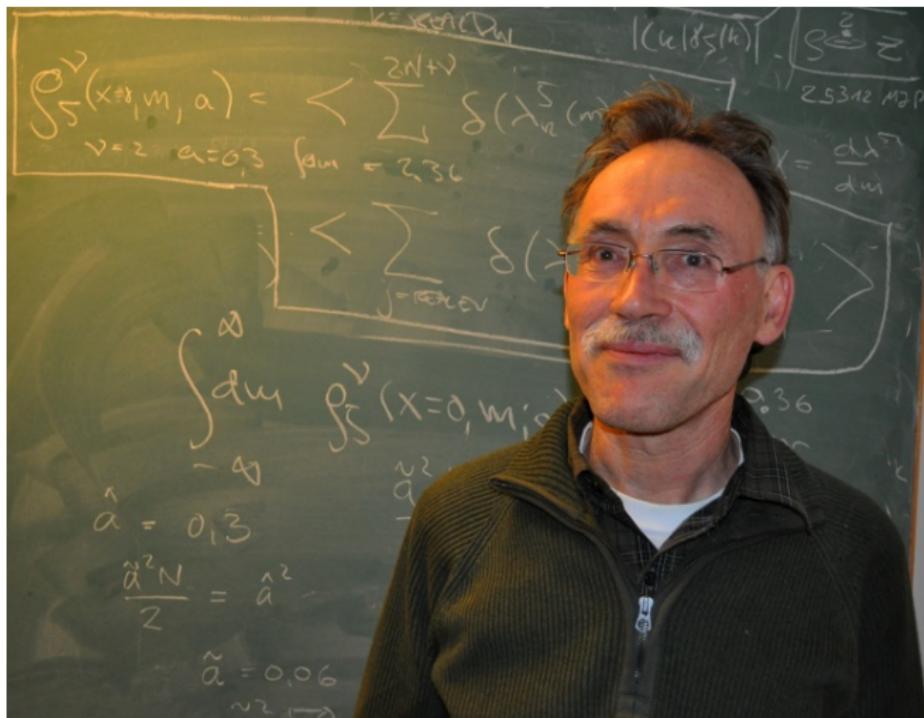


RMT in Sub Atomic Physics and Beyond



In Honor of Jac Verbaarschot's Birthday



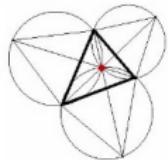
Random Matrix Realization of the Chern-Simons Term in 3D QCD-like Theories

Mario Kieburg

(ECT* Trento, Italy), 9th June 2019

Collaborators: Takuya Kanazawa & Jac Verbaarschot

Project started in Summer 2017 at CICC (Cuernavaca, Mexico)



Centro
Internacional de
Ciencias A.C.



Scientific gathering: “Correlations in
Time Series and Many-Body Systems”

Outline

- ▶ Chern-Simons Term in 3D
- ▶ Random Matrix Model
- ▶ Non-linear Sigma Model (Existence and Effects)
- ▶ Conclusions

Characteristic forms and geometric invariants

By SHIING-SHEN CHERN AND JAMES SIMONS*

Annals of Mathematics **99**, 48–69 (1974)

Chern-Simons Term in 3D ($A_\mu \in i\text{su}(N_c)$)

Yang-Mills Action:

$$S_{\text{YM}} = -\frac{1}{4g^2} \int d^3x \text{tr} F_{\mu\nu} F^{\mu\nu}$$

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} - i[A_\mu, A_\nu]$$

Chern-Simons Term:

$$S_{\text{CS}} = i \frac{k}{4\pi} \int d^3x \epsilon^{\mu\nu\lambda} \text{tr} (A_\mu A_{\lambda,\nu} - \frac{2}{3} A_\mu A_\nu A_\lambda)$$

- ▶ both terms are gauge invariant in the partition function when $k \in \mathbb{Z}$

but S_{YM} is real in Euclidean space-time

while S_{CS} is imaginary

Chern-Simons Term in 3D ($A_\mu \in i\text{su}(N_c)$)

$$S_{\text{CS}} = i \frac{k}{4\pi} \int d^3x \epsilon^{\mu\nu\lambda} \text{tr} (A_\mu A_{\lambda,\nu} - \frac{2}{3} A_\mu A_\nu A_\lambda)$$

- ▶ (bare) Chern-Simons level is quantized: $k \in \mathbb{Z}$
- ▶ dynamical quarks (say N_f with masses m_f) yield shift

$$\det(D + m_f) \stackrel{|m_f| \ll 1}{\approx} |\det(D + m_f)| e^{i\pi\eta(D)/2}$$

with the Dirac operator: $D = \gamma^\mu (\partial_\mu - iA_\mu)$
and η -invariant: $\eta(D) = \text{tr sign}(-iD)$

- ⇒ renormalized Chern-Simons level: $k_{\text{rn}} = k - N_f/2$

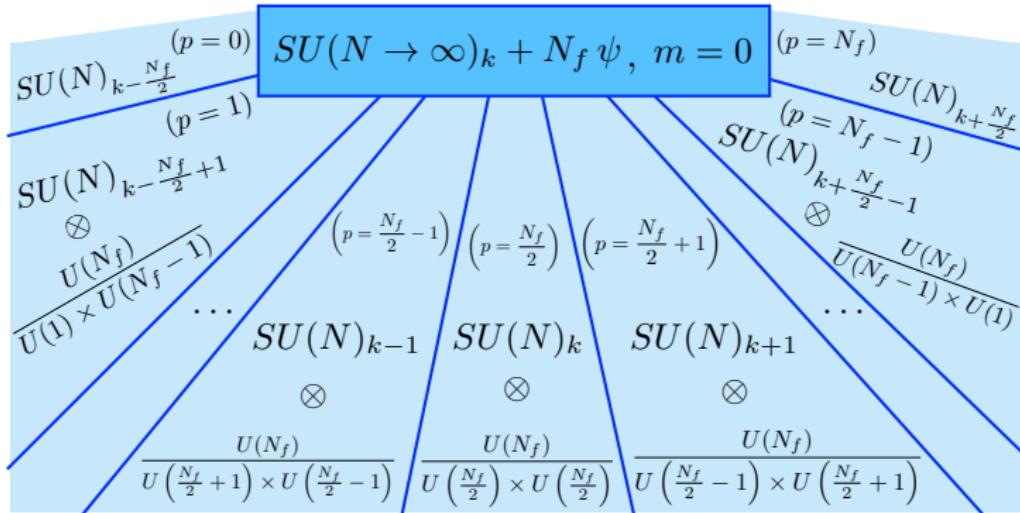
Sign problem has two sources!

What has been observed?

- ▶ symmetry breaking pattern at fixed k_{rn}

$$U(N_f) \rightarrow U(\lceil N_f/2 + k_{\text{rn}} \rceil) \times U(\lfloor N_f/2 - k_{\text{rn}} \rfloor)$$

- ⇒ first order phase transitions when varying large quark masses from positive to negative sign
- ▶ different phases also exist at vanishing masses



What does the spectral statistics look like?

**Monte Carlo simulations
do not work (sign problem)!**

We need an analytically feasible model!

Random Matrix Model



Drastic Simplification

- ▶ Dirac operator $\longrightarrow iH \in i \text{ Herm}(N)$
- ▶ Yang-Mills: $e^{-S_{\text{YM}}} \longleftrightarrow e^{-N \text{tr } H^2/2}$
- ▶ Chern-Simons: $e^{-S_{\text{CS}}} \longleftrightarrow e^{\alpha_2(\text{tr } H - 2ik)^2/2}$
- ▶ Chern-Simons level k and $\alpha_2 < 1$ can be chosen freely
- ▶ α_2 will be gauged to $\frac{N}{N + N_f + 1}$ in the end

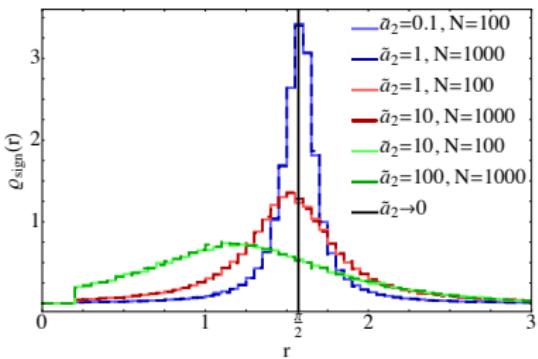
Renormalization of the Chern-Simons Level

- ▶ each of the N_f fermion determinants

$$\det(iH + m_f) \stackrel{|m_f| \ll 1}{\approx} |\det(iH + m_f)| e^{i\pi\eta(H)/2}$$

- ▶ phase from Chern-Simons part:

$$-2i\alpha_2 k \operatorname{tr} H \stackrel{N \gg 1}{\approx} -\pi i k \eta(H)$$



$$\rho_{\text{sign}}(r) = \left\langle \delta \left(r - \frac{\operatorname{tr} H}{\eta(H)} \right) \right\rangle$$

$$\alpha_2 = 1 - \tilde{\alpha}_2/N$$

$$\Rightarrow k_{\text{rn}} = k - N_f/2$$

Analytical Feasibility

- ▶ partition function:

$$Z_N^{N_f}(m; k) = \int d[H] e^{-N \text{tr} H^2/2 + \alpha_2 (\text{tr} H - 2ik)^2/2} \prod_{f=1}^{N_f} \det(iH + m_f)$$

- ▶ GUE partition function: $Z_{\text{GUE}}^{N_f}(m) = Z_N^{N_f}(m; k)|_{\alpha_2=0}$
- ▶ relation to GUE:

$$Z_N^{N_f}(m; k) \propto \int dx \exp \left[-\frac{(N_f + 1)N}{2} x^2 + 2i N k x \right] Z_{\text{GUE}}^{N_f}(m - ix)$$

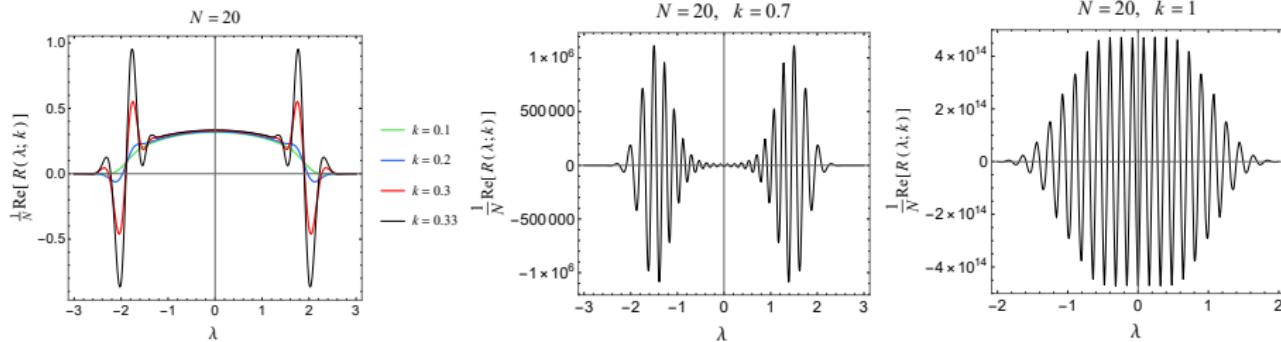
Similar for other spectral quantities!

Level Density

$$R(\lambda; M; k) \propto \exp \left[-\frac{\gamma N}{2} \lambda^2 - 2iN(1-\gamma)k\lambda \right] \prod_{f=1}^{N_f} (i\lambda + m)$$
$$\times \frac{Z_{N-1}^{N_f+2}(\text{diag}(m - i\gamma\lambda, -i\gamma\lambda, -i\gamma\lambda); k)}{Z_N^{N_f}(\text{diag}(m); k)}$$

with $\gamma = \frac{N_f + 1}{N_f + 2}$

Non-linear Sigma Model (Existence and Effects)



Quenched system ($N_f = 0$)

Technical Part (Map to Dual Space)

Evaluation of the partition function with supersymmetry method (Hubbard-Stratonovich)

$$Z_N^{N_f}(\text{diag}(m); k) \propto \int d[\tilde{H}] \exp \left[-\frac{N}{2} (\text{tr } \tilde{H} - 2k)^2 - \frac{N}{2} \text{tr } \tilde{H}^2 \right] \\ \times \det^N [\tilde{H} + \text{diag}(m_1, \dots, m_{N_f})]$$

with $\tilde{H} \in \text{Herm}(N_f)$

Action to be minimized

$$S = \frac{1}{2} (\text{tr } \tilde{H} - 2k)^2 + \frac{1}{2} \text{tr } \tilde{H}^2 - \text{tr } \ln(\tilde{H})$$

because of small masses $m_f \propto 1/N$

Technical Part (Saddle point analysis)

Saddle point equation

$$\frac{1}{\lambda_j} - \lambda_j = \sum_{j=1}^{N_f} \lambda_j - 2k = \widehat{2\lambda_0}$$

auxiliary variable

with λ_j eigenvalues of $\tilde{H} \in \text{Herm}(N_f)$

Saddle point solutions

$$\lambda_j = \lambda_0 + L_j \sqrt{\lambda_0^2 + 1}, \text{ with } L_j = \pm 1 \text{ and } \lambda_0 = \lambda_0(k, N_f, k_L)$$

$$\text{with } 2k_L = \sum_{j=1}^{N_f} L_j$$

Global minimum at those k_L closest to $k!$

Non-Linear Sigma Model

Global minimum at those k_L closest to $k!$

Lagrangian for zero-momentum part = RMT model

$$\mathcal{L}(U) = \omega \operatorname{tr} \begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_{N_f} \end{pmatrix} U \begin{pmatrix} \mathbf{1}_{\lceil N_f/2+k_L \rceil} & 0 \\ 0 & -\mathbf{1}_{\lfloor N_f/2-k_L \rfloor} \end{pmatrix} U^\dagger$$

with scaling factor $\omega = \sqrt{\lambda_0^2 + 1}$

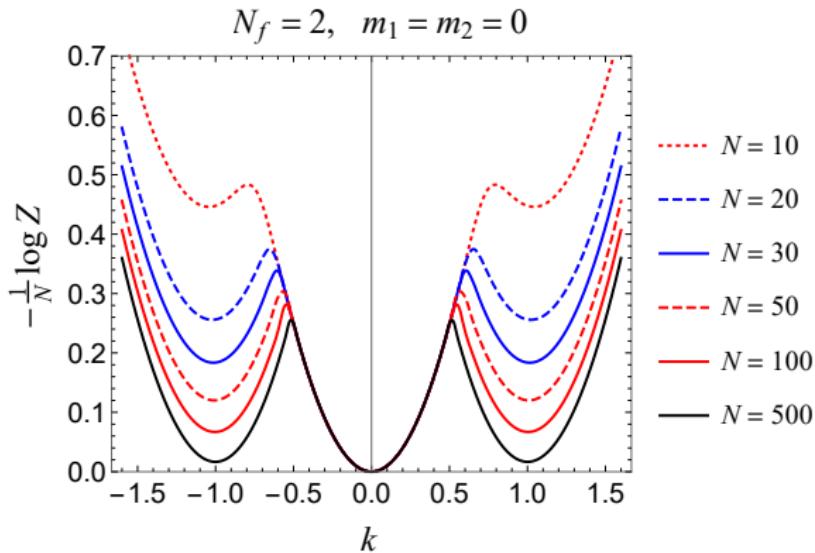
$$U \in \mathrm{U}(N_f)$$



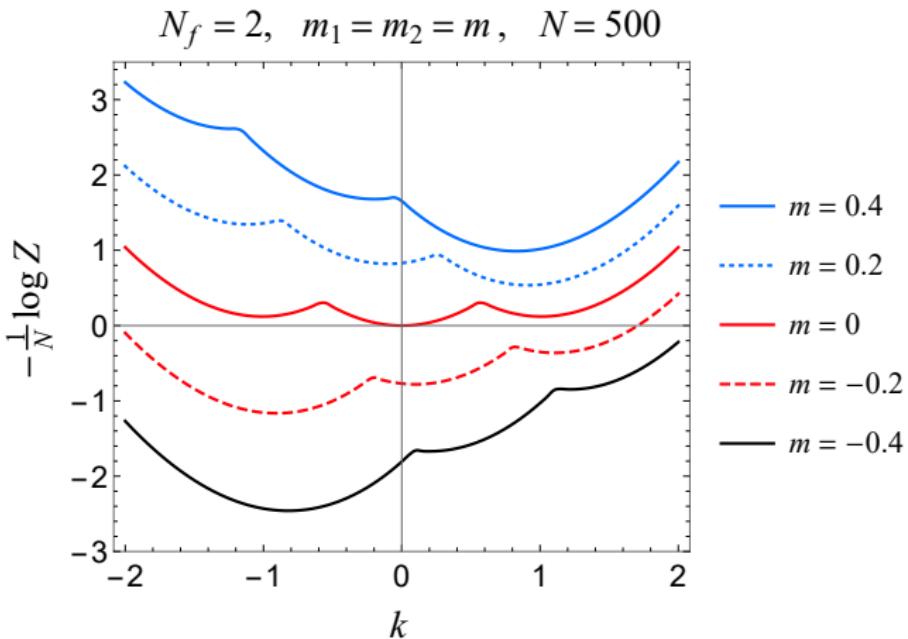
$$U \begin{pmatrix} \mathbf{1}_{\lceil N_f/2+k_L \rceil} & 0 \\ 0 & -\mathbf{1}_{\lfloor N_f/2-k_L \rfloor} \end{pmatrix} U^\dagger \in \frac{\mathrm{U}(N_f)}{\mathrm{U}(\lceil N_f/2+k_L \rceil) \times \mathrm{U}(\lfloor N_f/2-k_L \rfloor)}$$

Phase Transition Points

$$k = \left[1 + \frac{1}{24} \frac{1}{(N_f + 1)^2} + \frac{17}{1920} \frac{1}{(N_f + 1)^4} + \dots \right] \left(n + \frac{1}{2} \right), \quad n \in \mathbb{Z}$$



Masses of Order 1 Shift Points!



Curves are shifted for better visibility

Problem for Spectral Density

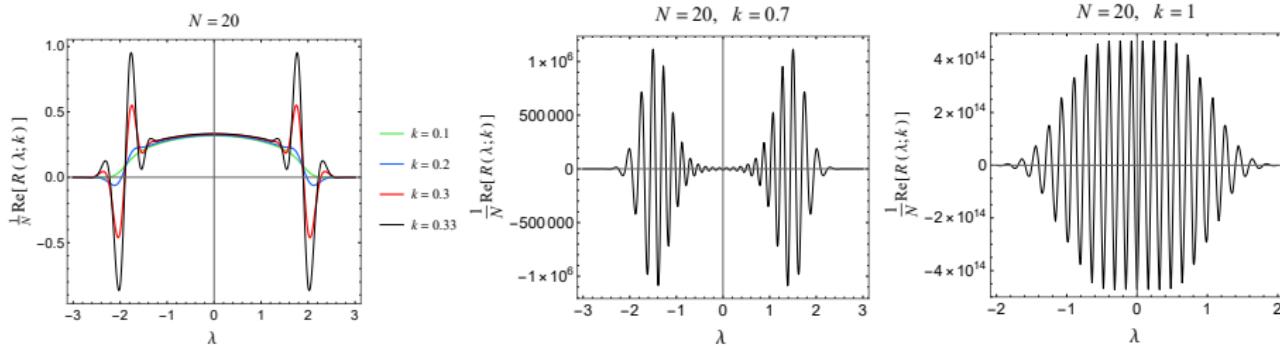
$$R(\lambda; M; k) \propto \exp \left[-\frac{(N_f + 1)N}{2(N_f + 2)} \lambda^2 - i \frac{2N}{N_f + 2} k \lambda \right] \prod_{f=1}^{N_f} (i\lambda + m)$$
$$\times \frac{Z_{N-1}^{N_f+2}(\text{diag}(m - i\gamma\lambda, -i\gamma\lambda, -i\gamma\lambda); k)}{Z_N^{N_f}(\text{diag}(m); k)}$$

- ▶ $Z_N^{N_f}$ has N_f transition points
- ▶ $Z_{N-1}^{N_f+2}$ has $N_f + 2$ transition points

Microscopic limit does not exist for $|k| > (N_f + 1)/2!$

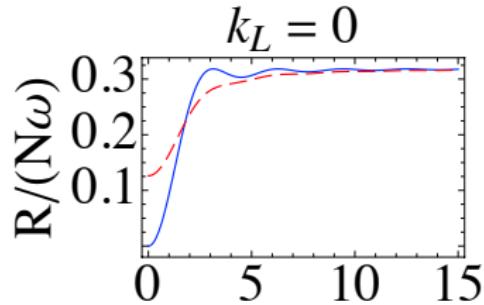
Problem for Spectral Density

Microscopic limit does not exist for $|k| > (N_f + 1)/2!$

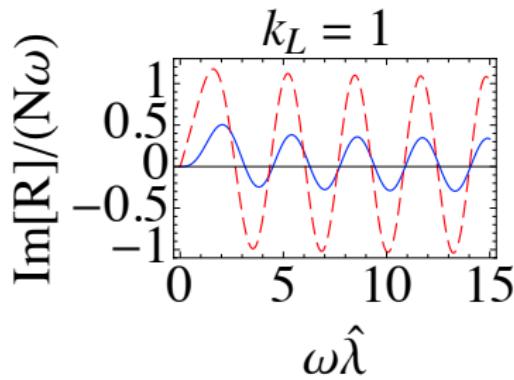
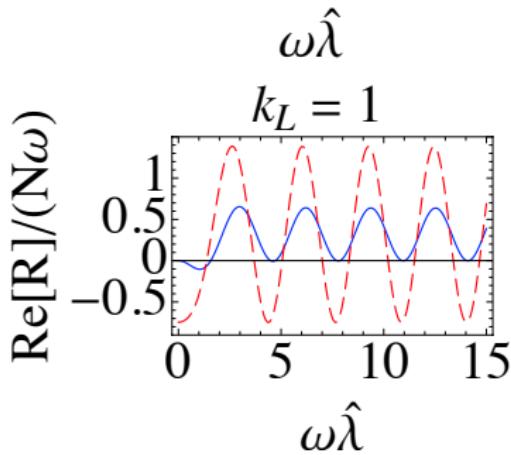


$$N_f = 0 \quad \implies \quad k_{\text{critical}} \approx 0.52$$

Spectral Density ($N_f = 2$)



= GUE-result
 $U(2) \rightarrow U(1) \times U(1)$



$\omega = \sqrt{\lambda_0^2 + 1}$ a scale factor; $Nm_1 = Nm_2 = 0$; $Nm_1 = -Nm_2 = 1.5$

Conclusions

Considered by us:

- ▶ complex representation:

$$U(N_f) \rightarrow U(\lceil N_f/2 + k_L \rceil) \times U(\lfloor N_f/2 - k_L \rfloor)$$

- ▶ real representation:

$$USp(2N_f) \rightarrow USp(\lceil N_f + 2k_L \rceil) \times USp(\lfloor N_f - 2k_L \rfloor)$$

- ▶ quaternion representation:

$$O(N_f) \rightarrow O(\lceil N_f/2 + k_L \rceil) \times O(\lfloor N_f/2 - k_L \rfloor)$$

Academic (Mathematical) Question:

- ▶ What is with the general setting?

- ▶ G a Lie group

- ▶ H a subgroup of G

- ▶ Is there a (signed) random matrix model with the spontaneous breaking of symmetry $G \rightarrow H$?

Many Thanks for your attention!

- ▶ Z. Komargodski and N. Seiberg [arXiv:1706.08755]
- ▶ T. Kanazawa, M. Kieburg, and J.J.M. Verbaarschot [arXiv:1904.03274]
- ▶ A. Armoni, T.T. Dumitrescu, G. Festuccia, and Z. Komargodski [arXiv:1905.01797]

Thanks to Cecile



Thanks to Cecile



Thanks to Kim and Tilo



Thanks to ECT*



**Especially to
Susan Driessen
and
and Jochen Wambach**

Thanks to All Participants



I Enjoyed your Company



During this Workshop!

You always inspire us, Jac!



Your Friends wish



you and Cecile all the best!