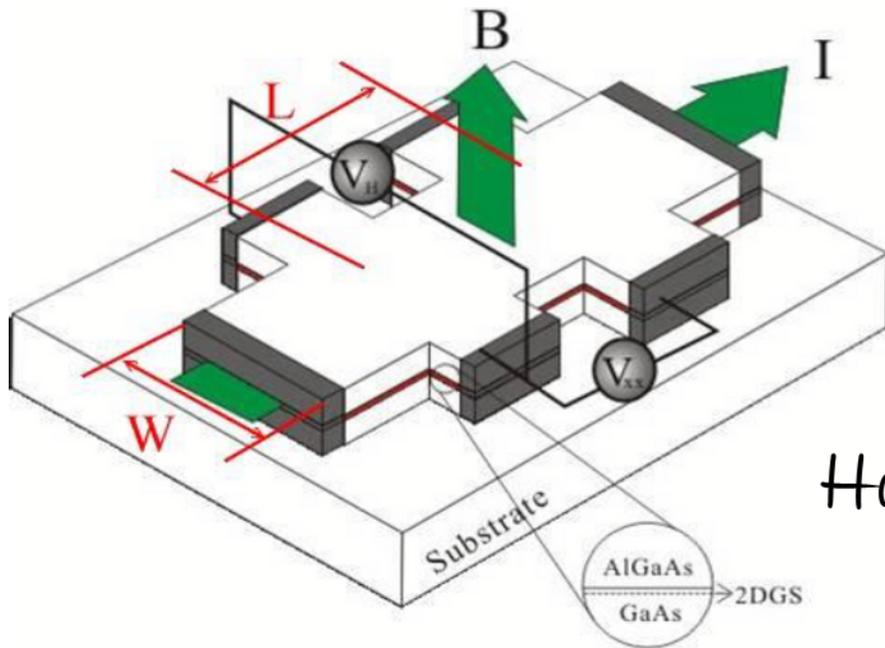


Particle-Hole Symmetries in Condensed Matter

Martin R. Zirnbauer (Cologne)
@ RMT in sub-atomic physics ...
ECT Trento (Aug 8, 2019)

Quantum Hall Effect



Two-dimensional electron gas at low temperature and in a strong magnetic field.

Hall resistance exhibits plateaus: $R_H = \frac{h}{\nu e^2}$.

$N_\phi = \frac{e}{h} \iint B = \#$ of states in lowest Landau level (LLL).

$\nu = N_e / N_\phi$ filling fraction; $\nu = 1/2$ half-filled LLL.

Particle-hole symmetry: $\sigma_{xx}(\nu) = \sigma_{xx}(1-\nu)$
is exact in the limit $\hbar\omega_c \rightarrow \infty$ and if interactions are two-body.

Particle-hole symmetry in the anomalous quantum Hall effect

S. M. Girvin

Surface Science Division, National Bureau of Standards, Washington, D.C. 20234

(Received 27 February 1984)

This paper explores the uses of particle-hole symmetry in the study of the anomalous quantum Hall effect. A rigorous algorithm is presented for generating the particle-hole dual of any state. This is used to derive Laughlin's quasihole state from first principles and to show that this state is exact in the limit $\nu \rightarrow 1$, where ν is the Landau-level filling factor. It is also rigorously demonstrated that the creation of m quasiholes in Laughlin's state with $\nu = 1/m$ is precisely equivalent to creation of one true hole. The charge-conjugation procedure is also generalized to obtain an algorithm for the generation of a hierarchy of states of arbitrary rational filling factors.

I. INTRODUCTION

The anomalous quantum Hall effect^{1,2} is one of the most striking many-body phenomena discovered in recent years. The Hall resistivity of a two-dimensional electron gas (inversion layer) in a high magnetic field at low temperatures exhibits quantized plateau values of the form $\rho_{xy} = h/e^2 i$, where i is a rational number $i = p/q$ with q odd. Associated with this quantization of the Hall resistivity is a marked decrease in the dissipation ($\rho_{xx} \rightarrow 0$). The latter suggests the

where the exponential factors have been lumped into the measure

$$d\mu(z) = \frac{dx dy}{2\pi l^2} e^{-|z|^2/2} . \quad (4)$$

Within (the N -particle version of) this space the variational wave functions proposed by Laughlin³ may be written

$$\psi_m(z_1, \dots, z_N) = \prod_{i < j} (z_i - z_j)^m , \quad (5)$$

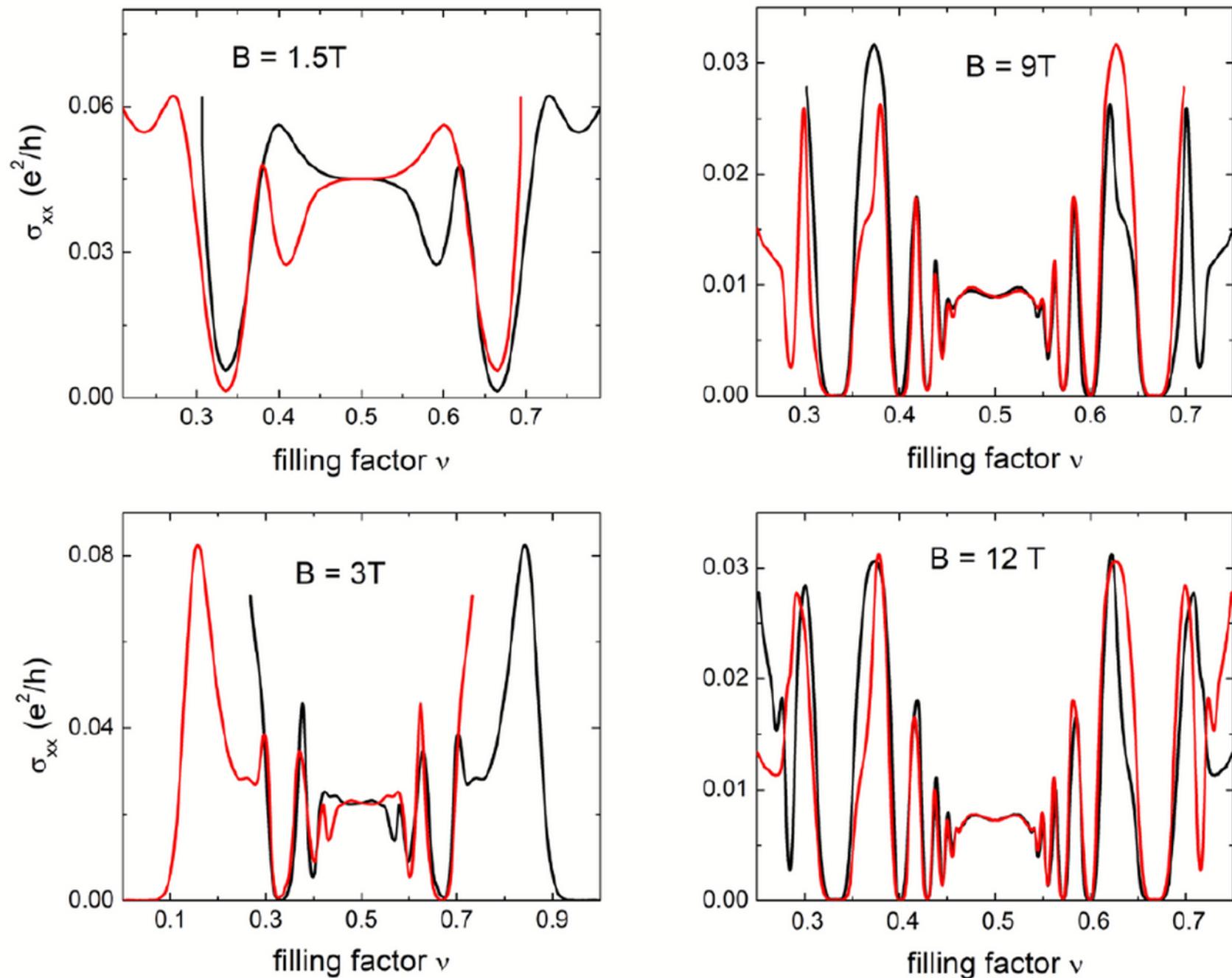


Figure 4. Examination of reflection symmetry in σ_{xx} over a large range of magnetic fields, from 1.5T to 12T. **Pan, Kang, Lilly, Reno, Baldwin, West, Pfeiffer, Tsui (March 2019)**

Is the Composite Fermion a Dirac Particle?

Dam Thanh Son

Kadanoff Center for Theoretical Physics, University of Chicago, Chicago, Illinois 60637, USA

(Received 19 February 2015; published 2 September 2015)

We propose a **particle-hole symmetric theory** of the Fermi-liquid ground state of a half-filled Landau level. This theory should be applicable for a Dirac fermion in the magnetic field at charge neutrality, as well as for the $\nu = \frac{1}{2}$ quantum Hall ground state of nonrelativistic fermions in the limit of negligible inter-Landau-level mixing. We argue that when particle-hole symmetry is exact, the composite fermion is a massless Dirac fermion, characterized by a Berry phase of π around the Fermi circle. We write down a tentative effective field theory of such a fermion and discuss the discrete symmetries, in particular, \mathcal{CP} . The Dirac composite fermions interact through a gauge, but non-Chern-Simons, interaction. The particle-hole conjugate pair of Jain-sequence states at filling factors $n/(2n+1)$ and $(n+1)/(2n+1)$, which in the conventional composite fermion picture corresponds to integer quantum Hall states with different filling factors, n and $n+1$, is now mapped to the same half-integer filling factor $n + \frac{1}{2}$ of the Dirac composite fermion. The Pfaffian and anti-Pfaffian states are interpreted as d -wave Bardeen-Cooper-Schrieffer paired states of the Dirac fermion with orbital angular momentum of opposite signs, while s -wave pairing would give rise to a particle-hole symmetric non-Abelian gapped phase. When particle-hole symmetry is not exact, the Dirac fermion has a \mathcal{CP} -breaking mass. The conventional fermionic Chern-Simons theory is shown to emerge in the nonrelativistic limit of the massive theory.

OUTLINE

0. Introduction

I. Particle-Hole Symmetry

II. Free fermions \leadsto Haldane phase

III. Half-filled lowest Landau level

The integer quantum Hall plateau transition is a current algebra after all

Martin R. Zirnbauer

Abstract

The scaling behavior near the transition between plateaus of the Integer Quantum Hall Effect (IQHE) has traditionally been interpreted on the basis of a two-parameter renormalization group (RG) flow conjectured from Pruisken's non-linear sigma model ($NL\sigma M$). Yet, the conformal field theory (CFT) describing the critical point remained elusive, and only fragments of a quantitative analytical understanding existed up to now. In the present paper we carry out a detailed analysis of the current-current correlation function for the conductivity tensor, initially in the Chalker-Coddington network model for the IQHE plateau transition and then in its exact reformulation as a supersymmetric vertex model. We develop a heuristic argument for the continuum limit of the non-local conductivity response function at criticality and thus identify a non-Abelian current algebra at level $n = 4$. Based on precise lattice expressions for the CFT primary fields we predict the multifractal scaling exponents of critical wavefunctions to be $\Delta_q = q(1 - q)/4$. The Lagrangian of the RG-fixed point theory for r retarded and r advanced replicas is proposed to be the $GL(r|r)_{n=4}$ Wess-Zumino-Witten model deformed by a truly marginal perturbation. The latter emerges from the $NL\sigma M$ by a natural scenario of spontaneous symmetry breaking.

I. On the notion of Particle-Hole Symmetry

Motivation: Tenfold Way (symmetry classes of disordered free fermions)

Altland & Z. (1996); Heinzner, Huckleberry & Z. (2004)

$$\begin{array}{c} \text{symmetry} \\ \text{group} \end{array} G \times \begin{array}{c} \text{Fock} \\ \text{space} \end{array} \mathcal{F} \longrightarrow \mathcal{F} = \wedge(V) \quad \text{fermions}$$

where $G_0 \subseteq G$ arbitrary group of unitary symmetries together with two distinguished anti-unitary generators:

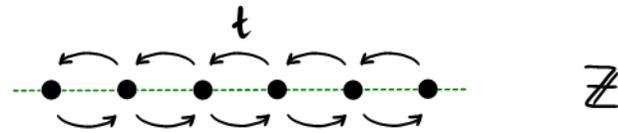
- time-reversal symmetry $T: \wedge^n(V) \longrightarrow \wedge^n(V)$
- **particle-hole symmetry** $C: \wedge^{\text{half}+q}(V) \longrightarrow \wedge^{\text{half}-q}(V)$

Goal: Classify G -symmetric quadratic Hamiltonians
(Hartree-Fock-Bogoliubov mean-field approximation, or "free fermions")

Thm (HHZ). G_0 -reduced block data $\overset{10}{\longleftrightarrow} \overset{10}{\text{Classical irred. symmetric spaces}}$
 $A, AI, AII, BDI, C, CI, CII, D, DII$

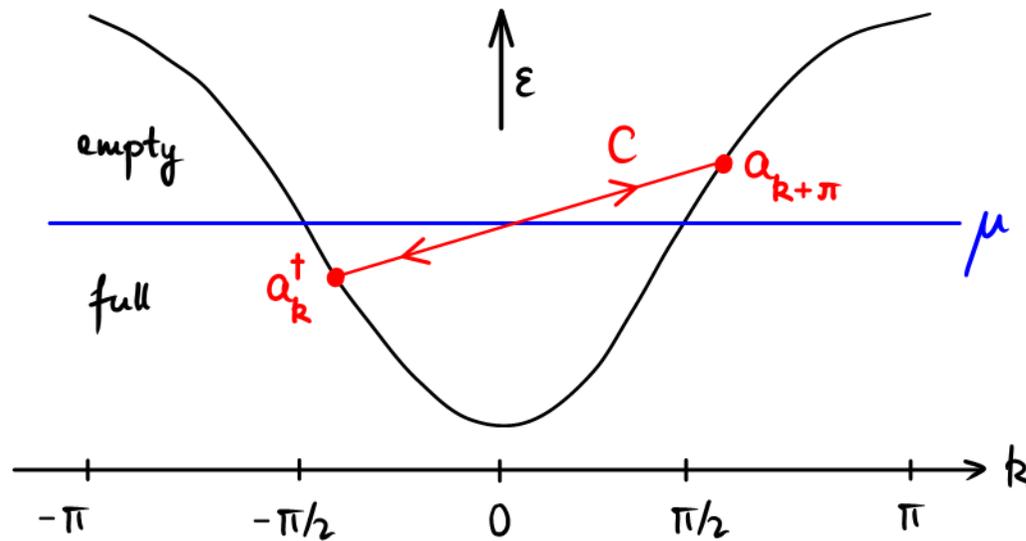
Cor. $H(k)$ (Fourier-Bloch) is invariably of one of the 10 types

Example 1: cosine band



$$H - \mu N = \sum_k \epsilon(k) a_k^\dagger a_k, \quad \epsilon(k) = -t \cos k,$$

has the particle-hole symmetry $a_k^\dagger \xleftrightarrow{C} a_{k+\pi}$.



More generally, any band $\epsilon(k+\pi) = -\epsilon(k)$ with $\int \epsilon(k) dk = 0$ has the same particle-hole symmetry.

Note. C is complex anti-linear: $C i C^{-1} = -i$.

Example 2: Hubbard model at half filling

$$\text{Hamiltonian } H = - \sum_{n \in \mathbb{Z}} \sum_{\sigma = \pm 1/2} (t a_{\sigma}^{\dagger}(n) a_{\sigma}(n+1) + \text{h.c.}) + \overset{u > 0}{U} \sum_n Q^2(n)$$

$$\text{Charge (normal-ordered) at site } n: \quad Q(n) = \frac{1}{2} \sum_{\sigma = \pm 1/2} (a_{\sigma}^{\dagger}(n) a_{\sigma}(n) - a_{\sigma}(n) a_{\sigma}^{\dagger}(n))$$

$$\text{Particle-hole transformation } C a_{\sigma}^{\dagger}(n) C^{-1} = (-1)^n a_{\sigma}(n)$$

$$\text{is a symmetry: } \left. \begin{array}{l} C Q(n) C^{-1} = -Q(n) \\ C t a_{\sigma}^{\dagger}(n) a_{\sigma}(n+1) C^{-1} = -\bar{t} a_{\sigma}(n) a_{\sigma}^{\dagger}(n+1) \end{array} \right\} C H C^{-1} = H$$

and leaves the ground state (at half filling) invariant.

Cor. Heisenberg antiferromagnetic quantum spin chain is particle-hole symmetric. **Note:** $C S^{\dagger}(n) C^{-1} = -S^{\dagger}(n)$
(spin operators)

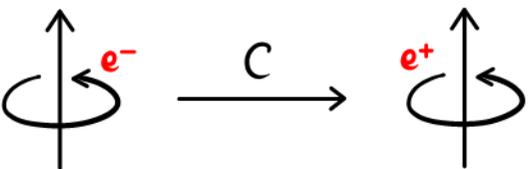
Dirac fermions

First quantization.

Dirac operator $H = mc^2\beta + c \sum_e \alpha_e p_e$, $p_e = \frac{\hbar}{i} \frac{\partial}{\partial x_e}$,
 satisfies $KHK^{-1} = -H$ for charge conjugation $K\psi = i\alpha_2 \bar{\psi}$ ← anti-linear
 (i.e. K inverts spectrum: $H\psi = E\psi \Rightarrow H K\psi = -E K\psi$).

Second quantization.

Charge conjugation is a unitary symmetry, $CHC^{-1} = H$,
 mapping electrons to positrons: $CiC^{-1} = +i$



Remark.



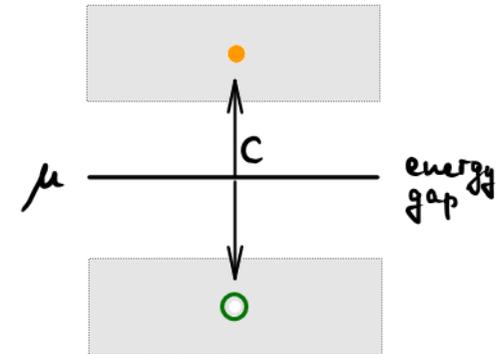
List of examples (particle-hole symmetry)

- Cosine band
- Su-Schrieffer-Heeger (SSH) model
- Kitaev-Majorana chain (BDI)
- Hubbard model
- Antiferromagnetic quantum spin chains (including Haldane phase)
- Half-filled lowest Landau level
- $\nu = 5/2$ fractional quantum Hall state
- Dirac Hamiltonian/vacuum

Gapped systems (insulators)

- μ : chemical potential (or Fermi energy)
- V_+ : positive-energy states ($E > \mu$), conduction bands
- V_- : negative-energy states ($E < \mu$), valence bands

- Fock space $\mathcal{F} = \wedge(V_+ \oplus V_-^*)$
 - ↙ single-particle excitations
 - ↗ single-hole excitations



- Dirac ket-to-bra bijection $\gamma: V_{\pm} \rightarrow V_{\pm}^*$, $|v\rangle \mapsto \langle v|$
(math: Fréchet-Riesz isomorphism $v \mapsto \langle v, \cdot \rangle$)
- Given an isomorphism $V_+ \xleftrightarrow{K} V_-$ one has a mapping

$$C: \mathcal{F} \xrightarrow{K} \wedge(V_- \oplus V_+^*) \xrightarrow{\gamma} \mathcal{F}$$

Defn. C is called a **particle-hole symmetry** if $CH = HC$.

Remark. K linear (**anti-linear**) $\iff C$ anti-linear (**linear**)
(cf. Dirac fermion) cosine band $a_k^+ \xleftrightarrow{C} a_{k+\pi}$

Gapless systems

- What if the energy excitation spectrum is gapless?

$$V = V_+ \oplus V_0 \oplus V_-$$

fermionic \nearrow zero modes (e.g. lowest Landau level;
boundary zero modes of spin chain)

- Assume $V_0 \cong \mathbb{C}^N$ (finite dimension N)

Defn. Let \mathcal{A} = algebra of (bounded) many-body operators.

Define particle-hole conjugation := anti-linear automorphism of \mathcal{A}
determined by $a_j \longleftrightarrow a_j^\dagger$

[where $a_j^\dagger \equiv \varepsilon(\nu)$, $\nu \in V_0$ and $a_j \equiv \iota(\varphi)$, $\varphi = \gamma\nu \in V_0^*$].

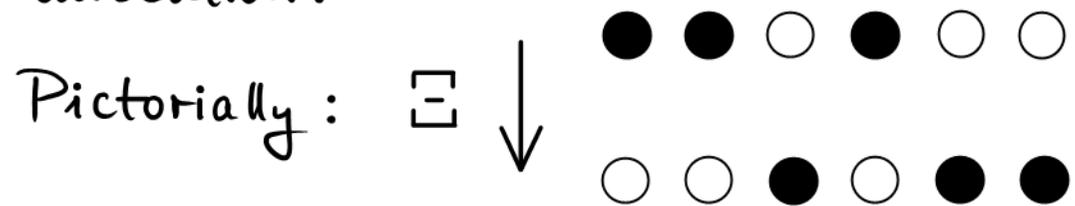
Question: does particle-hole conjugation lift to Fock space, i.e. does there exist $\Xi: \wedge^n(V_0) \rightarrow \wedge^{N-n}(V_0)$ such that $\Xi a_j \Xi^{-1} = a_j^\dagger$?

Particle-Hole Conjugation Lifted

Recall $V_0 \equiv V \cong \mathbb{C}^N$ with $N < \infty$.

- Fréchet-Riesz isomorphism $\gamma : V \rightarrow V^*$, $v \mapsto \langle v, \cdot \rangle$
on Fock space $\gamma_n : \Lambda^n(V) \rightarrow \Lambda^n(V^*) \cong \Lambda^n(V)^*$.
- A choice of generator $\Omega \in \Lambda^{\text{top}} = \Lambda^N(V)$ determines a
"wedge" isomorphism $\omega_n : \Lambda^n(V)^* \rightarrow \Lambda^{N-n}(V)$, $\Phi \mapsto \omega_n \Phi$,
by $(\omega_n \Phi) \wedge \Psi = \Phi(\Psi) \Omega_n$,
 $\Omega_0 \equiv \Omega$, $\Omega_n = (-1)^{N-n} \Omega_{n-1}$ ($n = 1, 2, \dots, N$).
- Particle-hole conjugation $\Xi : \Lambda^n(V) \xrightarrow{\gamma_n} \Lambda^n(V)^* \xrightarrow{\omega_n} \Lambda^{N-n}(V)$
is a complex **anti-linear** involution.

$$\Xi a_j \Xi^{-1} \stackrel{\checkmark}{=} a_j^\dagger$$



Two Facts

Lemma 1. Let X be any self-adjoint and Weyl-ordered one-body operator on $\Lambda(V)$,

$$\text{i.e. } X = \sum_{ij} X_{ij} (a_i^\dagger a_j - a_j a_i^\dagger) + \sum_{i < j} (\gamma_{ij} a_i^\dagger a_j^\dagger + \bar{\gamma}_{ij} a_j a_i), \quad X_{ij} = \bar{X}_{ji}.$$

Then X is odd under particle-hole conjugation: $\Xi X \Xi^{-1} = -X$.

Idea of proof.

- $O_p \rightarrow \Xi O_p \Xi^{-1}$ is an algebra automorphism.
- If $\gamma v = \langle v, \cdot \rangle \equiv \varphi$, then $\Xi a_v^\dagger \Xi^{-1} = a_\varphi$.
- Use CAR (canonical anti-commutation relations).

Remark. Ξ can never be a symmetry of any Fermi liquid.

Need spectrum-inverting isomorphism in addition: $C = \Xi K$.

Lemma 2. $\Xi^2 = (-1)^{N(N-1)/2} \text{Id}_{\Lambda(V)}$, $N = \dim(V)$.

[cf. time reversal $\tau^2 = (-1)^n \text{Id}_{\Lambda(V)}$]

Symmetry Protection of Zero Modes

Note. Zero modes typically killed by interactions.

Defn. A Hamiltonian H is said to be of **class AIII** if it commutes

with a group of symmetries $G = U(1)_Q \times \mathbb{Z}_2^C$
charge conservation \uparrow \uparrow particle-hole transformation

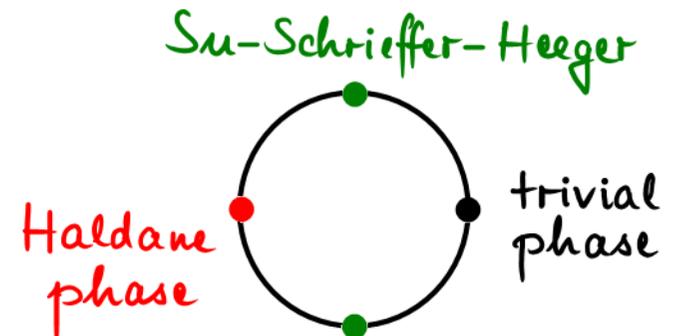
Q: What scenarios (in class AIII) for protected zero modes?

A: ① Unique ground state $\Psi = C\Psi$ ($C^2 = +1$)

② Kramers pair $\Psi \xleftrightarrow{C} \Psi'$ ($C^2 = -1$); Ψ, Ψ' same fermion parity

③④ C acts as a supersymmetry ($C^2 = \pm 1$); Ψ, Ψ' opposite fermion parity

Remark. \mathbb{Z}_4 classification of AIII-protected topological phases in one space dimension:



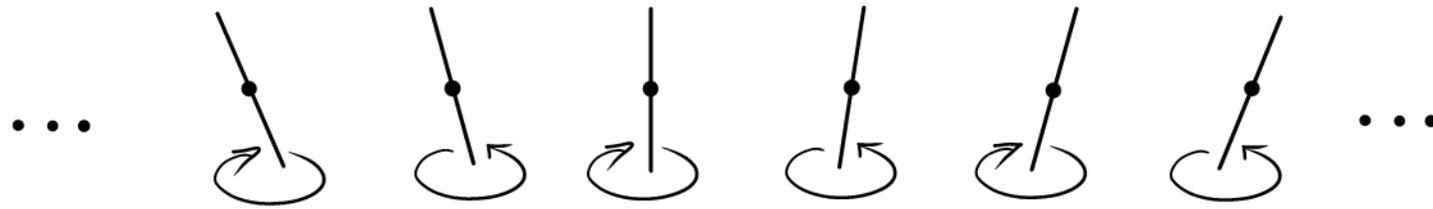
Short bibliographic history of class AIII

- F.J. Dyson, “*The dynamics of a disordered linear chain*”, Phys. Rev. **92** (1953) 1331-1338
- R. Gade, “*Anderson localization for sublattice models*”, Nucl. Phys. B **398** (1993) 499-515
- **J.J.M. Verbaarschot**, “*Spectrum of QCD Dirac operator and chiral random-matrix theory*”, Phys. Rev. Lett. **72** (1994) 2531-2533
- A. Altland & MRZ, “*Nonstandard symmetric classes in mesoscopic ... hybrid structures*”, Phys. Rev. B **55** (1997) 1142-1161
- H. Heinzner, A. Huckleberry & MRZ, “*Symmetry classes of disordered fermions*”, Commun. Math. Phys. **257** (2005) 725-771
- MRZ, “*Particle-hole symmetries: from Dyson’s Threefold Way to the Tenfold Way of disordered free fermions*”, arXiv:1910.xxxxx

II. Free fermions

\leadsto Haldane phase

Anti-ferromagnetic quantum spin chains



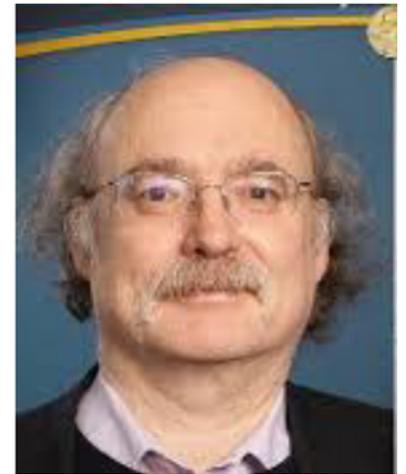
$$H = J \sum_n \mathbf{S}_n \cdot \mathbf{S}_{n+1} \quad (J > 0)$$

Spin 1/2 (Heisenberg chain, Bethe Ansatz):

gapless \leftarrow Lieb-Schultz-Mattis 1961

Spin 1: Haldane (1983) predicts excitation gap
from $O(3)$ nonlinear sigma model with
topological angle $\Theta = 2\pi|S|$

— neutron scattering experiments on CsNiCl_3



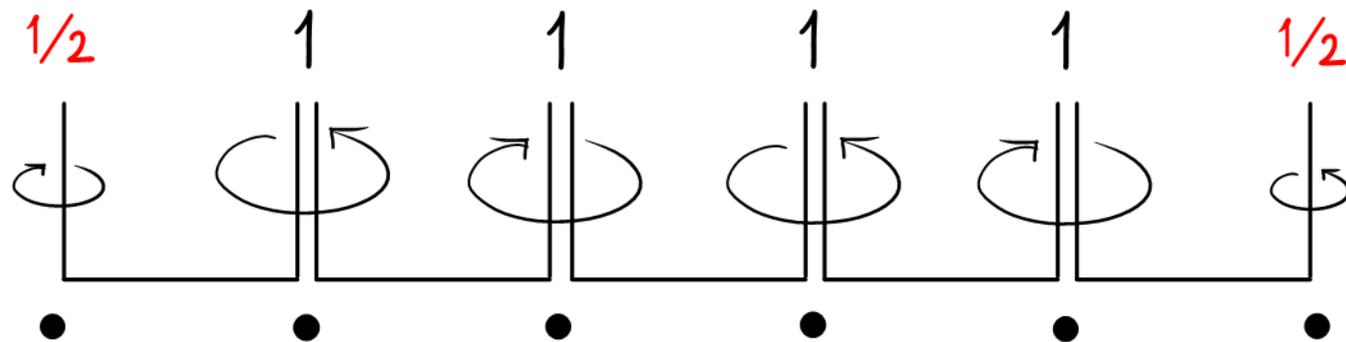
F. D. M. Haldane
Nobel Prize Physics 2016

Haldane phase (spin 1)

exponential decay of correlations (\Leftarrow mass gap), **BUT**

hidden topological order: edge excitations of spin 1/2 (!)
 \rightarrow "fractionalization"

Example: Affleck-Kennedy-Lieb-Tasaki (AKLT)



matrix-product state

Haldane phase new paradigm (beyond Landau-Ginzburg-Wilson)
for (short-range) topological order

Haldane phase as an SPT phase

symmetry-protected topological

Q: protection by what symmetry?

A1 (Pollmann-Berg-Turner-Oshikawa, 2010):

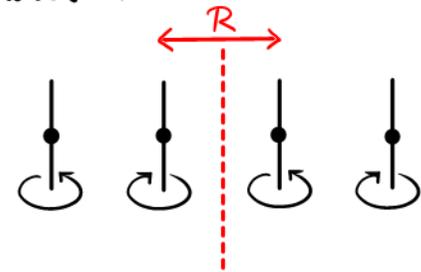
T (time reversal)

OR $\mathbb{Z}_2 \times \mathbb{Z}_2$ (dihedral group)

OR R (space reflection)

~~A1~~ (Anfuso-Rosch, 2007): local charge fluctuations!

A2 (Moudgalya-Pollmann, 2015): bond inversion
disorder?



A3 (MRZ, 2016; Verresen-Moesner-Pollmann, 2017): *particle-hole symmetry*

Def. Two Hamiltonians H_0 and H_1 are said to be in the same **topological phase** (or topologically equivalent, $H_0 \sim H_1$) if there exists a homotopy $[0, 1] \ni t \mapsto H(t)$, $H(0) = H_0$, $H(1) = H_1$ such that $H(t)$ has a unique ground state with a finite energy gap for excitations, for all t .

Def. A Hamiltonian H is said to be of symmetry class **AIII** if

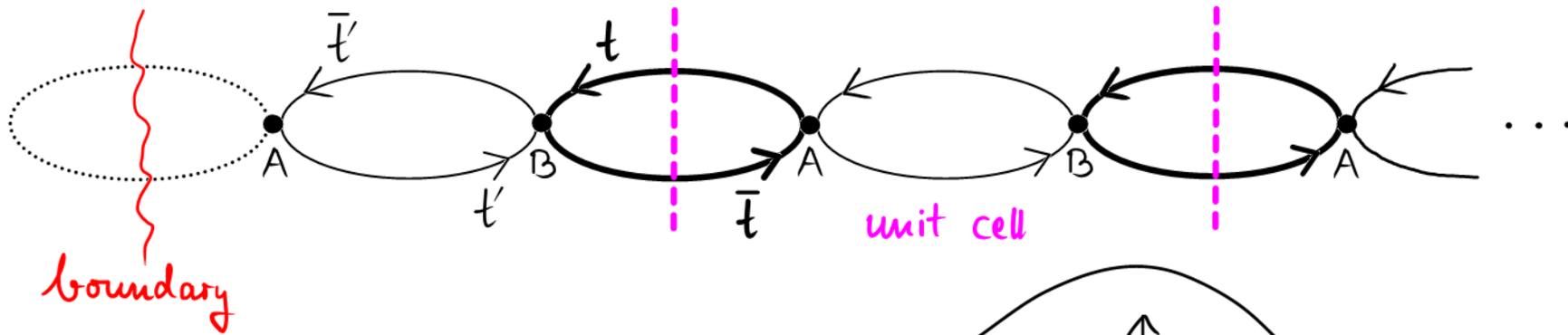
$$H = e^{i\theta Q} H e^{-i\theta Q} = C H C^{-1}.$$

(charge operator Q , particle-hole transformation C)

\leadsto **symmetry-protected** topological phase of class AIII

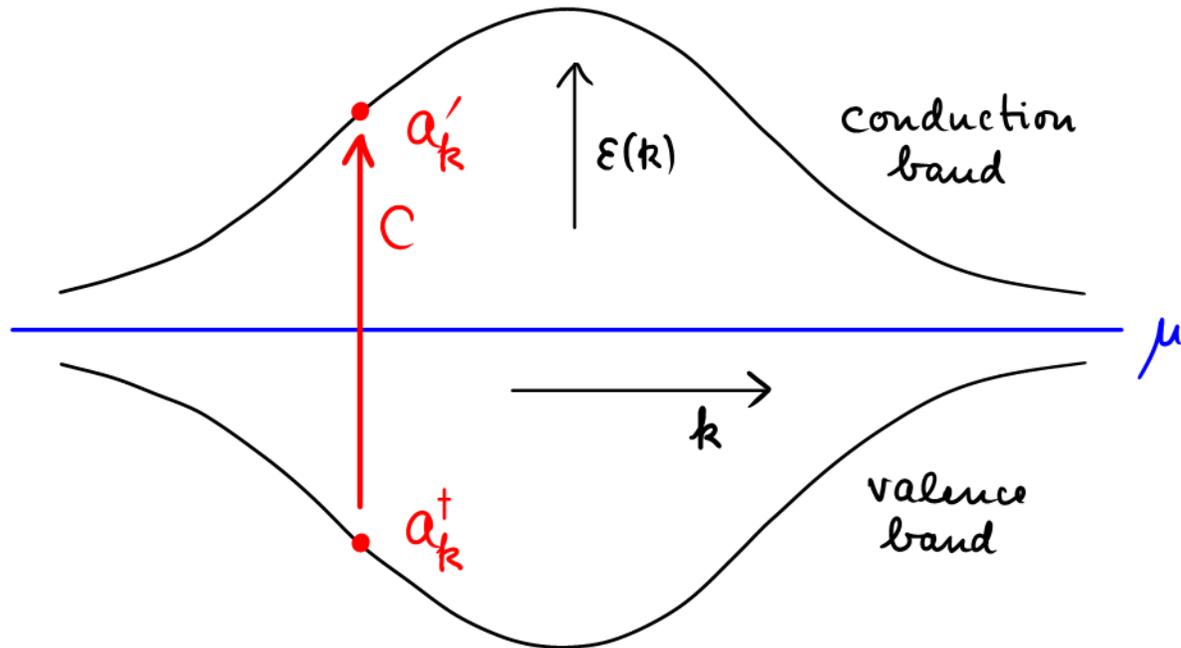
A3 (here): free fermion SPI phase
 in 1D with symmetry group $U(1)_Q \times \mathbb{Z}_2^C \overset{\text{AIII}}{\sim}$ Haldane phase

Su-Schrieffer-Heeger model ("polyacetylene"; class AIII)



$$|t| > |t'| \text{ (Peierls)}$$

\leadsto double the unit cell:

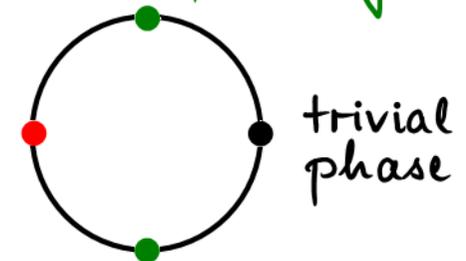


Valence band carries non-trivial topological invariant (e.g. from bulk-boundary correspondence).

Zero mode localized at boundary (cut strong bond):

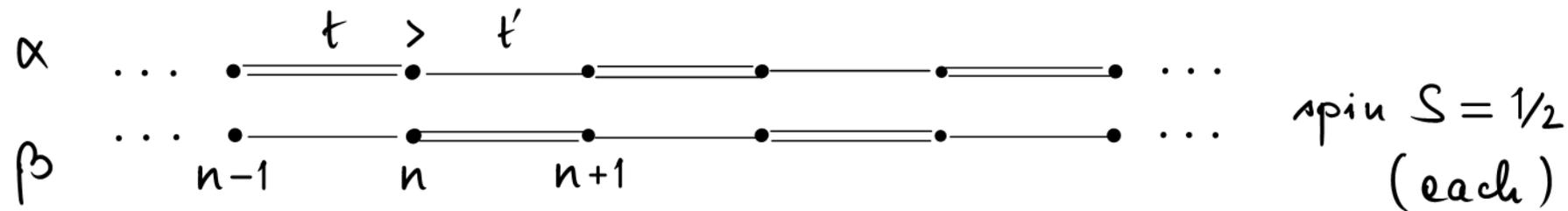
$$\psi(A_{n+1}) = -\frac{t'}{t} \psi(A_n), \quad \psi(B_n) = 0.$$

Su-Schrieffer-Heeger



From SSH to the Haldane phase

starting point: two chains of SSH



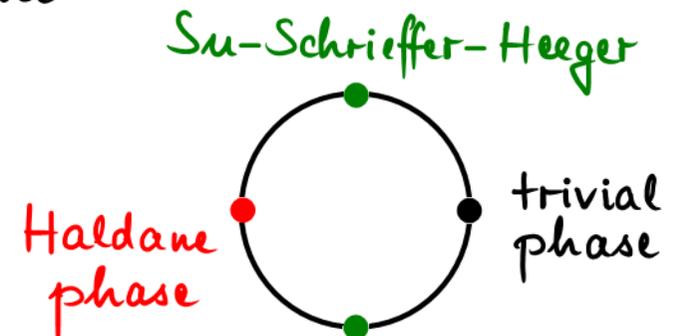
Symmetry group $G = U(1)_Q \times \mathbb{Z}_2^C$ (class AIII)

Recall $a^\dagger(n) \xrightarrow{C} (-1)^n a(n)$, $Q(n) \xrightarrow{C} -Q(n)$, $S(n) \xrightarrow{C} -S(n)$.

Hamiltonian (path in class AIII):

$$H(t, t', u, J) = H_{\text{free}} + u \sum_n Q^2(n) - J \sum_n S_\alpha(n) \cdot S_\beta(n)$$

"Hubbard"
"Hund's rule"



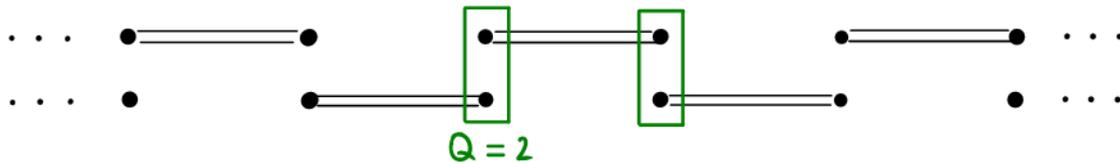
Deformation

Step 1. Turn off the hopping $t' \rightsquigarrow$ flat bands



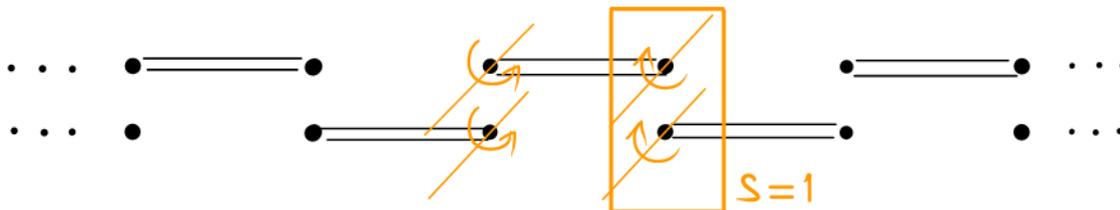
(energy gap stays open; topological invariant remains unchanged)

Step 2. Turn on the Hubbard coupling ($U \gg |t|$) \rightsquigarrow 2 charges/site



(singlet bonds form; energy gap stays open)

Step 3. Turn on the Hund's rule coupling ($J \sim U$) \rightsquigarrow $S=1$ on each site



(antiferromagnetic exchange coupling; energy gap stays open)

The resulting low-energy effective Hamiltonian is the $S=1$ Heisenberg chain
 \hookrightarrow Haldane phase

III. Half-filled lowest Landau level

Halperin, Lee, Read (1994)

Composite fermions (i.e. electrons with 2 fictitious magnetic flux quanta attached) experience zero net B-field and populate a Fermi sea (in free-fermion approximation).

△ Effective field theory with Lagrangian:

$$\mathcal{L} = i\psi^\dagger(\partial_t - iA_0 + ia_0)\psi - \frac{1}{2m}|(\partial_i - iA_i + ia_i)\psi|^2 + \frac{1}{2} \frac{1}{4\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + \dots$$

Phenomenologically quite successful!

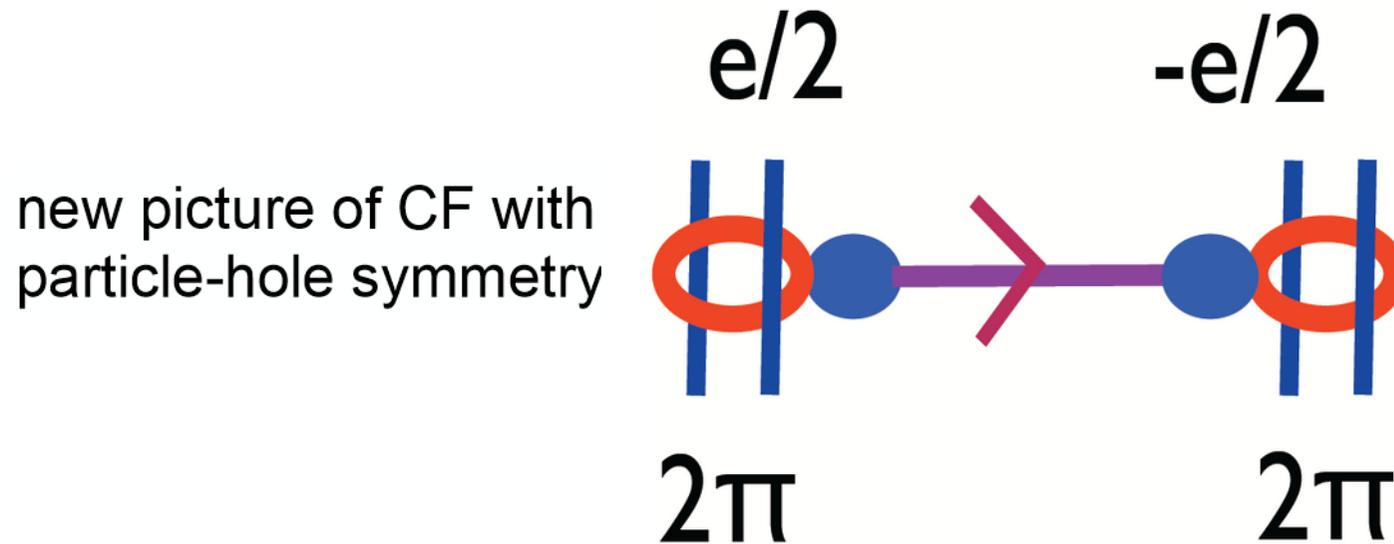
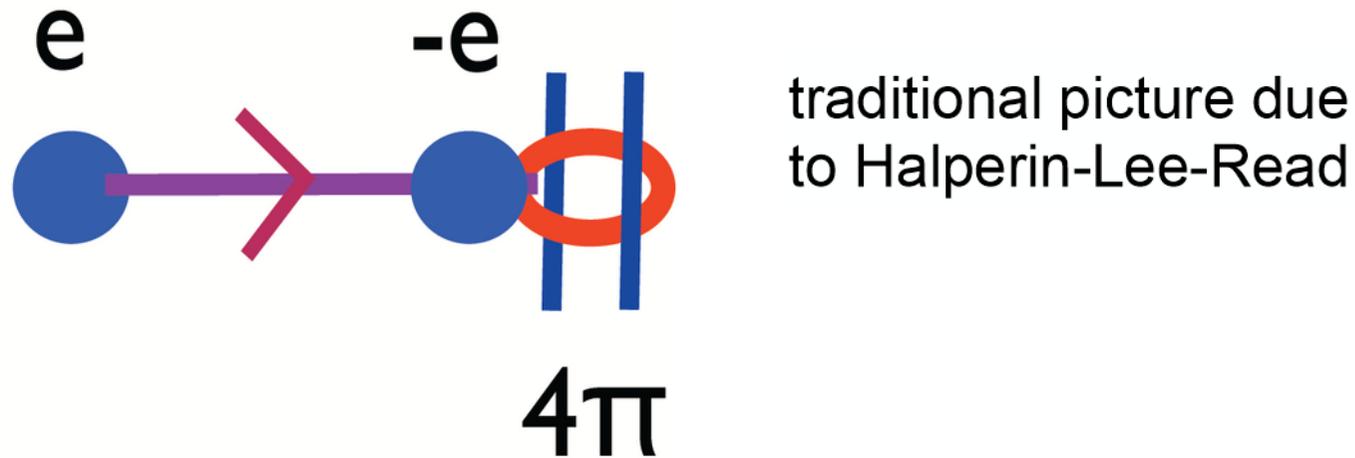
Problems: - mass renormalization?
- particle-hole symmetry?

Note. $H_{\text{eff}} = P_{LLL} V_2 P_{LLL}$, V_2 quadratic in charge density, is exactly particle-hole symmetric ...

Remark. Son's new theory resolves both of these issues!

Pictures of composite fermion as a dipole

Wang & Senthil (2016)



Son's Logic

- Realize lowest Landau level as zero-energy sector of Dirac fermion (in a homogeneous magnetic field):

$$S = \int d^3x i\bar{\Psi}\gamma^\mu(\partial_\mu - iA_\mu)\Psi + S_{\text{E.M.}}$$

- Switch to dual description (QED₃; fermionic particle-vortex duality)

by $\bar{\Psi}\gamma^\mu\Psi = J^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda$

($a_0 = \text{magnetization}$, $\epsilon^{ij}a_j = \text{electric polarization field}$):

$$S_{\text{eff}} = \int d^3x \left(i\bar{\psi}\gamma^\mu(\partial_\mu + 2ia_\mu)\psi + \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu a_\lambda \right) + S_{\text{E.M.}}$$

Remarks. — No mass term, no Chern-Simons term!

- Particle-hole symmetry (antilinear) is implemented as **CT**
(charge conjugation & time reversal)

Symmetry Considerations

- Electromagnetic gauge field $A = A_\mu dx^\mu$ is time-twisted:
 under time reversal $A \mapsto -\mathcal{T}^*A$ or $A_0 \mapsto +A_0, A_j \mapsto -A_j$
 and under parity $A \mapsto +\mathcal{P}^*A$ or $A_0 \mapsto +A_0, \dots$
- Charge 3-current $J = da$ is space-twisted. Hence
 time reversal : $a \mapsto +\mathcal{T}^*a$
 parity : $a \mapsto -\mathcal{P}^*a$ $\leadsto \int A \wedge J$ invariant

Consequence: First-quantized Hamiltonian for the fermionic vortex field

$$H = v\sigma_1(p_1 + 2a_1) + v\sigma_2(p_2 + 2a_2) + a_0$$

is odd under each of time reversal, parity, and charge conjugation.

Time reversal: $\psi \mapsto \overset{\mathbb{C}\text{-linear}}{\sigma_3} \psi, a_0 \mapsto -a_0, a_1 \mapsto +a_1, a_2 \mapsto +a_2.$

Parity ($x_2 \mapsto -x_2$): $\psi \mapsto \overset{\mathbb{C}\text{-antilinear}}{\bar{\psi}}, a_0 \mapsto -a_0, a_1 \mapsto -a_1, a_2 \mapsto +a_2.$

Charge conjugation: $\psi \mapsto \overset{\mathbb{C}\text{-antilinear}}{\sigma_1 \bar{\psi}}, a_\mu \mapsto -a_\mu.$

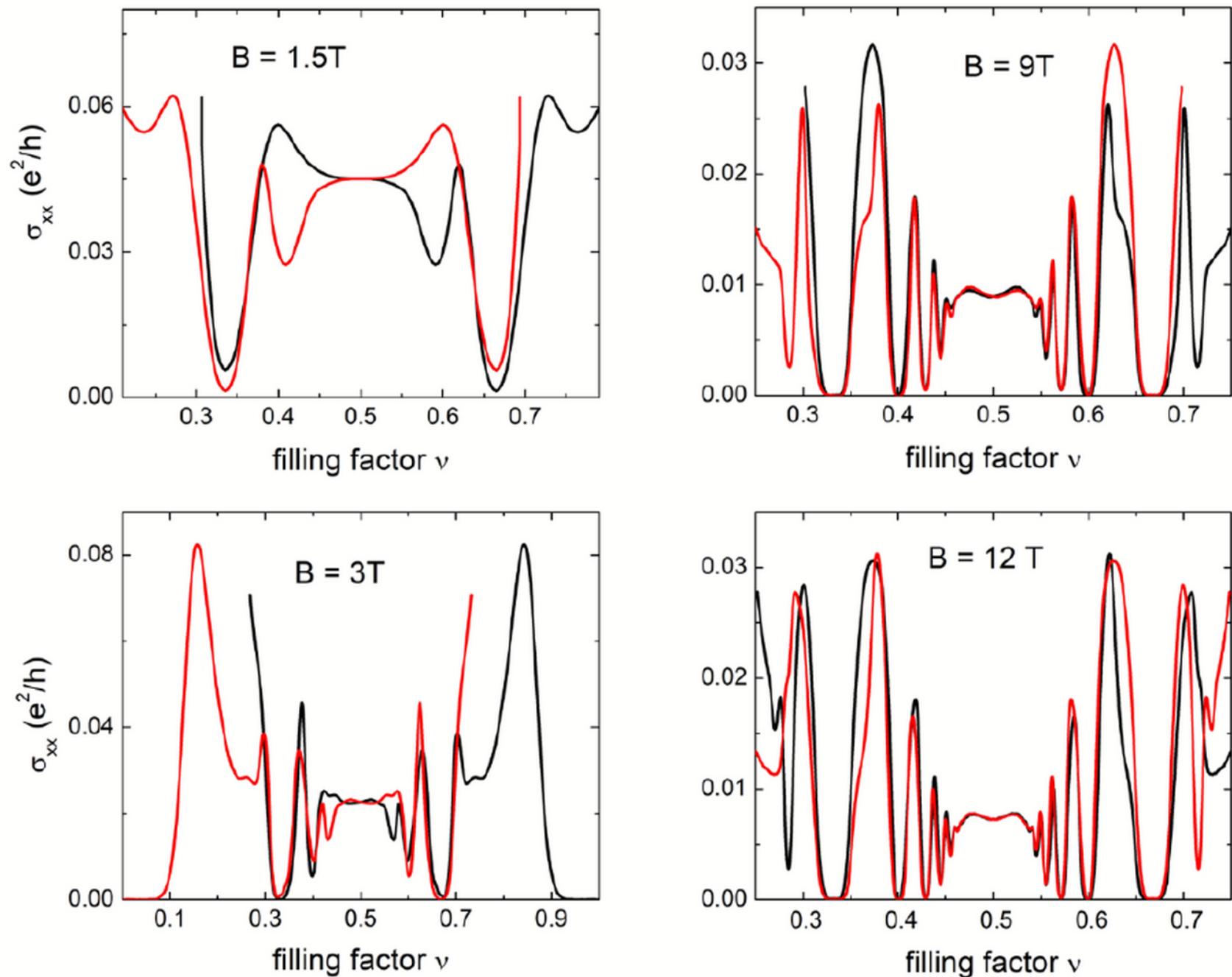


Figure 4. Examination of reflection symmetry in σ_{xx} over a large range of magnetic fields, from 1.5T to 12T.

Thank you!
(The End)