Quantum Chaos in Many-Body Theory

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Correlations of eigenvalues

Pair correlation function of $\rho(\lambda) = \sum_k \delta(\lambda - \lambda_k)$

$$\rho_{2c}(\lambda, \lambda') = \langle \rho(\lambda)\rho(\lambda') \rangle - \langle \rho(\lambda) \rangle \langle \rho(\lambda') \rangle = \langle \rho(\lambda) \rangle \delta(\lambda - \lambda') - \frac{\sin^2(\pi N(\lambda - \lambda'))}{(\pi N(\lambda - \lambda'))^2},$$

where we gave the result for the simplest random matrix theory (GUE).

- The level spacing is $1/(\pi N)$. So correlations become universal in units of the average level spacing.

Agreement with a specific RMT is determined by the non-unitary symmetries of the system.

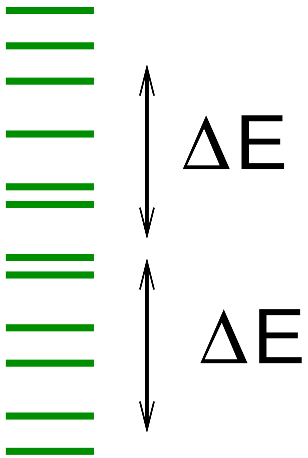
Universality of Spectral Correlations

- Spectral correlations of interacting systems are given by random matrix theory
- This is the strongest universal property in physics I know of – much stronger than the universality of critical exponents in second order phase transitions
- Separation of scales

Number Variance

$$\Sigma^2(n) = \int_{\Delta E}\int_{\Delta E} \rho_{2c}(\lambda,\lambda')d\lambda d\lambda', \qquad \int_{\Delta E} d\lambda \langle \rho(\lambda) \rangle = n.$$

- The δ function results in a linear term, $\Sigma^2(n) \sim n$.
- The $1/(\lambda-\lambda')^2$ term gives a logarithmic term , $\Sigma^2(n) \sim \frac{1}{\pi^2}\log n$.
- For uncorrelated eigenvalues we only have the delta function so that $\Sigma^2(n) = n$.

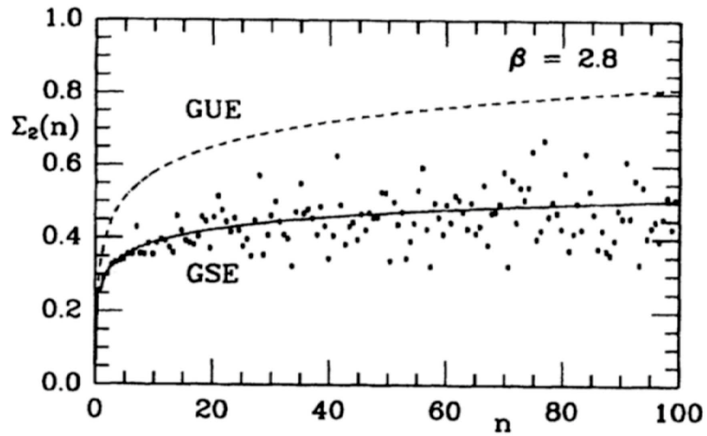


Characterization of Universal Random Matrix Behavior

- Short range repulsion of the eigenvalues as S^β .
- Spectral rigidity. The variance of the number of level in an interval containing n eigenvalues on average behaves as $(\beta/2\pi^2)\log n$ rather than n when correlations are absent.
- The classical limit of a theory with spectral correlations given by the Wigner-Dyson ensembles is a chaotic theory (Bohigas-Giannoni-Schmidt conjecture-1984).

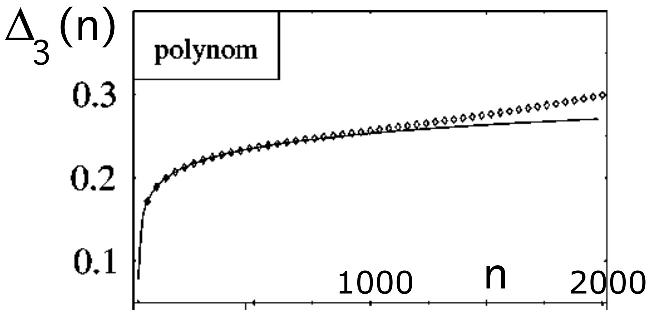
The reverse of the Bohigas-Giannoni-Schmidt conjecture also holds: if the system is not fully chaotic the level correlations deviate from the Wigner-Dyson results. Seligman-JV-Zirnbauer-1984

Spectral Correlations in Lattice QCD



Number variance for a single configuration Halasz-JV-1996.

Spectral Correlations in Lattice QCD

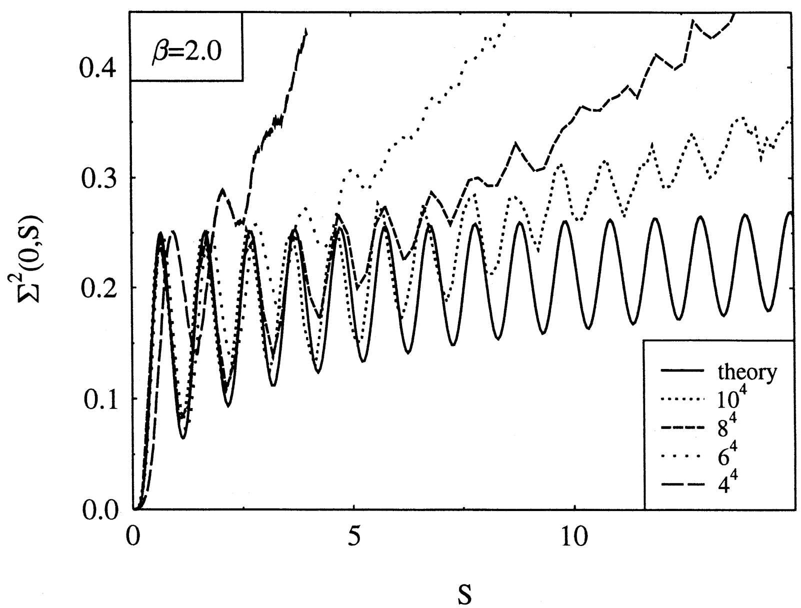


The Δ_3 statistic is obtained by integrating the number variance over a smoothening kernel.

$$\Delta_3(n) = \frac{2}{n^4}\int_0^n dr(n^3 - 2n^2 r + r^3)\Sigma^2(r).$$

Ensemble average of number variance for each configuration gives a very long range of agreement with RMT Guhr-Ma-Meyer-Wilke-1999

Thouless Energy in QCD



Thouless energy for various lattice volumes. The numerical data are consistent with a $\sqrt{V}$ -scaling.

Berbenni-Bitsch-Göckeler-Guhr-Jackson-Ma-Meyer-Schäfer-Weidenmüller-Wettig-Wilke-1998

Source for the Discrepancy

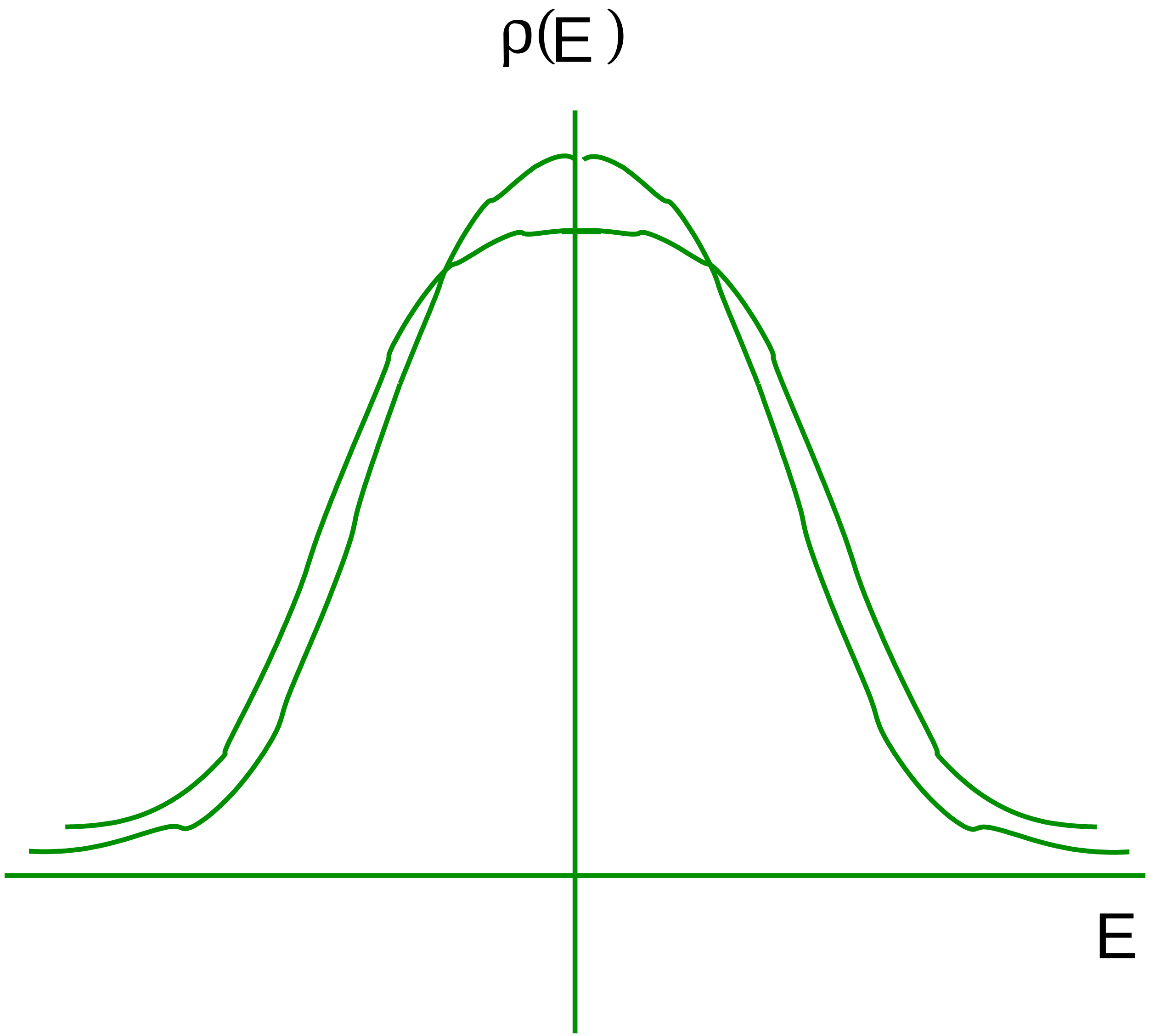
- When the number variance is calculated relative to the smoothened spectral density for each configuration, the range of agreement with RMT is very large.
- Early disagreement with RMT is found when calculating the number variance relative to the ensemble average of the spectral density.
- The scale of the departure from RMT is physical and is in agreement with QCD (as calculated from the chiral Lagrangian). It is given by $F_\pi^2/\Sigma\sqrt{V}$.
- The only conclusion can be that the deviation is due to fluctuations of the smoothened spectral density for each configuration.

Estimate of the Size of Collective Fluctuations

- The number of random variables for each lattice configuration is of order N^d.
- Elementary error analysis tell us that the relative inaccuracy in an observable is of order $1/N^{d/2}$
- For example the width of the spectral fluctuates relative the ensemble average $\bar{\sigma}$ as

$$\frac{\delta\sigma}{\bar{\sigma}} \sim \frac{1}{N^{d/2}}.$$

Collective Dipole Fluctuations



Because the spectral density is normalized to the total number of

Number Variance due to Collective Fluctuations

The average number of levels in an interval is given by

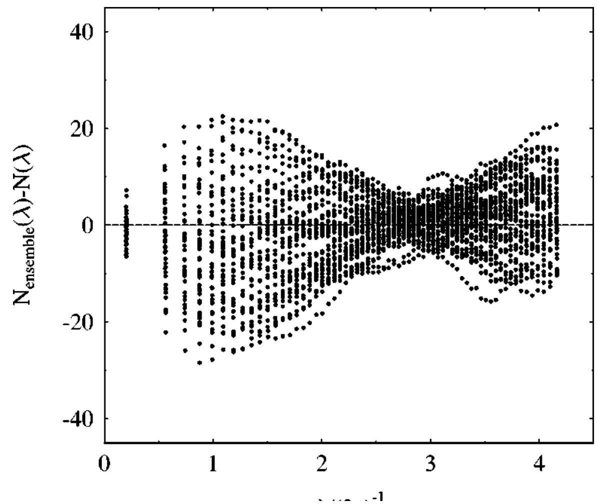
$$\bar{n} = \int_{\Delta E} \bar{\rho}(E) dE$$

The variance of the actual number of levels is

$$\Sigma^2(\bar{n}) = \overline{\left(\int_{\Delta E} \delta\rho(E) \right)^2} = \frac{\bar{n}^2}{N^d}.$$

- The Thouless scale is at $n_c \sim V^{1/2}/a^{d/2}$.
- The estimate obtained from the hooping term is $n_c \sim F_\pi^2 V^{(d-2)/d}$.
- Upper Critical Dimension: $d = 4$.
- Lower Critical Dimension: $d = 2$.

Collective Spectral Fluctuations



Guhr-Ma-Meyer-Wilke-1999

Difference of the total number of levels and the ensemble average of the total number of levels as a function of λ for QCD

- A linear deviation from the average number of levels gives a quadratic number variance.
- The quadratic term is entirely due to ensemble fluctuations.

III. Three Dimensional QCD

Number Variance

Conformality

Universality

Two Dimensional QCD

- Classification is different from 4d. In general, it depends on the dimensionality subject to Bott periodicity.
- According to the Coleman-Mermin-Wagner theorem, continuous symmetries cannot be spontaneously broken in two dimensions. So

$$\rho(E) \sim VE^{\alpha}.$$

Estimate of exponent

$$\alpha = 1/(2N^2 - 1), \quad N = 1, 2, \cdots$$

Nersesyan-Tvelik-Wenger-1994

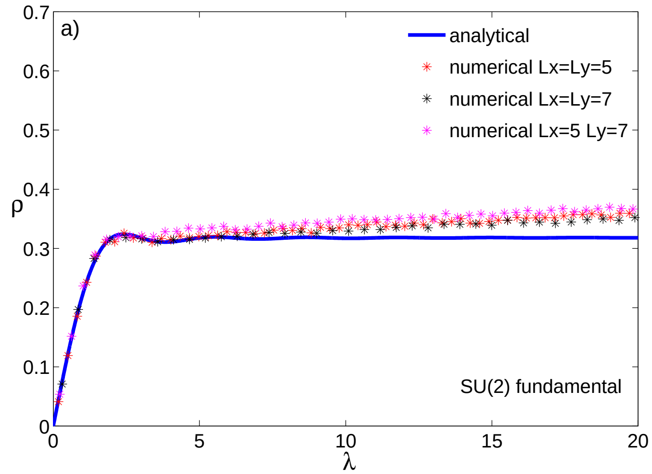
RMT Behavior and Conformal Spectra

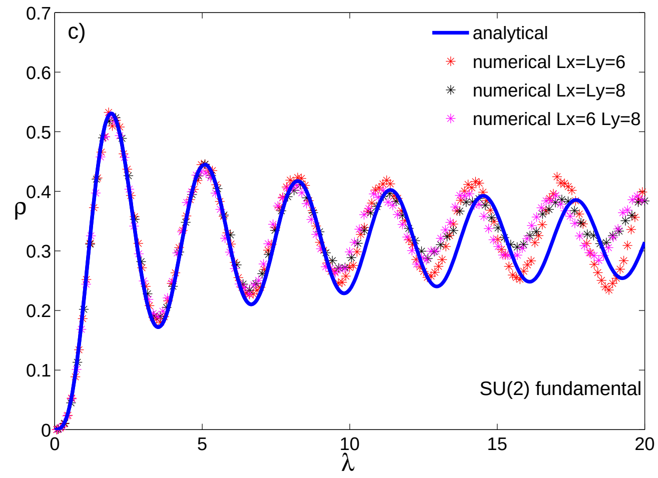
- $\rho(E) \sim VE^{\alpha}$
- $N(E) \sim VE^{\alpha+1}$

Eigenvalues scale with the volume as $\frac{1}{V^{1/(\alpha+1)}}$. Nontrivial microscopic limit

$$\rho_S(x) = \lim_{V\to\infty} \frac{1}{V^{1/(\alpha+1)}} \rho\left(\frac{x}{V^{1(\alpha+1)}}\right).$$

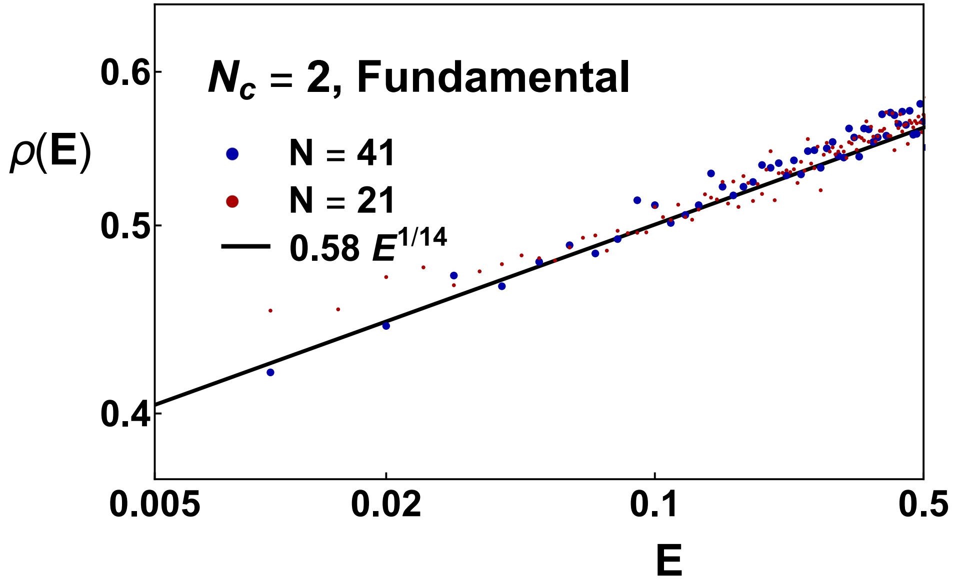
RMT Behavior in 2d QCD





Kieburg-JV-Zafeiropoulos-2014

RMT Universality and Spontaneous Symmetry Breaking



Spectral density of Dirac operator for lattice QCD with two colors and naive fermions. Kieburg-JV-2019

RMT Universality and Spontaneous Symmetry Breaking

$$Z(x,y) = \int dH P(H) \det{}^n(\bar{x}+r+H) \det{}^n(\bar{x}-r+H).$$

with $\bar{x} = (x+y)/2$ and $r = (x-y)/2$. This partition function has an U(2n) global symmetry which is spontaneously broken to U$(n)\times$U(n).

The U(2n) symmetry is broken by the ground state (or saddle-point) of the theory.

The low-energy effective partition function is given by

$$\int_{\sigma \in U(2n)/U(n)\times U(n)} d\sigma e^{-Nr\mathrm{Tr}\sigma}.$$

JV-Zirnbauer-1983,JV-Zahed-1993, Nishigaki-1996, Magnea-1999-2000, Nagao-Nishigaki-2000, Kanazawa-Kiebug-JV-2019

Universality of Spontaneous Symmetry Breaking

Theorem

The pattern of spontaneous symmetry breaking of a QCD-like theory is given by the breaking pattern of a random matrix theory with the same global symmetries in the limit of large matrix size.

IV. Many-Body Theory and the Sachdev-Ye-Kitaev Model

$N \ll 2^{-N}$

Sigma Model

SYK Model

How to Solve it

Average Spectral Density

Spectral Correlations

Universal Inaccuracy

Many-Body Theory

$$N \ll 2^N$$

Many-Body Theory

For a system of N particles

- Smallest eigenvalue: $\sim N$
- Number of levels: $\sim 2^N$
- Spacing of the levels: $\sim 2^{-N}$

Smoothness of the spectral density

- The number of parameters of a many-body system is at most of $O(N)$.
- The spectral density has structure on a scale of E_0/N, with E_0 a typical energy, e.g. the ground state energy.
- Yet there are 2^N levels.
- This gives an exponential separation of scales, and uninversal RMT fluctuations can persist for an exponentially large number of level spacings.

The Two-Body Random Ensemble aka Complex SYK

$$H = \sum_{\alpha\beta\gamma\delta} W_{\alpha\beta\gamma\delta} a^\dagger_\alpha a^\dagger_\beta a_\gamma a_\delta.$$

French-Wong-1970

Bohigas-Flores-1971

Mon-French-1975

The labels of the fermionic creation and annihilation operators run over N single particle states. The Hilbert space is given by all many particle states containing m particles with $m = 0, 1, \cdots, N$.

- $W_{\alpha\beta\gamma\delta}$ is Gaussian random.
- The Hamiltonian is particle number conserving.
- The matrix elements of the Hamiltonian are strongly correlated.

Brody-et-al-1981, JV-Zirnbauer-1984, Brown-Zelevinsky-Horoi-Frazier-1997, Izrailev-1990,Kota-2001,Benet-Weidenmüller-2002,Zelevinsky-Volya-2004, Borgonovi-Izrailev-Santos-Zelevinsky-2016

σ-Model Formulation of the TBRE

$$Z(x,y)\,\langle \det{}^n(H+x)\det{}^n(H+y)\rangle$$

The partition function has a U($2n$) symmetry which is spontaneously broken to U(n)$\times$U(n) by the saddle-point.

The σ-model is much more complicated now

$$Z=\int d\sigma e^{-\frac{1}{2}\mathrm{Tr}\sigma^2-\mathrm{Tr}\log(z-vR.\sigma)}, \qquad \sigma=\sigma_{ab}^{kl}.$$

with $z=(\underbrace{x,\cdots,x}_{n}\underbrace{y,\cdots,y}_{n})$, k and l are replica indices, and , a and b are many-body states. Information on the two-body interaction is contained in R

JV-Zirnbauer-1984

$$R_{abcd}=\sum_{\alpha\beta}\langle a|a_\alpha^\dagger a_\beta^\dagger|c\rangle\langle b|a_\alpha^\dagger a_\beta^\dagger|d\rangle.$$

Universality of correlations in the TBRM

- The saddle point solution is given by $\sigma_{ab}^{kl} = \delta_{ab}\delta^{kl}\bar{\sigma}$. This is the mean field solution of the GOE or GUE.
- We have again Goldstone modes $\sim \delta_{ab}$ diagonal in many body space correspoding to the spontaneous symmetry breaking U$(n)\times$U(n).
- The lowest order, two-point correlations are trially given by the universal RMT result. This is the case if we ignore coupling between massive and massless modes.
- The scaling $(x-y)/\Delta\lambda$ is manifest.
- However, we found large corrections due to these coupling terms.

Zirnbauer-JV-1984

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The Sachdev-Ye-Kitaev (SYK) Model

The two-body random ensemble from nuclear physics also has merged into the SYK model, where the fermion creation and annihilation operators are replaced by Majorana operators (in general q of them. For $q = 4$ the model is Mon-French-1975,Sachdev-Ye-1993,Kitaev-2015

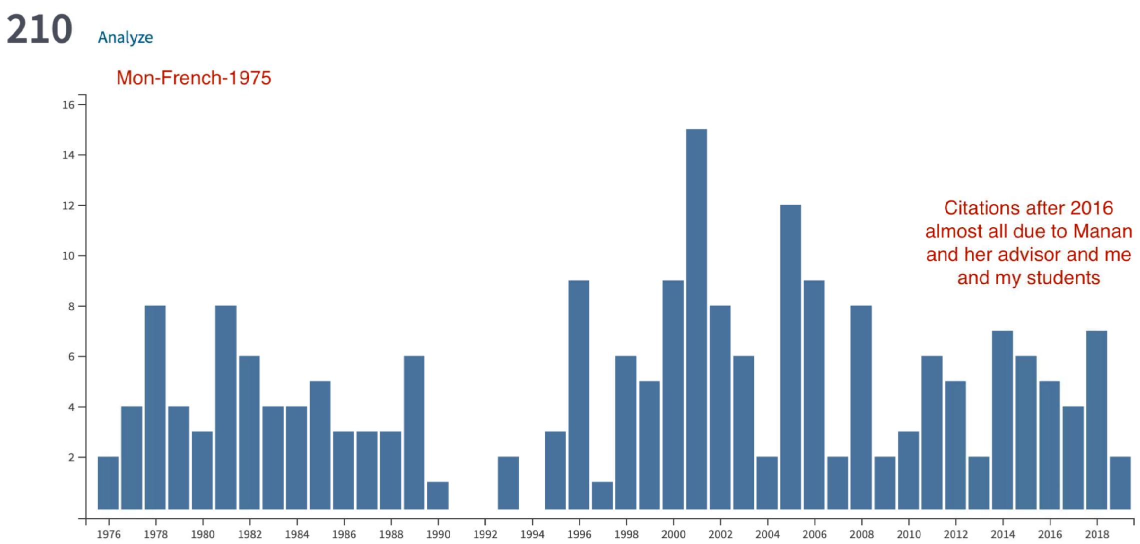
$$H = \sum_{\alpha<\beta<\gamma<\delta} W_{\alpha\beta\gamma\delta}\chi_\alpha\chi_\beta\chi_\gamma\chi_\delta, \qquad q = 4.$$

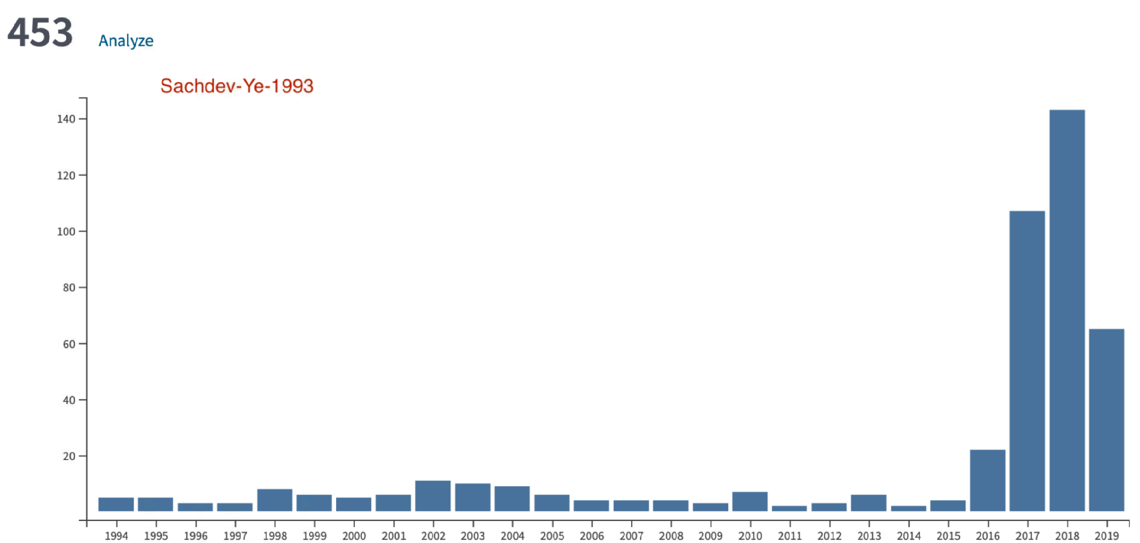
The Majorana operators satisfy the commutation relations

$$\{\chi_\alpha, \chi_\beta\} = \frac{1}{2}\delta_{\alpha\beta}, \quad \chi_{2k} = \frac{1}{\sqrt{2}}(a_k + a_k^\dagger), \qquad \chi_{2k-1} = \frac{i}{\sqrt{2}}(a_k - a_k^\dagger).$$

The two-body matrix elements are taken to be Gaussian distributed with variance that is chosen such that the ground state energy scales with N.

Mon and French versus Sachdev and Ye





Solving the SYK Model

- For $q = 2$ the model can be solved by diagonalizing the two-body matrix elements.
- Moment method. Moments have been calculated to order $1/N^4$ for arbitrary q. Mon-French-1975,Garcia-Garcia-Jia-JV-2018, Jia-JV-2018,Berkooz-Isachenkov-Narovlansky-Torrents-2018
- Formulating the model as a Feynman path integral. This makes it possible to take the large-N limit. Sachdev-Ye-1993,Kitaev-2015,Maldacena-Stanford-2016,Bagrets-Altland-kamenev-2016,Arevefa-Khramtsov-Volovich-2018,Wang-Bagrets-Chudnovskiy-Kamenev-2018
- Generating function $\langle \det(H+z) \rangle$. JV-Zirnbauer-1983,Benet-Weidenmüller-2002,Liu-Nowak-Zahed-2016,Altland-Bagrets-2018
- Representing the Majorana fermions as γ-matrices in N dimensions. This allows numerical diagonalization up to $N = 42$. Maldacena-Stanford-2015,Garcia-Garcia-JV-2016,Cotler-et-al-2016,Gur-Ari-Mahajan-Vaezi-2018

Path Integral Formulation

$$Z(\beta, s) = \sum_{k,l} e^{-\beta(E_k+E_l)+is(E_k-E_l)} = \int d\Sigma dG e^{-N\int_{\beta,s} dt dt' \mathcal{L}(\Sigma(t,t'),G(t,t'))}.$$

- The large parameter N appears as a prefactor which allows a saddle point approximation.
- Level correlations on the scale of the average level spacing follows from the long time behavior of the partition function.
- This seems not to be a natural formulation for obtaining level correlation.
- To obtain them we need the partition function at times 2^N which mixes up the large N expansion.

SYK Model and Quantum Gravity

- The saddle point equations are reparametrization invariant under Diff (S_1) but the saddle point solution is only Sl(2,R) invariant. This gives the Goldstone manifold Diff (S_1) /Sl(2,R).
- The action with with this pattern of symmetry breaking is the Schwarzian Action which is dual to two-dimensional (Jackiw-Teitelboim) gravity). Maldacena-Stanford-2016, Bagrets-Altland-Kamenev-2016-2017, Stanford-Witten-2017.
- The low-energy limit of the SYK model thus reduces to the Schwarzian action.
- The spectral density at low energies is $\rho(E) \sim e^{\sqrt{c(E-E_0)}}$.
- The SYK model is chaotic.

Banerjee-Altman-2017, Krihnan-Sanyal-Subramanian-2017, Caputa-etal-2017, Stanford-Witten-2017, Bagrets-Altland-Kamenev-2016, Garcia-Garcia-JV-2016, Ooguri-Vafa-2016, Kitaev-Su-2017,

V. Spectral Density

Scaling Limit

Analytical Formula

Low Temperature Expansion

The partition function is that of a system of $N/2$ interacting fermions. The low-temperature expansion is thus given by

$$\beta F = \beta E_0 + S + \frac{1}{2} cT,$$

where E_0 is the ground state energy, S is the entropy and cT the specific heat.

- E_0 , S and c are extensive.
- After a Laplace transform we obtain

$$\rho(E) = \int_{r-i\infty}^{r+i\infty} d\beta e^{\beta E} Z(\beta) = \int_{r-i\infty}^{r+i\infty} d\beta \beta^{-3/2} e^{\beta E} e^{-\beta E_0 + S + \frac{c}{2\beta}}$$
$$= \sinh(\sqrt{2c(E - E_))}).$$

Bethe formula for the nuclear level density. Bethe-1936, Cardy-1991

Scaling Limit of the Spectral Density of the SYK Model

- $N \gg q^2$: the eigenvalue density is point-wise a Gaussian.
  Mon-French-1971, Garcia-Garcia-JV-2016
- $q^2 \gg N$: the eigenvalue density is point-wise a semi-circle.
  Mon-French-1971, Benet-Weidenmüller-2002, Liu-Nowak-Zahed-2017
- q^2/N fixed for $N \to \infty$: This is a nontrivial scaling limit where the spectral density converges to the weight function of the Q-Hermite polynomials. Cotler-etal-2016, Garcia-Garcia-JV-2017.
  Garcia-Garcia-JV-2016

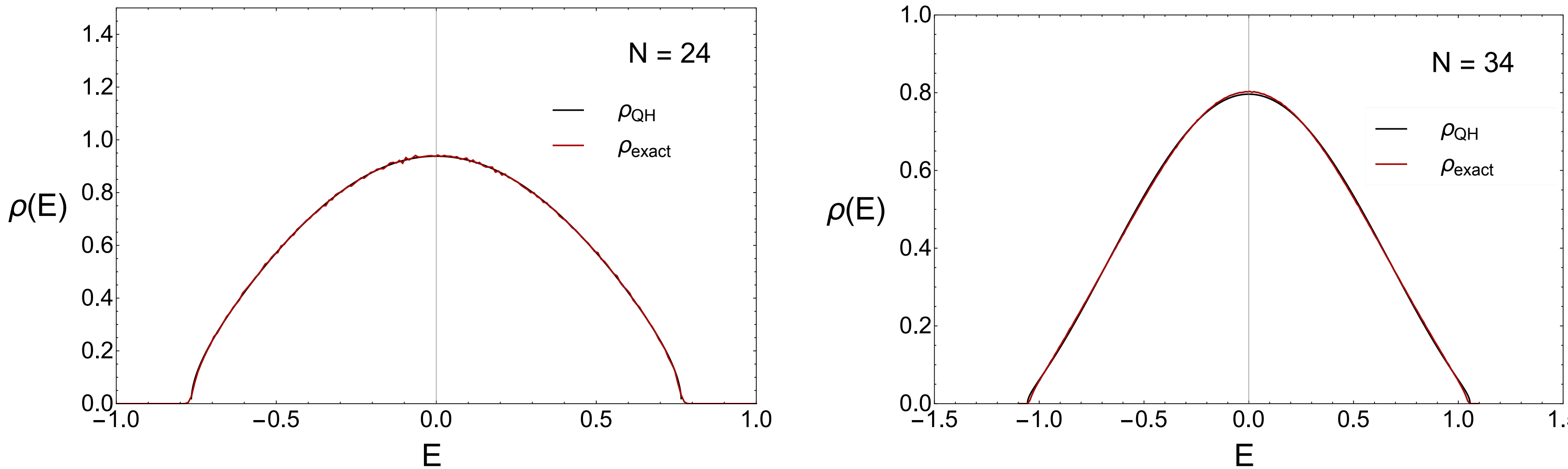
Q-Hermite Approximation

- In evaluating averages in the Q-Hermite approximation, the crossings of the Wick contractions as treated as independent.
- The spectral density is then determined by a single parameter $\eta = 2^{-N/2}\binom{N}{q}^{-1}\sum_\beta \mathrm{Tr}\Gamma_\alpha\Gamma_\beta\Gamma_\alpha\Gamma_\beta = \binom{N}{q}^{-1}\sum_{p=0}^{N}(-1)^p\binom{N-q}{q-p}\binom{q}{p}$, where $\Gamma_\alpha = \prod_{k=1}^{q}\chi_{\alpha_k}$. Garcia-Garcia-JV-2016

and is given by the weight function of the Q-Hermite polynomials, $\rho_{\mathrm{QH}}(E) = c_N\sqrt{1-(E/E_0)^2}\prod_{k=1}^{\infty}\left[1-4\frac{E^2}{E_0^2}\left(\frac{1}{2+\eta^k+\eta^{-k}}\right)\right]$ with $E_0 = 2\sigma/\sqrt{1-\eta}$ the ground state energy, and σ the variance of the spectral density.

Cotler-etal-2016, Garcia-Garcia-JV-2017.

Comparison with Numerical Results for $q = 4$



Comparison of the exact spectral density obtained by numerical diagonalization and the Q-Hermite result for the spectral density.

Garcia-Garcia-JV-2017

VI. Spectral Correlations

Maximum Chaos

Universal Correlations

Collective Spectral Modes

Univeral Inaccuracy

Thouless Energy

Upper Bound for Lyapunov Exponent

Lyapunov exponent λ_L

$$\Delta(t) \sim \Delta(0) e^{\lambda_L t}$$

Energy-time "uncertainty relation"

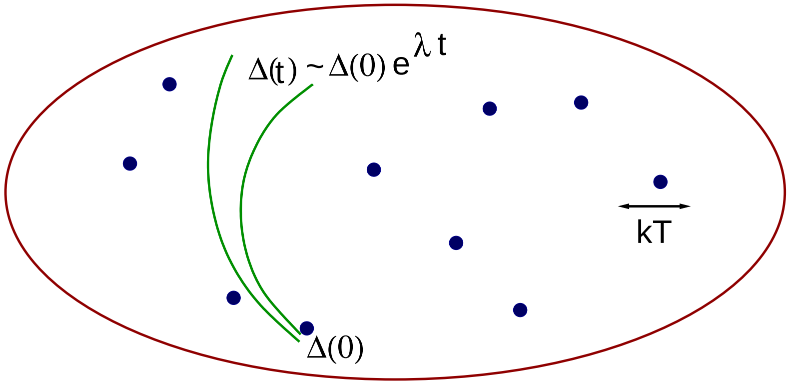
$$\begin{aligned} \Delta t \Delta E &\geq \frac{\hbar}{2} \\ \Delta t &\sim 1/\lambda, \qquad \Delta E \sim \pi k T \end{aligned}$$

So we have the bound

$$\lambda_L \leq \frac{2\pi k T}{\hbar}$$

Maldacena-Shenker-Stanford-2015

Of the same type as the η/S bound by Son.



Divergence of trajectories in a stadium at temperature T

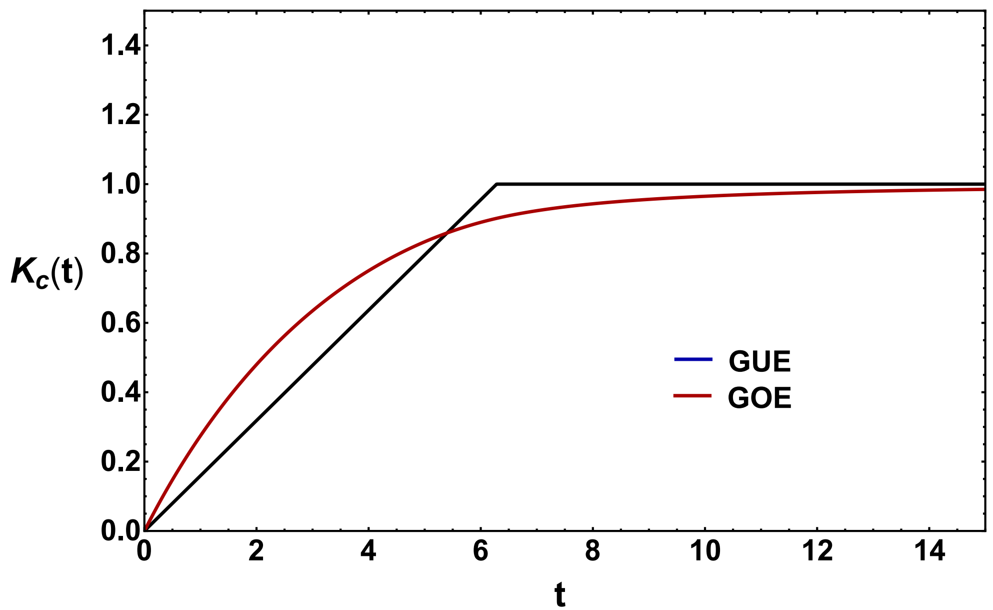
Correlations of eigenvalues

- Since the SYK model is maximally chaotic, the eigenvalue correlation should be given by RMT.
- This can be tested by comparing correlation functions.
- We will consider the pair correlation function.

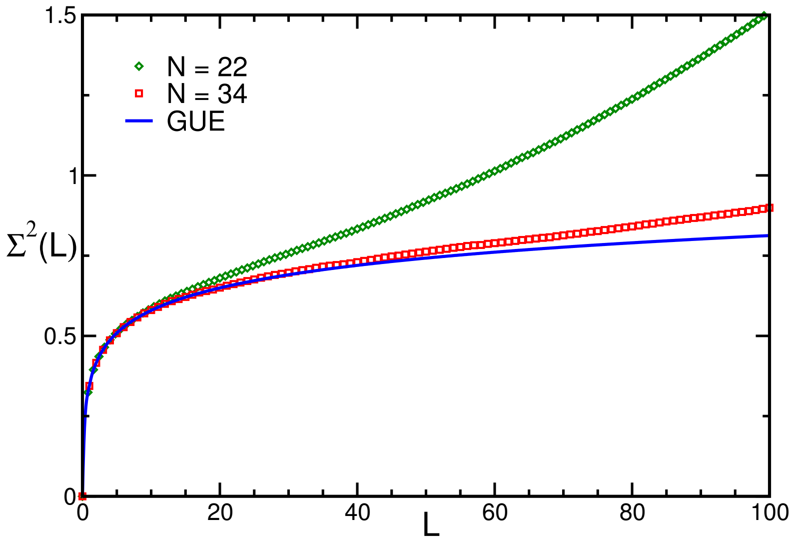
Spectral Form Factor

$$K_c(t) = \int dx dy e^{it(x-y)} \rho_{2c}(x,y) \frac{e^{-\frac{x^2+y^2}{2w^2}}}{\sqrt{\pi} w},$$

where we have added a regulator to remove finite size effects ($w \lesssim 2^{N/2}$) .

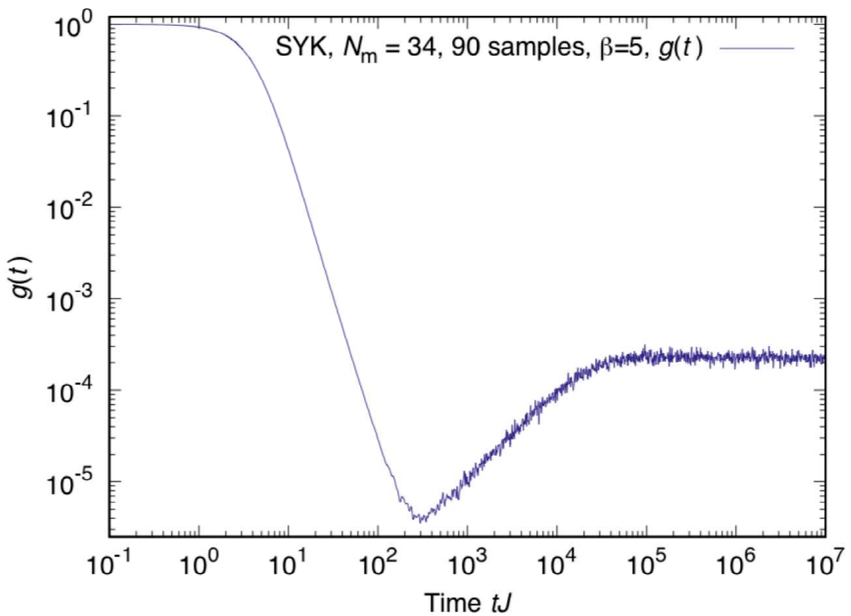


Number Variance and Spectral Form Factor



Number variance versus the length of the interval.

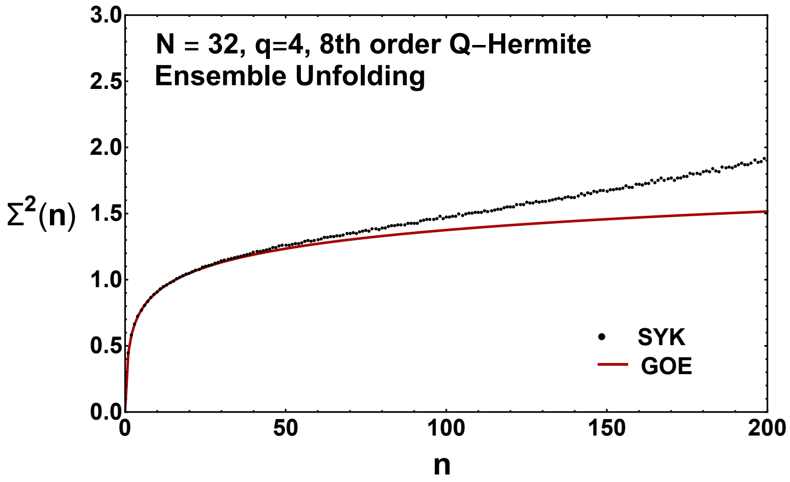
Garcia-Garcia-JV-
arXiv:1610.02363

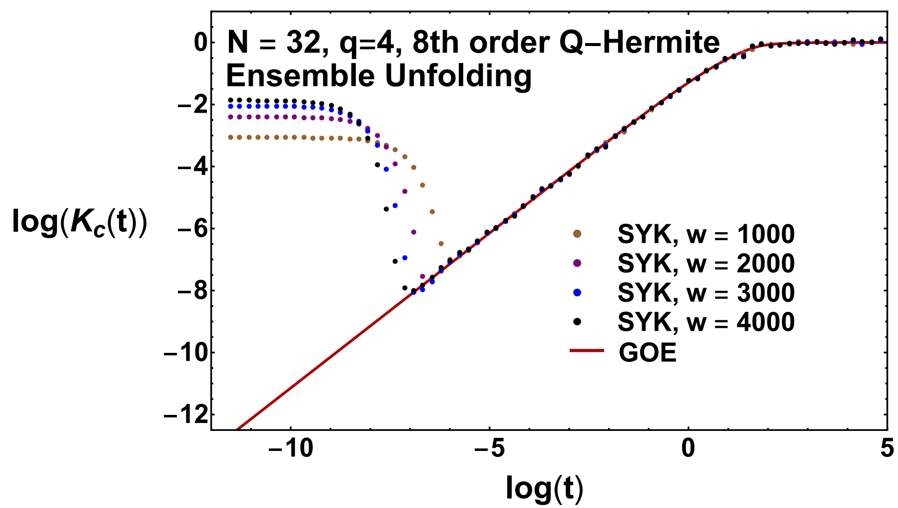


Spectral from factor versus time.

Cotler-Gur-Ari-Hanada-Polchinski-
Saad-Shenker-Streicher-Tezuka-
arXiv:1611.04650

Number Variance and Spectral Form Factor





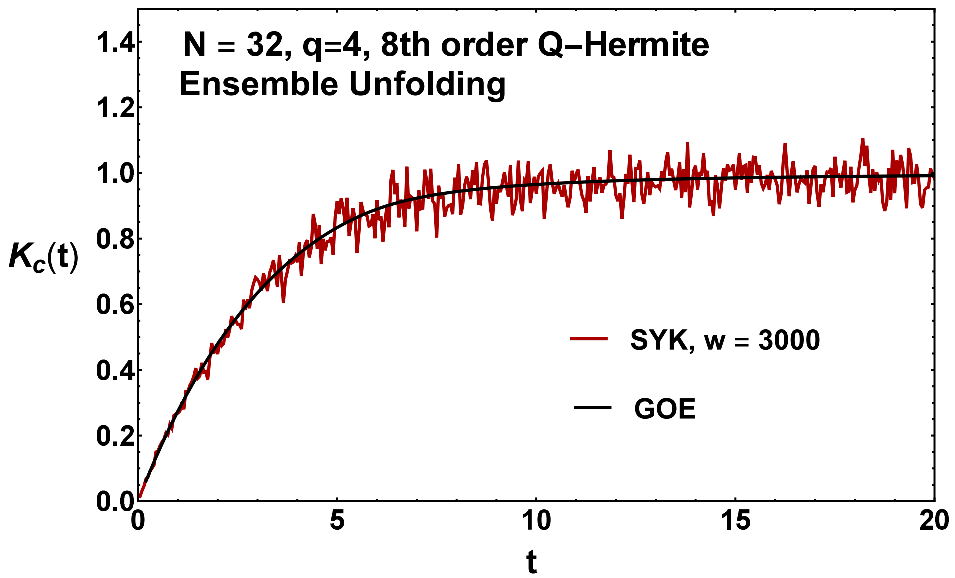
Number variance versus the length of the interval.

$$\Sigma^2(n) = \frac{1}{2\pi}\frac{n^2}{\pi}\int_{-\infty}^{\infty} K_c(t)\left(\frac{\sin(nt/2)}{nt/2}\right)^2.$$

Delon-Jost-Lombardi-1991

Connected part of the spectral from factor versus time. Note that the peak for small t is four orders of magnitude lower than in the figure on the previous slide. Garcia-Garcia-Jia-JV-2019

Form Factor on a Linear Scale



The spectral form factor of the SYK model on a linear scale.

Deviations from RMT

Deviations from RMT have two sources:

- Collective fluctuations from one realization to the next.
  Flores-Horoi-Muller-Seligman-2000, Altland-Bagrets-2018,
  Garcia-Garcia-Jia-JV-2018, Gharibyan-Hanada-Shenker-Tezuka-2018
- For a given realization, at some scale the spectral correlation see the details of the system. Berry-1984

Our aim is to eliminate the systematics to maximize the range of agreement with RMT.

In the previous figure, this scale behaves as N^2 .

N Versus $2^{N/2}$

- In a many-body system, we have two scales, N and $2^{N/2}$.

$$N \ll 2^{N/2}$$

- RMT universality arises in the microscopic limit were the correlation are expressed in units of the average level density, i.e. in the double scaling limit

$$(x - y)2^{N/2}.$$

One the questions of interest is what is the scale where spectral correlations of the SYK model start deviating from the result for the Wigner-Dyson ensembles.

In the previous figure, this scale behaves as N^2 .

Basic Problem

- The SYK model had $\binom{N}{q}$ independent matrix elements. Naive error analysis tells us that this results in fluctuations in observables of order $1/\binom{N}{q}^{1/2}$ going from one realization to the next.
- For example, the width of the spectrum for a given realization is given by the square root of the second moment. It varies by a relative error of $1/\binom{N}{q}^{1/2}$ going over the ensemble.
- More generally, generic ensemble fluctuations of the level density are of the same order, and deviations to the number variance become of order one at $\binom{N}{q}$ level spacings which is an exponential small fraction of the total number of levels.

Intrinsic Contribution to the Numbar Variance

Relative inaccuracy in level density of a single configuration

$$\frac{\delta\sigma}{\bar{\sigma}}.$$

This gives an inaccuracy in the level density

$$\frac{\delta\rho(E)}{\bar{\rho}(E)},$$

and contributes to the number variance of in interval ΔE containing $n = \int_{\Delta E} dE \bar{\rho}(E)$ on average,

$$\Sigma^2(n) = \left(\int_{\Delta E} dE \delta\rho(E) \right)^2 = 2 \binom{N}{q}^{-1} n^2.$$

The numerical coefficients requires a more precise calculation.

Flores-Horoj-Muller-Seligman-2001,Altland-Bagrets-2017,Saad-Shenker-Stanforld-2018,Garcia-Garcia-Jia-JV-2018

Collective Fluctuations of Spectra

Fluctuations of the variance are given by the moment,

$$\frac{\langle \mathrm{Tr} H^2 \mathrm{Tr} H^2 \rangle - \langle \mathrm{Tr} H^2 \rangle^2}{\langle \mathrm{Tr} H^2 \rangle^2} = \frac{6}{\binom{N}{q}}.$$

- Deviations from RMT are seen at a scale of $n \sim N^{q/4}$. This scale is known as the Thouless energy. This agrees with numerical results for $q = 3$ and $q = 4$.
  Altland-Bagrets-2017,Garcia-Garcia-Jia-JV-2018

Comment on σ Model result of Altland and Bagrets

$$R_2(\omega) = R_2^{\rm RMT}(\omega) + \frac{1}{2}\frac{\Delta^2}{\pi^2}\sum_k \binom{N}{k}\frac{1}{(i\omega - \epsilon(k))^2}$$

$$\gamma^2 = J^2\binom{N}{4}, \qquad \epsilon(k) = \gamma(S_k^{-1} - 1).$$

We have $S_k \sim 1/N^2$

$$\begin{aligned} R_2(\omega) &= R_2^{\rm RMT}(\omega) + 2^{-N}\sum_k \binom{N}{k}\frac{S_k^2}{(1 - i\omega S_k/\gamma)^2} \\ &= R_2^{\rm RMT}(\omega) + 2^{-N}\sum_k \binom{N}{k} S_k^2 \sum_n (n+1)(i\omega S_k/\gamma)^n \\ &= R_2^{\rm RMT}(\omega) + \frac{1}{2}\binom{N}{4}^{-1} + \sum_n a_n(i\omega/(\gamma N^2))^n. \end{aligned}$$

Expansion in Q-Hermite Polynomials

$$\rho_{\rm SYK}(E) = \rho_{\rm QH}(E) \left[\sum a_k H_k^Q(E/\sigma) \right].$$

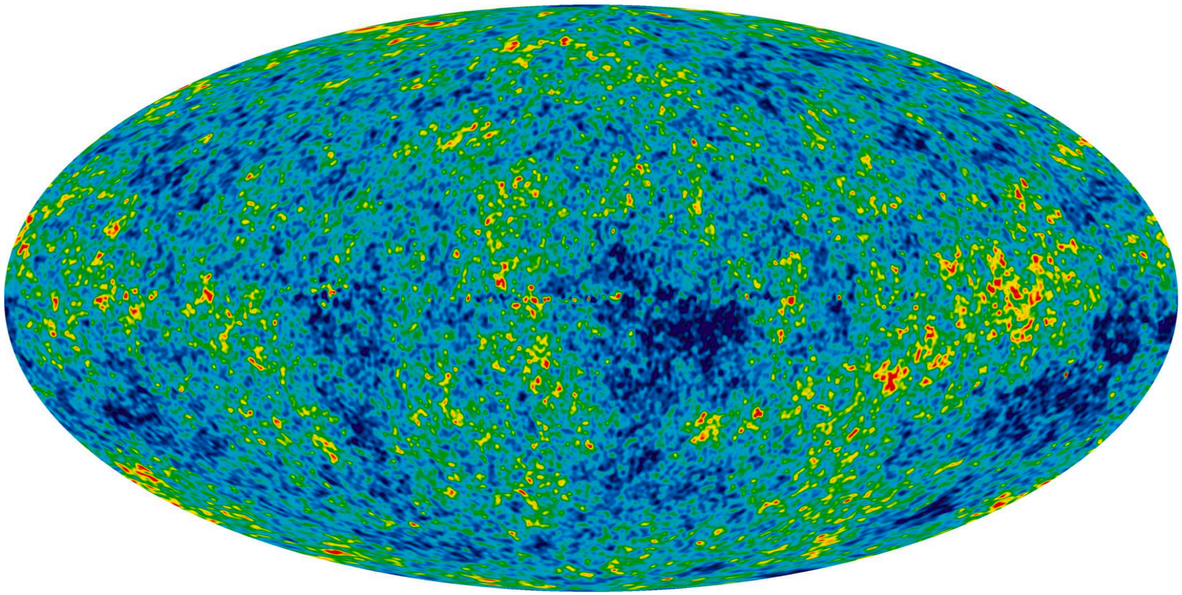
Both the ensemble average and the spectral density for a given configuration can be expanded this way. We have

$$a_0 = 1, \quad a_1 = 0,$$
$$\langle a_2 \rangle = 0, \quad \langle a_4 \rangle = 0, \quad \langle a_{2k+1} \rangle = 0$$

Numerically, the nonzero $|a_k| < 0.005$ and decreasing for larger k for $N = 32$ and $q = 4$.

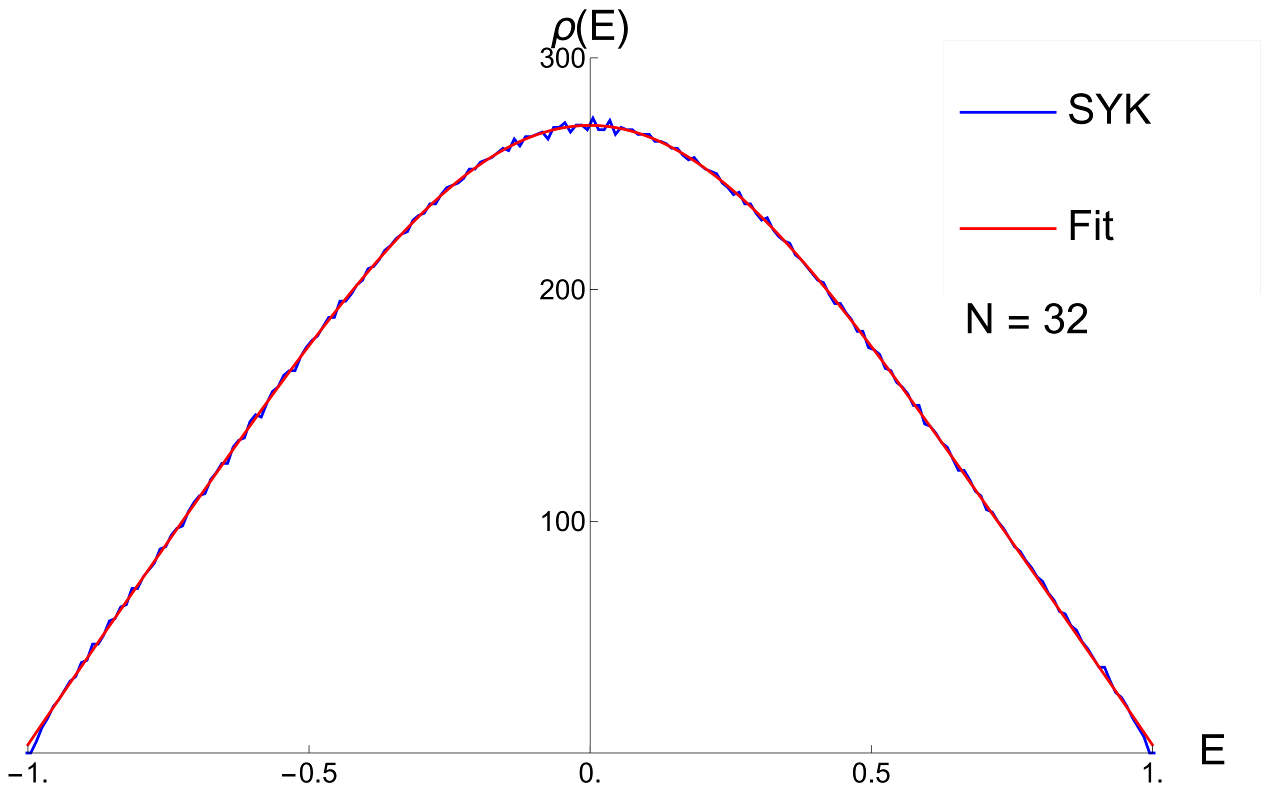
Long-wavelength fluctuations are contained in the covariances of the a_k for small k . RMT fluctuations are contained in the coefficients with $k \sim 2^{N/2}$. The wavelength of mode k is $2^{N/2}/k$ level spacings.

Separating out the Secular Behavior



After subtracting the spherical harmonics only a thermal spectrum is left. If we unfold configuration by configuration only RMT fluctuations remain.

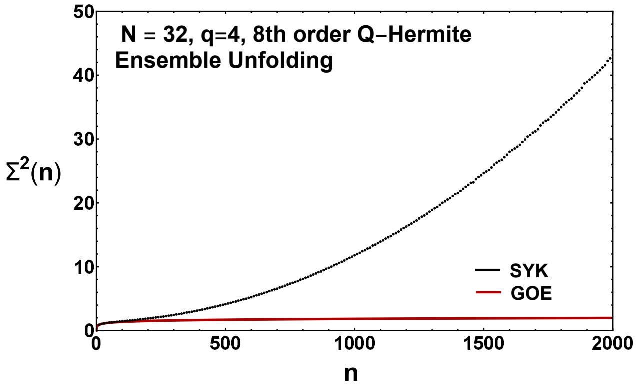
Extreme Unfolding

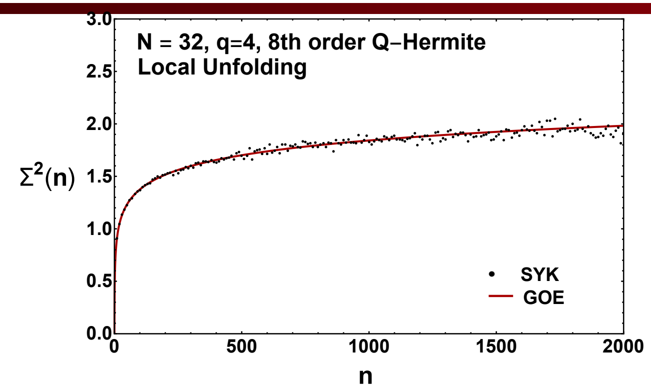


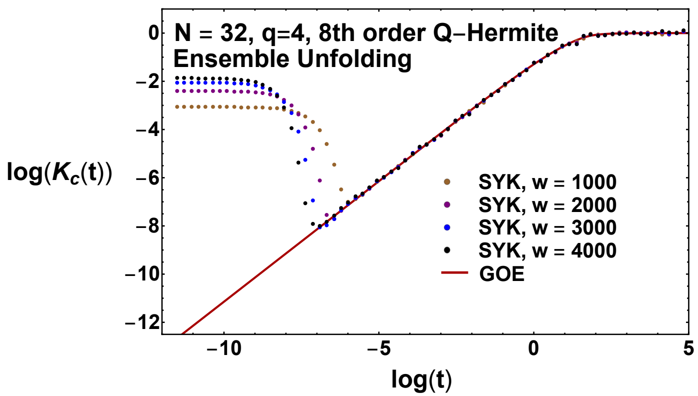
Five parameter fit to the average spectral density.

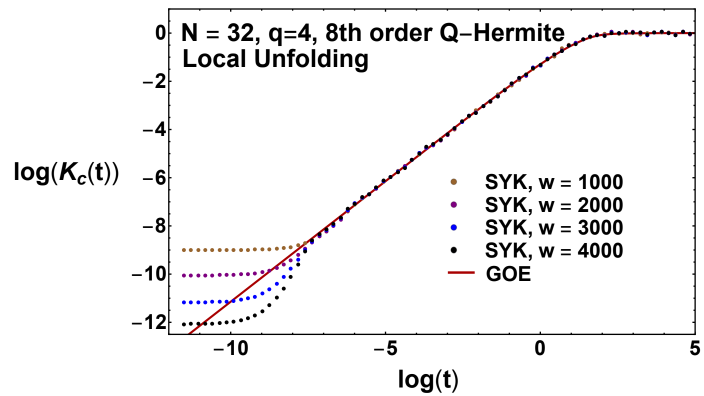
- The SYK model actually does not have any structure. So we should find RMT fluctuations to very large distance if the eliminate the collective fluctuations. This is achieved by unfolding configuration by configuration.

Effect of Local Unfolding





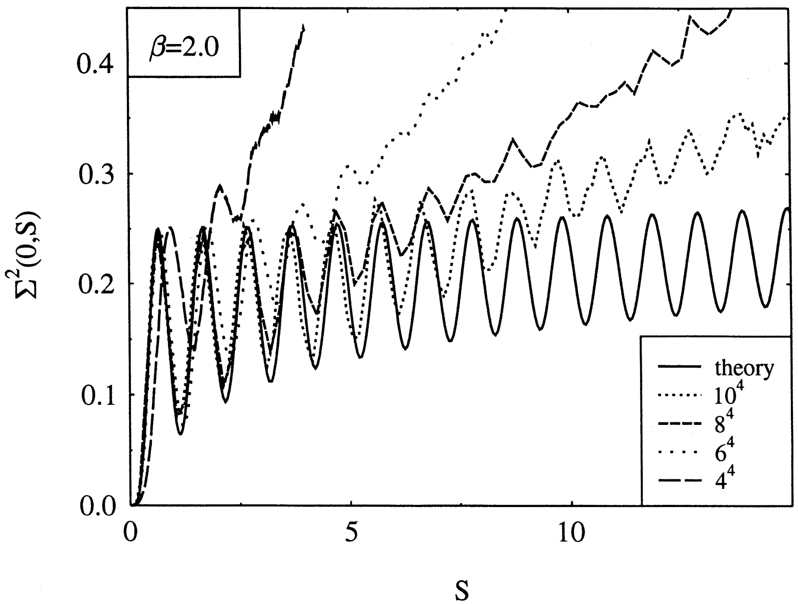


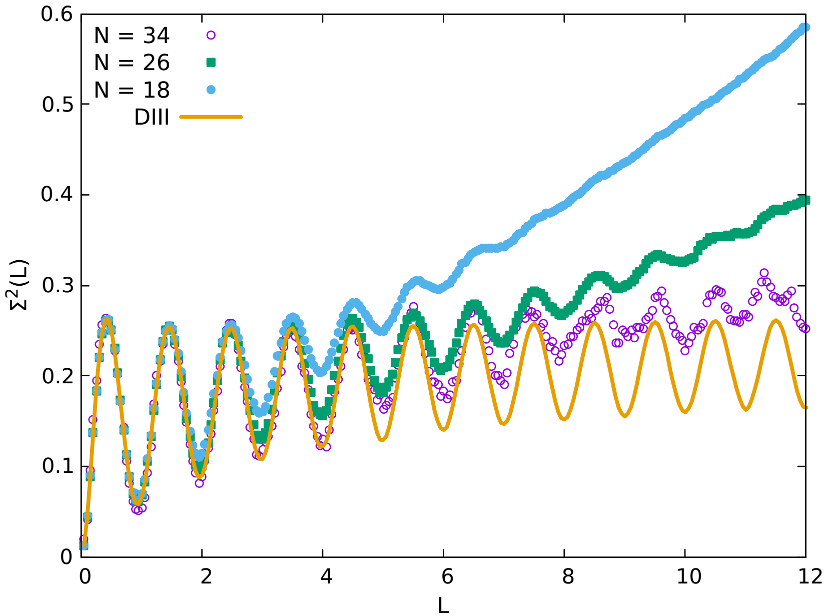


Number variance (upper) and spectral form factor (lower) of the unfolded, (ensemble – left and local –right) connected two-point function.

Garcia-Garcia-Jia-JV-2019

QCD versus SYK





Number variance for two-color QCD for various volumes Et-al-Wettig-1998 and the number variance of the SYK model for various numbers of fermions Garcia-Garcia-Jia-JV-2018. The Thouless energy is $\sim \sqrt{V}$ for QCD and $\sim N$ for the SYK model. See also, Kanazawa-Wettig-2018.

VII. Conclusions

- RMT behavior of spectral correlations is one of the strongest universality phenomena in nature, if not the strongest.

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- The agreement of correlations of Dirac eigenvalues with RMT is limited by collective fluctuations. The upper critical dimension is four.
- RMT fluctuation are likely to be universal in the conformal case as well. This is the case for 2d QCD.
- $N \ll 2^N$: Collective fluctuations limit RMT behavior to order N^2, but after subtracting them, RMT behavior is seen to $O(2^(N/2)$ level spacings.

Physics Goals for the Next 25 Years

- Many-Body Quantum Chaos
- Quantum Gravity
- Confinement
- Sign Problem
- Intelligent Computation
- Emergent Phenomena

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- Deviations from RMT are due to collective fluctuations of eigenvalues. There are no deviations when unfolding configuration by configuration. Two-point spectral correlations are not ergodic.
- These results are consistent with the idea that the SYK model is maximally chaotic.
- In a sense a black hole is dual to a compound nucleus.

Scale Fluctuations and Spectral Form Factor

$$K_c(t) = \sum_{k,l} e^{it(E_k - E_l)} - \left| \sum_k e^{itE_k} \right|^2$$

A scale transformation $E_k \to E_k(1+\delta)$ correspond to

$$K_c(t) \to K_c(t(1+\delta))$$

Averaging,

$$\langle K_c(t(1+\delta)) \rangle = K_c(t) + \frac{1}{2} K_c''(t) \langle \delta^2 \rangle$$

Only gives contributions at the kink or when $K_c(t)$ deviates from the random matrix result such that $K_c''(t) \neq 0$.

Interpretation

- The mass of the pseudo-Goldstone boson corresponding to z is given by $2\mathrm{Re}(z)\Sigma/F_\pi^2$.
- Random Matrix behavior is seen for z values where the pion Compton wavelength is much larger than the size of the box. Osborn-JV-1998.
- Low modes of Pion ripples on top of the vacuum correspond to collective fluctuations of the Dirac spectrum.