Epsilon-expansion for multi-scalar QFTs

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Based on:

A.Codello, M.S, G.P.Vacca, O.Zanusso

[arXiv:1809.05071 [hep-th]], Eur.Phys.J.C

[arXiv:1910.xxxxx [hep-th]], *Works in preparation*

Introduction

Scale invariant QFTs prove useful in the theoretical description of many physical phenomena

- *Critical phenomena (second order phase transitions)*
  - *Ferromagnet-paramagnet*
  - *Liquid-vapour*

  ⋮

Introduction

They appear in theoretical investigations in different areas of physics

- *Beyond the Standard Model*
  - *Conformal Technicolor*
  - *Dynamical electroweak symmetry breaking*
  - *Asymptotic safety scenario*

    ⋮

- *AdS/CFT corrspondence*

  ⋮

Introduction

Under general conditions, i.e. unitarity and Poincaré invariance, scale invariance is promoted to conformal invariance:

in 2d [Zamolodchikov '86]

in 4d [Luty, Polchinski, Rattazzi '12], (perturbative)

Introduction

Approaches to conformal invariant QFT

- *Lattice models*
- *Exact renormalization group*
- *Conformal bootstrap*
- *Epsilon expansion*

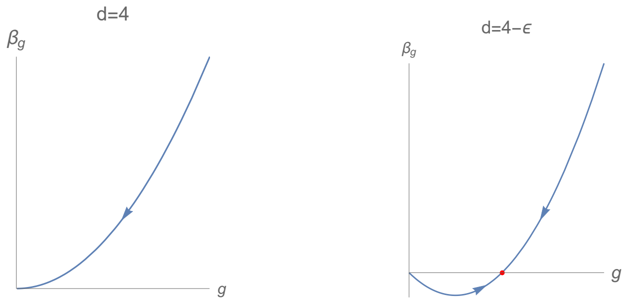
Introduction

Approaches to conformal invariant QFT

- *Lattice models*
- *Exact renormalization group*
- *Conformal bootstrap*
- *Epsilon expansion* $\left\{ \right.$
  - *renormalization group* [Wilson and Fisher 1972]
  - *conformal field theory* [Rychkov, Tan '15] $\cdots$

ϵ-expansion:

$$\mathcal{L} = \tfrac{1}{2}(\partial\phi)^2 + m^2\phi^2 + \tfrac{1}{4!}\,g\,\phi^4 \qquad d = 4-\epsilon$$



$$\beta_{m^2} \equiv \mu\frac{dm^2}{d\mu} = \frac{m^2 g}{(4\pi)^2} \qquad \beta_g \equiv \mu\frac{dg}{d\mu} = -\epsilon g + \frac{3g^2}{(4\pi)^2}$$

ϵ-expansion:

$$\xi \sim (T - T_c)^{-\nu} \qquad \nu = \frac{1}{2 - \gamma_m} \qquad \gamma_m = \frac{\mu}{m^2}\frac{dm^2}{d\mu} = \frac{g^*}{(4\pi)^2}$$

$$\beta_g = -\epsilon g + \frac{3g^2}{(4\pi)^2}$$

$$\frac{g}{(4\pi)^2} = \frac{\epsilon}{3} \qquad \Rightarrow \qquad \nu = \frac{3}{6 - \epsilon} = \begin{cases} 0.5 & \epsilon = 0 \\ 0.6 & \epsilon = 1 \end{cases}$$

compare to experimental value $\nu = 0.63$

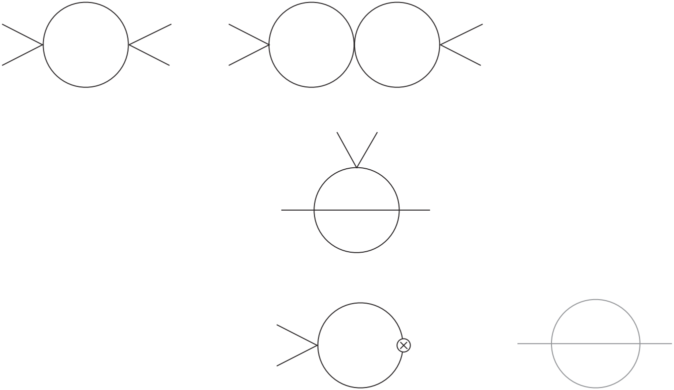
Functional perturbative RG

[Jack, Osborn '83], . . .
[O'Dwyer, Osborn '07]
[Codello, M.S, Vacca, Zanusso '17]
[Osborn, Stergiou '17]

Standard Renormalization group, but promoted to functional level
(here in dimensional regularization and $\overline{MS}$)

Functional perturbative RG

$$\mathcal{L} = \tfrac{1}{2}(\partial\phi)^2 + \tfrac{1}{4!}\, g\, \phi^4 \qquad d = 4 - \epsilon$$



$$\beta_g = -\epsilon g + 3g - 6g^3 + 4\gamma g \qquad \gamma = \frac{g^2}{12} \qquad g \to (4\pi)^2 g$$

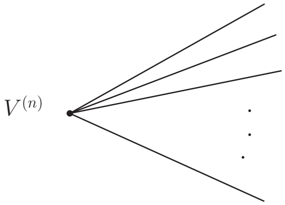
$$= -\epsilon g + 3g - \tfrac{17}{3} g^3$$

Functional perturbative RG

$$\mathcal{L} = \tfrac{1}{2}(\partial\phi)^2 + V(\phi)$$

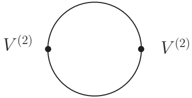
background-field method:

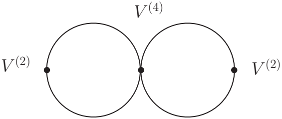
$$V(\phi + f) = \sum_{n=0}^{\infty} \frac{1}{n!} V^{(n)}(\phi)\, f^n$$

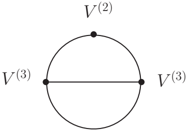


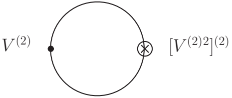
Functional perturbative RG

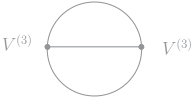
$$\mathcal{L} = \tfrac{1}{2}(\partial\phi)^2 + V(\phi) \qquad d = 4 - \epsilon$$











$$\beta_V = \tfrac{1}{2}(V^{(2)})^2 + \tfrac{1}{2}V^{(2)}(V^{(3)})^2 + \phi\gamma V^{(1)} \quad \gamma = \tfrac{1}{12}(V^{(4)})^2 \quad V \to (4\pi)^2 V$$

Functional perturbative RG

Allows to calculate (perhaps infinitely) many critical quantities in one shot

Functional perturbative RG

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + g_1\phi + g_2\phi^2 + g_3\phi^3 + g_4\phi^4 + \cdots \qquad d = 4$$

Dimensionful beta functions (dim.reg. and $\overline{MS}$)

$$\begin{aligned}
\beta_1 &= 12\, g_2 g_3 - 108\, g_3^3 - 288\, g_2 g_3 g_4 + 48\, g_1 g_4^2 \\
\beta_2 &= 24\, g_4 g_2 + 18\, g_3^2 - 1080\, g_3^2 g_4 - 480\, g_2 g_4^2 \\
\beta_3 &= 72\, g_4 g_3 - 3312\, g_3 g_4^2 \\
\beta_4 &= 72\, g_4^2 - 3264\, g_4^3 \\
&\vdots
\end{aligned}$$

Functional perturbative RG

Gives more insight into the structure of flow equations

Functional perturbative RG

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + V(\phi) \qquad d = 4$$

$$\beta_V = a(V^{(2)})^2 + bV^{(2)}(V^{(3)})^2$$

Functional perturbative RG

$$\mathcal{L} = \tfrac{1}{2}(\partial\phi)^2 + V(\phi) \qquad d = 4$$

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Functional perturbative RG

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$1 - \text{loop}$

Functional perturbative RG

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + V(\phi) \qquad d = 4$$

$$\beta_V = \frac{1}{2}(V^{(2)})^2 + \frac{1}{2}V^{(2)}(V^{(3)})^2$$

$$2 - \text{loops}$$

Multicritical theories

$$\mathcal{L} = \tfrac{1}{2}(\partial\phi)^2 + g\,\phi^m$$

$$[\phi^m] = d_c \qquad d_c = \frac{2m}{m-2} \qquad d = d_c - \epsilon$$

m	3	4	5	6	7	8	$\cdots$
d_c	6	4	10/3	3	14/5	8/3	$\cdots$

O'Dwyer, Osborn [arXiv:0708.2697 [hep-th]], Annals Phys
Codello, M.S, Vacca, Zanusso [arXiv:1705.05558 [hep-th]], Eur.Phys.J.C
Codello, M.S, Vacca, Zanusso [arXiv:1706.06887 [hep-th]], Phys.Rev.D

Multicritical theories

Notable examples ($d_c > 3$)

$m = 4 \quad \phi^4 \quad d = 4 - \epsilon$ *Ising*

$m = 3 \quad \phi^3 \quad d = 6 - \epsilon$ *Lee-Yang*

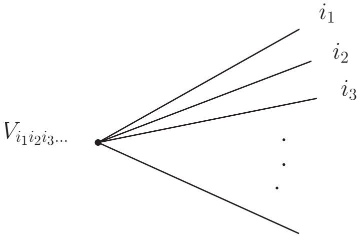
$m = 5 \quad \phi^5 \quad d = 10/3 - \epsilon$ *Tricritical Lee-Yang*

Functional perturbative RG

$$\mathcal{L} = \tfrac{1}{2}\partial\phi_i\partial\phi_i + V(\phi)$$

background-field method:

$$V(\phi + f) = \sum_{n=0}^{\infty} \frac{1}{n!} V_{i_1 i_2 \ldots}(\phi)\, f^{i_1} f^{i_2} \cdots$$



Functional perturbative RG

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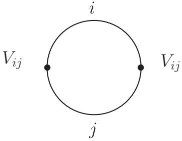

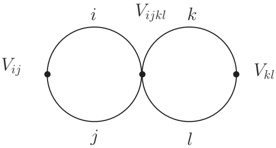

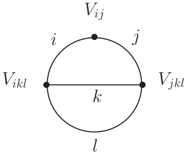

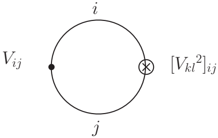

$$\beta_V = \frac{1}{2}V_{ij}V_{ij} + \frac{1}{2}V_{ij}V_{ikl}V_{jkl} + \phi_i\gamma_{ij}V_j$$

Renormalization group and epsilon expansion

for multi-scalar models

Quartic model with three flavours

One-loop beta function of dimensionless potential in $d = 4 - \epsilon$

$$\beta_v = -dv + \frac{d-2}{2}\phi_i v_i + \frac{1}{2} v_{ij} v_{ij}$$

Quartic model with three flavours

One-loop beta function of dimensionless potential in $d = 4 - \epsilon$

$$\beta_v = -dv + \frac{d-2}{2}\phi_i v_i + \frac{1}{2} v_{ij} v_{ij}$$

$$\beta_{ijkl} = -\epsilon\,\lambda_{ijkl} + 3\lambda_{ab(ij}\lambda_{kl)ab}, \qquad v(\phi) = \tfrac{1}{4!}\lambda_{ijkl}\,\phi_i\phi_i\phi_k\phi_l$$

Quartic model with three flavours

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Two-flavour model studied in [Osborn, Stergiou '17]

Quartic model with three flavours

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Concentrate here on the three-flavour model:

Quartic model with three flavours

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Concentrate here on the three-flavour model:

Kinetic term invariant under

$$U \in O(3), \qquad U^T U = 1$$

Quartic model with three flavours

One-loop beta function of dimensionless potential in $d = 4 - \epsilon$

$$\beta_v = -dv + \frac{d-2}{2}\phi_i v_i + \frac{1}{2} v_{ij} v_{ij}$$

$$\beta_{ijkl} = -\epsilon\, \lambda_{ijkl} + 3\lambda_{ab(ij}\lambda_{kl)ab}, \qquad v(\phi) = \tfrac{1}{4!}\lambda_{ijkl}\, \phi_i\phi_i\phi_k\phi_l$$

Concentrate here on the three-flavour model:

Under U couplings transform as

$$\lambda_{ijkl} \to U_{ia}U_{jb}U_{kc}U_{le}\lambda_{abce}$$

Quartic model with three flavours

$$\binom{4+3-1}{4} = 15 \qquad \text{couplings}$$

$$\begin{aligned} v(\phi) \; = \; & \frac{1}{4!}\big(\lambda_1\phi_1^4 + 4\lambda_2\phi_2\phi_1^3 + 4\lambda_3\phi_3\phi_1^3 + 6\lambda_4\phi_2^2\phi_1^2 + 6\lambda_7\phi_3^2\phi_1^2 \\ & +12\lambda_5\phi_2\phi_3\phi_1^2 + 4\lambda_6\phi_2^3\phi_1 + 4\lambda_{12}\phi_3^3\phi_1 + 12\lambda_{10}\phi_2\phi_3^2\phi_1 \\ & +12\lambda_8\phi_2^2\phi_3\phi_1 + \lambda_9\phi_2^4 + \lambda_{15}\phi_3^4 + 4\lambda_{14}\phi_2\phi_3^3 + 6\lambda_{13}\phi_2^2\phi_3^2 + 4\lambda_{11}\phi_2^3\phi_3\big) \end{aligned}$$

Quartic model with three flavours

$$[J_i, J_j] = i\,\epsilon_{ijk}\,J_k \qquad (J^2, J_3) \leftrightarrow (j, m)$$

$$\begin{array}{ccc} 15 & = & 1 \oplus 5 \oplus 9 \\ j & : & 0 \quad 2 \quad 4 \end{array}$$

$$r_9 = \begin{pmatrix} \sqrt{2}\,(3\,(\lambda_1 + 2\lambda_4 - 8\lambda_7 + \lambda_9 - 8\lambda_{13}) + 8\lambda_{15}) \\ 30\,(\lambda_5 + \lambda_{11}) - 40\lambda_{14} \\ 40\lambda_{12} - 30\,(\lambda_3 + \lambda_8) \\ -10\,(\lambda_1 - 6\lambda_7 - \lambda_9 + 6\lambda_{13}) \\ -20\,(\lambda_2 + \lambda_6 - 6\lambda_{10}) \\ 70\,(\lambda_{11} - 3\lambda_5) \\ 70\,(\lambda_3 - 3\lambda_8) \\ 35\,(\lambda_1 - 6\lambda_4 + \lambda_9) \\ 140\,(\lambda_2 - \lambda_6) \end{pmatrix} \equiv \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \\ g_7 \\ g_8 \\ g_9 \end{pmatrix}$$

Quartic model with three flavours

$$m = 0, \pm 1, \pm 2, \pm 3, \pm 4$$

$$J_3 \cdot \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \\ g_7 \\ g_8 \\ g_9 \end{pmatrix} = \begin{pmatrix} 0 \\ g_2 \\ -g_3 \\ 2g_4 \\ -2g_5 \\ 3g_6 \\ -3g_7 \\ 4g_8 \\ -4g_9 \end{pmatrix}$$

Quartic model with three flavours

$$J_3 \cdot \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \\ g_7 \\ g_8 \\ g_9 \end{pmatrix} = i \begin{pmatrix} 0 \\ -g_3 \\ g_2 \\ -2g_5 \\ 2g_4 \\ -3g_7 \\ 3g_6 \\ -4g_9 \\ 4g_8 \end{pmatrix}$$

Quartic model with three flavours

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$$\exp(-i\theta J_3): \quad \begin{array}{ccc} g_1 & \longrightarrow & g_1 \\ \begin{pmatrix} g_2 \\ g_3 \end{pmatrix} & \longrightarrow & R(\theta) \begin{pmatrix} g_2 \\ g_3 \end{pmatrix} \\ \begin{pmatrix} g_4 \\ g_5 \end{pmatrix} & \longrightarrow & R(2\theta) \begin{pmatrix} g_4 \\ g_5 \end{pmatrix} \\ \begin{pmatrix} g_6 \\ g_7 \end{pmatrix} & \longrightarrow & R(3\theta) \begin{pmatrix} g_6 \\ g_7 \end{pmatrix} \\ \begin{pmatrix} g_8 \\ g_9 \end{pmatrix} & \longrightarrow & R(4\theta) \begin{pmatrix} g_8 \\ g_9 \end{pmatrix} \end{array}$$

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Quartic model with three flavours

$$r_5 = \begin{pmatrix} \sqrt{2}\,(\lambda_1 + 2\lambda_4 - \lambda_7 + \lambda_9 - \lambda_{13} - 2\lambda_{15}) \\ 6\,(\lambda_5 + \lambda_{11} + \lambda_{14}) \\ -6\,(\lambda_3 + \lambda_8 + \lambda_{12}) \\ -3\,(\lambda_1 + \lambda_7 - \lambda_9 - \lambda_{13}) \\ -6\,(\lambda_2 + \lambda_6 + \lambda_{10}) \end{pmatrix} \equiv \begin{pmatrix} g_{10} \\ g_{11} \\ g_{12} \\ g_{13} \\ g_{14} \end{pmatrix}$$

Quartic model with three flavours

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$$J_3 \cdot \begin{pmatrix} g_{10} \\ g_{11} \\ g_{12} \\ g_{13} \\ g_{14} \end{pmatrix} = i \begin{pmatrix} 0 \\ -g_{12} \\ g_{11} \\ -2g_{14} \\ 2g_{13} \end{pmatrix}$$

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Quartic model with three flavours

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$$r_1 = \lambda_1 + 2\lambda_4 + 2\lambda_7 + \lambda_9 + 2\lambda_{13} + \lambda_{15} \;\equiv\; g_{15}$$

Quartic model with three flavours

$$\beta(R\cdot g) = R\cdot\beta(g) \qquad\qquad \beta(g*) = 0 \;\Rightarrow\; \beta(R\cdot g*) = 0$$

$O(3)$-related fixed points are equivalent

One may use this freedom to set some couplings to zero
and simplify the fixed point equations

$$\begin{pmatrix} g_{11} \\ g_{12} \end{pmatrix} \longrightarrow R \begin{pmatrix} g_{11} \\ g_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ g'_{12} \end{pmatrix}$$

Quartic model with three flavours

Alternative representation (N flavours):

$$\lambda_{ijkl} = \kappa\delta_{(ij}\delta_{kl)} + \rho_{(ij}\delta_{kl)} + \sigma_{ijkl}, \qquad \rho_{ll} = 0, \quad \sigma_{ijll} = 0$$

$$\kappa = \frac{3}{N(N+2)}\lambda_{aabb}, \qquad \rho_{ij} = \frac{6}{N+4}\left(\lambda_{ijaa} - \frac{1}{N}\delta_{ij}\lambda_{aabb}\right)$$

$$\sigma_{ijkl} = \lambda_{ijkl} - \frac{6}{N+4}\delta_{(ij}\lambda_{kl)aa} + \frac{3}{(N+2)(N+4)}\lambda_{aabb}\delta_{(ij}\delta_{kl)}$$

For three-flavour model one can make the following identification

$$\kappa \leftrightarrow r_1, \qquad \rho_{ij} \leftrightarrow r_5, \qquad \sigma_{ijkl} \leftrightarrow r_9$$

Use freedom to diagonalize ρ_{ij} $\Leftrightarrow$ set $g_{11} = g_{12} = g_{14} = 0$

Quartic model with three flavours

$$
\begin{aligned}
\beta_1 &= -\epsilon\, g_1 + \frac{27 g_1^2}{980\sqrt{2}} + g_1\left(\frac{4 g_{15}}{5} - \frac{10\sqrt{2} g_{10}}{49}\right) + \frac{3 g_8^2}{2450\sqrt{2}} + \frac{57}{49}\sqrt{2} g_{10}^2 \\
\beta_8 &= -\epsilon\, g_8 + \frac{1}{140} g_8\left(3\sqrt{2} g_1 + 40\sqrt{2} g_{10} + 112 g_{15}\right) \\
\beta_{10} &= -\epsilon\, g_{10} - \frac{47 g_{10}^2}{147\sqrt{2}} + g_{10}\left(\frac{57 g_1}{490\sqrt{2}} + \frac{19 g_{15}}{15}\right) - \frac{g_1^2}{196\sqrt{2}} + \frac{g_8^2}{2450\sqrt{2}} \\
\beta_{15} &= -\epsilon\, g_{15} + \frac{11 g_{15}^2}{15} + \frac{g_1^2}{280} + \frac{g_8^2}{4900} + \frac{19 g_{10}^2}{84}
\end{aligned}
$$

Quartic model with three flavours

Fully interacting fixed points

N	Anomalous dimensions	Symmetry	Name
1	$\frac{1}{108}$	$\mathbb{Z}_2$	Ising
2	$\left(\frac{1}{100}, \frac{1}{100}\right)$	$O(2)$	O(2)
3	$\left(\frac{5}{484}, \frac{5}{484}, \frac{5}{484}\right)$	$O(3)$	O(3)
	$\left(\frac{5}{486}, \frac{5}{486}, \frac{5}{486}\right)$	$(\mathbb{Z}_2)^3 \rtimes S_3$	cubic
	$(0.01004, 0.01042, 0.01042)$	$\mathbb{Z}_2 \times O(2)$	biconical

$$V_{3,O(3)} = \frac{1}{80}\left(\phi_1^2 + \phi_2^2 + \phi_3^2\right)^2$$

$$V_{3,\mathrm{cubic}} = \frac{1}{108}\Big(\phi_1^4 + \phi_2^4 + \phi_3^4 + 3(\phi_1^2\phi_2^2 + \phi_1^2\phi_3^2 + \phi_2^2\phi_3^2)\Big)$$

$$V_{3,\mathrm{biconical}} = 0.0105352\,(\phi_1^2 + \phi_2^2)^2 + 0.0287745\,\phi_3^2(\phi_1^2 + \phi_2^2) + 0.00843668\,\phi_3^4$$

Quartic model with three flavours

Fully interacting fixed points

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2	$\left(\frac{1}{100}, \frac{1}{100}\right)$	$O(2)$	O(2)
3	$\left(\frac{5}{484}, \frac{5}{484}, \frac{5}{484}\right)$	$O(3)$	O(3)
	$\left(\frac{5}{486}, \frac{5}{486}, \frac{5}{486}\right)$	$(\mathbb{Z}_2)^3 \rtimes S_3$	cubic
	$(0.01004, 0.01042, 0.01042)$	$\mathbb{Z}_2 \times O(2)$	biconical

(O(3) and cubic:) $\Big\}\ \lambda_{ijaa} \propto \delta_{ij}$

$$V_{3,O(3)} = \frac{1}{80}\left(\phi_1^2 + \phi_2^2 + \phi_3^2\right)^2$$

$$V_{3,\text{cubic}} = \frac{1}{108}\Big(\phi_1^4 + \phi_2^4 + \phi_3^4 + 3(\phi_1^2\phi_2^2 + \phi_1^2\phi_3^2 + \phi_2^2\phi_3^2)\Big)$$

$$V_{3,\text{biconical}} = 0.0105352\,(\phi_1^2 + \phi_2^2)^2 + 0.0287745\,\phi_3^2(\phi_1^2 + \phi_2^2) + 0.00843668\,\phi_3^4$$

Quartic model with three flavours

Fully interacting fixed points

N	Anomalous dimensions	Symmetry	Name
1	$\frac{1}{108}$	$\mathbb{Z}_2$	Ising
2	$\left(\frac{1}{100}, \frac{1}{100}\right)$	$O(2)$	O(2)
3	$\left(\frac{5}{484}, \frac{5}{484}, \frac{5}{484}\right)$	$O(3)$	O(3)
	$\left(\frac{5}{486}, \frac{5}{486}, \frac{5}{486}\right)$	$(\mathbb{Z}_2)^3 \rtimes S_3$	cubic
	$(0.01004, 0.01042, 0.01042)$	$\mathbb{Z}_2 \times O(2)$	biconical

$\lambda_{ijaa} \not\propto \delta_{ij}$

$$V_{3,O(3)} = \frac{1}{80}\left(\phi_1^2 + \phi_2^2 + \phi_3^2\right)^2$$

$$V_{3,\text{cubic}} = \frac{1}{108}\Big(\phi_1^4 + \phi_2^4 + \phi_3^4 + 3(\phi_1^2\phi_2^2 + \phi_1^2\phi_3^2 + \phi_2^2\phi_3^2)\Big)$$

$$V_{3,\text{biconical}} = 0.0105352\,(\phi_1^2 + \phi_2^2)^2 + 0.0287745\,\phi_3^2(\phi_1^2 + \phi_2^2) + 0.00843668\,\phi_3^4$$

Cubic model with three flavours

One-loop beta function of dimensionless potential in $d = 6 - \epsilon$

$$\beta_v = -dv + \frac{d-2}{2}\phi_i v_i + \phi_i \gamma_{ij} v_j - \frac{2}{3} v_{ij} v_{jk} v_{ki}$$

Cubic model with three flavours

One-loop beta function of dimensionless potential in $d = 6 - \epsilon$

$$\beta_v = -dv + \frac{d-2}{2}\phi_i v_i + \phi_i \gamma_{ij} v_j - \frac{2}{3} v_{ij} v_{jk} v_{ki}$$

$$v(\phi) = \frac{1}{3!}\,\lambda_{ijk}\,\phi_i\phi_j\phi_k, \qquad \gamma_{ij} = \frac{1}{3}\lambda_{iab}\lambda_{jab}$$

$$\beta_{ijk} = -\frac{1}{2}\epsilon\,\lambda_{ijk} + \lambda_{abc}\lambda_{ab(i}\lambda_{jk)c} - 4\lambda_{iab}\lambda_{jbc}\lambda_{kca}$$

Cubic model with three flavours

One-loop beta function of dimensionless potential in $d = 6 - \epsilon$

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Three-flavour model:

Cubic model with three flavours

One-loop beta function of dimensionless potential in $d = 6 - \epsilon$

$$\beta_v = -dv + \frac{d-2}{2}\phi_i v_i + \phi_i \gamma_{ij} v_j - \frac{2}{3} v_{ij} v_{jk} v_{ki}$$

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Three-flavour model:

Kinetic term invariant under

$$U \in O(3), \qquad U^T U = 1$$

Cubic model with three flavours

One-loop beta function of dimensionless potential in $d = 6 - \epsilon$

$$\beta_v = -dv + \frac{d-2}{2}\phi_i v_i + \phi_i \gamma_{ij} v_j - \frac{2}{3} v_{ij} v_{jk} v_{ki}$$

$$v(\phi) = \frac{1}{3!}\,\lambda_{ijk}\,\phi_i\phi_j\phi_k, \qquad \gamma_{ij} = \frac{1}{3}\lambda_{iab}\lambda_{jab}$$

$$\beta_{ijk} = -\frac{1}{2}\epsilon\,\lambda_{ijk} + \lambda_{abc}\lambda_{ab(i}\lambda_{jk)c} - 4\lambda_{iab}\lambda_{jbc}\lambda_{kca}$$

Three-flavour model:

Under U couplings transform as

$$\lambda_{ijk} \to U_{ia}U_{jb}U_{kc}\lambda_{abc}$$

Cubic model with three flavours

One-loop beta function of dimensionless potential in $d = 6 - \epsilon$

$$\beta_v = -dv + \frac{d-2}{2}\phi_i v_i + \phi_i \gamma_{ij} v_j - \frac{2}{3} v_{ij} v_{jk} v_{ki}$$

$$v(\phi) = \frac{1}{3!}\,\lambda_{ijk}\,\phi_i\phi_j\phi_k, \qquad \gamma_{ij} = \frac{1}{3}\lambda_{iab}\lambda_{jab}$$

$$\beta_{ijk} = -\frac{1}{2}\epsilon\,\lambda_{ijk} + \lambda_{abc}\lambda_{ab(i}\lambda_{jk)c} - 4\lambda_{iab}\lambda_{jbc}\lambda_{kca}$$

Three-flavour model:

$$\binom{3+3-1}{3} = 10 \qquad \text{couplings}$$

$$v = \frac{1}{6}\big(\lambda_1\phi_1^3 + 3\lambda_2\phi_2\phi_1^2 + 3\lambda_3\phi_3\phi_1^2 + 3\lambda_4\phi_2^2\phi_1 + 3\lambda_7\phi_3^2\phi_1 + 6\lambda_5\phi_2\phi_3\phi_1 + \lambda_6\phi_2^3 + \lambda_{10}\phi_3^3 + 3\lambda_9\phi_2\phi_3^2 + 3\lambda_8\phi_2^2\phi_3\big)$$

Cubic model with three flavours

move to basis split into irreps:

$$10 = 7 \oplus 3$$

$$r_7 = \begin{pmatrix} 2\sqrt{2}\,(3\,(\lambda_3+\lambda_8)-2\lambda_{10}) \\ \lambda_2+\lambda_6-4\lambda_9 \\ -\lambda_1-\lambda_4+4\lambda_7 \\ 10\,(\lambda_8-\lambda_3) \\ -20\lambda_5 \\ 5\,(\lambda_6-3\lambda_2) \\ 5\,(\lambda_1-3\lambda_4) \end{pmatrix} \equiv \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \\ g_7 \end{pmatrix}, \qquad J_3 \cdot \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \\ g_7 \end{pmatrix} = i \begin{pmatrix} 0 \\ -g_3 \\ g_2 \\ -2g_5 \\ 2g_4 \\ -3g_7 \\ 3g_6 \end{pmatrix}$$

$$r_3 = \begin{pmatrix} \sqrt{2}\,(\lambda_3+\lambda_8+\lambda_{10}) \\ \lambda_2+\lambda_6+\lambda_9 \\ -\lambda_1-\lambda_4-\lambda_7 \end{pmatrix} \equiv \begin{pmatrix} g_8 \\ g_9 \\ g_{10} \end{pmatrix}, \qquad J_3 \cdot \begin{pmatrix} g_8 \\ g_9 \\ g_{10} \end{pmatrix} = i \begin{pmatrix} 0 \\ -g_{10} \\ g_9 \end{pmatrix}$$

Cubic model with three flavours

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$$10 = 7 \oplus 3$$

$|m| = 0$ - transforms trivially under $\exp(-i\theta J_3)$

$$r_7 = \begin{pmatrix} 2\sqrt{2}\,(3\,(\lambda_3+\lambda_8) - 2\lambda_{10}) \\ \lambda_2+\lambda_6-4\lambda_9 \\ -\lambda_1-\lambda_4+4\lambda_7 \\ 10\,(\lambda_8-\lambda_3) \\ -20\lambda_5 \\ 5\,(\lambda_6-3\lambda_2) \\ 5\,(\lambda_1-3\lambda_4) \end{pmatrix} \equiv \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \\ g_7 \end{pmatrix}, \qquad J_3 \cdot \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \\ g_7 \end{pmatrix} = i \begin{pmatrix} 0 \\ -g_3 \\ g_2 \\ -2g_5 \\ 2g_4 \\ -3g_7 \\ 3g_6 \end{pmatrix}$$

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Cubic model with three flavours

move to basis split into irreps:

$$10 = 7 \oplus 3$$

linear combination of $|m| = 1$ couplings
transform under $\exp(-i\theta J_3)$ with $R(|m|\theta) = R(\theta)$

$$r_7 = \begin{pmatrix} 2\sqrt{2}\,(3\,(\lambda_3 + \lambda_8) - 2\lambda_{10}) \\ \lambda_2 + \lambda_6 - 4\lambda_9 \\ -\lambda_1 - \lambda_4 + 4\lambda_7 \\ 10\,(\lambda_8 - \lambda_3) \\ -20\lambda_5 \\ 5\,(\lambda_6 - 3\lambda_2) \\ 5\,(\lambda_1 - 3\lambda_4) \end{pmatrix} \equiv \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \\ g_7 \end{pmatrix}, \qquad J_3 \cdot \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \\ g_7 \end{pmatrix} = i \begin{pmatrix} 0 \\ -g_3 \\ g_2 \\ -2g_5 \\ 2g_4 \\ -3g_7 \\ 3g_6 \end{pmatrix}$$

$$r_3 = \begin{pmatrix} \sqrt{2}\,(\lambda_3 + \lambda_8 + \lambda_{10}) \\ \lambda_2 + \lambda_6 + \lambda_9 \\ -\lambda_1 - \lambda_4 - \lambda_7 \end{pmatrix} \equiv \begin{pmatrix} g_8 \\ g_9 \\ g_{10} \end{pmatrix}, \qquad J_3 \cdot \begin{pmatrix} g_8 \\ g_9 \\ g_{10} \end{pmatrix} = i \begin{pmatrix} 0 \\ -g_{10} \\ g_9 \end{pmatrix}$$

Cubic model with three flavours

move to basis split into irreps:

$$10 = 7 \oplus 3$$

linear combination of $|m| = 2$ couplings
transform under $\exp(-i\theta J_3)$ with $R(|m|\theta) = R(2\theta)$

$$r_7 = \begin{pmatrix} 2\sqrt{2}\,(3\,(\lambda_3 + \lambda_8) - 2\lambda_{10}) \\ \lambda_2 + \lambda_6 - 4\lambda_9 \\ -\lambda_1 - \lambda_4 + 4\lambda_7 \\ 10\,(\lambda_8 - \lambda_3) \\ -20\lambda_5 \\ 5\,(\lambda_6 - 3\lambda_2) \\ 5\,(\lambda_1 - 3\lambda_4) \end{pmatrix} \equiv \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \\ g_7 \end{pmatrix}, \qquad J_3 \cdot \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \\ g_7 \end{pmatrix} = i \begin{pmatrix} 0 \\ -g_3 \\ g_2 \\ -2g_5 \\ 2g_4 \\ -3g_7 \\ 3g_6 \end{pmatrix}$$

$$r_3 = \begin{pmatrix} \sqrt{2}\,(\lambda_3 + \lambda_8 + \lambda_{10}) \\ \lambda_2 + \lambda_6 + \lambda_9 \\ -\lambda_1 - \lambda_4 - \lambda_7 \end{pmatrix} \equiv \begin{pmatrix} g_8 \\ g_9 \\ g_{10} \end{pmatrix}, \qquad J_3 \cdot \begin{pmatrix} g_8 \\ g_9 \\ g_{10} \end{pmatrix} = i \begin{pmatrix} 0 \\ -g_{10} \\ g_9 \end{pmatrix}$$

Cubic model with three flavours

move to basis split into irreps:

$$10 = 7 \oplus 3$$

linear combination of $|m| = 3$ couplings

transform under $\exp(-i\theta J_3)$ with $R(|m|\theta) = R(3\theta)$

$$r_7 = \begin{pmatrix} 2\sqrt{2}\,(3\,(\lambda_3+\lambda_8)-2\lambda_{10}) \\ \lambda_2+\lambda_6-4\lambda_9 \\ -\lambda_1-\lambda_4+4\lambda_7 \\ 10\,(\lambda_8-\lambda_3) \\ -20\lambda_5 \\ 5\,(\lambda_6-3\lambda_2) \\ 5\,(\lambda_1-3\lambda_4) \end{pmatrix} \equiv \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \\ g_7 \end{pmatrix}, \qquad J_3 \cdot \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \\ g_7 \end{pmatrix} = i \begin{pmatrix} 0 \\ -g_3 \\ g_2 \\ -2g_5 \\ 2g_4 \\ -3g_7 \\ 3g_6 \end{pmatrix}$$

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Cubic model with three flavours

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$$r_7 = \begin{pmatrix} 2\sqrt{2}\,(3\,(\lambda_3+\lambda_8) - 2\lambda_{10}) \\ \lambda_2+\lambda_6-4\lambda_9 \\ -\lambda_1-\lambda_4+4\lambda_7 \\ 10\,(\lambda_8-\lambda_3) \\ -20\lambda_5 \\ 5\,(\lambda_6-3\lambda_2) \\ 5\,(\lambda_1-3\lambda_4) \end{pmatrix} \equiv \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \\ g_7 \end{pmatrix}, \qquad J_3 \cdot \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \\ g_7 \end{pmatrix} = i \begin{pmatrix} 0 \\ -g_3 \\ g_2 \\ -2g_5 \\ 2g_4 \\ -3g_7 \\ 3g_6 \end{pmatrix}$$

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$$r_7 = \begin{pmatrix} 2\sqrt{2}\,(3\,(\lambda_3 + \lambda_8) - 2\lambda_{10}) \\ \lambda_2 + \lambda_6 - 4\lambda_9 \\ -\lambda_1 - \lambda_4 + 4\lambda_7 \\ 10\,(\lambda_8 - \lambda_3) \\ -20\lambda_5 \\ 5\,(\lambda_6 - 3\lambda_2) \\ 5\,(\lambda_1 - 3\lambda_4) \end{pmatrix} \equiv \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \\ g_7 \end{pmatrix}, \qquad J_3 \cdot \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \\ g_7 \end{pmatrix} = i \begin{pmatrix} 0 \\ -g_3 \\ g_2 \\ -2g_5 \\ 2g_4 \\ -3g_7 \\ 3g_6 \end{pmatrix}$$

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Cubic model with three flavours

$$r_3 \;=\; \text{fundamental representation of } O(3)$$

$$r_3 = \begin{pmatrix} g_8 \\ g_9 \\ g_{10} \end{pmatrix} \longrightarrow R \begin{pmatrix} g_8 \\ g_9 \\ g_{10} \end{pmatrix} = \begin{pmatrix} g_8' \\ 0 \\ 0 \end{pmatrix}, \qquad O(3) \longrightarrow O(2)$$

Cubic model with three flavours

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$$r_7 = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \\ g_7 \end{pmatrix} \qquad \exp(-i\theta J_3) : \quad \begin{array}{ccc} g_1 & \longrightarrow & g_1 \\ \begin{pmatrix} g_2 \\ g_3 \end{pmatrix} & \longrightarrow & R(\theta) \begin{pmatrix} g_2 \\ g_3 \end{pmatrix} = \begin{pmatrix} 0 \\ g_3' \end{pmatrix} \\ \begin{pmatrix} g_4 \\ g_5 \end{pmatrix} & \longrightarrow & R(2\theta) \begin{pmatrix} g_4 \\ g_5 \end{pmatrix} \\ \begin{pmatrix} g_6 \\ g_7 \end{pmatrix} & \longrightarrow & R(3\theta) \begin{pmatrix} g_6 \\ g_7 \end{pmatrix} \end{array}$$

Cubic model with three flavours

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Use $O(3)$ freedom to set $g_2 = g_9 = g_{10} = 0$

Cubic model with three flavours

$$\begin{aligned}\beta_1 \;=\; & -\frac{\epsilon}{2}g_1 + \frac{1}{400}\left(g_1^3+2g_1\left(6g_3^2+3g_4^2+3g_5^2-2g_6^2-2g_7^2\right)+12\sqrt{2}g_3\left(g_4g_7-g_5g_6-g_3g_4\right)\right)\\ & +\frac{1}{25}\left(3g_1^2+17g_3^2-2g_4^2-2g_5^2-g_6^2-g_7^2\right)g_8-\frac{43}{50}g_1g_8^2+\frac{4}{5}g_8^3\end{aligned}$$

$$\begin{aligned}\beta_2 \;=\; & -\frac{1}{400}\left((12g_3^2+g_4^2-g_5^2)g_6+2g_4g_5g_7-\sqrt{2}g_1\left(g_4g_6+g_5\left(2g_3+g_7\right)\right)\right)\\ & +\frac{1}{75\sqrt{2}}g_8\left(g_4g_6+g_5\left(g_7-18g_3\right)\right)\end{aligned}$$

$$\vdots$$

$$\begin{aligned}\beta_{10} \;=\; & \frac{1}{1200}\left(24g_3^3+16g_7g_3^2+2g_1^2g_3+\left(6g_4^2+6g_5^2-8(g_6^2+g_7^2)\right)g_3\right.\\ & - \left. 4g_4g_5g_6+\sqrt{2}g_1\left(3g_5g_6+g_4\left(g_3-3g_7\right)\right)+2(g_4^2-g_5^2)g_7\right)\\ & - \frac{11}{1200}\,g_8\left(2g_1g_3+\sqrt{2}\left(g_4\left(3g_3+g_7\right)-g_5g_6\right)\right)+\frac{1}{5}g_3g_8^2\end{aligned}$$

Cubic model with three flavours

Fully interacting fixed points

N	Anomalous dimensions / ϵ	Symmetry	Name
1	$-\frac{1}{18}$	None	Lee-Yang
2	$\left(\frac{1}{6}, \frac{1}{6}\right)$	S_3	3-state Potts
	$\left(-\frac{61}{998}, -\frac{25}{499}\right)$	$\mathbb{Z}_2$	Z2
3	$\left(-\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}\right)$	S_4	4-state Potts
	$(0.167418, 0.093267, 0.093267)$	$O(2)$	O(2)
	$\left(-\frac{533}{7994}, -\frac{401}{7994}, -\frac{401}{7994}\right)$	S_3	S3
	$\left(\frac{289}{1698}, \frac{157-3\sqrt{561}}{1698}, \frac{157+3\sqrt{561}}{1698}\right)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	
	$(0.168983, 0.1653200, -0.059256)$	$\mathbb{Z}_2$	
	$(-0.063434, -0.055612, -0.047844)$	$\mathbb{Z}_2$	

Cubic model with three flavours

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	$(0.168983, 0.1653200, -0.059256)$	$\mathbb{Z}_2$	
	$(-0.063434, -0.055612, -0.047844)$	$\mathbb{Z}_2$	

Cubic model with three flavours

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	$\left(\frac{289}{1698}, \frac{157-3\sqrt{561}}{1698}, \frac{157+3\sqrt{561}}{1698}\right)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	
	$(0.168983, 0.1653200, -0.059256)$	$\mathbb{Z}_2$	
	$(-0.063434, -0.055612, -0.047844)$	$\mathbb{Z}_2$	

Fixed points do not split at NLO

Cubic model with three flavours

$$V_{3,S_4} = \frac{i}{2}\phi_1\phi_2\phi_3$$

$$V_{3,O(2)} = 0.187016\,\phi_3\left(\phi_1^2+\phi_2^2-0.420329\,\phi_3^2\right)$$

$$V_{3,S_3} = \frac{i\left(13\sqrt{26}\phi_3^3+30\sqrt{26}\left(\phi_2^2+\phi_1^2\right)\phi_3-\sqrt{1009}\,\phi_1\left(\phi_1^2-3\phi_2^2\right)\right)}{12\sqrt{11991}}$$

$$V_{3,\mathbb{Z}_2\times\mathbb{Z}_2} = \frac{\phi_3\left(3\sqrt{17}(\phi_2^2-\phi_1^2)-3\sqrt{33}(\phi_2^2+\phi_1^2)+4\sqrt{33}\phi_3^2\right)}{12\sqrt{566}}$$

$$\begin{aligned} V_{3,\mathbb{Z}_2-\mathrm{I}} &= \phi_2(0.25879\phi_1^2-0.09099\phi_2^2+0.01418\phi_3^2) \\ &\quad + i\phi_3(0.06156\phi_2^2-0.07055\phi_1^2-0.06948\phi_3^2) \end{aligned}$$

$$\begin{aligned} V_{3,\mathbb{Z}_2-\mathrm{II}} &= \frac{i}{6}\left(\phi_3(0.67445\phi_2^2+0.72878\phi_1^2+0.28095\phi_3^2)\right. \\ &\quad \left.+\,\phi_2\left(0.46865\phi_1^2-0.058091\phi_3^2-0.13685\phi_2^2\right)\right) \end{aligned}$$

Quintic model with two flavours

Three-loop beta function of dimensionless potential in $d = 10/3 - \epsilon$

$$\beta_v = -dv + \frac{d-2}{2}\phi_i v_i + \phi_i \gamma_{ij} v_j + \frac{1}{3} v_{ijkl} v_{ijkm} v_{lm} - \frac{3}{2} v_{ijk} v_{ilm} v_{jklm}$$

$$v = \frac{1}{5!}\,\lambda_{ijklm}\,\phi_i\phi_j\phi_k\phi_l\phi_m, \qquad \gamma_{ij} = \frac{1}{60}\lambda_{iabcd}\lambda_{jabcd}$$

$$\begin{aligned}\beta_{ijklm} \;=\; & -\frac{3}{2}\epsilon\,\lambda_{ijklm} + \frac{1}{12}\lambda_{abcde}\lambda_{abcd(i}\lambda_{jklm)e} \\ & + \frac{20}{3}\lambda_{abcd(i}\lambda_{klm|de|}\lambda_{j)abce} - 45\,\lambda_{abe(ij}\lambda_{|abcd|m}\lambda_{kl)cde}\end{aligned}$$

Quintic model with two flavours

$$\text{Two flavours}: \qquad \binom{5+2-1}{5} = 6 \quad \text{couplings}$$

$$v = \frac{1}{120}\left(\lambda_1\phi_1^5 + 5\lambda_2\phi_2\phi_1^4 + 10\lambda_3\phi_2^2\phi_1^3 + 10\lambda_4\phi_2^3\phi_1^2 + 5\lambda_5\phi_2^4\phi_1 + \lambda_6\phi_2^5\right)$$

Under $O(2)$: 6-dim representation is split into three 2-dim irreps

$$r_1 = \begin{pmatrix} \lambda_1 + 2\lambda_3 + \lambda_5 \\ \lambda_2 + 2\lambda_4 + \lambda_6 \end{pmatrix} \equiv \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

$$r_3 = \begin{pmatrix} -\lambda_1 + 2\lambda_3 + 3\lambda_5 \\ -3\lambda_2 - 2\lambda_4 + \lambda_6 \end{pmatrix} \equiv \begin{pmatrix} g_3 \\ g_4 \end{pmatrix}$$

$$r_5 = \begin{pmatrix} \lambda_1 - 10\lambda_3 + 5\lambda_5 \\ 5\lambda_2 - 10\lambda_4 + \lambda_6 \end{pmatrix} \equiv \begin{pmatrix} g_5 \\ g_6 \end{pmatrix}$$

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$$r_5 = \begin{pmatrix} \lambda_1 - 10\lambda_3 + 5\lambda_5 \\ 5\lambda_2 - 10\lambda_4 + \lambda_6 \end{pmatrix} \equiv \begin{pmatrix} g_5 \\ g_6 \end{pmatrix} \longrightarrow R(5\theta)\begin{pmatrix} g_5 \\ g_6 \end{pmatrix}$$

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$$r_3 = \begin{pmatrix} -\lambda_1 + 2\lambda_3 + 3\lambda_5 \\ -3\lambda_2 - 2\lambda_4 + \lambda_6 \end{pmatrix} \equiv \begin{pmatrix} g_3 \\ g_4 \end{pmatrix} \longrightarrow R(3\theta)\begin{pmatrix} g_3 \\ g_4 \end{pmatrix}$$

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Quintic model with two flavours

$$\beta_1 = \frac{1}{64}\left(-1099 g_3 g_6^2 - 2 g_4\left(612 g_3 + 1099 g_5\right) g_6 + g_5(-612 g_3^2 + 1099 g_5 g_3 + 612 g_4^2)\right)$$

$$\begin{aligned}\beta_2 \;=\; & -\frac{3}{2} g_2 + \frac{1}{384}\left(g_2^3 - \left(753 g_3^2 + 753 g_4^2 + 3670(g_5^2 + g_6^2)\right) g_2\right. \\ & \left. -6594\, g_4 g_6^2 + 6 g_4 g_5\left(1099 g_5 - 1224 g_3\right) + 12\left(306(g_3^2 - g_4^2) + 1099 g_3 g_5\right) g_6\right)\end{aligned}$$

$$\begin{aligned}\beta_3 \;=\; & -\frac{3}{2} g_3 + \frac{1}{640}\left(-2445 g_3^3 - 251 g_2^2 g_3 - \left(2445 g_4^2 + 13466(g_5^2 + g_6^2)\right) g_3\right. \\ & \left. +4 g_2\left(\left(612 g_3 + 1099 g_5\right) g_6 - 612 g_4 g_5\right) + 3672(g_5^3 - 3 g_5 g_6^2)\right)\end{aligned}$$

Quintic model with two flavours

$$\beta_4 = -\frac{3}{2}g_4 + \frac{1}{640}\left(-3672g_6^3 - 13466g_4g_6^2 + 11016g_5^2g_6 - 251g_2^2g_4 \right.$$
$$\left. +g_4\big(-13466g_5^2 - 2445(g_3^2+g_4^2)\big) + 2g_2\big(1099g_5^2 - 1224g_3g_5 - 1099g_6^2 - 1224g_4g_6\big)\right)$$

$$\beta_5 = -\frac{3}{2}g_5 + \frac{1}{1920}\Big(-1835g_5g_2^2 + \big(6594g_4g_5 + g_3(6594g_6 - 3672g_4)\big)g_2$$
$$-68(243g_3 + 389g_5)g_6^2 + g_5\big(-26452g_5^2 + 16524g_3g_5 - 20199(g_3^2+g_4^2)\big)$$
$$+33048g_4g_5g_6\Big)$$

$$\beta_6 = -\frac{3}{2}g_6 + \frac{1}{1920}\Big(-26452g_6^3 - 16524g_4g_6^2$$
$$-\big(1835g_2^2 + 6594g_4g_2 + 26452g_5^2 + 20199(g_3^2+g_4^2) + 33048g_3g_5\big)g_6$$
$$+16524g_4g_5^2 + 6g_2\big(306(g_3^2 - g_4^2) + 1099g_3g_5\big)\Big)$$

Quintic model with two flavours

Fully interacting fixed points

N	Anomalous dimensions	Symmetry	Name
1	$-\frac{1}{1530}$	None	Tricritical Lee-Yang
2	$\left(\frac{3}{10}, \frac{3}{10}\right)$	D_5	Pentagonal
	$\left(-\frac{1}{978}, -\frac{1}{978}\right)$	S_3	Potts3
	$(-0.000589, -0.000718)$	$\mathbb{Z}_2$	Z_2

$$V_{2,\mathrm{Pentagonal}} = \frac{1}{80}\phi_2\left(5\phi_1^4 - 10\phi_2^2\phi_1^2 + \phi_2^4\right)$$

$$V_{2,\mathrm{Potts}} = \frac{i\phi_2\left(\phi_2^2 - 3\phi_1^2\right)\left(\phi_1^2 + \phi_2^2\right)}{48\sqrt{163}}$$

$$V_{2,\mathrm{Z}_2} = -i\,\phi_2\,(0.000739091\,\phi_2^4 + 0.00605538\,\phi_1^2\phi_2^2 + 0.00248364\,\phi_1^4)$$

Quintic model with two flavours

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Epsilon expansion from conformal field theory

for multi-scalar models

ϵ expansion from CFT

- *Conformal invariance* $(d = d_c - \epsilon)$

$$\langle O_a(x) O_b(y) \rangle = \frac{c_a\,\delta_{ab}}{|x-y|^{2\Delta_a}} \qquad \Delta_a = a\delta + \gamma_a$$

$$\langle O_a(x) O_b(y) O_c(z) \rangle = \frac{C_{abc}}{|x-y|^{\Delta_a+\Delta_b-\Delta_c}\,|y-z|^{\Delta_b+\Delta_c-\Delta_a}\,|z-x|^{\Delta_c+\Delta_a-\Delta_b}}$$

- *Schwinger-Dyson equation*

$$\left\langle \frac{\delta S}{\delta\phi}(x)\; O_1(y)\, O_2(z) \ldots \right\rangle = 0$$

ϵ expansion from CFT

$$V = \frac{1}{m!} V_{i_1 \cdots i_m} \phi_{i_1} \cdots \phi_{i_m}, \qquad d_c = \frac{2m}{m-2}$$

$$\text{Number of monomials for } N \text{ fields } = \binom{m+N-1}{m}$$

Field anomalous dimension

$$\langle\tilde{\phi}_i(x)\tilde{\phi}_j(y)\rangle = \frac{\tilde{c}\delta_{ij}}{|x-y|^{2\Delta_i}}, \qquad \tilde{\phi}_i = R_{ik}\phi_k, \qquad R^T R = 1$$

Field anomalous dimension

$$\langle\tilde{\phi}_i(x)\tilde{\phi}_j(y)\rangle = \frac{\tilde{c}\delta_{ij}}{|x-y|^{2\Delta_i}}, \qquad \tilde{\phi}_i = R_{ik}\phi_k, \qquad R^T R = 1$$

$$\Box_x \Box_y \langle\tilde{\phi}_i(x)\tilde{\phi}_j(y)\rangle = \Box_x \Box_y \frac{\tilde{c}\delta_{ij}}{|x-y|^{2\Delta_i}}, \qquad \Box\phi_i = V_i$$

Field anomalous dimension

$$\langle\tilde{\phi}_i(x)\tilde{\phi}_j(y)\rangle = \frac{\tilde{c}\delta_{ij}}{|x-y|^{2\Delta_i}}, \qquad \tilde{\phi}_i = R_{ik}\phi_k, \qquad R^T R = 1$$

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$$\langle\tilde{V}_i(x)\tilde{V}_j(y)\rangle = \Box_x^2\frac{\tilde{c}}{|x-y|^{2\Delta_i}}\ \delta_{ij}$$

Field anomalous dimension

at leading order:

$$\frac{c^{m-1}}{|x-y|^{2(m-1)\delta_c}}\,\tilde{V}_{ai_1\cdots i_{m-1}}\,\tilde{V}_{bi_1\cdots i_{m-1}}\Big|_{\phi=0} = \frac{16\,\delta_c(\delta_c+1)\gamma_a}{|x-y|^{2\delta_c+4}}\,c\,\delta_{ab}$$

$$\delta_c = \frac{m}{m-2}, \qquad c = \frac{1}{4\pi}\frac{\Gamma\left(\delta_c\right)}{\pi^{\delta_c}}$$

Field anomalous dimension

at leading order:

$$\frac{c^{m-1}}{|x-y|^{2(m-1)\delta_c}}\,\tilde{V}_{ai_1\cdots i_{m-1}}\,\tilde{V}_{bi_1\cdots i_{m-1}}\Big|_{\phi=0}=\frac{16\,\delta_c(\delta_c+1)\gamma_a}{|x-y|^{2\delta_c+4}}\,c\,\delta_{ab}$$

$$\delta_c=\frac{m}{m-2},\qquad c=\frac{1}{4\pi}\frac{\Gamma\left(\delta_c\right)}{\pi^{\delta_c}}$$

simplified and written in ϕ_i basis:

$$\boxed{\gamma_{ab}=\frac{(m-2)^2}{32m!}\,c^{m-2}V_{ai_1i_2\cdots i_{m-1}}V_{bi_1i_2\cdots i_{m-1}}}$$

$$\gamma_{ab}=R_{ac}R_{bc}\gamma_c,\qquad \mu\frac{d}{d\mu}\phi_a=-\gamma_{ab}\,\phi_b$$

Anomalous dimensions of higher order operators ($m = 2n$)

$$\left\langle \phi(x)\phi^k(y)\phi^{k+1}(z)\right\rangle = \frac{C_{1,k,k+1}}{|x-y|^{\Delta_1+\Delta_k-\Delta_{k+1}}|y-z|^{\Delta_k+\Delta_{k+1}-\Delta_1}|z-x|^{\Delta_1+\Delta_{k+1}-\Delta_k}}$$

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$$\left\langle \Box_x\phi(x)\phi^k(y)\phi^{k+1}(z)\right\rangle = \Box_x \frac{C_{1,k,k+1}}{|x-y|^{\Delta_1+\Delta_k-\Delta_{k+1}}|y-z|^{\Delta_k+\Delta_{k+1}-\Delta_1}|z-x|^{\Delta_1+\Delta_{k+1}-\Delta_k}}$$

@ LO

$$\frac{g}{(2n-1)!}\frac{C^{\text{free}}_{2n-1,k,k+1}}{|x-y|^2|y-z|^{\frac{2k}{n-1}-2}|z-x|^{\frac{2n}{n-1}}} = \frac{2}{n-1}(\gamma_{k+1}-\gamma_k)\frac{C^{\text{free}}_{1,k,k+1}}{|x-y|^2|y-z|^{\frac{2k}{n-1}-2}|z-x|^{\frac{2n}{n-1}}}$$

Anomalous dimensions of higher order operators ($m = 2n$)

$$\gamma_{k+1} - \gamma_k = \frac{2}{(n-2)!\,n!}\,\frac{k!}{(k-n+1)!}c^{n-1}g + O(g^2)\,, \quad c = \frac{1}{4\pi}\,\frac{\Gamma\left(\delta_c\right)}{\pi^{\delta_c}} \qquad k \geq n-1$$

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$$\gamma_{k+1} - \gamma_k = \frac{2}{(n-2)!\, n!} \frac{k!}{(k-n+1)!} c^{n-1} g + O(g^2)\,, \quad c = \frac{1}{4\pi} \frac{\Gamma(\delta_c)}{\pi^{\delta_c}} \qquad k \geq n-1$$

$$\gamma_k = \frac{2(n-1)}{n!^2} \frac{k!}{(k-n)!} c^{n-1} g + O(g^2)\,, \qquad k \geq n$$

Anomalous dimensions of higher order operators ($m = 2n$)

$$\begin{aligned} \phi^k \quad &\longrightarrow \quad \mathcal{S}_k \equiv S_{i_1 \cdots i_k} \phi_{i_1} \cdots \phi_{i_k} \\ \gamma^k \quad &\longrightarrow \quad \gamma_k^S \end{aligned}$$

Anomalous dimensions of higher order operators ($m = 2n$)

$$\phi^k \quad \longrightarrow \quad \mathcal{S}_k \equiv S_{i_1 \cdots i_k} \phi_{i_1} \cdots \phi_{i_k}$$

$$\gamma^k \quad \longrightarrow \quad \gamma_k^S$$

$$\Big\langle \phi(x) \phi^k(y) \phi^{k+1}(z) \Big\rangle \quad \longrightarrow \quad \Big\langle \phi_i(x)\, \mathcal{S}_k(y) \mathcal{S}_{k+1}(z) \Big\rangle$$

Anomalous dimensions of higher order operators ($m = 2n$)

$$(\gamma^S_{k+1} - \gamma^S_k)\, S_{ii_1\cdots i_k} S_{i_1\cdots i_k} = c_{n,k}\, V_{ii_1\cdots i_n j_1\cdots j_{n-1}} S_{j_1\cdots j_{n-1} l_1\cdots l_{k+1-n}} S_{l_1\cdots l_{k+1-n} i_1\cdots i_n}$$

$$c_{n,k} = \frac{c^{n-1}}{2(n-2)!n!} \frac{k!}{(k-n+1)!}$$

Anomalous dimensions of higher order operators ($m = 2n$)

$$(\gamma^S_{k+1} - \gamma^S_k)\, S_{ii_1\cdots i_k} S_{i_1\cdots i_k} = c_{n,k}\, V_{ii_1\cdots i_n j_1\cdots j_{n-1}} S_{j_1\cdots j_{n-1} l_1\cdots l_{k+1-n}} S_{l_1\cdots l_{k+1-n} i_1\cdots i_n}$$

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$$\gamma_1 = \mathcal{O}(V^2), \qquad c_{n,k} = 0 \quad k < n-1$$

$$\gamma^S_n S_{ii_1\cdots i_{n-1}} S_{i_1\cdots i_{n-1}} \overset{\text{LO}}{=} c_{n,n-1} V_{ii_1\cdots i_n j_1\cdots j_{n-1}} S_{j_1\cdots j_{n-1}} S_{i_1\cdots i_n}$$

Anomalous dimensions of higher order operators ($m = 2n$)

$$(\gamma^S_{k+1} - \gamma^S_k)\, S_{ii_1\cdots i_k} S_{i_1\cdots i_k} = c_{n,k}\, V_{ii_1\cdots i_n j_1\cdots j_{n-1}} S_{j_1\cdots j_{n-1} l_1\cdots l_{k+1-n}} S_{l_1\cdots l_{k+1-n} i_1\cdots i_n}$$

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Anomalous dimensions of higher order operators ($m = 2n$)

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Anomalous dimensions of higher order operators ($m = 2n$)

$$(\gamma^S_{k+1} - \gamma^S_k)\, S_{ii_1\cdots i_k} S_{i_1\cdots i_k} = c_{n,k}\, V_{ii_1\cdots i_n j_1\cdots j_{n-1}} S_{j_1\cdots j_{n-1} l_1\cdots l_{k+1-n}} S_{l_1\cdots l_{k+1-n} i_1\cdots i_n}$$

$$c_{n,k} = \frac{c^{n-1}}{2(n-2)!n!} \frac{k!}{(k-n+1)!}$$

$$\boxed{\gamma^S_{n+l} S_{i_1\cdots i_{n+l}} = \frac{(n-1)c^{n-1}}{2n!^2} \frac{(n+l)!}{l!}\, V_{j_1\cdots j_n (i_1\cdots i_n} S_{i_{n+1}\cdots i_{n+l}) j_1\cdots j_n}} \quad l \geq 0$$

Anomalous dimensions of higher order operators ($m = 2n$)

$$(\gamma^S_{k+1} - \gamma^S_k)\, S_{ii_1\cdots i_k} S_{i_1\cdots i_k} = c_{n,k}\, V_{ii_1\cdots i_n j_1\cdots j_{n-1}} S_{j_1\cdots j_{n-1} l_1\cdots l_{k+1-n}} S_{l_1\cdots l_{k+1-n} i_1\cdots i_n}$$

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Can be solved once the symmetry is known

Anomalous dimensions of higher order operators ($m = 2n$)

$$(\gamma^S_{k+1} - \gamma^S_k)\, S_{ii_1\cdots i_k} S_{i_1\cdots i_k} = c_{n,k}\, V_{ii_1\cdots i_n j_1\cdots j_{n-1}} S_{j_1\cdots j_{n-1} l_1\cdots l_{k+1-n}} S_{l_1\cdots l_{k+1-n} i_1\cdots i_n}$$

$$c_{n,k} = \frac{c^{n-1}}{2(n-2)!n!} \frac{k!}{(k-n+1)!}$$

$$\boxed{\gamma^S_{n+l} S_{i_1\cdots i_{n+l}} = \frac{(n-1)c^{n-1}}{2n!^2} \frac{(n+l)!}{l!} V_{j_1\cdots j_n (i_1\cdots i_n} S_{i_{n+1}\cdots i_{n+l}) j_1\cdots j_n}} \quad l \geq 0$$

Hypercubic theories:
Antipin, Bersini [arXiv:1903.04950 [hep-th]]

Anomalous dimensions of higher order operators ($m = 2n$)

$$\big\langle \phi_i(x)\, \mathcal{S}_k(y) \mathcal{S}_{k+1}(z) \big\rangle \qquad \mathcal{S}_{2n-1} = \mathcal{V}_i = \Box \phi_i$$

$$k = 2n-2: \quad \big\langle \phi_i(x) \mathcal{S}_{2n-2}(y) \mathcal{V}_j(z) \big\rangle$$

$$k = 2n-1: \quad \big\langle \phi_i(x) \mathcal{V}_j(y) \mathcal{S}_{2n}(z) \big\rangle$$

Anomalous dimensions of higher order operators ($m = 2n$)

$$\big\langle \phi_i(x)\,\mathcal{S}_k(y)\mathcal{S}_{k+1}(z)\big\rangle \qquad \mathcal{S}_{2n-1} = \mathcal{V}_i = \Box\phi_i$$

$$k = 2n-2: \quad \big\langle \phi_i(x)\mathcal{S}_{2n-2}(y)\mathcal{V}_j(z)\big\rangle$$

$$k = 2n-1: \quad \big\langle \phi_i(x)\mathcal{V}_j(y)\mathcal{S}_{2n}(z)\big\rangle$$

$$(\gamma^i_{2n-1} - \gamma^S_{2n-2})\,V_{ijI_{2n-2}}S_{I_{2n-2}} = c_{n,2n-2}\,V_{iI_nJ_{n-1}}S_{J_{n-1}K_{n-1}}V_{K_{n-1}I_nj}$$

$$(\gamma^S_{2n} - \gamma^i_{2n-1})\,S_{jI_{2n-1}}V_{iI_{2n-1}} = c_{n,2n-1}\,V_{jI_nJ_{n-1}}V_{iJ_{n-1}K_n}S_{K_nI_n}$$

Anomalous dimensions of higher order operators ($m = 2n$)

$$\left\langle \phi_i(x)\,\mathcal{S}_k(y)\mathcal{S}_{k+1}(z)\right\rangle \qquad \mathcal{S}_{2n-1} = \mathcal{V}_i = \Box\phi_i$$

$$k = 2n-2: \quad \left\langle \phi_i(x)\mathcal{S}_{2n-2}(y)\mathcal{V}_j(z)\right\rangle$$

$$k = 2n-1: \quad \left\langle \phi_i(x)\mathcal{V}_j(y)\mathcal{S}_{2n}(z)\right\rangle$$

$$\gamma^i_n,\, \ldots,\, \gamma^i_{2n-2},\, \gamma^i_{2n-1},\, \gamma^i_{2n},\, \gamma^i_{2n+1},\, \gamma^i_{2n+2},\, \ldots$$

Anomalous dimensions of higher order operators ($m = 2n$)

$$\langle \phi_i(x)\, \mathcal{S}_k(y) \mathcal{S}_{k+1}(z) \rangle \qquad \mathcal{S}_{2n-1} = \mathcal{V}_i = \Box \phi_i$$

$$k = 2n-2: \quad \langle \phi_i(x) \mathcal{S}_{2n-2}(y) \mathcal{V}_j(z) \rangle$$

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$$\gamma^i_n,\, \ldots,\, \gamma^i_{2n-2},\, \gamma^i_{2n-1},\, \gamma^i_{2n},\, \gamma^i_{2n+1},\, \gamma^i_{2n+2},\, \ldots$$

Anomalous dimensions of higher order operators ($m = 2n$)

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$$k = 2n-1: \quad \langle \phi_i(x) \mathcal{V}_j(y) \mathcal{S}_{2n}(z) \rangle$$

$$\gamma^i_n, \ldots, \gamma^i_{2n-2}, \gamma^i_{2n-1}, \gamma^i_{2n}, \gamma^i_{2n+1}, \gamma^i_{2n+2}, \cdots$$

Anomalous dimensions of higher order operators ($m = 2n$)

$$\big\langle \phi_i(x)\, \mathcal{S}_k(y)\mathcal{S}_{k+1}(z)\big\rangle \qquad \mathcal{S}_{2n-1} = \mathcal{V}_i = \Box\phi_i$$

$$k = 2n-2: \quad \big\langle \phi_i(x)\mathcal{S}_{2n-2}(y)\mathcal{V}_j(z)\big\rangle \;=\; \Box_z\big\langle \phi_i(x)\mathcal{S}_{2n-2}(y)\phi_j(z)\big\rangle$$

$$k = 2n-1: \quad \big\langle \phi_i(x)\mathcal{V}_j(y)\mathcal{S}_{2n}(z)\big\rangle \;=\; \Box_y\big\langle \phi_i(x)\phi_j(y)\mathcal{S}_{2n}(z)\big\rangle$$

$$\Box_x\Box_y\big\langle \phi^i(x)\phi^j(y)\mathcal{S}_{2n-2}(z)\big\rangle$$

$$\Box_x\Box_y\big\langle \phi^i(x)\phi^j(y)\mathcal{S}_{2n}(z)\big\rangle$$

Anomalous dimensions of higher order operators ($m = 2n$)

$$\Box_x \Box_y \big\langle \phi^i(x)\phi^j(y)\mathcal{S}_{2n-2}(z)\big\rangle$$

$$\Box_x \Box_y \big\langle \phi^i(x)\phi^j(y)\mathcal{S}_{2n}(z)\big\rangle$$

$$\begin{aligned}(\gamma^i_{2n-1} - \gamma^S_{2n-2})\, V_{ijI_{2n-2}} S_{I_{2n-2}} &= c_{n,2n-2}\, V_{iI_n J_{n-1}} S_{J_{n-1}K_{n-1}} V_{K_{n-1}I_n j} \\ (\gamma^S_{2n} - \gamma^i_{2n-1})\, S_{jI_{2n-1}} V_{iI_{2n-1}} &= c_{n,2n-1}\, V_{jI_n J_{n-1}} V_{iJ_{n-1}K_n} S_{K_n I_n}\end{aligned}$$

Anomalous dimensions of higher order operators ($m = 2n$)

$$\Box_x \Box_y \langle \phi^i(x)\phi^j(y)\mathcal{S}_{2n-2}(z)\rangle$$

$$\Box_x \Box_y \langle \phi^i(x)\phi^j(y)\mathcal{S}_{2n}(z)\rangle$$

$$[\Box\phi_i] = [\phi_i] + 2$$

$$\begin{aligned}
(\gamma^i_{2n-1} - \gamma^S_{2n-2})\, V_{ijI_{2n-2}} S_{I_{2n-2}} &= c_{n,2n-2}\, V_{iI_n J_{n-1}} S_{J_{n-1}K_{n-1}} V_{K_{n-1}I_n j} \\
(\gamma^S_{2n} - \gamma^i_{2n-1})\, S_{jI_{2n-1}} V_{iI_{2n-1}} &= c_{n,2n-1}\, V_{jI_n J_{n-1}} V_{iJ_{n-1}K_n} S_{K_n I_n}
\end{aligned}$$

Anomalous dimensions of higher order operators ($m = 2n$)

$$\Box_x \Box_y \langle \phi^i(x) \phi^j(y) \mathcal{S}_{2n-2}(z) \rangle$$

$$\Box_x \Box_y \langle \phi^i(x) \phi^j(y) \mathcal{S}_{2n}(z) \rangle$$

$$[\Box \phi_i] = [\phi_i] + 2$$

$$(\gamma^i_{2n-1} - \gamma^S_{2n-2})\, V_{ijI_{2n-2}} S_{I_{2n-2}} = c_{n,2n-2}\, V_{iI_n J_{n-1}} S_{J_{n-1}K_{n-1}} V_{K_{n-1}I_n j}$$

$$(\gamma^S_{2n} - \gamma^i_{2n-1})\, S_{jI_{2n-1}} V_{iI_{2n-1}} = c_{n,2n-1}\, V_{jI_n J_{n-1}} V_{iJ_{n-1}K_n} S_{K_n I_n}$$

Anomalous dimensions of higher order operators ($m = 2n$)

$$\Box_x \Box_y \big\langle \phi^i(x)\phi^j(y)\mathcal{S}_{2n-2}(z)\big\rangle$$

$$\Box_x \Box_y \big\langle \phi^i(x)\phi^j(y)\mathcal{S}_{2n}(z)\big\rangle$$

$$(\gamma^S_{2n} - \gamma^i_{2n-1})\, S_{jI_{2n-1}} V_{iI_{2n-1}} = c_{n,2n-1}\, V_{jI_n J_{n-1}} V_{iJ_{n-1}K_n} S_{K_n I_n}$$

Anomalous dimensions of higher order operators $(m = 2n)$

$$\Box_x \Box_y \left\langle \phi^i(x)\phi^j(y)\mathcal{S}_{2n-2}(z)\right\rangle$$

$$\Box_x \Box_y \left\langle \phi^i(x)\phi^j(y)\mathcal{S}_{2n}(z)\right\rangle$$

$$(\gamma^S_{2n} - \gamma^i_{2n-1})\, S_{jI_{2n-1}} V_{iI_{2n-1}} = c_{n,2n-1}\, V_{jI_n J_{n-1}} V_{iJ_{n-1}K_n} S_{K_n I_n}$$

Anomalous dimensions of higher order operators ($m = 2n$)

$$\Box_x \Box_y \left\langle \phi^i(x)\phi^j(y)\mathcal{S}_{2n-2}(z)\right\rangle$$

Fixed point equations ($m = 2n$)

$$\Box_x \Box_y \left\langle \phi^i(x)\phi^j(y)\mathcal{S}_{2n-2}(z)\right\rangle$$

$$\Downarrow$$

$$\Big((n-1)\epsilon - \gamma^S_{2n-2}\Big)\, V_{ijL_{2n-2}} S_{L_{2n-2}} = \frac{(2n-3)!c^{n-1}}{n!(n-2)!^2} V_{iI_{n-1}J_n} V_{jJ_nK_{n-1}} S_{K_{n-1}I_{n-1}}$$

Fixed point equations ($m = 2n$)

$$\Box_x \Box_y \langle \phi^i(x) \phi^j(y) \mathcal{S}_{2n-2}(z) \rangle$$

$$\Downarrow$$

$$\Big((n-1)\epsilon - \gamma^S_{2n-2} \Big) V_{ijL_{2n-2}} S_{L_{2n-2}} = \frac{(2n-3)!c^{n-1}}{n!(n-2)!^2} V_{iI_{n-1}J_n} V_{jJ_nK_{n-1}} S_{K_{n-1}I_{n-1}}$$

$$\gamma^S_{2n-2} S_{i_1 \cdots i_{2n-2}} = \frac{(n-1)c^{n-1}}{2n!^2} \frac{(2n-2)!}{(n-2)!} V_{j_1 \cdots j_n (i_1 \cdots i_n} S_{i_{n+1} \cdots i_{2n-2}) j_1 \cdots j_n}$$

Fixed point equations ($m = 2n$)

$$\Box_x \Box_y \langle \phi^i(x) \phi^j(y) \mathcal{S}_{2n-2}(z) \rangle$$

$$\Downarrow$$

$$\left((n-1)\epsilon - \gamma^S_{2n-2}\right) V_{ijL_{2n-2}} S_{L_{2n-2}} = \frac{(2n-3)!c^{n-1}}{n!(n-2)!^2} V_{iI_{n-1}J_n} V_{jJ_nK_{n-1}} S_{K_{n-1}I_{n-1}}$$

$$\gamma^S_{2n-2} S_{i_1 \cdots i_{2n-2}} = \frac{(n-1)c^{n-1}}{2n!^2} \frac{(2n-2)!}{(n-2)!} V_{j_1 \cdots j_n (i_1 \cdots i_n} S_{i_{n+1} \cdots i_{2n-2}) j_1 \cdots j_n}$$

Fixed point equations ($m = 2n$)

$$\Box_x \Box_y \big\langle \phi^i(x) \phi^j(y) \mathcal{S}_{2n-2}(z) \big\rangle$$

$$\Downarrow$$

$$\boxed{0 = (1-n)\epsilon\, V_{i_1 \cdots i_{2n}} + \frac{(n-1)(2n)!}{4n!^3} c^{n-1} V_{j_1 \cdots j_n (i_1 \cdots i_n} V_{i_{n+1} \cdots i_{2n}) j_1 \cdots j_n}}$$

Structure constants

$$m = 2\ell + 1$$

$$\Box_x : \qquad \Big\langle \phi_i(x)\mathcal{S}_{2p-1}(y)\tilde{\mathcal{S}}_{2q-1}(z)\Big\rangle \;\propto\; C_{\phi_i \mathcal{S}_{2p-1}\tilde{\mathcal{S}}_{2q-1}}$$

$$q + p \geq \ell + 1, \qquad |q - p| \leq \ell$$

$$\Downarrow$$

$$C_{\phi_i \mathcal{S}_{2p-1}\tilde{\mathcal{S}}_{2q-1}} \;=\; \#\, V_{ik_1\cdots k_t l_1\cdots l_r}\, S_{l_1\cdots l_r j_1\cdots j_s}\, \tilde{S}_{j_1\cdots j_s k_1\cdots k_t}$$

$$\# \;=\; \frac{(2\ell-1)^2(2p-1)!(2q-1)!}{4(4(p-q)^2-1)(\ell+p-q)!(\ell+q-p)!(p+q-\ell-1)!}\, c^{\ell+p+q-1}$$

$$2\ell = r + t, \quad 2p - 1 = r + s, \quad 2q - 1 = s + t$$

Structure constants

$$m = 3 \quad (d_c = 6)$$

$$\Box_x : \qquad \langle \phi_i(x)\phi_j(y)\phi_k(z)\rangle \quad \propto \quad C_{\phi_i\phi_j\phi_k}$$

$$\Downarrow$$

$$C_{\phi_i\phi_j\phi_k} = -\frac{c^2}{4}V_{ijk}$$

Fixed point equations ($m = 3$)

$$m = 3 \quad (d_c = 6)$$

$$\Box_x \Box_y \Box_z : \qquad \langle \phi_i(x) \phi_j(y) \phi_k(z) \rangle \;\propto\; C_{\phi_i \phi_j \phi_k}$$

$$\Downarrow$$

$$\boxed{48\, \gamma_{a(i} V_{jk)a} - 8\epsilon\, V_{ijk} = c\, V_{iab} V_{jbc} V_{kca}}$$

Summary

- *We have discussed epsilon expansion within both RG and CFT frameworks, from a functional perspective, for multi-scalar models*
- *Resorting to a convenient basis (irreps of kinetic term symm), theory space has been fully explored seeking fixed points in* $d = 4 - \epsilon$ *and* $d = 6 - \epsilon$ *up to* $N = 3$, *and in* $d = 10/3 - \epsilon$ *for* $N = 2$
- *We have shown that there are no fixed points beyond known ones in* $d = 4 - \epsilon$ *with three flavours, and found new fixed points in* $d = 6 - \epsilon$ *three flavour and* $d = 10/3 - \epsilon$ *two-flavour models*

Summary

- *The method may prove useful for extensions to theories with higher number of flavours*
- *From the CFT approach we have explained how critical quantities can be obtained and especially found general eigenvalue equations that determine the scaling operators and their anomalous dimensions, and which can be solved once the symmetry is known*
- *Relying solely on conformal symmetry, we have derived general constraint ("fixed point") equations for even models and for the* $d_c = 6$ *model. Can this be extended to higher order odd models?*

Thank You