Universität Bielefeld

SFB 1283

Eigenvector correlations in the complex and quaternion Ginibre ensemble: Determinantal structure and universality

Gernot Akemann

RMT in Sub Atomic Physics and Beyond: Jac *65!

ECT* Trento 08.08.2019

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+ R. Tribe, A. Tsareas & O. Zaboronski RMTA [arXiv: 1903.09016]
+ Y.-P. Förster & M. Kieburg to appear

My first visit to SUNY

QCD$_3$ and the Replica Method

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[c] The Niels Bohr Institute
Blegdamsvej 17
DK-2100 Copenhagen Ø
Denmark

Plan of the talk

I) Eigenvectors: Main questions and motivation

II) Review of Chalker and Mehlig

III) Eigenvector correlations as determinantal point processes

IV) Universal large-N limits and link to density correlations

V) Summary and open questions

Motivation: standard random matrix paradigm

- strongly coupled/ many-body Hamiltonian $\mathcal{H}$

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- complex non-Hermitian Hamiltonian/Dirac $\mathcal{H} \neq \mathcal{H}^\dagger$:

  - stability of complex systems [May 1972]
  - scattering in open quantum systems [Fyodorov, Sommers '03]
  - quantum field theories with chemical potential [Stephanov '96]

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- <u>rôle of eigenvectors:</u> [cf. talk by M. Nowak]

  - sensitive to perturbations
  - time dependent Brownian motion in Ginibre: [Burda et al. '14]

  $\rightarrow$ coupled evolution of $\lambda_\alpha, L_\alpha, R_\alpha$ ($\neq$ GUE or normal J)

Setup: Eigenvectors in the Ginibre ensembles

- complex/quaternion Ginibre ensemble
  $J_{ij} \in \mathcal{N}_{\mathbb{C}/\mathbb{H}}(0,1)$: $\langle J_{ij}\, J^*_{kl} \rangle = \delta_{ik}\delta_{jl}$, all other zero

  $N \times N$ independent Gaussian matrix elements
- distribution of all matrix elements $\boxed{\mathcal{P}(J) \sim \exp\left[-\mathrm{Tr}\, JJ^\dagger\right]}$

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- Correlations of λ_α well understood

Reminder: Complex eigenvalue correlations I [Ginibre '65]

complex Ginibre ensemble = 2D Coulomb gas $\sim |\Delta_N(\Lambda)|^{\beta=2}$

- joint eigenvalue density $P(\Lambda) \sim \prod_{\alpha=1}^{N} w(\lambda_\alpha) \prod_{j>k}^{N} |\lambda_j - \lambda_k|^2$

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- density $\rho_1(\lambda) = K_N(\lambda, \lambda)$, and $\lim_{N\to\infty} \to$ circular law

Reminder: Complex eigenvalue correlations II

[Ginibre '65, Mehta '94, Kanzieper '02]

<u>quaternion Ginibre ensemble</u> $\neq$ |Vandermonde|^4

- joint eigenvalue density
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- example density $\rho_1(\lambda) = |\lambda - \lambda^*| w(\lambda) \kappa_N(\lambda, \lambda)$,

  large-N limit = circular law for Gauß

Approach of Chalker & Mehlig

- consider expectation values $\langle \mathcal{O} \rangle := \int [dJ] \mathcal{O}(J) \mathcal{P}(J)$ of the **overlapp matrix** $\boxed{\mathcal{O}_{\alpha\beta} = (L_\alpha, L_\beta)(R_\alpha, R_\beta)}$

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  $$\boxed{D_{11}(z) := \langle \sum_{\alpha=1}^{N} \mathcal{O}_{\alpha\alpha} \delta(z - \lambda_\alpha) \rangle = N \langle \mathcal{O}_{11} \delta(z - \lambda_1) \rangle}$$

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- both can be expressed in terms of integrals over $\lambda_{\alpha \geq 1,2}$ for complex [Chalker, Mehlig '98, '99] & quaternion Ginibre [Förster '18]

Methods and results for alternative quantities

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- $\boxed{\langle |\mathcal{O}_{12}|^2 \rangle}$ and $\boxed{\langle \mathcal{O}_{11}\mathcal{O}_{22} \rangle}$ [Bourgade, Dubach '18]
- $\boxed{|(L_\alpha, L_\beta)|}$ correlations of angles [Beynach-Georges, Zeitouni '18]

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- **real and quaternionic Ginibre** eigenvector correlations cf. [Dubach '18; Förster '18] & products [Burda, Spisak, Vivo '16]

Joint densities - complex Ginibre

- Schur decomposition $\boxed{J = U(\Lambda + T)U^\dagger}$
  where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$ complex eigenvalues
  T complex strictly upper triangular, $U \in U(N)$
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with **Vandermonde determinant**

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- eigenvectors depend on $T \Rightarrow$ average over T nontrivial

  $\langle \mathcal{O} \rangle_T := \int [dT] \mathcal{O}(\Lambda, T) P(T)$

Performing the T-average

<u>making T-dependence explicit:</u> $J \to \begin{pmatrix} \lambda_1 & T_{12} & \dots & T_{1N} \\ 0 & \lambda_2 & & \dots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda_N \end{pmatrix}$

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- **Result:** averages over Λ and T factorise

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Performing the T-average

<u>making T-dependence explicit:</u> $J \to \begin{pmatrix} \lambda_1 & T_{12} & \dots & T_{1N} \\ 0 & \lambda_2 & & \dots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda_N \end{pmatrix}$

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- holds whenever T-average remains Gauß, e.g. in

  - induced or elliptic Ginibre, and quasi-harmonic potentials

Diagonal overlaps as complex eigenvalue averages

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\begin{aligned}
D_{11}(z_1) &= N\langle \mathcal{O}_{11}\delta(z_1-\lambda_1)\rangle \\
&= \frac{N\, e^{-|z_1|^2}}{Z_N}\int d^2\lambda_2\cdots d^2\lambda_N |\Delta_N(z_1,\lambda_2,\ldots)|^2 \\
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- same structure as complex eigenvalue correlations, with **new weight** $w_{11}(\lambda_l) = e^{-|\lambda_l|^2}\left(1+|z_1-\lambda_l|^2\right)$
- define D_{11} with more constraints $\Rightarrow D_{12}$:
  $D_{11}(z_1,z_2) = (N-1)$ above $\times\, \delta(z_2-\lambda_2)$ etc. for k-points

Off-diagonal from diagonal overlap

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\begin{aligned}
D_{12}(z_1, z_2) &= \frac{-N(N-1)e^{-|z_1|^2-|z_2|^2}}{Z_N}\int d^2\lambda_3 \cdots d^2\lambda_N |\Delta_{N-3}(\lambda_3, \ldots)|^2 \\
&\times \prod_{l=3}^{N} e^{-|\lambda_l|^2}(z_1^* - \lambda_l^*)(z_2 - \lambda_l)\Big((z_1 - \lambda_l)(z_2^* - \lambda_l^*) + 1\Big)
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- **Lemma1**[ATTZ '19]: Define $\hat{\mathcal{I}} f(z_1, z_1^*, z_2, z_2^*) = f(z_1, z_2^*, z_2, z_1^*)$

$$\Rightarrow \boxed{D_{12}(z_1, z_2) = \frac{-e^{-|z_1-z_2|^2}}{1-|z_1-z_2|^2}\hat{\mathcal{I}} D_{11}(z_1, z_2)}$$

Orthogonal polynomial approach to overlaps

- $D_{11}(z_1)$ partition function wrt weight $w_{11}(z_1, \lambda)$
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- almost like Ginibre, heuristic argument (translational invariance) leads to large-N result

Determinantal structiure of the conditional overlap

- **Theorem 1** [A,Tribe, Tsareas, Zaboronski '19]

$$\boxed{D_{11}(\lambda_1,\ldots,\lambda_k) = \frac{1}{\pi} e^{-|\lambda_1|^2} f_{N-1}(|\lambda_1|^2) \det_{\mathbf{2}\leq i,j\leq k}\left[K_{11}(\lambda_i^*,\lambda_j)\right]}$$

where $e_p(x) = \sum_{l=0}^{p} \frac{x^l}{l!}$ exponential polynomials, $x = |\lambda_1|^2$,

$f_{N-1}(x) = Ne_{N-1}(x) - xe_{N-2}(x)$, determines $k = 1$

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- exact result for finite N $\forall k$
- $D_{12}(\lambda_1,\ldots,\lambda_k)$ can be defined similarly and follows from a generalisation of Lemma 1 to all $k > 2$

Idea of the proof

- $\exists$ alternative way to express the kernel for arbitrary weight:

  moment matrix $M_{ij} := < u^i, u^j > = \int d^2\lambda (\lambda^*)^i \lambda^j w(\lambda)$

  $\Rightarrow \boxed{K(u^*, v) = (w(u)w(v))^{\frac{1}{2}} \sum_{i,j=0}^{N-1} (u^*)^i (M^{-1})_{ij} v^j}$

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  $\Rightarrow \quad P_k(u) = \sum_{l=0}^{k} (L^{*-1})_{kl} u^l = \sum_{l=0}^{k} \lambda_1^{k-l} \frac{f_l(|\lambda_1|^2)}{f_k(|\lambda_1|^2)} u^l$

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- a very tedius calculation leads to a form that is amenable to the large-N limit

Large-N limits: Global vs. local regime

- Reminder eigenvalues density correlations:

  - global density: circular law
  - local bulk kernel $K(u^*, v) = \frac{1}{\pi} \exp[-|u|^2 - |v|^2 + u^* v]$
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- **Corollary 1** Local bulk limit (origin) for conditional overlap

$$\boxed{\lim_{N\to\infty} \frac{1}{N} D_{11}(\lambda_1, \ldots, \lambda_k) = \frac{1}{\pi} \det_{2\leq i,j\leq k}[K_{11}^{\mathrm{bulk}}(\lambda_i^*, \lambda_j)]}$$

$$K_{11}^{\mathrm{bulk}}(u^*, v) = \frac{1}{\pi}(1 - |u - \lambda_1|^2) e^{-|u-\lambda_1|^2} \frac{d}{dz} \left(\frac{e^z - 1}{z} \right) \Big|_{z=(u^*-\lambda_1^*)(v-\lambda_1)}$$

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$$\boxed{\lim_{N\to\infty} \frac{1}{N} D_{11}(\lambda_1, \ldots, \lambda_k) = \frac{1}{\pi} \det_{2\le i,j\le k}[K_{11}^{\mathrm{bulk}}(\lambda_i^*, \lambda_j)]}$$

$$K_{11}^{\mathrm{bulk}}(u^*, v) = \frac{1}{\pi}(1 - |u - \lambda_1|^2) e^{-|u-\lambda_1|^2} \frac{d}{dz} \left(\frac{e^z - 1}{z} \right) \Big|_{z=(u^*-\lambda_1^*)(v-\lambda_1)}$$

- is universal throughout the bulk, agrees with kernel for complex eigenvalues of truncated unitary ensemble

Large-N limits: Global vs. local regime

- Reminder eigenvalues density correlations:

  - global density: circular law
  - local bulk kernel $K(u^*, v) = \frac{1}{\pi} \exp[-|u|^2 - |v|^2 + u^* v]$
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- example for off-diagonal overlapp: [Chalker, Mehlig '99] heuristic $D_{12}^{\mathrm{bulk}}(\lambda_1, \lambda_2) = \frac{1}{\pi^2 |\lambda_1 - \lambda_2|^4} \left(1 - (1 + |\lambda_1 - \lambda_2|^2) e^{-|\lambda_1 - \lambda_2|^2} \right)$

Large-N continued

- **Corollary 2** Local edge scaling limit for conditional overlap

$$\boxed{\lim_{N\to\infty} \frac{1}{\sqrt{N}} D_{11}\left(e^{i\theta}(\sqrt{N}+\lambda_1)\right) = \frac{e^{-\frac{1}{2}(\lambda_1+\lambda_1^*)^2} - \frac{\sqrt{2\pi}}{2}(\lambda_1+\lambda_1^*)\mathrm{erfc}\frac{(\lambda_1+\lambda_1^*)}{\sqrt{2}}}{\sqrt{2\pi^3}}}$$

- for $k \geq 2$: $\times$ det of edge kernel

Large-N continued

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- **Corollary 3** Algebraic decay of overlaps

  When all eigenvalues in the bulk have a large separation: $|\lambda_i - \lambda_j| \geq L \; \forall i \neq j = 1, \ldots, k$ we have

$$\boxed{D_{11}^{\mathrm{bulk}}(\lambda_1, \ldots, \lambda_k) = \frac{1}{\pi^k} \prod_{l=2}^{k} \left(1 - \frac{1}{|\lambda_1 - \lambda_l|^4}\right) + \mathcal{O}(e^{-L^2})}$$

Large-N continued

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- in contract to exponential decay of eigenvalue correlation functions

Limiting relation in the bulk

$$D_{11}^{\text{bulk}}(\lambda_1, \ldots, \lambda_k) =$$

$$\prod_{l=2}^{k} \left(\frac{-1 - |\lambda_1 - \lambda_l|^2}{|\lambda_1 - \lambda_l|^4} \right) \left(1 - |\lambda_1 - \lambda_l|^2 - (\lambda_1 - \lambda_l) \frac{\partial}{\partial \lambda_l} \right)$$

$$\rho_k^{\text{bulk}}(\lambda_1, \ldots, \lambda_k)$$

- this hints at the possibility that the known universality of complex eigenvalue correlation functions could be transferred to the overlaps

Summary . . .

- diagonal and off-diagonal overlapp are part of a DPP
- corresponding kernels computed explicitly for finite-N
- local large-N limits in bulk and at edge follow

. . . and open questions

- further explicit examples for which the DPP remains intact, e.g. form induced Ginibre (rectangular matrices, chiral Ginibre, quaternion $\sim$?
- the computation of the kernel at finite-N is difficult to generalise: requires inversion of moment matrix M
- $\exists$ general relation eigenvalue - eigenvector correlations for large-N $\Rightarrow$ universality?

Happy Birthday Jac !