The Thermodynamics and Chaos of the Sachdev-Ye-Kitaev Model

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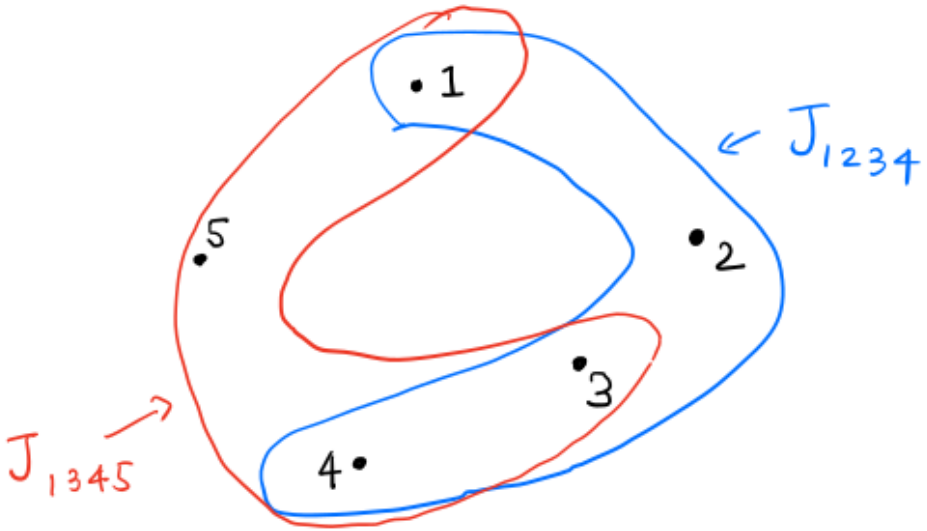
The SYK model

The SYK (Sachdev-Ye-Kitaev) model describes a disordered four-body interaction among N Majoranas in $0+1$ dimension:

$$H(J_{ijkl}) = \sum_{1 \le i < j < k < l \le N} J_{ijkl} \gamma_i \gamma_j \gamma_k \gamma_l,$$

where J_{ijkl} are Gaussian distributed and γ_i's are Dirac matrices.



A schematic diagram with $N = 5$

Why SYK model?

- High energy side: a highly solvable model for AdS/CFT duality, sheds light on the black hole information paradox. Dual theory includes a sector of Jackiw-Teitelboim gravity (stay tuned for Antonio's talk).
- Condensed matter side: a simple model for many-body quantum chaos and thermalization. (This talk)

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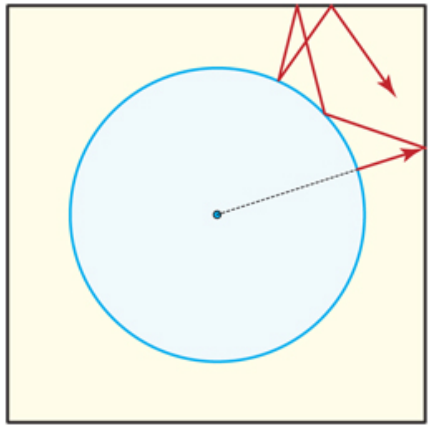
(Single-particle) quantum chaos and random matrix

Single particle chaos: Sinai's billiard



Classical signature of chaos: nonzero Lyapunov exponent.
Quantum signature of chaos?

Bohigas-Giannoni-Schmit conjecture

Bohigas-Giannoni-Schmit (BGS) conjecture: the unfolded quantum energy levels of a classically chaotic system correlate according to random matrix theory (RMT) predictions (at the distance of a few level spacings).

Sinai billiard example:

- energy levels: $-\frac{\hbar^2}{2m}\nabla^2\psi + V(x)\psi = E_i\psi$.
- spectral density: $\rho(x) = \sum_i \delta(x - E_i)$.
- average spectral density: $\langle\rho(x)\rangle$.
- correlation: $\langle\rho(x)\rho(y)\rangle - \langle\rho(x)\rangle\langle\rho(y)\rangle$.
- unfolding : $\rho(x) \to \frac{\rho(x)}{\langle\rho(x)\rangle}$
- unfolded correlation: $\frac{\langle\rho(x)\rho(y)\rangle}{\langle\rho(x)\rangle\langle\rho(y)\rangle} - 1$.
- BGS conjecture: $\frac{\langle\rho(x)\rho(y)\rangle}{\langle\rho(x)\rangle\langle\rho(y)\rangle} - 1 = \frac{\langle\rho(x)\rho(y)\rangle_{RMT}}{\langle\rho(x)\rangle_{RMT}\langle\rho(y)\rangle_{RMT}} - 1$

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Random matrix theory

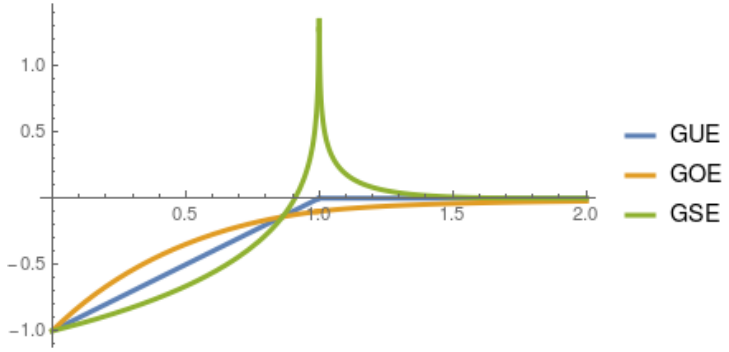
Hermitian matrices randomly distributed according to

$$P(H)dH = \exp\left(-n\mathrm{Tr}H^2\right)dH$$

- Real symmetric H: Gaussian Orthogonal Ensemble (GOE)
- Quaternionic H: Gaussian Symplectic Ensemble (GSE)
- No constraint on H (other than Hermiticity): Gaussian Unitary Ensemble (GUE)

Random matrix theory

- Spectral form factor: Fourier transform of the unfolded correlation function $\frac{\langle\rho(x)\rho(y)\rangle}{\langle\rho(x)\rangle\langle\rho(y)\rangle} - 1$.
- RMT spectral form factor:



- Sinai's billiard form factor = GOE form factor. (BGS conjecture satisfied)
- Proof of BGS conjecture: semiclassical analysis of long periodic orbits. (Berry 1985, Müller-Heusler-Braun-Haake-Altland 2004)

Back to SYK

- Goal: understand the SYK model from an RMT perspective.
- Need: $\langle \rho \rangle_{SYK}$ and $\langle \rho\rho \rangle_{SYK}$.
- Method: $\langle \rho \rangle_{SYK}$ – analytic/combinatorial; $\langle \rho\rho \rangle_{SYK}$ – numerical.

The q-body SYK model

For analytic study, it is advantageous to generalize the four-body interaction to a q-body interaction:

$$H(J_\alpha) = (\sqrt{-1})^{q(q-1)/2} \sum_{1 \le i_1 < i_2 < \cdots < i_q \le N} J_{i_1 i_2 \cdots i_q} \gamma_{i_1} \gamma_{i_2} \cdots \gamma_{i_q}$$
$$=: \sum_\alpha J_\alpha \Gamma_\alpha,$$

with

$$\alpha = \{i_1, i_2, \ldots, i_q\},\ 1 \le i_1 < i_2 < \cdots < i_q \le N,$$

and

$$\Gamma_\alpha = (\sqrt{-1})^{q(q-1)/2} \gamma_{i_1} \gamma_{i_2} \cdots \gamma_{i_q}.$$

Again J_α follows a Gaussian distribution.

Average spectral density: moment method

- Average spectral density:

$$\langle \rho(E) \rangle = \left\langle \sum_{k=1}^{2^{\lfloor \frac{N}{2} \rfloor}} \delta(E - E_k) \right\rangle .$$

- Moments:

$$M_{2p} := \int dE \langle \rho(E) \rangle E^{2p} = \left\langle \mathrm{Tr} H(J_\alpha)^{2p} \right\rangle$$

  Odd moments $M_{2p+1} = 0$ due to $J_\alpha \to -J_\alpha$ symmetry.
- The collection of M_{2p} uniquely determines $\langle \rho(E) \rangle$.

Wick contractions and chord diagrams

- Since J_α is Gaussian, we can use Wick theorem to compute the averaging:

$$M_{2p} = \sum \text{all Wick contractions with } 2p\ \Gamma\text{'s}\ .$$

Each contraction is a trace over a product of $2p$ Γ's.

- For example, one contraction that contributes to M_6 is

$$\sum_{\alpha_1,\alpha_2,\alpha_3} \mathrm{Tr}(\Gamma_{\alpha_1}\Gamma_{\alpha_2}\Gamma_{\alpha_3}\Gamma_{\alpha_2}\Gamma_{\alpha_3}\Gamma_{\alpha_1}),$$

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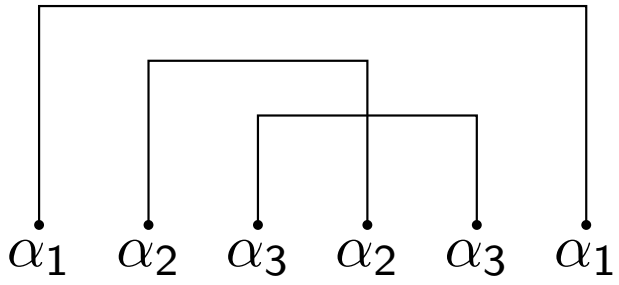

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Wick contractions and chord diagrams

- The $2p$-th moment contains $(2p-1)!!$ contractions/chord diagrams.
- We denote a chord diagram by G, the corresponding contraction value is denoted by η_G, hence Wick theorem can be restated as

$$M_{2p} = \sum_{i=1}^{(2p-1)!!} \eta_{G_i}.$$

- η_G is determined by how the chords intersect each other.

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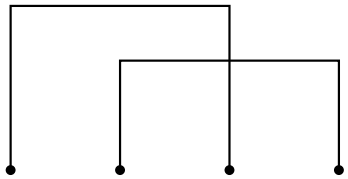
Q-Hermite approximation

- A special role will be played by the following contraction:

$\eta :=$  $= (-1)^q \left(1 - \frac{2q^2}{N} + \frac{2q^2(q-1)^2}{N^2} + \cdots\right)$

- When chord intersections do not form close loops, we have the identity [García-García-YJ-Verbaarschot 2018]:

$$\eta_G = \eta^{n_c},$$

where n_c is the number of intersections in G. For example:

 $= \eta^2$

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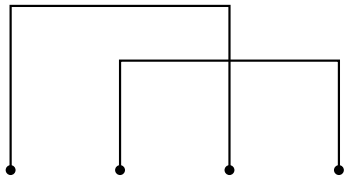
$\eta :=$  $= (-1)^q \left(1 - \frac{2q^2}{N} + \frac{2q^2(q-1)^2}{N^2} + \cdots\right)$

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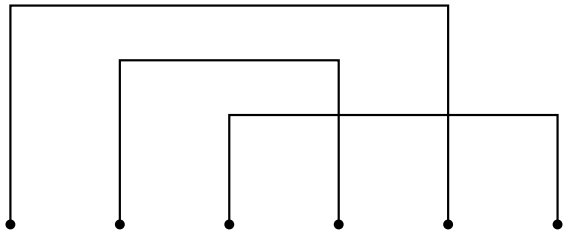
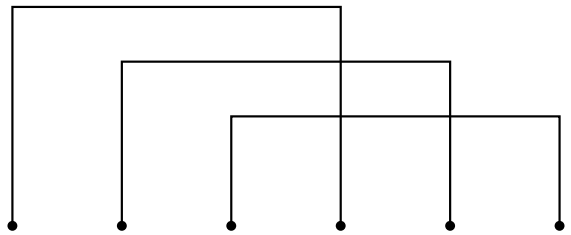
 $= \eta^2$

Q-Hermite approximation

- When there are close loops,

$$\eta_G \approx \eta^{n_c}$$

is still a very good approximation. For example:



$\approx \eta^3$

[Erdős-Schröder 2014, Cotler et al. 2017, García-García-Verbaarschot 2017, García-García-YJ-Verbaarschot 2018]

- Corrections to this approximation has been studied to $1/N^3$ order. [García-García-YJ-Verbaarschot 2018, YJ-Verbaarschot 2018]

Q-Hermite approximation

- The approximated moments then are

$$M_{2p}(\eta) \approx \sum_{i=1}^{(2p-1)!!} \eta^{n_{c_i}}.$$

This is the generating function for chord intersections and is well studied in mathematics community [Touchard 1952, Riordan 1975, Ismail-Stanton-Viennot 1987].

- The unique spectral density that corresponds to the approximated moments is

$$\rho_{QH}(E) = c_N \sqrt{1-(E/E_0)^2} \prod_{k=1}^{\infty} \left[1 - 4\frac{E^2}{E_0^2}\left(\frac{1}{2+\eta^k+\eta^{-k}}\right)\right].$$

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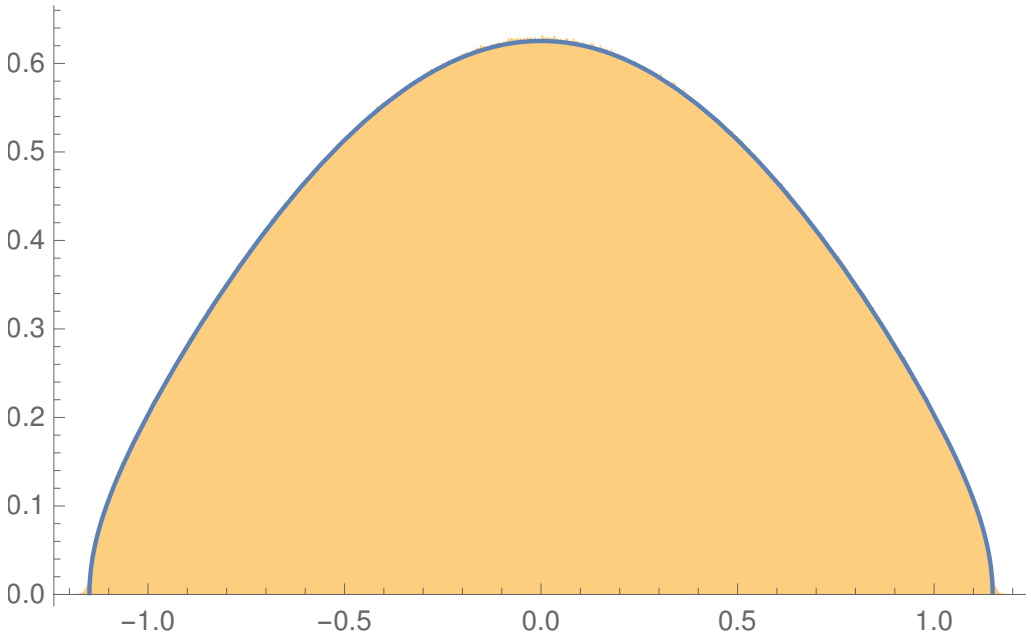
Q-Hermite approximation

Compare $\rho_{QH}(E)$ with numerical data ($N = 24, q = 4$):



Orange: histogram of the numerical eigenvalues (8000 realizations);
Blue curve: $\rho_{QH}(E)$ (taking first 10 terms in the infnite product).

We can see ρ_{QH} approximates $\langle\rho\rangle_{SYK}$ quite well.

Properties of $\rho_{QH}(E)$

- $\rho_{QH}(E)$ interpolates between Gaussian distribution and semicircle distribution:
  Large N and fixed q: $\eta \to 1$, $\rho_{QH}(E) \to$ Gaussian.
  Large N and $q \sim N$: $\eta \to 0$, $\rho_{QH}(E) \to$ semicircle.
- Near the ground state $E \approx E_0$, the Q-Hermite density simplifies [Cotler et al. 2017, García-García-Verbaarschot 2017]:

$$\rho_{QH}(E) \approx e^{S_0} \sinh(\alpha\sqrt{1 - E/E_0}),$$

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Properties of $\rho_{QH}(E)$

- Both S_0 and α are proportional to $N \Rightarrow$ Level spacings are exponentially small in N near the ground state: typically not the case in Fermi liquid models.
- S_0 is the zero-temperature entropy, α can be interpreted as the pion decay constant of the low-energy Goldstone theory of the SYK model (Schwarzian theory). [Maldacena-Stanford 2016]
- In the double scaling limit q^2/N fixed, $N \to \infty$, ρ_{QH} becomes the exact spectral density, and more single-trace observables can be studied analytically. [Berkooz-Isachenkov-Narovlansky-Torrents 2018]

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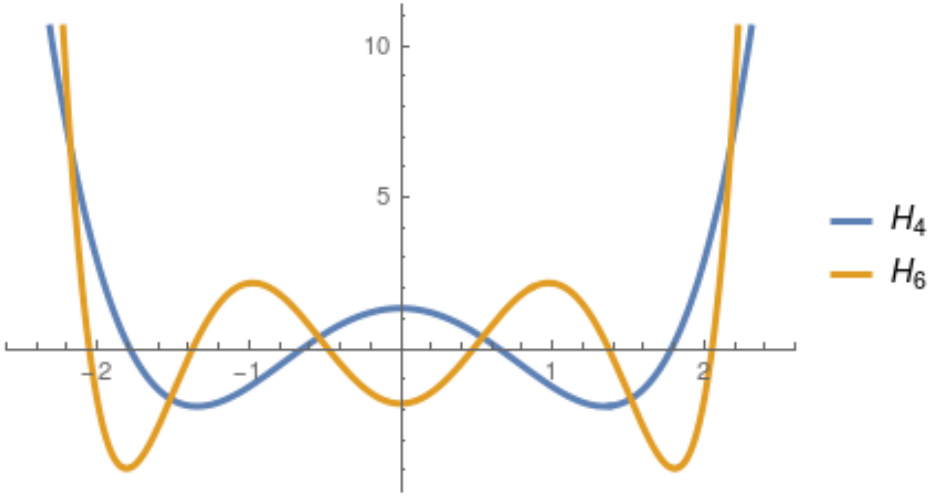
Correlations of eigenvalues

We will use $\rho_{QH}(E)$ to unfold SYK eigenvalues since $\rho_{QH}(E)$ is an accruate approximation to $\langle\rho(E)\rangle_{SYK}$. The spectral fluctuations around $\rho_{QH}(E)$ can be naturally expanded in terms of the Q-Hermite polynomials H_n^η [García-García-YJ-Verbaarschot, in preparation]:

$$\frac{\rho(E)}{\rho_{QH}(E)} = 1 + \sum_{k=1}^{\infty} a_k H_{2k}^{\eta}(E).$$

Note the odd polynomials do not appear because $\rho(E) = \rho(-E)$.



Collective fluctuations of the spectra

What do we expect to see? Generically there are both short-range and long-range fluctuations:

- Short-range fluctuations: a few level spacings ($\sim e^{-N}$), will be responsible for RMT behaviour (BGS conjecture).
- Long-range fluctuations:
  - Translations forbidden by symmetry $\rho(E) = \rho(-E)$.
  - Scale fluctuations ("breathing" mode): fluctuations of the distribution width realization by realization, can be estimated by the variance of variances

$$\langle (\mathrm{Tr} H^2)^2 \rangle - \langle \mathrm{Tr} H^2 \rangle^2 = \frac{2}{\binom{N}{q}} \sim N^{-q}$$

Note $N^{-q} >> e^{-N}$. This is to be contrasted with random matrix models, where the scale fluctuation and level spacing are both $\sim N^{-1}$.

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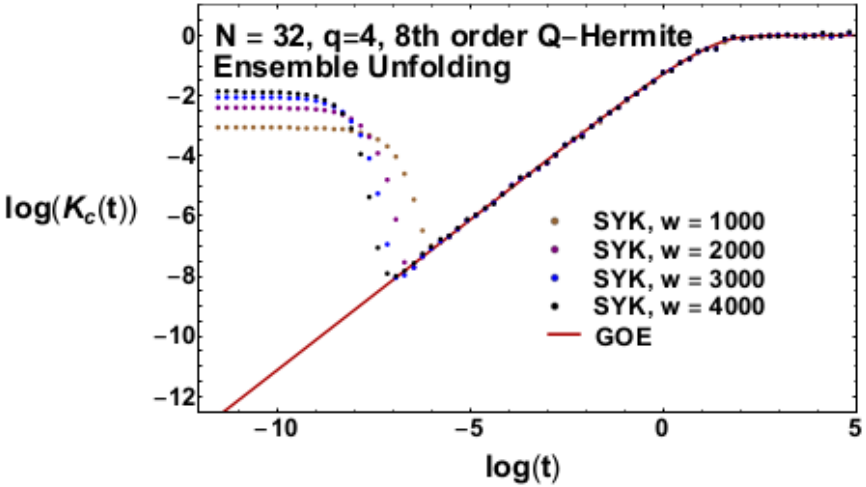
$$\langle (\mathrm{Tr} H^2)^2 \rangle - \langle \mathrm{Tr} H^2 \rangle^2 = \frac{2}{\binom{N}{q}} \sim N^{-q}$$

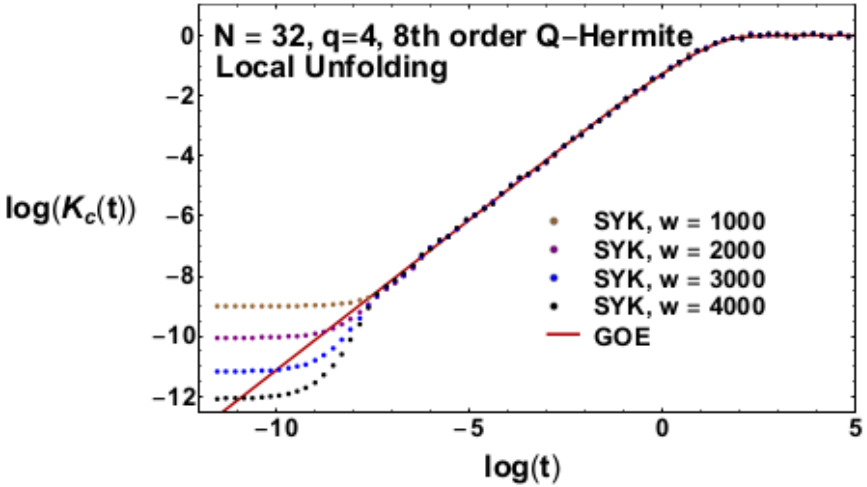
Note $N^{-q} >> e^{-N}$. This is to be contrasted with random matrix models, where the scale fluctuation and level spacing are both $\sim N^{-1}$.

Spectral form factor of SYK

The eigenvalue statistics will get closer to RMT result if long-range modes are discarded realization by realization, that is, consider the correlations of

$$\frac{\rho(E)}{\rho_{QH}(E)} - \sum_{k=0}^{M} a_k H_{2k}^{\eta}(E).$$

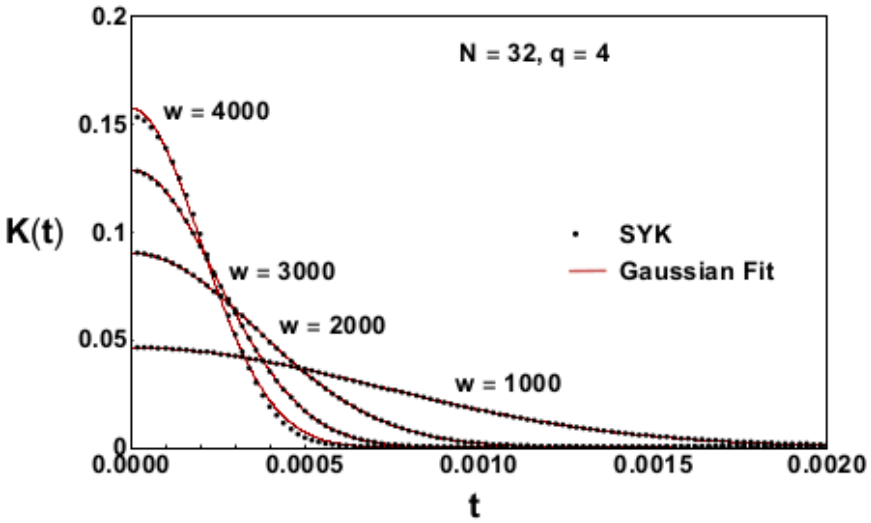




The SYK form factor converges very well to RMT form factor after subtracting mere first eight modes! (Recall $N^{-q} >> e^{-N}$.)

Spectral form factor of SYK

We can also study the deviation from RMT before substracting long-range modes:



The early-time form factor is well fitted by Gaussian when there is a finite regulator w, and tends to a delta function as the regulator is removed ($w \to \infty$).

Aside: SYK as a enumeration machine

- There are $(2p-1)!!$ chord diagrams with p chords (as in $\langle \mathrm{Tr} H^{2p} \rangle$).
- A much more nontrivial enumeration:

$$\sum_{i=1}^{(2p-1)!!} (\#\text{ of chord intersections})_i = \frac{1}{3}\binom{p}{2}(2p-1)!!$$

- Simple-looking but nontrivial to prove. [Touchard 1952, Riordan 1975, Flajolet-Noy 2000].
- Claim: The SYK model provides an alternative proof, and gives (infinitely) more such identities.
- The correct language for generalization is *intersection graph*.

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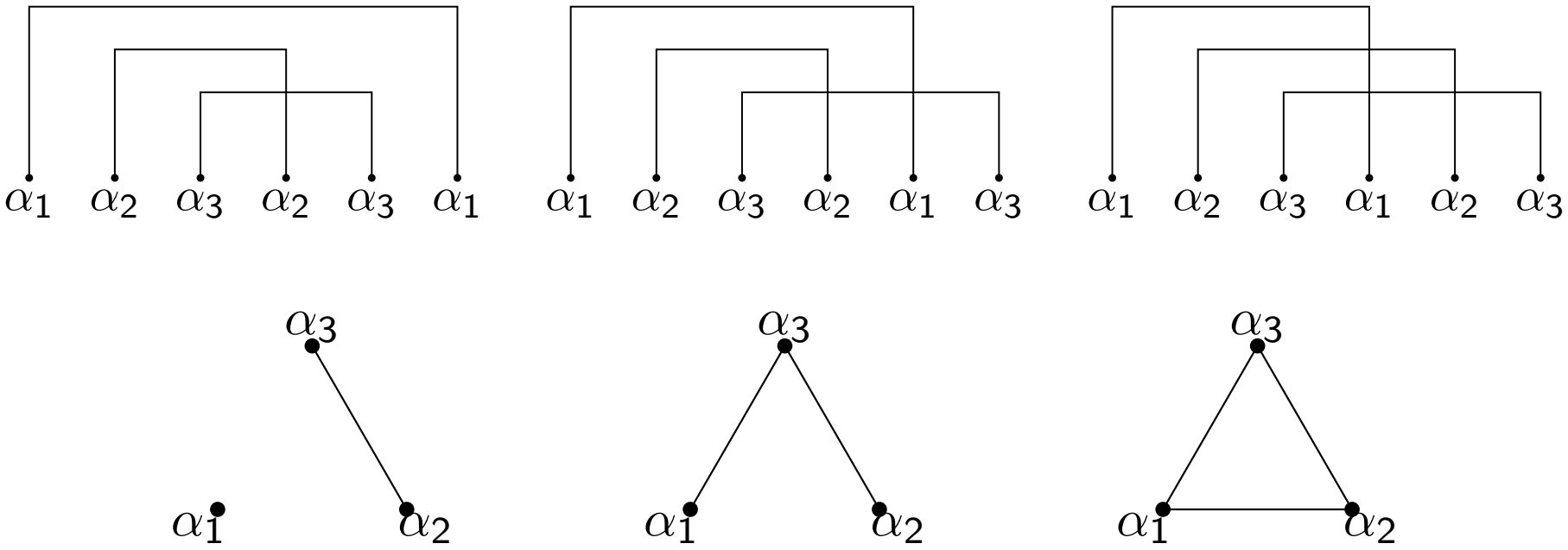
- Translate chord diagrams to intersection graphs:
  1. Represent each chord by a vertex.
  2. Connect two vertices by an edge if the corresponding chords intersect each other.



Aside: SYK as a enumeration machine

- In terms of intersection graphs, the previous chord intersection identity can be rewritten as

$$\sum_{i=1}^{(2p-1)!!} (\#\bullet\!\!-\!\!\bullet)_i = \frac{1}{3}\binom{p}{2}(2p-1)!!.$$

- The SYK model generates the following (and infinitely more)[García-García-YJ-Verbaarschot 2018, YJ-Verbaarschot 2018]:

$$\sum_{i=1}^{(2p-1)!!} (\#\bullet\!\!-\!\!\bullet)_i = \frac{1}{3}\binom{p}{2}(2p-1)!!$$

$$\sum_{i=1}^{(2p-1)!!} (\#\triangle)_i = \frac{1}{15}\binom{p}{3}(2p-1)!!$$

$$\sum_{i=1}^{(2p-1)!!} \left(5\#\boxtimes + \#\boxslash - \#\square\right)_i = \frac{1}{15}\binom{p}{4}(2p-1)!!$$

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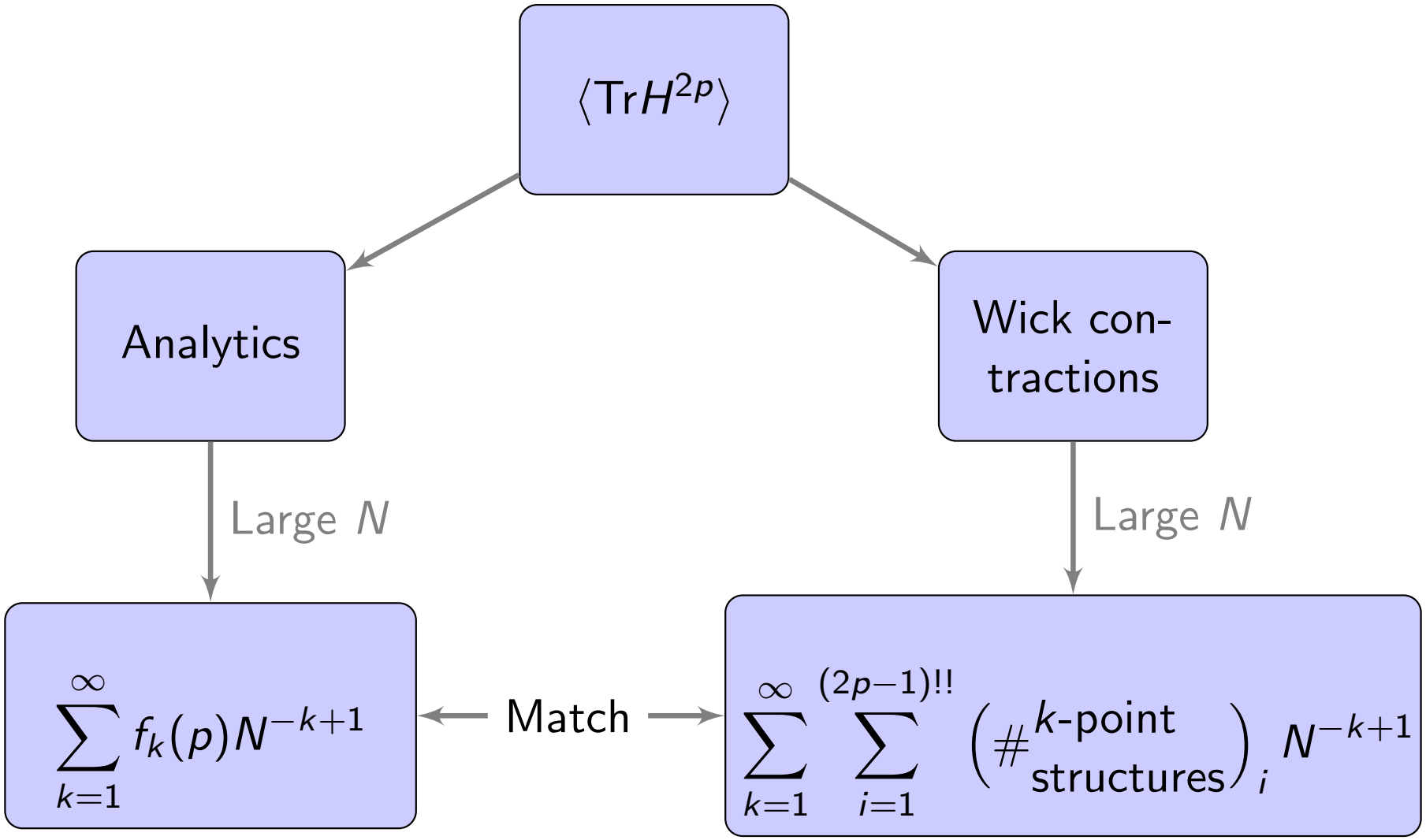
Sketch of the proof:



Aside: SYK as a enumeration machine

We can apply the same technique to the supersymmetric SYK model (odd q), and get a graded version of the enumerations:

$$\sum_{i=1}^{(2p-1)!!} (-1)^{(\#\bullet\!\!-\!\!\bullet)_i} (\#\bullet\!\!-\!\!\bullet)_i = -\binom{p}{2}$$

$$\sum_{i=1}^{(2p-1)!!} (-1)^{(\#\bullet\!\!-\!\!\bullet)_i} \left(\#\triangle\right)_i = -\binom{p}{3}$$

$$\sum_{i=1}^{(2p-1)!!} (-1)^{(\#\bullet\!\!-\!\!\bullet)_i} \left(5\#\boxtimes + \#\boxslash - \#\square\right)_i = -\binom{p}{4}$$

Recap and outlook

- BGS conjecture relates chaos to random matrix, well understood in the case of single-particle chaos via long periodic orbits. No such precise understanding exists for many-body chaos. $\Longrightarrow$ SYK may help due to its high degree of solvablity.
- We have a quite precise understanding of the average spectral density of SYK, via Q-Hermite density ρ_{QH}. It has the distinct feature that level spacings are exponentially small in N near ground state.
- The numerical study of the spectral form factor confirms SYK has RMT behviour at late time. If the first few long-range fluctuations are subtracted, it becomes RMT all the way. (needs an explanation)
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What next?

- Toward a many-body BGS:
  - An analytic understanding of spectral form factor. ($\langle \mathrm{Tr} H^m \mathrm{Tr} H^n \rangle$? Sigma model?)
  - What is the classical limit of SYK?
- A long-shot question: in the UV of SYK we have a (intersection graph, $1/N$) expansion, in the IR we have a (Jakiw-Teitelboim) gravitational (genus, e^{-N}) expansion [Shenker-Stanford-Saad 2019], how does the UV renormalize to the IR exactly?

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Last but not least

Happy 65, Jac!