Enter the Eigenvectors!

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RMT in Sub Atomic Physics and Beyond
A Conference in Honor of Jac Verbaarschot's 65th Birthday
ECT*, Trento, August 8th, 2019

Jac and me

- in February 1987 I met Jac in Kraków (Skyrmions and Anomalies Workshop at the Mogilany Palace)
- from 1987-1989 we were postdocs at NTG Stony Brook
- from 1992-1994 we were again together at NTG Stony Brook
- in 1995 I became aware of the power of freeness for RMT
- since then, we meet each other on various RMT meetings

Port Jefferson, LI, 1987?

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Not only BBQ...

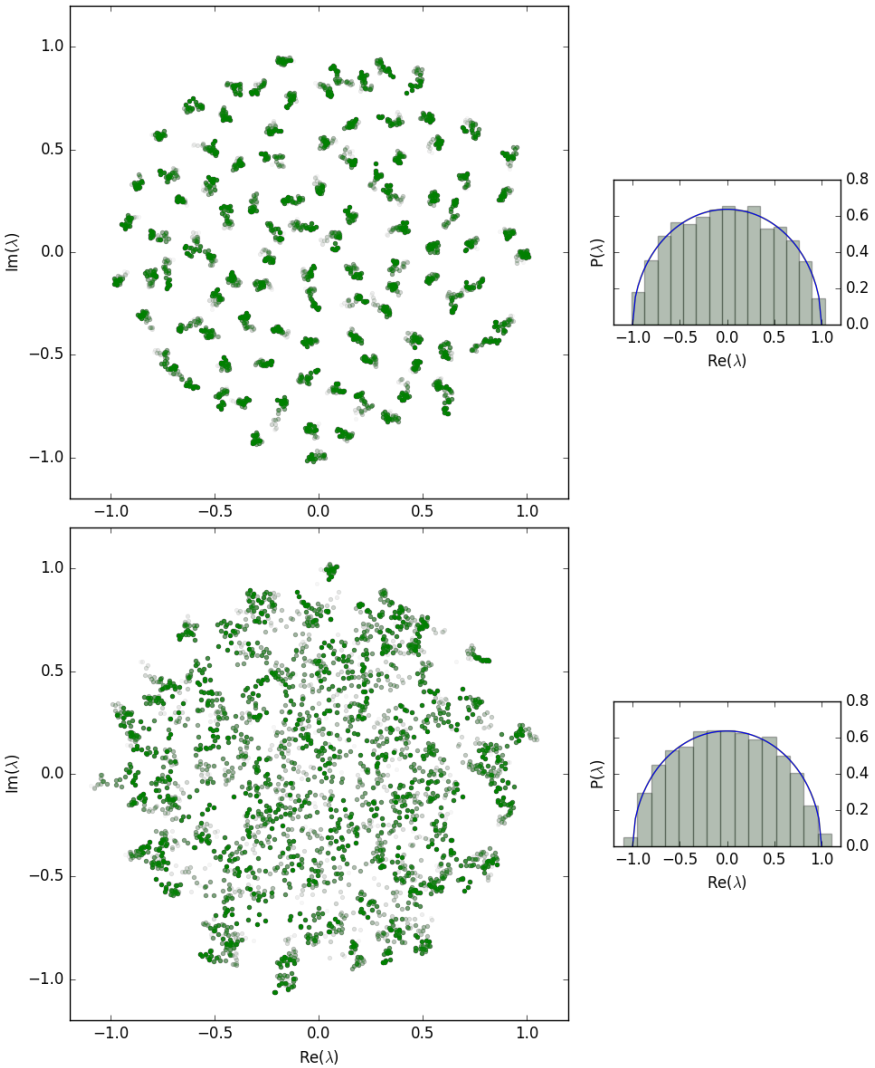
1. *Flavor Mixing in the Instanton Vacuum* M.A. Nowak, J.J.M. Verbaarschot, I. Zahed (SUNY, Stony Brook). Aug 15, 1988. 33 pp. Published in Nucl.Phys. B324 (1989) 1-33
2. *Pseudoscalars in the Instanton Liquid Model* R. Alkofer, M. A. Nowak (SUNY, Stony Brook), J.J.M. Verbaarschot (CERN), I. Zahed (SUNY, Stony Brook). Sep 1989. 5 pp. Published in Phys.Lett. B233 (1989) 205-209
3. *Instantons and Chiral Dynamics* M. A. Nowak (SUNY, Stony Brook), J.J.M. Verbaarschot (CERN), I. Zahed (SUNY, Stony Brook). Jun 1989. 8 pp. Published in Phys.Lett. B228 (1989) 251-258
4. *OZI Rule and Instantons* M. A. Nowak (SUNY, Stony Brook), J.J.M. Verbaarschot (CERN), I. Zahed (SUNY, Stony Brook). May 1989. 5 pp. Published in Phys.Lett. B226 (1989) 382-386
5. *Chiral Fermions in the Instanton Vacuum at Finite Temperature* M. A. Nowak, J.J.M. Verbaarschot, I. Zahed (CERN SUNY, Stony Brook). Jan 1989. 12 pp. Published in Nucl.Phys. B325 (1989) 581-592
6. *Is the Nucleon Strange?* M. A. Nowak (SUNY, Stony Brook), J.J.M. Verbaarschot (CERN), I. Zahed (SUNY, Stony Brook). Sep 19, 1988. 5 pp. Published in Phys.Lett. B217 (1989) 157-161
7. *Chiral symmetry breaking and instantons* M. A. Nowak, J.J.M. Verbaarschot, I. Zahed (SUNY, Stony Brook). 1996. 10 pp. Published in In *Columbus 1988, Relativistic nuclear many-body physics* 144-153
8. *Finite temperature correlators in the Schwinger model* A. Fayyazuddin, T.H. Hansson (Stockholm U.), M. A. Nowak, J.J.M. Verbaarschot, I. Zahed (SUNY, Stony Brook). Dec 1993. 33 pp. Published in Nucl.Phys. B425 (1994) 553-578

All the Best to You and Cecile, from Ewa and me...

Plan

- Why to look at eigenvectors?
- Proper objects to look at
- Main results (FRV)
- A brief overview on the derivation
- Applications
- Conclusions and perspectives

Coulomb gas versus Ginibre Ensemble [Grela, Warchoł; 2018]



Allegory of the Cave [Plato, Republic, 380BC]

Plato realizes that the humankind can think, speak etc without (so far as they acknowledge) any awareness of the realms of the **Forms**

In the allegory, Plato likens people untutored in the Theory of Forms to prisoners chained in a cave, unable to turn their heads. All they can see is the wall of the cave. Behind them burns a fire. Between the fire and the prisoners there is a parapet, along which puppeteers can walk. The puppeteers, who are behind the prisoners, hold up puppets that cast shadows on the wall of the cave. The prisoners are unable to see these puppets, the real objects, that pass behind them. What the prisoners see and hear are shadows and echoes cast by objects that they do not see.

Plato's Allegory of the Cave, Jan Saenredam, 1605

Challenge of the main paradigm of the random matrix models....

Setting the stage: reminder from algebra

A matrix X is non-normal iff $XX^\dagger \neq X^\dagger X$.
If a non-normal matrix can be diagonalized, it possesses two sets of eigenvectors: right $|R_k\rangle$ (column) and left $\langle L_k|$ (rows), satisfying eigenequations

$$\langle L_k| X = \langle L_k| \lambda_k, \qquad X |R_k\rangle = \lambda_k |R_k\rangle$$

The diagonalization is via similarity transformation $X = S\Lambda S^{-1}$ with S and S^{-1} encoding eigenvectors $X = \sum_k |R_k\rangle \lambda_k \langle L_k|$.
The eigenvectors are not orthogonal $\langle R_k|R_l\rangle \neq \delta_{kj}$ but biorthogonal $\langle L_k|R_j\rangle = \delta_{kj}$ ($\Leftrightarrow S^{-1}S = \mathbf{1}$).
Resolution of identity $\sum_k |R_k\rangle \langle L_k| = \mathbf{1}$ ($\Leftrightarrow SS^{-1} = \mathbf{1}$).

Non-hermitian case - large N - electrostatic analogy

Analytic methods break down, since spectra are complex
$\rho(z) = \frac{1}{N}\left\langle \sum_i \delta^{(2)}(z-\lambda_i)\right\rangle$.

- Electrostatic potential [Girko;1984],[Brown;1986],[Sommers et al.;1988]
  $\phi(z,\bar{z}) \equiv \lim_{\epsilon\to 0}\lim_{N\to\infty}\left\langle \frac{1}{N}\mathrm{tr}\ln[|z-X|^2+\epsilon^2]\right\rangle$
- Green's function (electric field)
  $g = \partial_z\phi = \lim_{\epsilon\to 0}\lim_{N\to\infty}\left\langle \frac{1}{N}\mathrm{tr}\frac{\bar{z}-X^\dagger}{|z-X|^2+\epsilon^2}\right\rangle$
- Gauss law $\rho(z,\tau) = \frac{1}{\pi}\partial_{\bar{z}}g|_{\epsilon=0} = \frac{1}{\pi}\frac{\partial^2\phi}{\partial z\partial\bar{z}}|_{\epsilon=0}$
  Proof: $\delta^{(2)}(z) = \lim_{\epsilon\to 0}\frac{1}{\pi}\frac{\epsilon^2}{(|z|^2+\epsilon^2)^2}$

Linearization trick [Janik, MAN, Papp, Wambach, Zahed; 1997]

- $\phi(z,\bar{z}) \equiv \lim_{\epsilon\to 0}\lim_{N\to\infty}\left\langle \frac{1}{N}\mathrm{tr}\ln[|z-X|^2+\epsilon^2]\right\rangle$
  $= \lim_{\epsilon\to 0}\lim_{N\to\infty}\left\langle \frac{1}{N}\ln D_N\right\rangle$ where
  $D_N(z,\bar{z},\epsilon) = \det(Z\otimes \mathbf{1}_N - \mathcal{X})$ with
  $$Z = \begin{pmatrix} z & i\epsilon \\ i\epsilon & \bar{z} \end{pmatrix} \qquad \mathcal{X} = \begin{pmatrix} X & 0 \\ 0 & X^\dagger \end{pmatrix}$$
- $\mathcal{G}(z,\bar{z}) = \frac{1}{N}\left\langle \mathrm{btr}\frac{1}{(Z-\mathcal{X})}\right\rangle = \begin{pmatrix} \mathcal{G}_{11} & \mathcal{G}_{1\bar{1}} \\ \mathcal{G}_{\bar{1}1} & \mathcal{G}_{\bar{1}\bar{1}} \end{pmatrix}$
  $$\mathrm{btr}\begin{pmatrix} A & B \\ C & D \end{pmatrix}_{2N\times 2N} \equiv \begin{pmatrix} \mathrm{tr}A & \mathrm{tr}B \\ \mathrm{tr}C & \mathrm{tr}D \end{pmatrix}_{2\times 2}$$
- $\mathcal{G}_{11} = g(z,\bar{z})$ yields spectral function
- $\mathcal{G}_{1\bar{1}}\cdot\mathcal{G}_{\bar{1}1}$ yields elements of a certain eigenvector correlator [Savin,Sokolov; 1997],[Chalker,Mehlig;1998].

Loophole in the standard arguments?

- For non-hermitian matrices X, we have left and right eigenvectors $X = \sum_k \lambda_k |R_k >< L_k|$ where $X|R_k >= \lambda_k |R_k >$ and $< L_k|X = \lambda_k < L_k|$
- $< L_j|R_k >= \delta_{jk}$, but $< L_i|L_j > \neq 0$ and $< R_i|R_j > \neq 0$.
- $D_N = \det(Z - \mathcal{X}) = \det[S^{-1}(Z - \mathcal{X})S] =$ $\det \begin{pmatrix} z\mathbf{1}_N - \Lambda & -i\epsilon L^\dagger L \\ i\epsilon R^\dagger R & \bar{z}\mathbf{1}_N - \bar{\Lambda} \end{pmatrix}$
- Spectrum (Λ) entangled with diagonal part of the overlap of eigenvectors $O_{ij} \equiv < L_i|L_j >< R_j|R_i >$.
- Naive limit $\epsilon \to 0$ kills the entanglement leading to incomplete description of the non-hermitian RM

Cure: Hidden variable

We promote $i\epsilon$ to full, complex-valued dynamical variable.

Then, *orthogonal direction* w unravels the eigenvector correlator $O(z) = \frac{1}{N^2}\left\langle \sum_k O_{kk}\delta^{(2)}(z-\lambda_k)\right\rangle$, where $O_{ij} =< L_i|L_j >< R_j|R_i >$ and $|L_i >$ ($|R_i >$) are left (right) eigenvectors of X. [Janik, MAN, Noerenberg, Papp, Zahed; 1999]

Replacing $Z = \begin{pmatrix} z & i\epsilon \\ i\epsilon & \bar{z} \end{pmatrix}$ by **quaternion** $Q = \begin{pmatrix} z & i\bar{w} \\ iw & \bar{z} \end{pmatrix}$ provides algebraic generalization of free random variables calculus for nonhermitian RMM. [Janik, MAN, Papp,Wambach, Zahed; 1997], [Feinberg, Zee; 1997],[Jarosz, MAN; 2006], [Belinschi, Sniady, Speicher; 2015].

"Quaternionization trick"

- $\phi(z,w) \equiv \lim_{N\to\infty} \left\langle \frac{1}{N}\mathrm{tr}\ln[|z-X|^2 + w\bar{w}] \right\rangle$
- $\mathcal{G}(z,w) = \frac{1}{N}\left\langle \mathrm{btr}\frac{1}{(Q-\mathcal{X})} \right\rangle = \begin{pmatrix} \partial_z\phi & \partial_{iw}\phi \\ \partial_{i\bar{w}}\phi & \partial_{\bar{z}}\phi \end{pmatrix}$
- On top of the vector ęlectric field" $G(z,w) = \partial_z\phi$ we have second vector, "velocity" field $V(z,w) = \partial_{iw}\phi$.
- $\rho(z) = \frac{1}{\pi}\partial_{\bar{z}}G(z,0) = \frac{1}{\pi}\partial_{z\bar{z}}\phi(z,0)$ gives spectral function
- $O(z) = \frac{1}{\pi}|V(z,0)|^2$ yields Chalker-Mehlig eigenvector correlator
- $\partial_{iw}G(z,w) = \partial_z V(z,w)$ so both vector fields are intertwined.

This construction can be written formally in the resolvent form

$$\mathcal{G} = \left\langle (Q - \mathcal{X})^{-1} \right\rangle, \qquad G(z) = \frac{1}{N} \mathrm{bTr} \mathcal{G}$$

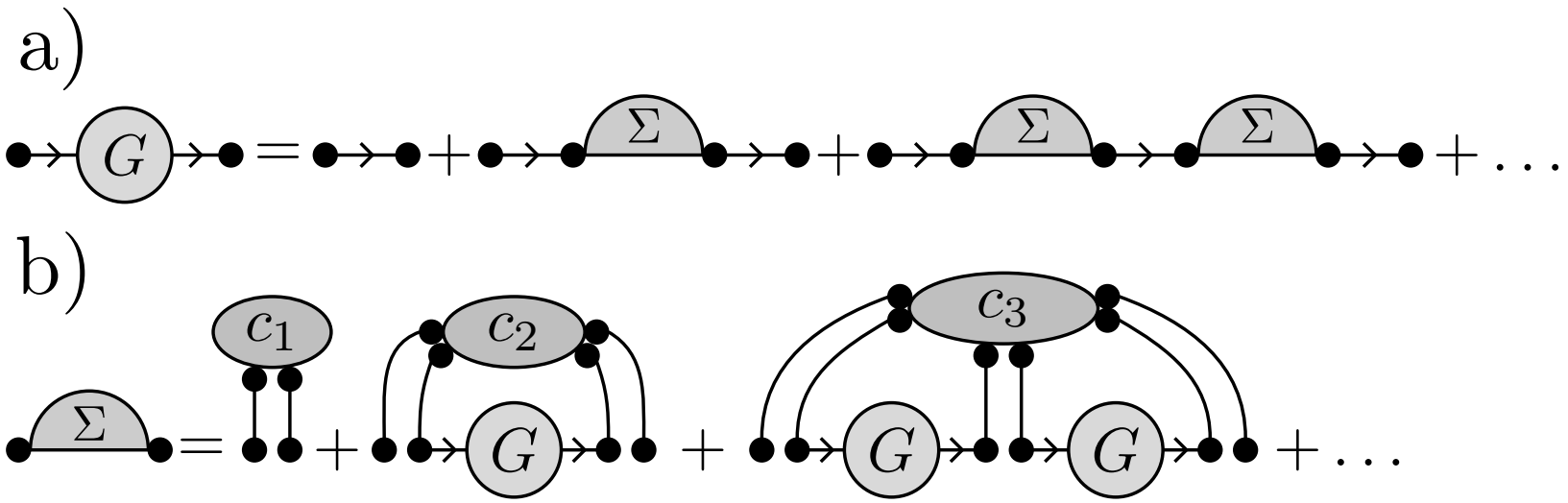
with

$$Q = \begin{pmatrix} z & i\bar{w} \\ iw & \bar{z} \end{pmatrix}, \qquad \mathcal{X} = \begin{pmatrix} X & 0 \\ 0 & X^\dagger \end{pmatrix}$$

Moment expansion

$$\mathcal{G} = Q^{-1} + \left\langle Q^{-1} \mathcal{X} Q^{-1} \right\rangle + \left\langle Q^{-1} \mathcal{X} Q^{-1} \mathcal{X} Q^{-1} \right\rangle + \dots$$

Large N limit: planar diagrams $\longrightarrow$ Schwinger-Dyson equation



Note that all ingredients are quaternions, i.e.

$$\begin{aligned} R_{\alpha\beta} &= c^{(1)}_{\alpha}\delta_{\alpha\beta} + c^{(2)}_{\alpha\beta}Q_{\alpha\beta} + \sum_{\mu\in\{1,\bar{1}\}} c^{(3)}_{\alpha\mu\beta}Q_{\alpha\mu}Q_{\mu\beta} + \\ &+ \sum_{\mu,\nu\in\{1,\bar{1}\}} c^{(4)}_{\alpha\mu\nu\beta}Q_{\alpha\mu}Q_{\mu\nu}Q_{\nu\beta} + \ldots \end{aligned}$$

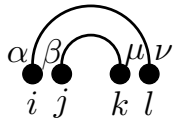
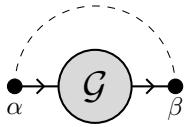
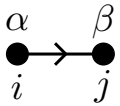
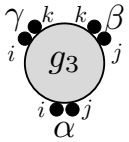
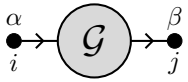
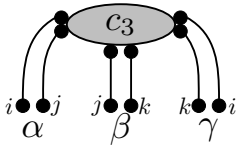
where

$$c^{(k)}_{\alpha_1\alpha_2\ldots\alpha_k} = \left\langle \frac{1}{N}\mathrm{Tr}X^{\alpha_1}X^{\alpha_2}\ldots X^{\alpha_k} \right\rangle_c$$

and we adopt notation when $\bar{1}$ index means hermitian conjugate $\dagger$, 1 means no conjugation, $Q_{11} = z$ etc.
Formally, $\Sigma(Q) = R(G(Q))$ equivalent to the 2 by 2 matrix equation

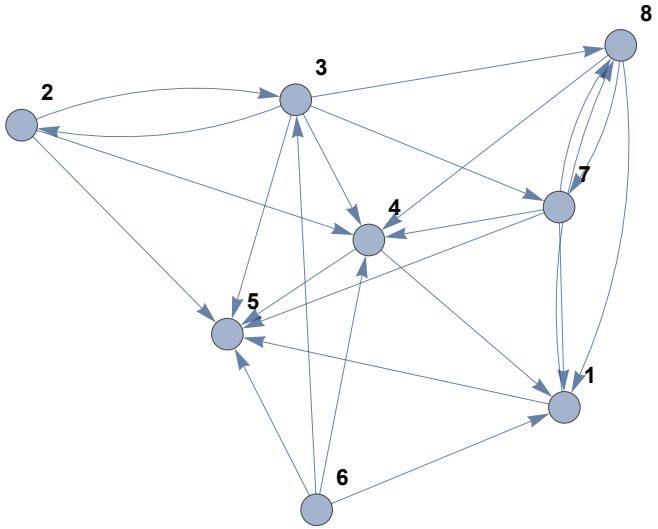
$$R(G(Q)) + \frac{1}{G(Q)} = Q$$

in analogy to one-dimensional $R(G(z)) + 1/G(z) = z$.
Note that the R-diagonal case offers a tremendous simplification.

propagator	$\langle \mathcal{X}_{\alpha\beta,ij}\mathcal{X}_{\mu\nu,jk}\rangle_G$		Green's function	$G_{\alpha\beta} = \frac{1}{N}\mathrm{bTr}\mathcal{G}$	
horizontal line	$(Q^{-1})_{\alpha\beta}\delta_{ij}$		vertex	$Ng_3X^{\alpha}_{ij}X^{\beta}_{jk}X^{\gamma}_{ki}$	
resolvent	$\mathcal{G} = \left\langle (\mathcal{Q}-\mathcal{X})^{-1}_{\alpha\beta,ij}\right\rangle$		cumulant	$\left\langle X^{\alpha}_{ij}X^{\beta}_{jk}X^{\gamma}_{ki}\right\rangle_c$	

Tablica: Diagrammatic representation of the basic expressions in the moment expansion of the resolvent. The propagator represents the averages with respect to the Gaussian potential. An exemplary vertex is drawn for the theory which contains the cubic interaction $Ng_3\mathrm{Tr}X^{\alpha}X^{\beta}X^{\gamma}$ in the potential. A cumulant (dressed vertex) represents a sum over all connected diagrams connected to the baseline. Its structure in matrix indices (Latin letters) is the same as that of the vertex, because the propagators are the Kronecker deltas in this indices. The dashed line without arrows represent summation over Latin indices only.

Stability - temporal changes of networks seen as perturbations



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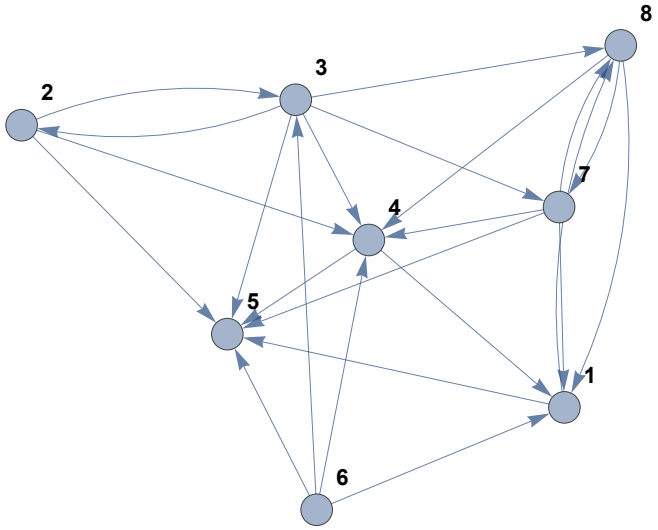

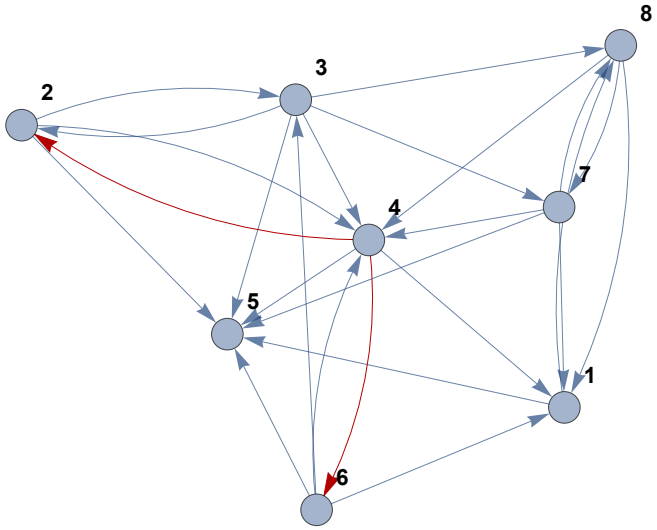

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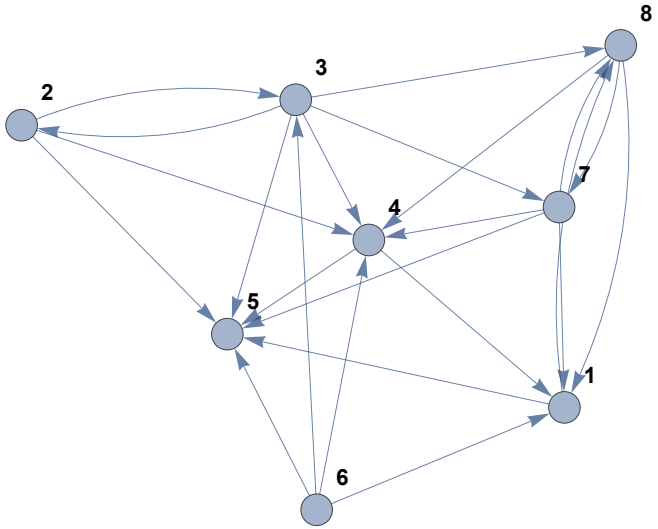

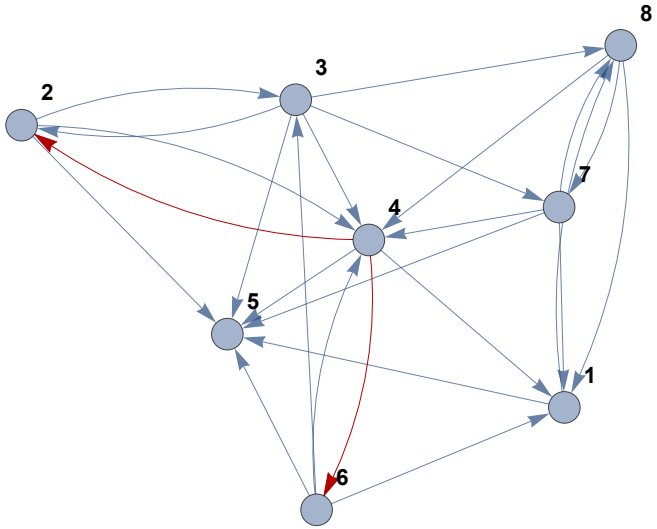

Adjacency matrix: $A \to A' = A + P$ How does the spectrum change? In first order perturbation theory

$$\lambda'_k = \lambda_k + \langle L_k | P | R_k \rangle + \mathcal{O}(||P||^2)$$

Upper bound $|\delta\lambda_k| \leqslant ||L_k|| \cdot ||R_k|| \cdot ||P|| = ||P|| \sqrt{\langle L_k | L_k \rangle \langle R_k | R_k \rangle}$.
Eigenvalue condition number [Wilkinson 1965]

Transient behavior

Dynamical system $\dot{x}_i = f_i(\vec{x})$. Linearization around some fixed point:

$$\frac{d}{dt}\delta y_i(t) = \sum_{j=1}^{N} J_{ik}\delta y_k(t)$$

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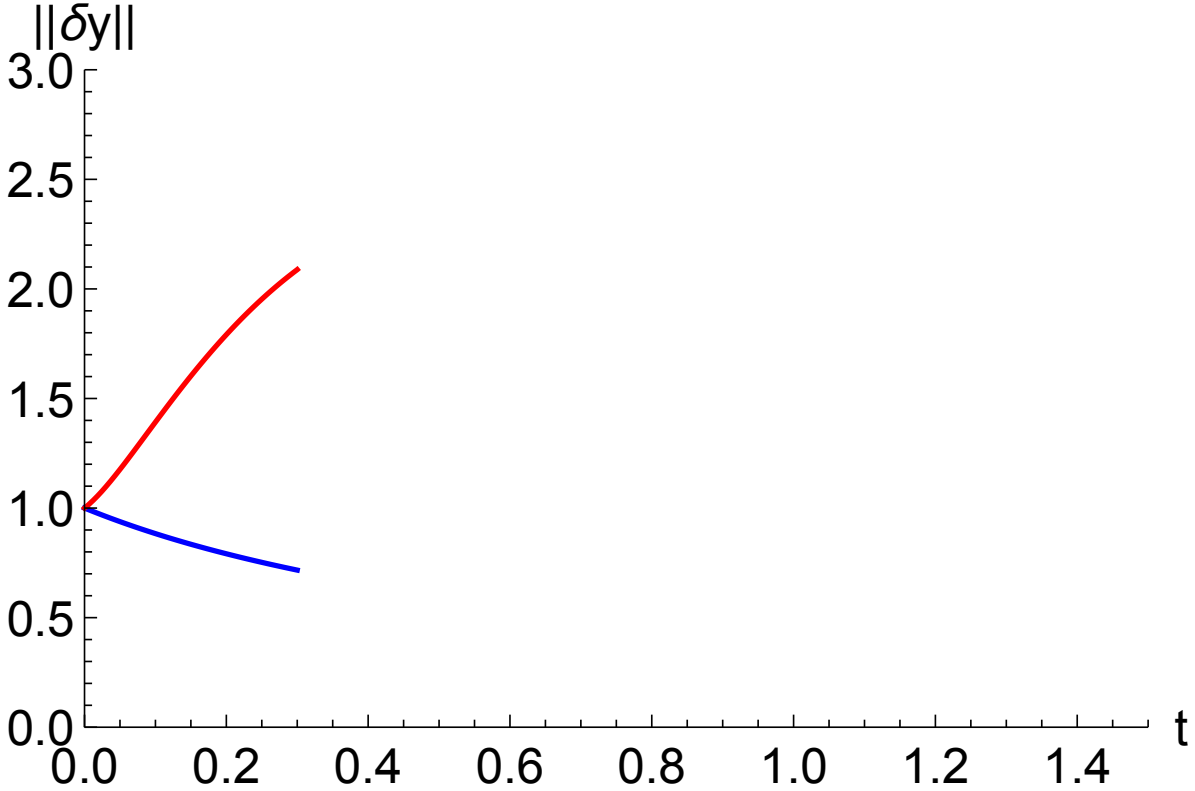

$$\delta y(0) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)^T$$

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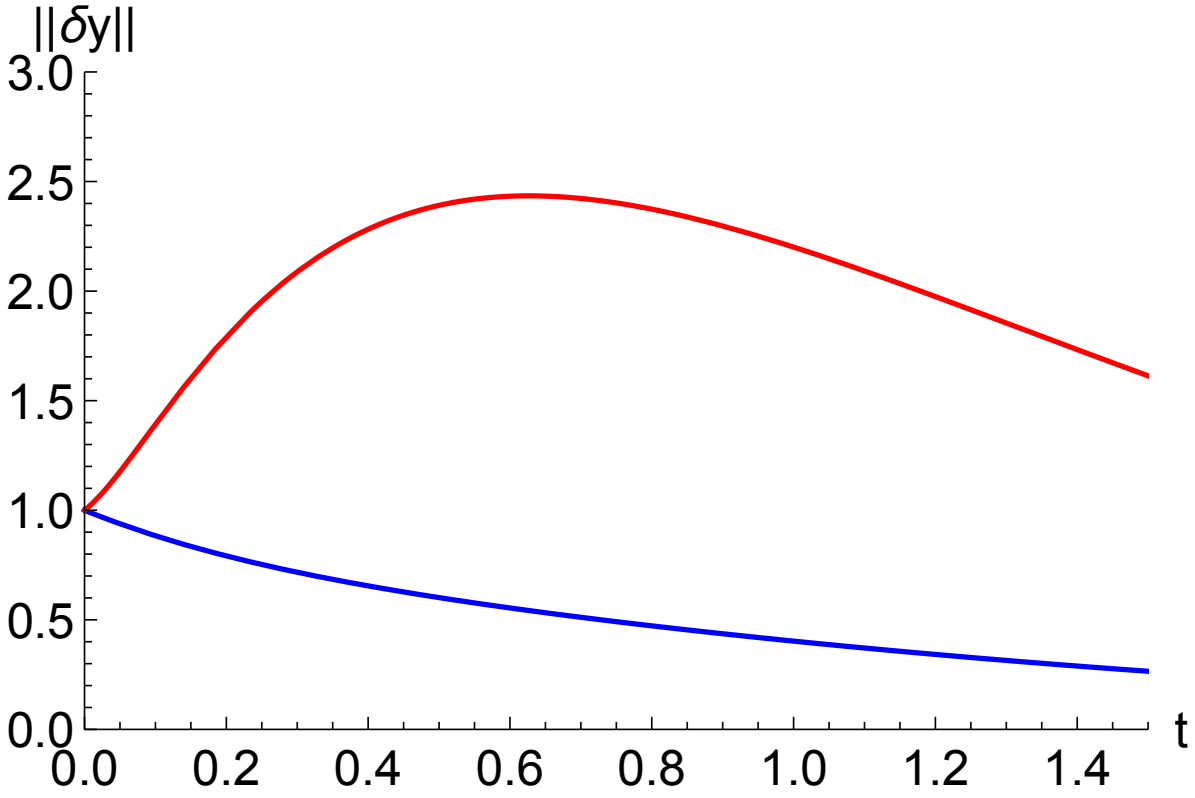

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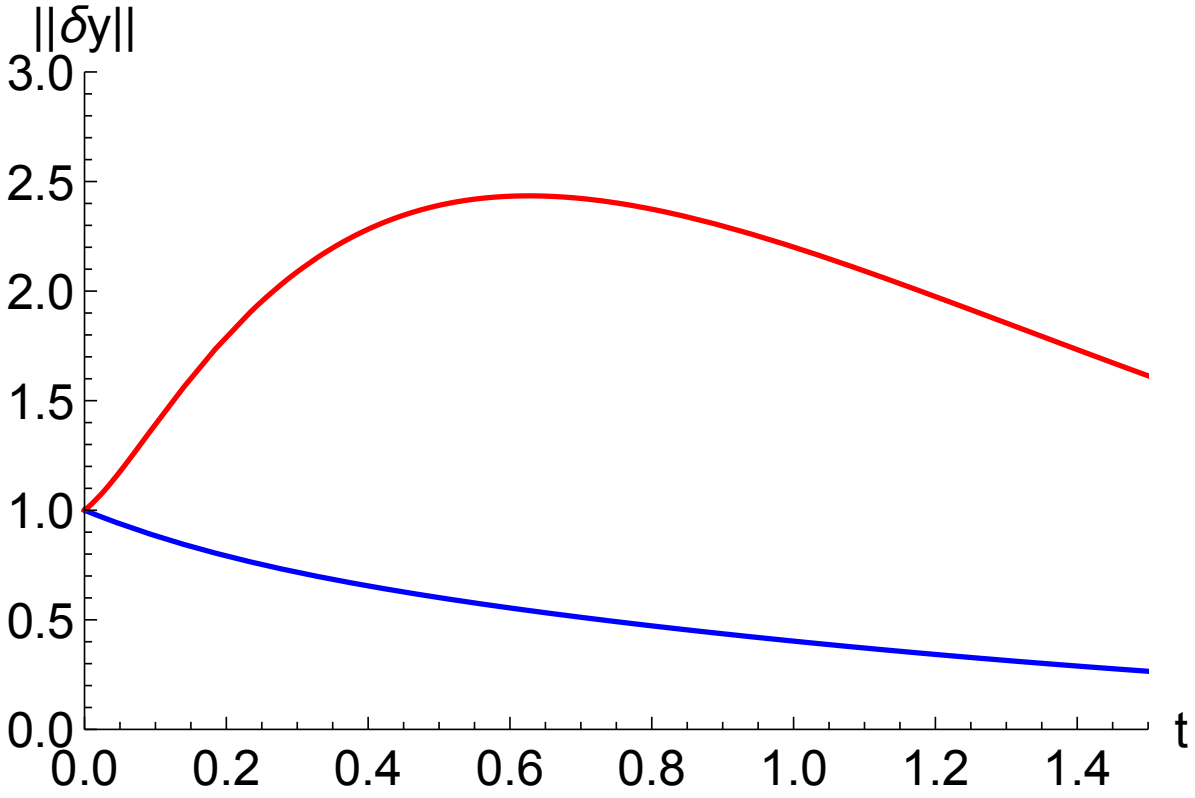


$$\delta y(0) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)^T$$

$$\begin{pmatrix} -1 & 10 \\ 0 & -2 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix}$$

Why does it happen?

Formal solution $\delta\vec{y} = e^{tJ}\delta\vec{y}_0$. Spectral decomposition yields

$$||\delta y(t)||^2 = \sum_{j,k=1}^{N} e^{(\lambda_j+\bar{\lambda}_k)t} \underbrace{\langle \delta y_0 | L_k \rangle} \langle R_k | R_j \rangle \underbrace{\langle L_j | \delta y_0 \rangle}.$$

- All eigenmodes are coupled to each other
- Imaginary part produce oscillations (interference)
- Amplification by the unrestricted norm of eigenvectors

For normal matrices (eigenvectors are orthogonal) eigenmodes decouple

$$||\delta y(t)||^2 = \sum_{k=1}^{N} e^{2t\mathrm{Re}\lambda_k} |\langle L_k | \delta y_0 \rangle|^2.$$

How to address the problem of eigenvectors correctly?

Biorthogonality $\langle L_k | R_j \rangle = \delta_{kj}$, completeness $\sum_k |R_k\rangle \langle L_k| = \mathbf{1}$
Invariant under rescaling $|R_k\rangle \to c_k |R_k\rangle$ and $\langle L_k| \to \langle L_k| c_k^{-1}$
The simplest invariant quantity: matrix of overlaps
$O_{ij} = \langle L_i | L_j \rangle \langle R_j | R_i \rangle$ Chalker Mehlig [1998]
Weighted density

$$D(z,w) = \left\langle \frac{1}{N} \sum_{j,k=1}^{N} O_{jk} \delta(z-\lambda_j)\delta(w-\lambda_k) \right\rangle = \tilde{O}_1(z)\delta(z-w) + O_2(z,w)$$

with

$$\tilde{O}_1(z) = \left\langle \frac{1}{N} \sum_k O_{kk} \delta^{(2)}(z-\lambda_k) \right\rangle \quad \left(O_1 = \frac{1}{N} \tilde{O}_1 \right),$$

$$O_2(z,w) = \left\langle \frac{1}{N} \sum_{j \neq k} O_{jk} \delta^{(2)}(z-\lambda_j) \delta^{(2)}(w-\lambda_k) \right\rangle$$

Sum rules: $\sum_j O_{ij} = 1 \Rightarrow \int d^2 w D(z,w) = \rho(z)$.
[Walter,Starr 2015]

2-point eigenvector functions and Bethe-Salpeter equations

Natural candidate

$$\mathfrak{h}(z_1, \bar{z}_2) = \frac{1}{N}\mathrm{Tr}(z_1 - X)^{-1}(\bar{z}_2 - X^\dagger)^{-1} = \frac{1}{N}\sum_{k,l} O_{kl}\frac{1}{(z_1 - \lambda_k)(\bar{z}_2 - \bar{\lambda}_l)}$$

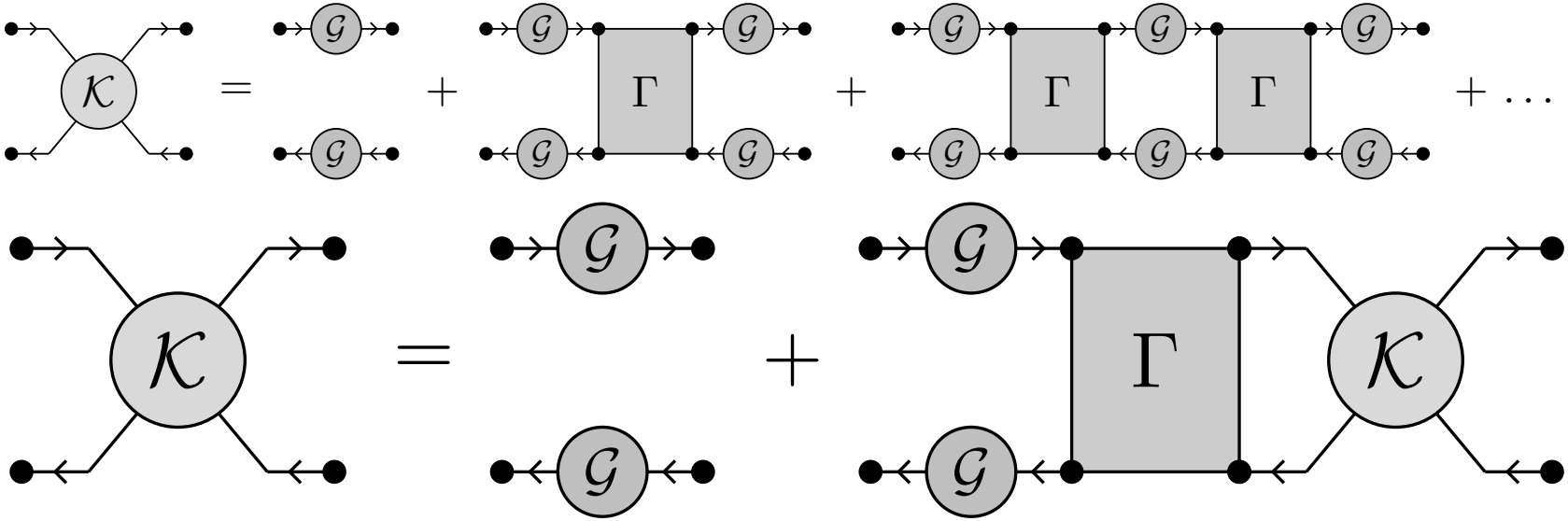
Same problems $\Rightarrow$ regularization $+$ linearization

$$\mathcal{K} = \left\langle (Q - \mathcal{X})^{-1} \otimes (P^T - \mathcal{X}^T)^{-1} \right\rangle$$

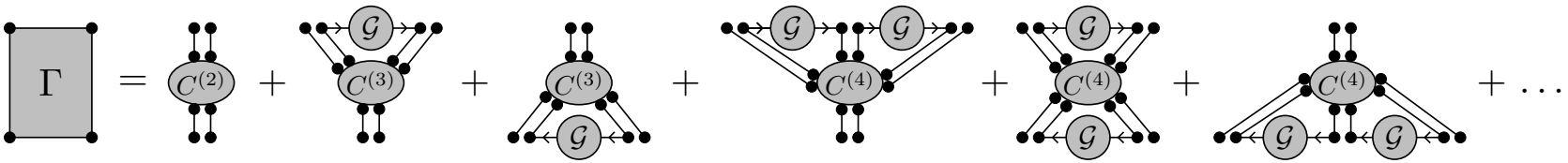
$+$ proper contraction of indices (like a block-trace) $\Rightarrow 4 \times 4$ matrix. One of its entries is the object of our interest.
Note, some analogy to the freeness of the II kind for hermitian ensembles, where $G(z_1, z_2) = \frac{1}{N^2}\mathrm{Tr}(z_1 - X)^{-1}\mathrm{Tr}(z_2 - X)^{-1}$

Details in Arxiv: [1801.02526]

Moment expansion $\longrightarrow$ planar diagrams



$$K(Q,P) = G(Q) \otimes G^T(P)\,(\mathbf{1}_4 + \Gamma(Q,P)K(Q,P))$$



Overview on the progress

- Ginibre finite N + large N Chalker, Mehlig ['98,'00]
- Quantum scattering ensemble Mehlig, Fyodorov, Frahm, Schomerus, Beenakker (et al.) ['00-'03]
- O_1 in large N for unitarily invariant matrices MAN et al. ['99]
- Eigenvector non-orthogonality can be experimentally probed Fyodorov, Savin ['11], Legrand et al. ['14]
- Crucial role in diffusion processes on matrices Kraków group ['14], Dubach, Burgade ['18], Grela, Warchoł ['18]
- O_1 for biunitarily invariant ensembles (Single ring theorem +) Belinschi, MAN, Speicher, Tarnowski ['17]
- Full distribution of O_{ii} Bourgade, Dubach ['18], Fyodorov ['17]
- O_2 in large N for unitarily invariant matrices; special case biuintarily invariant ensembles MAN, Tarnowski [JHEP06(2018)152] ← **TODAY**
- Extention to multi-point functions and calculation for the Ginibre Crawford, Rosenthal ['18]
- Determinantal structure Akemann et al ['19] ← **TODAY**

Main results

Complex matrices with unitary invariance $P(X) = P(UXU^\dagger)$

$$\underbrace{K(Q,P)}_{two-point} = \underbrace{G(Q) \otimes G^T(P)}_{1-point} \left(\mathbf{1}_4 + \underbrace{\Gamma(Q,P)}_{cumulants} K(\underbrace{Q,P}_{quaternions}) \right)$$

Traced product of resolvents

$\mathfrak{h}(z_1, \bar{z}_2) = \left\langle \frac{1}{N} \mathrm{Tr}(z_1 - X)^{-1} (\bar{z}_2 - X^\dagger)^{-1} \right\rangle$

$$\mathfrak{h}(z_1, \bar{z}_2) = \frac{\mathfrak{g}(z_1)\bar{\mathfrak{g}}(\bar{z}_2)}{1 - \mathfrak{g}(z_1)\bar{\mathfrak{g}}(\bar{z}_2)\mathcal{R}_{1\bar{1}}(\mathrm{diag}(\mathfrak{g}(z_1), \bar{\mathfrak{g}}(\bar{z}_2))}$$

Dunford-Taylor integral

$$\left\langle \frac{1}{N} \mathrm{Tr} f(X) g(X^\dagger) \right\rangle = \frac{1}{(2\pi i)^2} \oint \oint f(z_1) g(\bar{z}_2) \mathfrak{h}(z_1, \bar{z}_2) dz_1 d\bar{z}_2$$

For transients take $f = g = \exp(Xt)$.

Biunitarily invariant ensembles

pdf invariant under transformation $X \to UXV^\dagger$ with $U, V \in U(N)$.
Symmetry transformations bring to the SVD canonical form $\to$ all observables are determined by the distribution of singular values.
Spectrum is rotationally invariant $\rho(z, \bar{z}) = \rho(r = |z|)$.
$F(r) = 2\pi \int_0^r \rho(r) r dr, \quad r^2 = |z|^2$

- [Feinberg, Zee '97], [Guionnet, Krishnapur, Zeitouni, 2011] Large N: single ring theorem
- $S_{XX^\dagger}(F(r) - 1) = \frac{1}{r^2}$ Haagerup, Larsen ['00]
- Exact finite N mapping between jpdfs Kieburg, Kosters ['17]
- $O_1(r) = \frac{1}{\pi} \frac{F(r)(1-F(r))}{r^2}$ Belinschi, Speicher, MAN, Tarnowski ['17]
- $$O_2(z_1, z_2) = \frac{1}{\pi} \partial_{\bar{z}_1} \partial_{z_2} \frac{\bar{z}_1(z_1 - z_2) O_1(r_1) + z_2(\bar{z}_1 - \bar{z}_2) O_1(r_2)}{|z_1 - z_2|^2 [F(r_1) - F(r_2)]}$$
- $\mathfrak{h}(z_1, \bar{z}_2) = \frac{1}{z_1 \bar{z}_2 - r_{out}^2}$

Examples

- Ginibre

$$O_2(z_1, z_2) = \frac{-1}{\pi^2} \frac{1 - z_1 \bar{z}_2}{|z_1 - z_2|^4}$$

- Truncated unitary. $U(N+L) \to N \times N$, $N, L \to \infty$ with $\kappa = \frac{L}{N}$ fixed.

$$O_2(z_1, z_2) = \frac{1}{\pi^2} \frac{-1 + z_1 \bar{z}_2 (1+\kappa)}{|z_1 - z_2|^4}.$$

- Spherical ensemble (ratio of two Ginibres)

$$O_2(z_1, z_2) = \frac{1}{\pi^2} \frac{-1}{|z_1 - z_2|^4}$$

- Product of 2 Ginibres

$$O_2(z_1, z_2) = -\frac{1}{\pi^2} \frac{2(|z_1| + |z_2|)(z_1 \bar{z}_2 + |z_1 z_2|) - |z_1 + z_2|^2 - 4|z_1 z_2|}{4|z_1 z_2||z_1 - z_2|^4}$$

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Towards microscopic universality

Microscopic scaling is not accessible within this framework.
Hermitian models: singularities of two-point Green's functions are heralds of non-trivial scaling limits.
Sum rule:

$$N O_1(z) + \int O_2(z,w) d^2 w = \rho(z)$$

Microscopic scaling: $w = z + \frac{u}{\sqrt{N\rho(z)}}$

$$d^2 w \to \frac{d^2 u}{N\rho(z)}, \qquad O_2 = \frac{-1}{\pi^2} \frac{P(z,w)}{|z-w|^4} \to N^2 \rho^2(z) \frac{P(z,z)}{|u|^4}$$

Explicit calculations:

$$\lim_{w \to z} P(z,w) = \frac{O_1(z)}{\rho(z)}.$$

Applications: Rajan-Abbott model (2006)

- Network of N neurons represented by weighted adjacency matrix. Matematically, $X = J + M$ where deterministic $M = |u >< m|$ represents synaptic activity and random $J = X\Lambda$. Excitatory and inhibitory neurons, subjected to global balance $< m|u >= f_E\mu_E + f_I\mu_I = 0$. Additional local balance - elements of each row sum to zero, i.e. $J|u >= 0$, where $|u >= (1, 1, ...1)^T$. Spectra of J and $J + M$ are the identical.
- E. Gudowska-Nowak, MAN, D. Chialvo, J. Ochab, W. Tarnowski - hep 1805.03592 (submitted to Neural Computation). Explicit application of FRV to real neural systems. Local balance triggers huge instabilities in eigenvectors, leading to transient behavior. We conjecture, that this phenomenon is responsible for speeding the dynamics and synchronization. In real systems, local balance may come from self-organized criticality.

Conclusions

- Full formalism for calculations of the two-point eigenvector function in large N limit for complex non-Hermitian matrices with unitarily invariant pdf.
- Challenging applications I
  - NI (Natural Intelligence) - brain science (Brain Initiative (USA), Human Brain Project (EU), ...
  - AI (Artificial Intelligence, in particular deep learning)
  - NI $\leftrightarrow$ AI mutual feedback - Bio-inspired Artificial Neural Networks Programme in Kraków (2019-2022)
- Challenging applications II
  'Light in the tunnel for finite density Euclidean lattice?'[Liu, MAN, Zahed, Nucl. Phys. B (2016)]
- Unexplored mathematics of non-normal matrices, ranging from macroscopic description (FRV) to microscopic universality