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Bielefeld Seoul
International Research Training Group 2235

RMT in Sub-Atomic Physics and Beyond

Universal Broadening of Zero Modes: A General Framework and Identification

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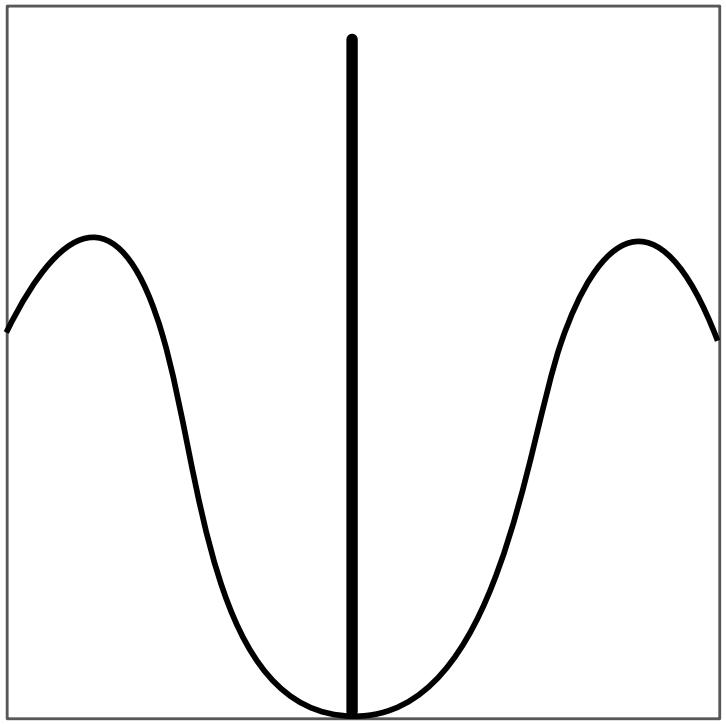
Bielefeld University

Joint work with
Mario Kieburg and Kim Splittorff
[arXiv:1902:01733]

The Model

The Model



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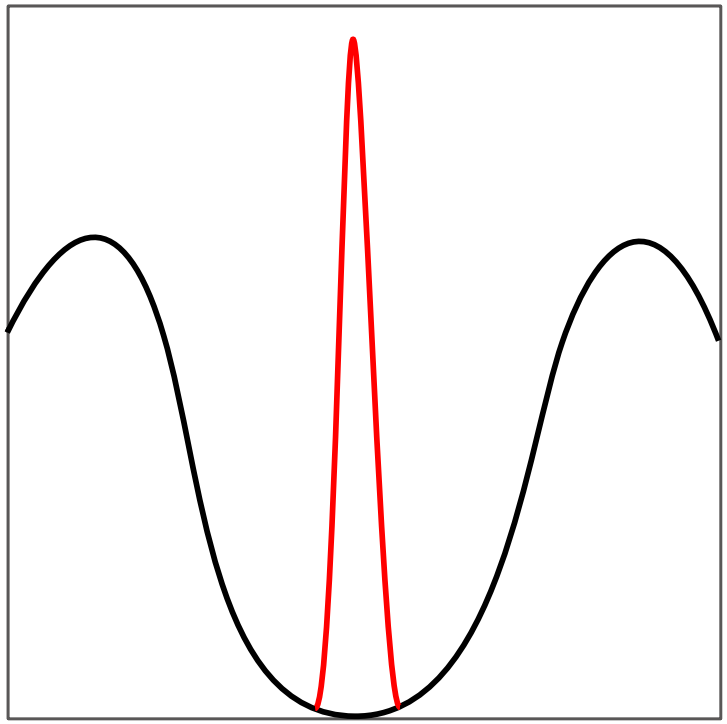

$$K = \underbrace{A}_{\text{Pure System with zero mode}} + \underbrace{\alpha U S U^{\dagger}}_{\text{Small perturbation}}$$

The Model

Motivation: Physical Systems

- Topological modes, but more general framework
- The behaviour was seen before in several specific systems
  - Wilson-Dirac Operator in finite-volume lattice-QCD

$$D = \begin{pmatrix} 0 & W \\ W^\dagger & 0 \end{pmatrix} + a \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

[Akemann, Damgaard, Splittorff, Verbaarschot [arXiv:1012.0752]]
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  - Coupled Chiral Systems

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- Difficult to distinguish topological and non-topological modes

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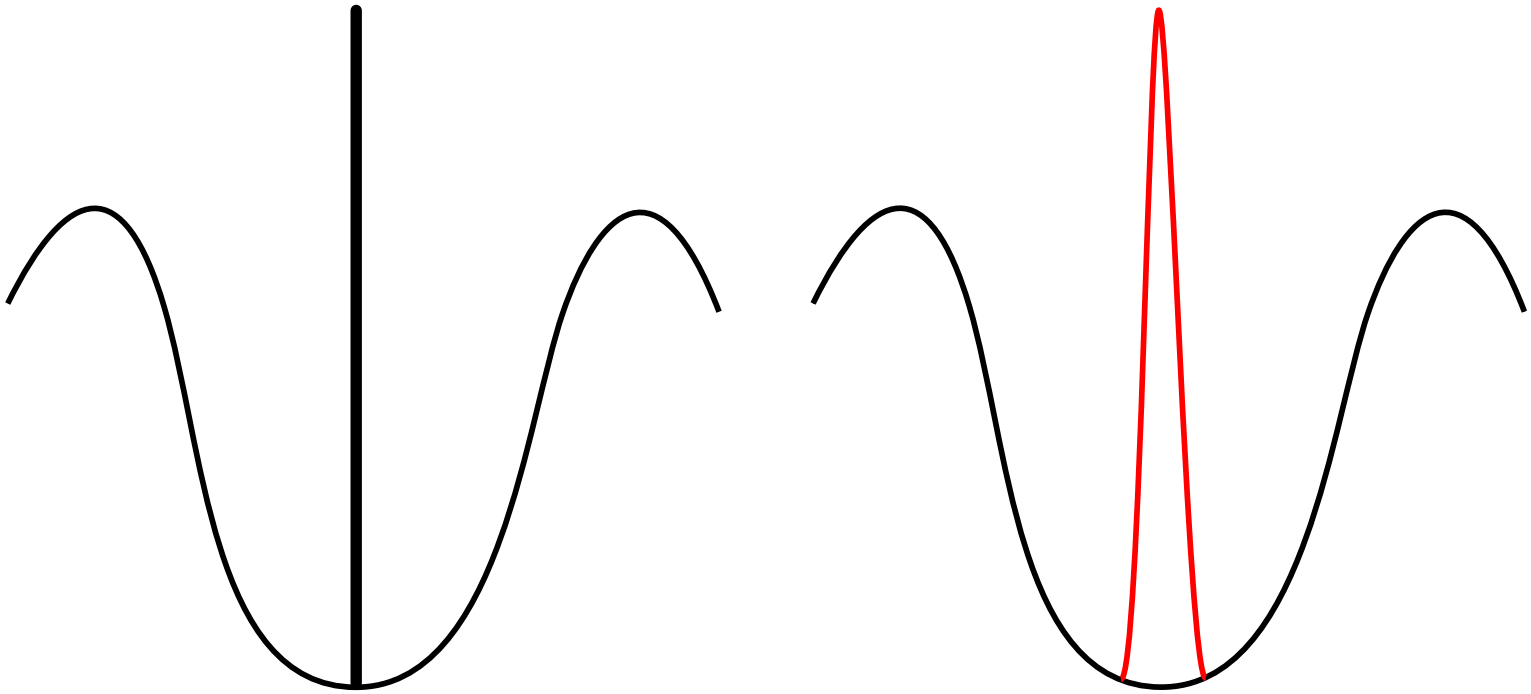

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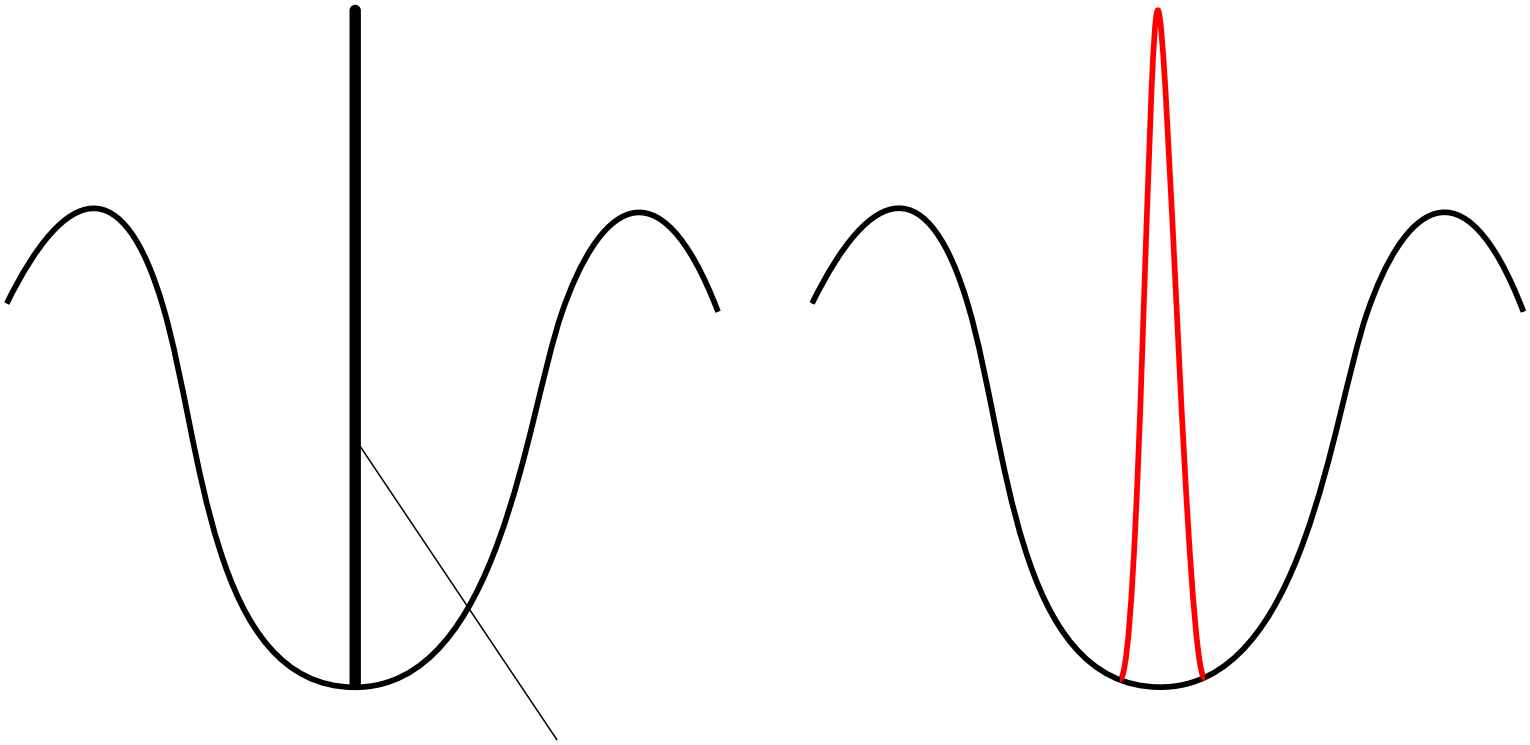

Single mode feels full ensemble S.

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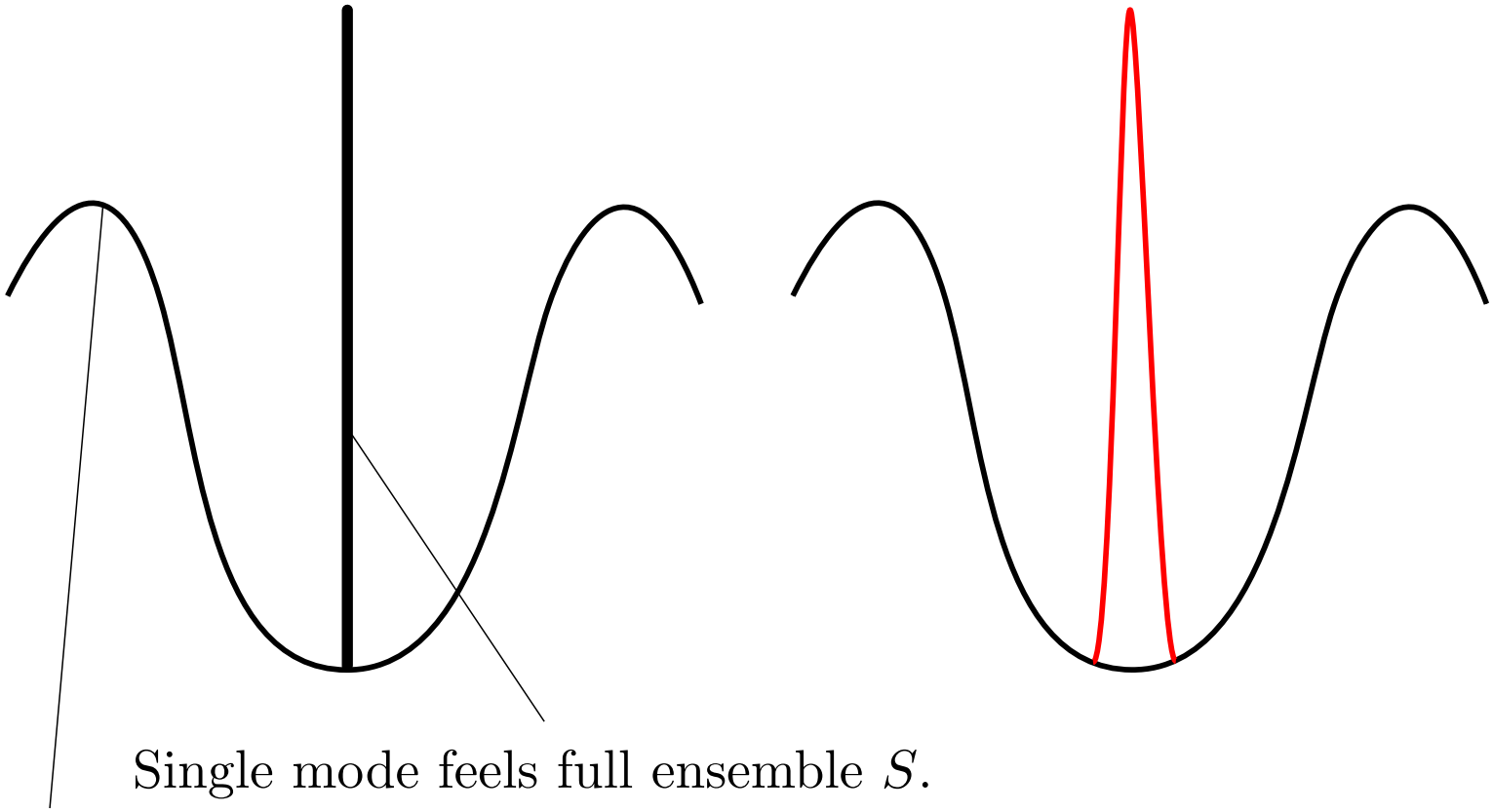

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Bulk only perturbed at higher orders.

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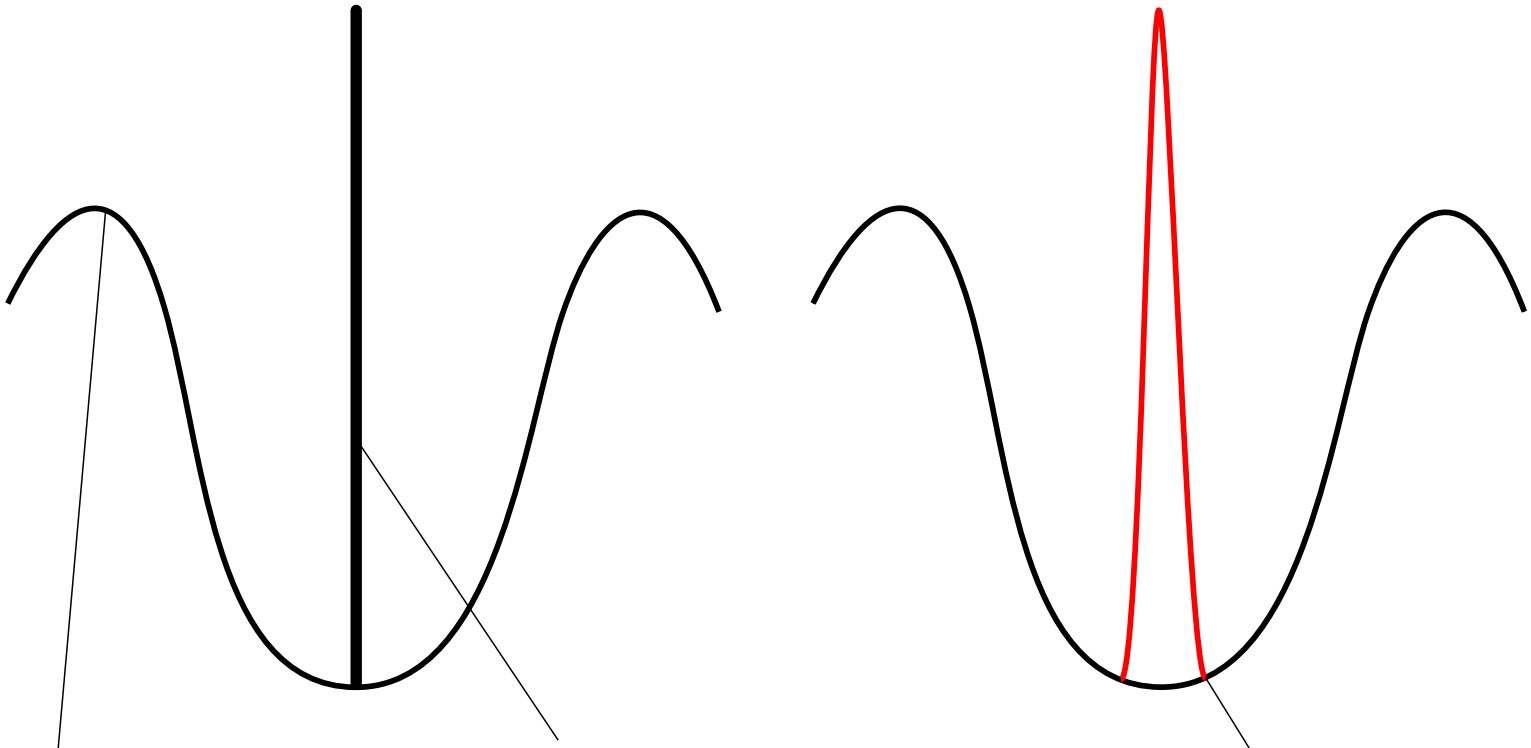

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Bulk only perturbed at higher orders.

Breaks down if the perturbation touches the bulk.

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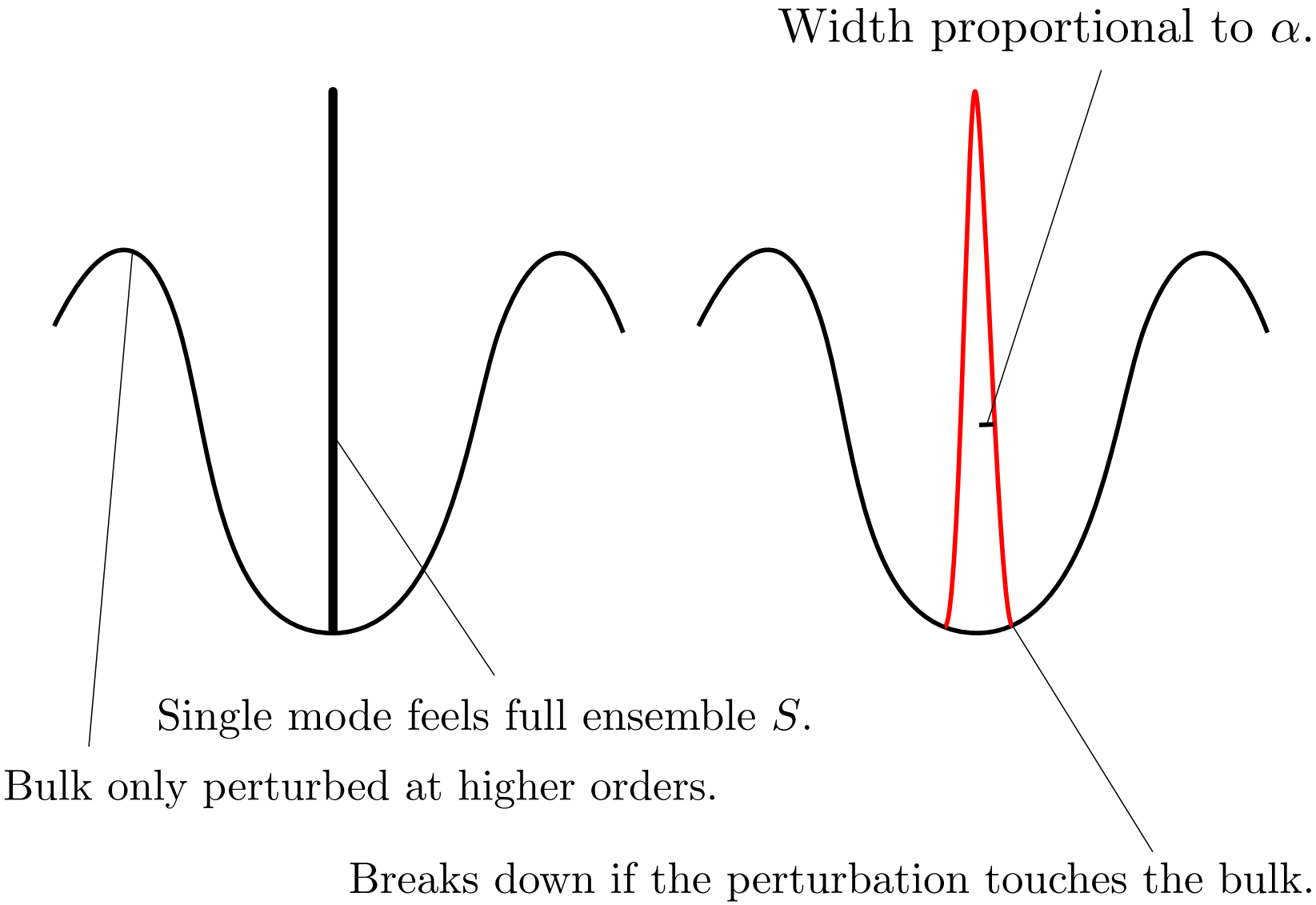

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Set-up

- A and S deterministic.
- Broadening comes from averaging over change of basis.

$$K = \underbrace{A}_{\text{Fixed}} + \alpha \overbrace{U}^{\text{Random}} \underbrace{S}_{\text{Fixed}} \overbrace{U^{\dagger}}^{\text{Random}}$$

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Decoupling of Spectrum

Decoupling of Spectrum

Eigenvalue Equation

- Choice of basis

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\begin{aligned}
A &= \left(\begin{array}{c|c} A' = \mathrm{diag}(\lambda_{\nu+1}, \ldots, \lambda_N) & 0_{(N-\nu)\times\nu} \\ \hline 0_{\nu\times(N-\nu)} & 0_{\nu\times\nu} \end{array}\right) \\
USU^\dagger &= \left(\begin{array}{c|c} S_1 & S_2 \\ \hline S_2^\dagger & S_3 \end{array}\right) \ , \ S_3 \text{ corresponds to zero modes} \\
U &= \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \ , \ U_1 \text{ is } (N-\nu)\times N \ , \ U_2 \text{ is } \nu \times N
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Former zero modes are determined by S_3.

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Decoupling of Spectrum

Conditions

- Perturbation small enough

$$\alpha = o\left(\frac{1}{||S||_{\mathrm{op}}}\sqrt{\frac{N}{\mathrm{Tr}(A')^{-2}}}\right)$$

- Centred

$$\mathrm{Tr}S = 0$$

- Sufficient mixing for limit

$$\lim_{N\to\infty} q = \infty,$$

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Central Limit Theorem for Matrices

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- Possible for all Altland-Zirnbauer classes
- Result for non-chiral classes is

$$\lim_{N\to\infty} p(S_3) \quad \propto \quad \exp\left[-\frac{\gamma N^2 \mathrm{Tr} {S_3}^2}{2\alpha^2 \mathrm{Tr} S^2}\right]$$

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Central Limit Theorem for Matrices

Non-chiral Classes

- Consider $S' = \kappa S_3$, with $\kappa = N/\sqrt{\mathrm{Tr} S^2}$

$$p(S') = \int_{\mathcal{K}_\nu} d\mu(U_2)\delta\left(S' - \kappa U_2 S U_2^\dagger\right)$$

- Rewrite Haar-measure

$$\int_{\mathcal{K}_\nu} d\mu(U_2) f(U_2) = \frac{\int_{\mathcal{G}_\nu} dU_2 f(U_2)\delta(\mathbf{1}_\nu - U_2 U_2^\dagger)}{\int_{\mathcal{G}_\nu} dU_2 \delta(\mathbf{1}_\nu - U_2 U_2^\dagger)}.$$

- Rewrite δ-function

$$\delta(X) \propto \lim_{\epsilon \to 0} \int_{\mathcal{H}_\nu} dH \exp[-\epsilon \mathrm{Tr} H^2 + i \mathrm{Tr} X H]$$

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- Starting point is therefore

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p(S') \quad \propto \quad & \lim_{\epsilon\to 0}\int_{\mathcal{G}_\nu} dU_2 \int_{\mathcal{P}_\nu} dP \int_{\mathcal{H}_\nu} dH \exp[-\epsilon \mathrm{Tr} H^2 + i\mathrm{Tr}(\overbrace{S' - \kappa U_2 S U_2^\dagger}^{\text{From spectrum}})H] \\
& \times \exp[\epsilon\gamma N \mathrm{Tr}(\mathbf{1}_\nu - iP)^2 + \gamma N \mathrm{Tr}(\underbrace{\mathbf{1}_\nu - U_2 U_2^\dagger}_{\text{From Haar-measure}})(\mathbf{1}_\nu - iP)]
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- Interchange integrals and perform U_2-integral

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- After rescaling $P \to P/\sqrt{\gamma N}$ the saddlepoint is

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- Expanding the determinant

$$\ln \det{}^{-\gamma}\left[\mathbf{1}_{N\nu} + i\frac{S}{\gamma\sqrt{\mathrm{Tr} S^2}} \otimes H\right] \quad = \quad \gamma \sum_{j=1}^{\infty} \frac{1}{j} \mathrm{Tr}\left(-i\frac{S}{\gamma\sqrt{\mathrm{Tr} S^2}}\right)^j \mathrm{Tr} H^j$$

- Higher orders vanish

$$\left|\frac{\mathrm{Tr}(S)^j}{(\mathrm{Tr} S^2)^{j/2}}\right| \leq \frac{||S||_{\mathrm{op}}^{j-2} \mathrm{Tr} S^2}{(\mathrm{Tr} S^2)^{j/2}} = \frac{1}{q^{j-2}} \overset{N\to\infty}{\to} 0$$

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- Higher orders vanish

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Central Limit Theorem for Matrices

Non-chiral Classes

- After rescaling $P \to P/\sqrt{\gamma N}$ the saddlepoint is

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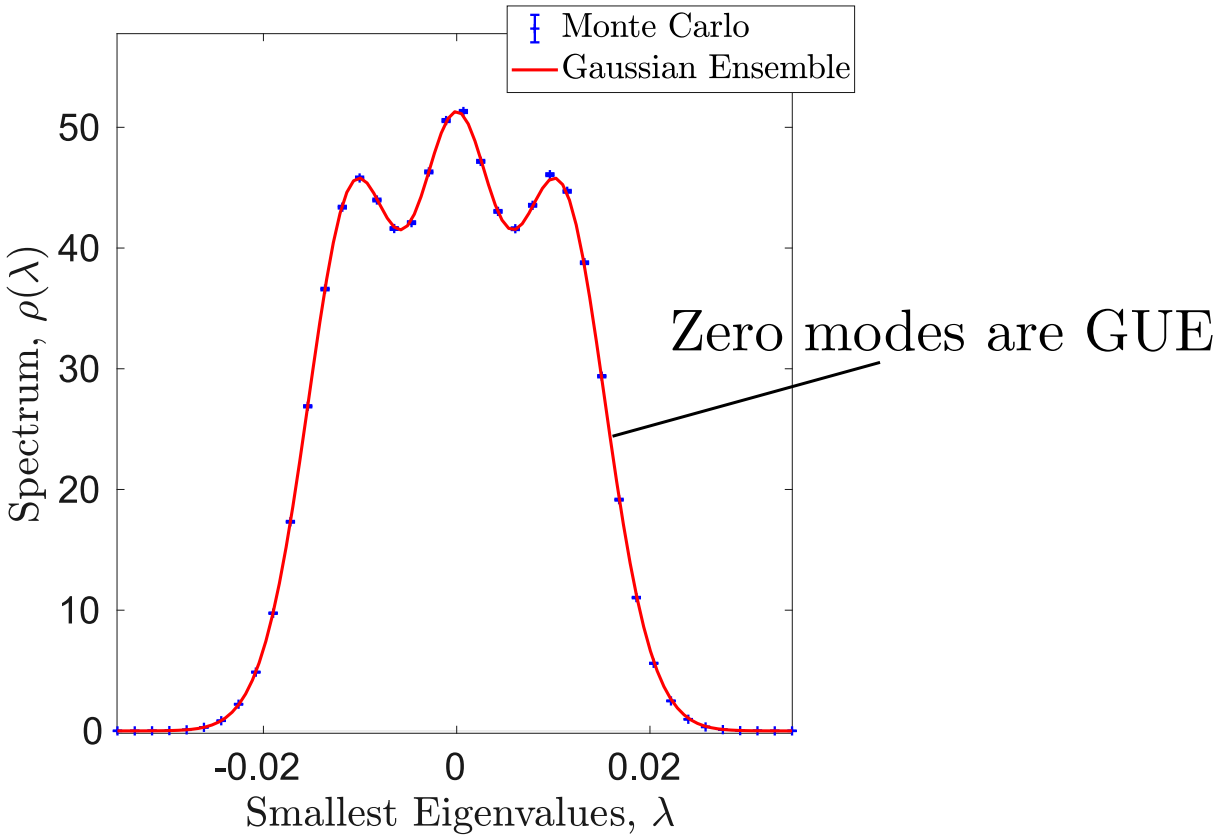

$$K = \begin{pmatrix} 0 & M \\ M^\dagger & 0 \end{pmatrix} + \alpha U S U^\dagger$$

Scaling and Application

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- Assume $\mathrm{Tr} S^2 \sim N$ and α fixed, the broadened modes scale as $\sqrt{N}$
- From field theory $N \sim V$
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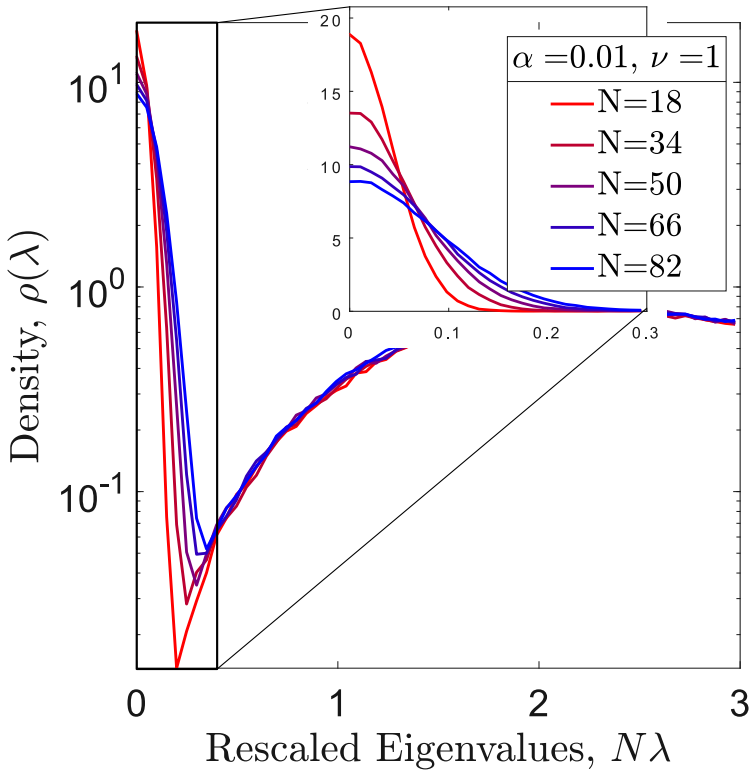

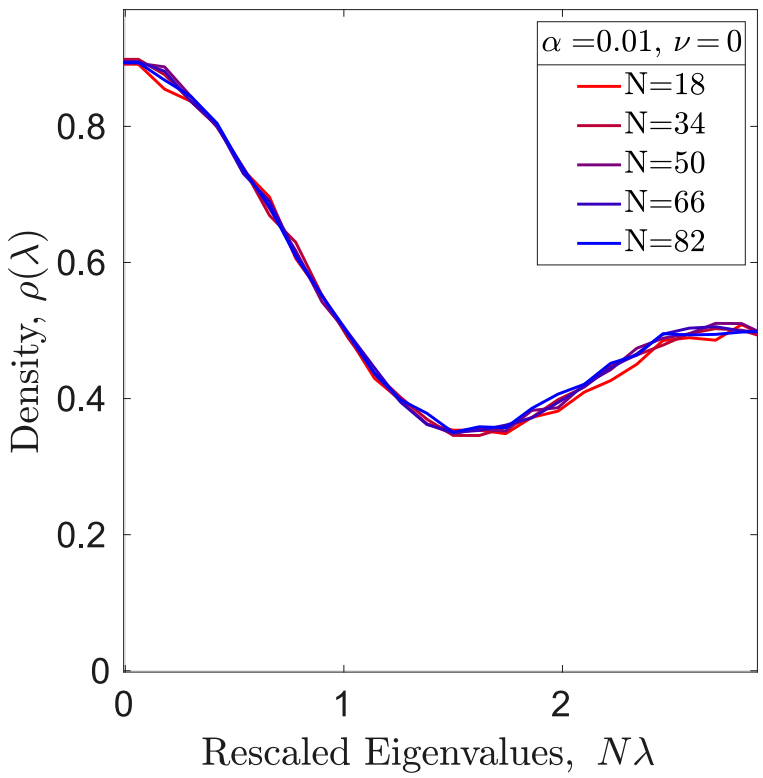

$$K = \begin{pmatrix} iM & 0 \\ 0 & -iM \end{pmatrix} + \alpha O \begin{pmatrix} 0 & iW \\ -iW^T & 0 \end{pmatrix} O^T$$

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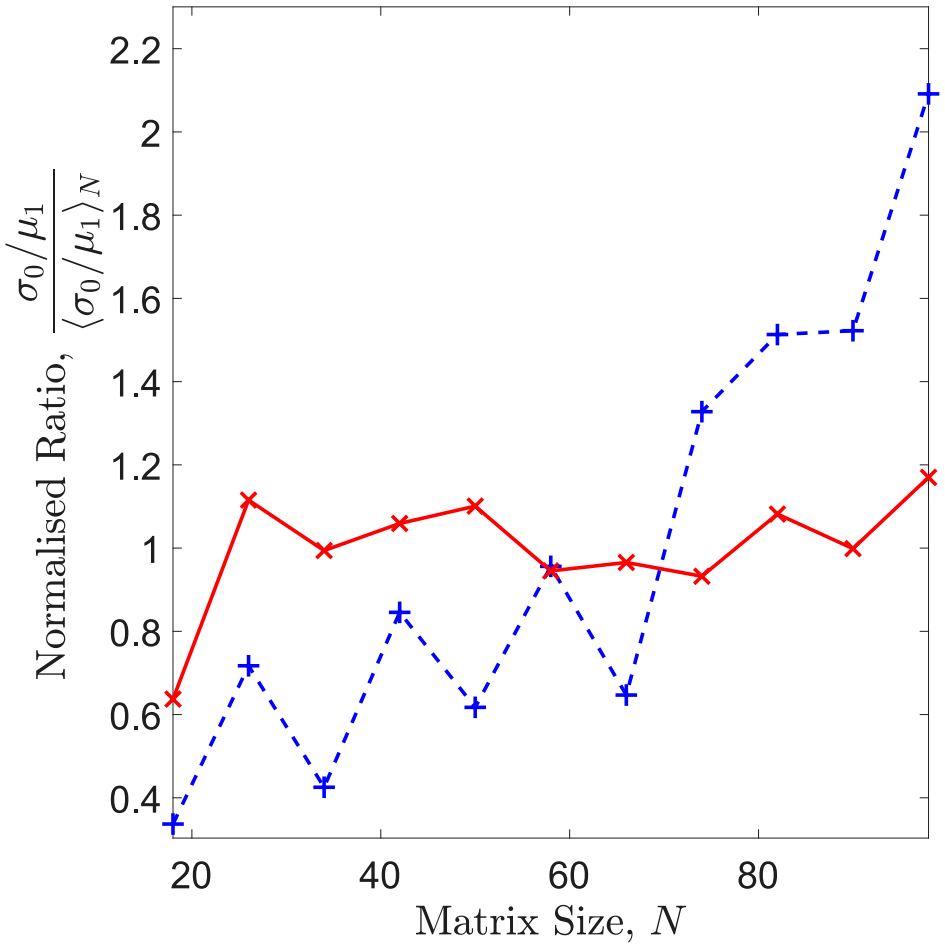

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Thank you for your time!

Extra Slides

Eigenvalue Equation

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\begin{aligned}
0 &= \det(K - \lambda \mathbf{1}_N) \\
&= \det\left(\begin{array}{c|c} A' + \alpha S_1 - \lambda \mathbf{1}_{N-\nu} & \alpha S_2 \\ \hline \alpha S_2^\dagger & \alpha S_3 - \lambda \mathbf{1}_\nu \end{array}\right) \\
&= \det(A' - \lambda \mathbf{1}_{N-\nu}) \det\left(\begin{array}{c|c} \mathbf{1}_{N-\nu} + \alpha (A' - \lambda \mathbf{1}_{N-\nu})^{-1} S_1 & \alpha (A' - \lambda \mathbf{1}_{N-\nu})^{-1} S_2 \\ \hline \alpha S_2^\dagger & \alpha S_3 - \lambda \mathbf{1}_\nu \end{array}\right) . \\
&= \det\left(A' - \lambda \mathbf{1}_{N-\nu}\right) \det\left(\mathbf{1}_{N-\nu} + \alpha (A' - \lambda \mathbf{1}_{N-\nu})^{-1} U_1 S U_1^\dagger\right) \\
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Extra Slides

Eigenvalue Equation

- Write as Neumann sum

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- Insert this

$$\begin{aligned} & \alpha U_2 S \Big(\mathbf{1}_{N-\nu} - \alpha U_1 \left(\mathbf{1}_{N-\nu} + \alpha \left(A' - \lambda \mathbf{1}_{N-\nu}\right)^{-1} U_1 S U_1^\dagger\right)^{-1} \\ & \times \left(A' - \lambda \mathbf{1}_{N-\nu}\right)^{-1} U_1 S \Big) U_2^\dagger \\ = & \alpha U_2 S \left(\mathbf{1}_{N-\nu} + \sum_{j=1}^{\infty} (-\alpha)^j \left[U_1^\dagger \left(A' - \lambda \mathbf{1}_{N-\nu}\right)^{-1} U_1 S\right]^j\right) U_2^\dagger \\ = & \alpha U_2 S \left[\mathbf{1}_{N-\nu} + \alpha U_1^\dagger \left(A' - \lambda \mathbf{1}_{N-\nu}\right)^{-1} U_1 S\right]^{-1} U_2^\dagger . \end{aligned}$$

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Extra Slides

Eigenvalue Equation

- Consider the squared norm of $\alpha U_1^\dagger (A')^{-1} U_1 S \left|\chi\right\rangle$

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Former zero modes are determined by S_3.

Extra Slides

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