General Relativity without RMT

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STREAMLINES AND CONFORMAL INVARIANCE IN YANG–MILLS THEORIES

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- Outlook

Color-ordered Yang-Mills Amplitudes

Tree-level amplitudes with n particles in the adjoint rep.

$$\mathcal{A}(1,2,\ldots,n) = \sum_{P(2,3,\ldots,n)} \mathrm{Tr}(T^{a_1}T^{a_2}\ldots T^{a_n})A(1,2,\ldots,n)$$

Lots of identities involving $A(1,2,\ldots,n)$

Examples: simple identities like cyclicity and reflections:

$$\begin{aligned} A(1,2,\ldots,n) &= A(2,3,\ldots,n,1) \\ A(1,2,\ldots,n) &= (-1)^n A(n,n-1,\ldots,1) \end{aligned}$$

- plus many more.

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- plus many more. The basis of operators is only of size $(n-3)$!

String theory gives gravity – gauge theory relations

Some of the most amazing relations between gravity and gauge theory amplitudes were derived by Kawai, Lewellen and Tye (KLT) from string theory.

A closed string amplitude factorizes into a product of two open string amplitudes.

In the field theory limit this translates into relations between gravity amplitudes and Yang-Mills amplitudes.

These relations have also been proven directly from field theory
[E. Bjerrum-Bohr, PHD, B. Feng, T. Sondergaard]

KLT-relations: Examples

Let an n-point gravity amplitude be denoted by $M(1, 2, \ldots, n)$.

For 4-point amplitudes (let $s_{12} = (p_1 + p_2)^2$ etc.):

$$M(1, 2, 3, 4) \;=\; s_{12} A(1, 2, 3, 4) A(3, 4, 2, 1)$$

For higher n one gets a *sum* of terms on the right hand side.

The precise statement

Define

$$X_n^{(n_+,n_-)} = \sum_{\gamma,\beta\in S_{n-3}} A(1,\beta_{2,n-2},n-1,n)\widetilde{\mathcal{S}}[\beta_{2,n-2}|\gamma_{2,n-2}]\widetilde{A}(1,n-1,\gamma_{2,n-2},n)$$

where $\widetilde{\mathcal{S}}$ is a 'momentum kernel' depending on external momenta s_{ij}.

n_+ (n_-) denotes the number of positive (negative) helicity legs in A which is changed to negative (positive) helicity legs in $\widetilde{A}$.

- When $n_+ = n_- = 0,\ X_n^{(0,0)} = M(1,2,\ldots,n)$.
- When $n_+ \neq n_-,\ X_n^{(n_+,n_-)} = 0$.

Gravity from Gauge Theory

Pictorially:

It is as if two gluons of helicity $+1$ generate one graviton of helicity $+2$.

The momentum kernel in the middle miraculously cancels all unwanted double poles.

Examples

Consider a 4-point amplitude with $(n_+, n_-) = (0, 1)$:

$$0 = s_{12} A(1^-, 2^-, 3^+, 4^+) \widetilde{A}(3^+, 4^+, 2^+, 1^-).$$

Actually, this reproduces a well-known 'MHV rule'.

Examples

A new and non-trivial 5-point example with $(n_+, n_-) = (1, 0)$:

$$
\begin{aligned}
0 = s_{12} A(1^-, 2^-, 3^+, 4^+, 5^+) \big[& s_{13} \widetilde{A}(4^+, 5^+, 2^-, 3^-, 1^-) \\
& + (s_{13} + s_{23}) \widetilde{A}(4^+, 5^+, 3^-, 2^-, 1^-) \big] \\
& + s_{13} A(1^-, 3^+, 2^-, 4^+, 5^+) \big[s_{12} \widetilde{A}(4^+, 5^+, 3^-, 2^-, 1^-) \\
& + (s_{12} + s_{23}) \widetilde{A}(4^+, 5^+, 2^-, 3^-, 1^-) \big].
\end{aligned}
$$

A physical interpretation

Every time we have

$$X_n^{(n_+,n_-)} = 0$$

we have a new non-linear identity among gauge theory amplitudes.

How can we understand these new identities?

A flipped helicity on an external leg produces (+ 1 - 1 = 0) a *scalar* leg. This corresponds to gravity amplitudes with a single scalar: **it vanishes**.

In this way the gravity – gauge theory relation can be used to deduce identities in Yang-Mills theory alone!

Back to gravity

Perturbative gravity amplitudes are not worse, computationally, than Yang-Mills amplitudes

To calculate tree-level gravity amplitudes we simply 'square' gauge theory amplitudes

An analog story holds at loop-level due to the method of *unitarity cuts*

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Gravity is an example where this is not true

Basically, it is because gravity couples to the energy-momentum tensor. Inverse factors of $\hbar$ fly around and cancel $\hbar$'s in numerators

Think of this: in the Klein-Gordon equation $\hbar$ follows the mass:

$$(\partial^2 + m^2/\hbar^2)\phi(x) \;=\; 0$$

so, for example, contributions proportional to m^n can cancel $\hbar$'s!

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The starting point is then the 2nd quantized field theory based on

$$\mathcal{S} = \int d^4x \sqrt{-g} \left[\frac{1}{16\pi G} R + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{m^2}{2} \phi^2 \right] .$$

where R is the curvature and we expand the metric $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$

At tree-level and in the static limit this trivially gives Newton's law

$$V(r) = Gm_1m_2/r$$

after a Fourier transform

Loop level: a basis of integrals

One-loop integrals can conveniently be expanded in a 'basis':

Boxes, Triangles, Bubbles, and rational terms

Every one-loop amplitude $= A \cdot \mathrm{Box} + B \cdot \mathrm{Triangle} + C \cdot \mathrm{Bubble} + D \cdot \mathrm{Rational}$

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All of classical general relativity is sitting in the Triangles!

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Now start drawing all possible Feynman diagrams...

Only diagrams of loops that only connect with the external scalar legs can survive $\hbar \to 0$

A few equations explain this

$$I_{triangle} = \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2 + i\epsilon} \frac{1}{(\ell+q)^2 + i\epsilon} \frac{1}{(\ell+p_1)^2 - m_1^2 + i\epsilon}$$

Where does this integral pick up the classical contribution?

Consider $|\vec{\ell}| \ll m_1$, i.e. $(\ell + p_1)^2 - m_1^2 = \ell^2 + 2\ell \cdot p_1 \simeq 2m_1\ell_0$

Perform the ℓ_0 integral by closing the contour in upper half-plane:

$$\int_{|\vec{\ell}| \ll m} \frac{d^3\vec{\ell}}{(2\pi)^3} \frac{i}{4m} \frac{1}{\vec{\ell}^2} \frac{1}{(\vec{\ell} + q)^2} = -\frac{i}{32m|\vec{q}|} \, .$$

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Tree level:

$$M_1 = -\frac{16\pi G}{q^2}\left(m_1^2 m_2^2 - 2(p_1 \cdot p_4)^2 - (p_1 \cdot p_4) q^2\right)$$

1-loop level:

$$M_2 = -i(8\pi G)^2 \left(\frac{c(m_1, m_2) I_{\triangleright}(p_1, q)}{\left(q^2 - 4m_1^2\right)^2} + \frac{c(m_2, m_1) I_{\triangleright}(p_4, -q)}{\left(q^2 - 4m_2^2\right)^2} \right)$$

with

$$c(m_1, m_2) = (q^2)^5 + (q^2)^4 \left(6p_1 \cdot p_4 - 10m_1^2\right) + \mathcal{O}\left((q^2)^3\right)$$

- a pretty complicated expression!

Now we can calculate the interaction potentials

At leading order in q^2,

$$M_2 = \frac{6\pi^2 G^2}{|\vec{q}|}(m_1 + m_2)(5(p_1 \cdot p_4)^2 - m_1^2 m_2^2) + O(|\vec{q}|)$$

A few pages later, the effective Hamiltonian:

$$\begin{aligned} H = & \frac{\vec{p}_1^2}{2m_1} + \frac{\vec{p}_4^2}{2m_2} - \frac{\vec{p}_1^4}{8m_1^3} - \frac{\vec{p}_4^4}{8m_2^3} \\ & - \frac{Gm_1m_2}{r} - \frac{G^2 m_1 m_2 (m_1 + m_2)}{2r^3} \\ & - \frac{Gm_1m_2}{2r}\left(\frac{3\vec{p}_1^2}{m_1^2} + \frac{3\vec{p}_4^2}{m_2^2} - \frac{7\vec{p}_1 \cdot \vec{p}_4}{m_1 m_2} - \frac{(\vec{p}_1 \cdot \vec{r})(\vec{p}_4 \cdot \vec{r})}{m_1 m_2 r}\right), \end{aligned} \tag{1}$$

The effective equations of motion corresponding to this were first derived in 1938 by *Einstein, Infeld and Hoffmann*

So much effort to re-derive a result from 1938?

We now know how to turn the crank, and go to arbitrarily high order!

The expansion never involves acceleration and higher derivatives

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From virial theorem:

$$p^2/2m \sim GMm/r$$

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The new challenge is to try to do better than that!

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Let us turn to the *eikonal limit*

New: The Post-Minkowskian (PM) expansion

Define

$$M(\vec{b}) \equiv \int d^2\vec{q} e^{-i\vec{q}\cdot\vec{b}} M(\vec{q})$$

We then find

$$M(\vec{b}) = 4p(E_1 + E_2)(e^{i\chi(\vec{b})} - 1)$$

and $\chi(\vec{b})$ is the scattering function. For tree-level and 1-loop:

$$\chi_i(b) = \frac{1}{2\sqrt{\hat{M}^4 - 4m_1^2 m_2^2}} \int \frac{d^2\vec{q}}{(2\pi)^2} e^{-i\vec{q}\cdot\vec{b}} \mathcal{M}_i(\vec{q})$$

where $\hat{M}^2 \equiv M^2 - m_1^2 - m_2^2$ and $M^2 \equiv s$

New: The Post-Minkowskian (PM) expansion

Now we have the scattering angle!

$$2\sin(\theta/2) = \frac{-2M}{\sqrt{\hat{M}^4 - 4m_1^2 m_2^2}} \frac{\partial}{\partial b} \left(\chi_1(b) + \chi_2(b)\right)$$

Doing the integrals,

$$2\sin(\theta/2) \;=\; \frac{4GM}{b}\Big(\frac{\hat{M}^4 - 2m_1^2 m_2^2}{\hat{M}^4 - 4m_1^2 m_2^2} + \frac{3\pi}{16}\frac{G(m_1+m_2)}{b}\frac{5\hat{M}^4 - 4m_1^2 m_2^2}{\hat{M}^4 - 4m_1^2 m_2^2}\Big).$$

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Now we know how to generalize this to arbitrary high order!

A Post-Minkowskian 2-body interaction potential

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Cheung, Rothstein and Solon have proposed a Post-Minkowskian potential

They define an effective theory with a scalar-scalar potential $\sim \phi\phi V \phi\phi$.

To fix V they demand that the scattering amplitude matches that of the full theory to the given order in G

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$$\hat{G}(z) = (z - \hat{\mathcal{H}})^{-1} \qquad \hat{T}(z) = \hat{V} + \hat{V}\hat{G}(z)\hat{T}$$

$$\lim_{\epsilon \to 0} \langle p | \hat{T}(E_p + i\epsilon) | p' \rangle = \mathcal{M}(p, p')$$

A Lippmann-Schwinger equation:

$$\mathcal{M}(p, p') = V(p, p') + \int \frac{d^3k}{(2\pi)^3} \frac{\mathcal{M}(p, k) V(k, p')}{E_p - E_k + i\epsilon}$$

Our approach

Now invert:

$$V(p,p') = \mathcal{M}(p,p') - \int \frac{d^3k}{(2\pi)^3} \frac{\mathcal{M}(p,k)\mathcal{M}(k,p')}{E_p - E_k + i\epsilon} + ..$$

The corrections are *Born subtractions*

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The corrections are *Born subtractions*

It agrees exactly with that of Cheung et al.

Things move fast

Calculations have very recently been pushed to 2-loop order by Bern et al.

With the formalism established it can be pushed to much higher order.

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- It pays to calculate classical GR from quantum loops
- Quantum Field Theory reproduce both PN and PM expansions in GR
- The formalism is ready – now time to compute!