \delta_{\Omega,\Omega'} G_{\lambda,c} \right) + \\ (-)^{I+K} \langle I',K',\lambda,-K-K' | I,-K \rangle \left(\delta_{vv'} \tilde{G}_\lambda(\Omega,-\Omega') + \delta_{\Omega,-\Omega'} G_{\lambda,c} \right) \end{bmatrix} \right\|^2$$

Electric transitions within a rotational band are dominated by the cluster contribution. For Z'_2 symmetry in the ground state band of ^{9}Be

$$B(E2;3/2,I' \to 3/2,I) = \left(Ze\beta^2 \sqrt{\frac{5}{4\pi}} \right)^2 \left\langle I',3/2,2,0 \middle| I,3/2 \right\rangle^2$$

$$Q^{(2)}(3/2,I) = \left(\sqrt{\frac{16\pi}{5}} Ze\beta^2 \sqrt{\frac{5}{4\pi}} \right) \left\langle I,3/2,2,0 \middle| I,3/2 \right\rangle \left\langle I,I,2,0 \middle| I,I \right\rangle$$

Magnetic transitions and magnetic moments are dominated by the single particle contribution

$$\mu^{(1)}(3/2,I) = \sqrt{\frac{4\pi}{3}} \left\langle I,3/2,1,0 \middle| I,3/2 \right\rangle \left\langle I,I,1,0 \middle| I,I \right\rangle G_1(3/2)$$

EVIDENCE FOR Z'_2 SYMMETRY

^{9}Be

	Exp (NDT)	Th (CSM)
$Q(3/2^-)$	5.288(38)	5.30
$B(E2;3/2^- \longrightarrow 5/2^-)$	40.5(30)	35.9
$B(E2;3/2^- \longrightarrow 7/2^-)$	18(8)	20.0
$\mu(3/2^-)$	-1.1778(9)	-1.13

Quadrupole moment in efm^2 and B(E2) values in e^{2}fm^4. Magnetic moment in μ_N.

Electric transitions along a rotational band are dominated by the cluster contribution. For D_{3h} symmetry

$$G_{\lambda,c} = Z\beta^{\lambda}\sqrt{\frac{2\lambda+1}{4\pi}}c_{\lambda}$$

$$c_0 = 1, c_2 = 1/2, c_3 = \sqrt{5/8}, c_4 = 3/8$$

EVIDENCE FOR D'_{3h} SYMMETRY

		Exp (NDT)	Th (CSM)
^{12}C (D_{3h})	$B(E2;2^+_1 \longrightarrow 0^+_1)$	4.65 ± 0.26	4.8
	$B(E3;3^-_1 \longrightarrow 0^+_1)$	12 ± 2	7.6
^{13}C (D'_{3h})	$B(E2;3/2^- \longrightarrow 1/2^-)$	3.5 ± 0.8	4.8
	$B(E2;5/2^- \longrightarrow 1/2^-)$	3.1 ± 0.2	3.2
	$B(E3;5/2^+ \longrightarrow 1/2^-)$	10 ± 4	4.3

B(EL) values in W.u.

FORM FACTORS

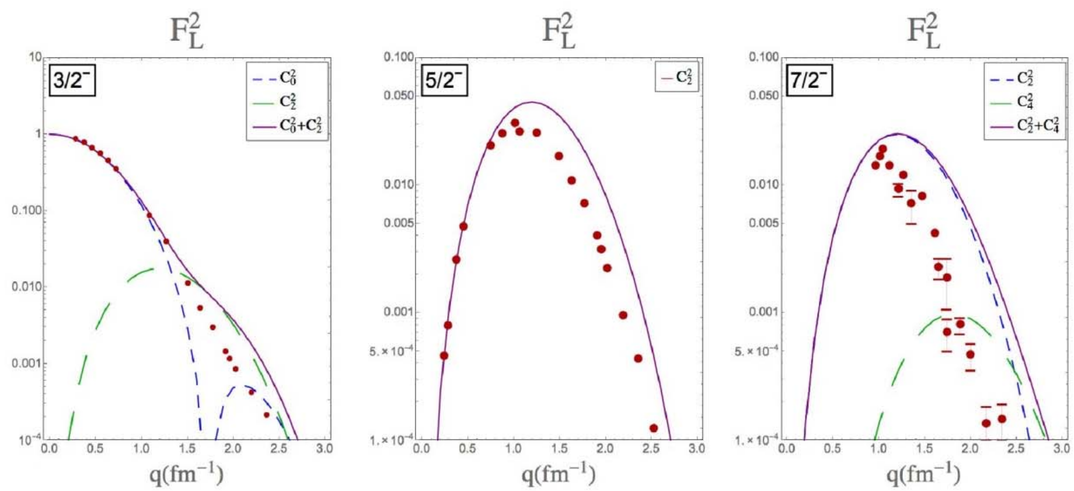
$$F_\lambda = F_\lambda^{s.p.} + F_\lambda^{c}$$

Electric transitions are dominated by the cluster contribution

$$F_\lambda^c(q; J, K \to J', K')$$
$$= \delta_{K,K'} Z \sqrt{\frac{2\lambda+1}{4\pi}} c_\lambda \langle J, K, \lambda, 0 \mid J', K' \rangle \, j_\lambda(q\beta) e^{-q^2/4\alpha}$$

Electric longitudinal form factors

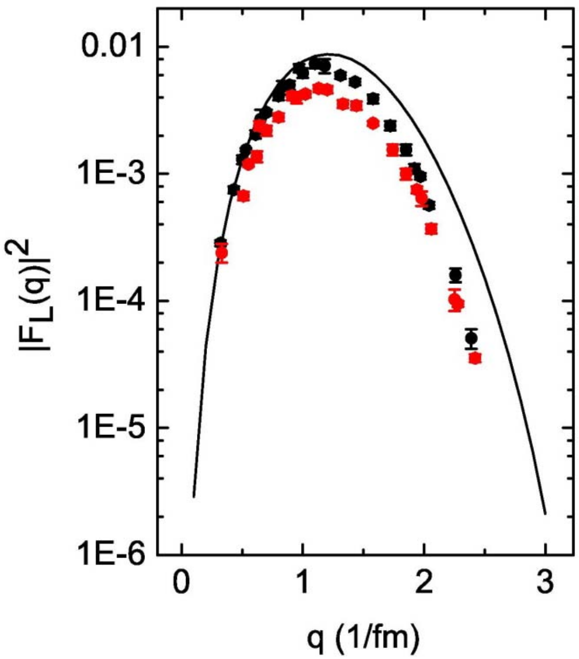
^{9}Be



Elastic 3/2⟶3/2 Inelastic 3/2⟶5/2 and 3/2⟶7/2

Electric longitudinal form factors

^{13}C



$1/2 \longrightarrow 3/2$ (black) and $1/2 \longrightarrow 5/2$ (red)

D'_{3h} symmetry predicts equal form factors !
Experimentally verified!

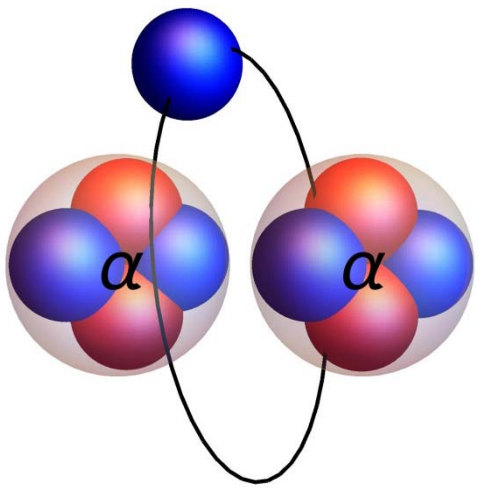
SUMMARY AND CONCLUSIONS

Symmetry provides analytic expressions for energy levels, electromagnetic transition rates and form factors in $k\alpha$+x nuclei. Evidence has been presented for k=2,3,4;x=0 (^{8}Be,^{12}C,^{16}O), and k=2,3;x=1 (^{9}Be,^{9}B;^{13}C,^{13}N).

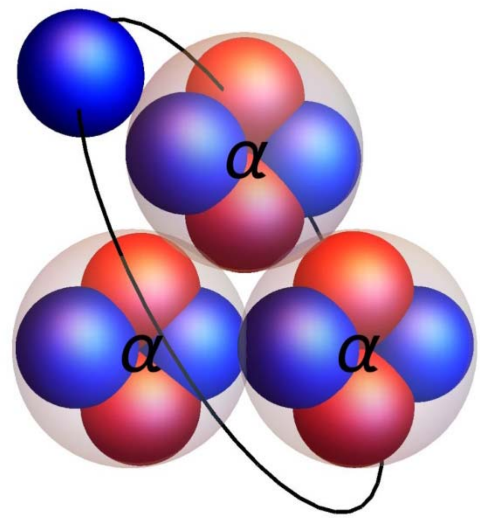
All properties are calculated in terms of one parameter, β, the distance of the α-particles from the center of mass.
This parameter can be fixed either from the moment of inertia or from the charge radius.

Particularly interesting is the picture that emerges for nuclei with k α particles plus x nucleons,

^{9}Be



^{13}C



confirming von Oertzen original suggestion.

The results may be used for comparison with experiment and as benchmarks for *ab initio* calculations of light nuclei.

ADDITIONAL REMARKS

- Signatures of symmetry in clustering are properties of spectra and electromagnetic transition rates, especially parity doubling, unusual structure of the rotational bands, and enhanced E2, E3, E4, E5, … transitions.
- Clustering appears to extend at least to angular momentum J=6 in k=2,3,4 systems, and up to excitation energies of the order of 25 MeV.
- Clustering appears to extend up to A=16. [Recent unpublished work on k=5 and k=6 systems appears to indicate that also A=20 and A=24 nuclei have the cluster structure suggested by Brink].
- Clustering appears to survive the addition of nucleons, x=1. [Recent unpublished work on systems with x=2, ^{10}Be, and ^{14}C appears to indicate that systems with k=2 are also clusterized.]

Further confirmation of clustering in light nuclei must come from new experiments.

The ACM and its extension CSM make definite (simple) predictions for where states should occur and what are their decay properties, both in $k\alpha$-structures and in $k\alpha + x$ nucleon structures. These experiments could include electron scattering, photon scattering, particle transfer and electromagnetic decay rates.

Symmetry has been for many years one of the primary tools to study systems in molecular, atomic, nuclear and particle physics.

It appears to be so also for cluster physics with a combination of both discrete and continuous symmetries!