Universality in the ϵ-expansion

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Universality in the ϵ-expansion

- The primary goal of RG analysis is the study of **universality classes** (UC) in arbitrary dimension
- **IR**: **classify phase transition** and dictate the structure of phase diagrams
- **UV**: needed to construct continuum limits, **asymptotic freedom and safety**
- Universal quantitative properties: **CFT data**
- In general a universality class (UC) is characterized by an **upper critical dimension** d_c where it is described by mean field theory (MFT) and close to which we can perform the **ϵ-expansion**
- Most UCs have **fractional** d_c and appear for the first time at a loop order $L_{\rm LO}$ **generally bigger than one... many UCs have been missed!**
- Lesson from Non-Perturbative RG (NPRG): **be functional**! Smart way to access **CFT data in the ϵ-expansion**
- **Multicomponent world still mostly uncharted!**

Functional Perturbative RG (FPRG)

- Action for a general single component scalar theory in d-dimensions

$$S = \int \mathrm{d}^d x \left\{ \frac{1}{2}(\partial\phi)^2 + V(\phi) \right\}$$

- Effective action in the **loop expansion** ($G_{xy} \equiv (S^{(2)})^{-1}_{xy}$)

$$\Gamma = S + \frac{1}{2}\mathrm{Tr}\log S^{(2)} + \frac{1}{8} S^{(4)}_{xyzw} G_{xy} G_{zw} - \frac{1}{12} S^{(3)}_{xyz} S^{(3)}_{abc} G_{xa} G_{yb} G_{zc} + \cdots$$

- We will use dimensional regularization $\overline{\mathrm{MS}}$ scheme in

$$d = d_c - \epsilon$$

In single scalar perturbation theory d_c labels `universality classes`

- **Beta functionals** β_V, β_Z describing the RG flow of the potential V and of the wave function renormalization functional Z can be computed from the $\frac{1}{\epsilon}$-poles of the effective action

$$\Gamma_{\mathrm{div}} \sim -\frac{1}{\epsilon} \int \mathrm{d}^{d_c} x \left\{ \beta_V(\phi) + \beta_Z(\phi) \frac{1}{2} (\partial\phi)^2 + \cdots \right\}$$

Which are the possible d_c?

$L=1$	$L=2$	$L=3$	$L=4$	$L=5$	$L=6$
2	3	$\frac{8}{3}$	$\frac{5}{2}$	$\frac{12}{5}$	$\frac{7}{3}$
4	4	$\frac{10}{3}$	3	$\frac{14}{5}$	$\frac{8}{3}$
6	5	4	$\frac{7}{2}$	$\frac{16}{5}$	3
	6	$\frac{14}{3}$	4	$\frac{18}{5}$	$\frac{10}{3}$
		$\frac{16}{3}$	$\frac{9}{2}$	4	$\frac{11}{3}$
		6	5	$\frac{22}{5}$	4
			$\frac{11}{2}$	$\frac{24}{5}$	$\frac{13}{3}$
			6	$\frac{26}{5}$	$\frac{14}{3}$
				$\frac{28}{5}$	5
				6	$\frac{16}{3}$
					$\frac{17}{3}$
					6

d_c where $\frac{1}{\epsilon}$-poles are present at a given loop order L

Landau-Ginzburg lagrangians

$L = 1$	$L = 2$	$L = 3$	$L = 4$	$L = 5$	$L = 6$
φ^{∞}					
φ^4	φ^6	φ^8	φ^{10}	φ^{12}	φ^{14}
φ^3	φ^4	φ^5	φ^6	φ^7	φ^8
		φ^4			φ^6
	φ^3		φ^4		φ^5
				φ^4	
		φ^3			φ^4
			φ^3		
				φ^3	
					φ^3

Only integer Landau-Ginzburg lagrangians are allowed: **even** and **odd** models

Multicritical models

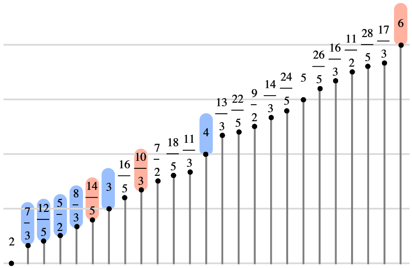
- Landau-Ginzburg for multicritical models

$$S[\phi] = \int \mathrm{d}^d x \Big\{ \frac{1}{2}\partial_\mu \phi \partial^\mu \phi + \frac{g}{m!}\phi^m \Big\}$$

  for m a natural number $m \geq 3$

- **Even models** $m = 2n$: they are the so-called unitary multicritical models which are protected by a $\mathbb{Z}_2$ parity ($\phi \to -\phi$) and include both the `Ising` ($m = 4$) and `Tricritical` ($m = 6$) universality classes as the first special cases
- **Odd models** $m = 2n + 1$: represents a sequence of multicritical non-unitary theories which are protected by a generalization of parity $\mathcal{P}T : S[\phi] \to S[-\phi]^\star$ and include the `Lee-Yang` universality class ($m = 3$) as first example
- Even models are known to interpolate in $d = 2$ with the unitary minimal CFTs $\mathcal{M}(p, p + 1)$ for $p = 1 + m/2$, which arise from the representations of the infinite dimensional Virasoro algebra
- Similarly, there are speculations (Zambelli+Zanusso) pointing at the fact that the non-unitary models might interpolate with the sequence of minimal non-unitary multicritical theories $\mathcal{M}(2, m + 2)$ studied in (Belavin+Belavin+Litvinov+Pugai+Zamolodchikov). This is established for the `Lee-Yang` case $m = 3$ (Cardy)

Minimal models and their range of existence



Allowed $2 \leq d_c \leq 6$ with integer LG Lagrangian with $L_{\mathrm{LO}} \leq 6$: unitary and non-unitary minimal models

- A UC is generally present in all dimensions $2 \leq d \leq d_c$ and in particular for all integer dimensions it the range (caveat: if it does not become first order...)
- The only non-trivial UCs in $d = 3$ are $d_c = 6, 4, \frac{10}{3}$ while the only non-trivial in $d = 4$ is $d_c = 6$
- Universal quantities $U(d)$ are such in all the range $2 \leq d \leq d_c$. $U(2)$ is known from CFT, $U(d_c)$ from MFT (i.e. `Gaussian`), $U'(d_c)$ and in some cases $U''(d_c)$ from ϵ-expansion. General form of $U(d)$ demands non-perturbative methods

Beta functionals: functional form of β_V

- **We can determine the functional form of the beta functional β_V given d_c**
- We first determine the *vertex set* at the given loop order (i.e. the set of possible vertices of a diagram contributing a $\frac{1}{\epsilon}$-pole). The L–loop contribution to the functional β_V is a monomial built out of the possible vertices in the vertex set of order L
- For example, the two loop vertex set is $\{(V^{(3)})^2\}$ while at three loop it is $\{(V^{(3)})^4, (V^{(3)})^2 V^{(4)}, (V^{(4)})^2\}$
- At a given loop order the full vertex set is

$$\left\{(V^{(3)})^{2(L-1)}, \ldots, (V^{(L+1)})^2\right\}$$

since a given graph must satisfy

$$V - P + L = 1 \quad \Rightarrow \quad V = P - (L-1)$$

due to the fact that they are all closed vacuum diagrams, and topology demands them to have Euler character one

- We thus need to consider the partitions of $2P$ of length V for $P_{\min} = L + 1 \leq P \leq P_{\max} = 3(L-1)$
- It's easy in this way to construct the $L = 4$ vertex set

$$\left\{(V^{(3)})^6, (V^{(3)})^4 V^{(4)}, (V^{(3)})^2 (V^{(4)})^2, (V^{(3)})^3 (V^{(5)}), (V^{(4)})^3, V^{(3)} V^{(4)} V^{(5)}, (V^{(3)})^2 (V^{(6)}), (V^{(5)})^2\right\}$$

as one can also check explicitly from the knowledge of four loop diagrams

Beta functionals: functional form of β_V

- Once the vertex set is known we can determine which monomials (with additional $(V^{(2)})^k$ insertions) contribute to β_V for given d_c and L since the dimension of a vertex is

$$[V^{(n)}] = [V][\phi]^{-n} = d - n\left(\frac{d}{2} - 1\right) = d\left(1 - \frac{n}{2}\right) + n$$

  and each monomial must match the dimension of β_V which is obviously d_c

- Clearly at one loop the beta functional can only be of the form $(V^{(2)})^{\frac{d_c}{2}}$ and can be present only when d_c is even. At two loop it must be of the form $(V^{(2)})^k(V^{(3)})^2$ with k integer solution of

$$2\left[d_c\left(1 - \frac{3}{2}\right) + 3\right] + 2k = d_c$$

  At three loop the vertex set contains three elements. One consider the following relations

$$
\begin{array}{ll}
(V^{(2)})^k(V^{(3)})^4 & 2k + 4\left[d_c\left(1 - \frac{3}{2}\right) + 3\right] = d_c \\
(V^{(2)})^k(V^{(3)})^2 V^{(4)} & 2k + 2\left[d_c\left(1 - \frac{3}{2}\right) + 3\right] + \left[d_c\left(1 - \frac{4}{2}\right) + 4\right] = d_c \\
(V^{(2)})^k(V^{(4)})^2 & 2k + 2\left[d_c\left(1 - \frac{4}{2}\right) + 4\right] = d_c
\end{array}
$$

  and looks for solutions with $k \in \mathbb{N}$

- Ising $(d_c = 4)$

$$\beta_V = \underbrace{a(V^{(2)})^2}_{L=1} + \underbrace{bV^{(2)}(V^{(3)})^2}_{L=2} + \underbrace{c_1(V^{(3)})^4 + c_2 V^{(2)}(V^{(3)})^2 V^{(4)} + c_3(V^{(2)})^2(V^{(4)})^2}_{L=3} + \ldots$$

- Lee-Yang $(d_c = 6)$

$$\begin{aligned}\beta_V &= \underbrace{a(V^{(2)})^3}_{L=1} + \underbrace{b(V^{(2)})^3(V^{(3)})^2}_{L=2} \\ &\quad + \underbrace{c_1(V^{(2)})^3(V^{(3)})^4 + c_2(V^{(2)})^4(V^{(3)})^2 V^{(4)} + c_3(V^{(2)})^5(V^{(4)})^2}_{L=3} + \ldots\end{aligned}$$

- Tricritical $(d_c = 3)$

$$\beta_V = \underbrace{a(V^{(3)})^2}_{L=2} + \underbrace{b_1 V^{(3)} V^{(4)} V^{(5)} + b_2(V^{(4)})^3 + b_3 V^{(2)}(V^{(5)})^2}_{L=4} + \ldots$$

- Similar expression can be easily derived for any d_c and for any order $\mathrm{N}^k\mathrm{LO}$

Beta functionals: determining the LO and NLO coefficients

- **The LO and NLO order contributions are scheme independent** To see this we just take $V(\phi) = \frac{g}{m!}\phi^m$ so that the beta functionals are projected to the beta functions of the respective critical coupling which we already know has LO and NLO universal beta function coefficients.
- **These coefficients are the *universal ϵ-expansion data* of the theory**
- **Example**: computing `Ising` LO and NLO universal coefficients

$$\beta_V = a(V'')^2 + bV''(V''')^2$$

Even if they can be easily determined by matching with standard perturbative RG here we re-compute them in a different way offered by the functional constraint which does not involve the critical `Ising` coupling $\frac{\lambda_4}{4!}\varphi^4$

1. The LO contribution can be extracted from the free theory alone $V = \frac{\lambda_2}{2}\varphi^2$ if we look at the flow of the vacuum energy since $\beta_0 = a\lambda_2^2$. This is trivial since the one loop contribution to the vacuum energy is just one half the number of degrees of freedom $a = \frac{1}{2}\frac{1}{(4\pi)^2}$
2. The NLO coefficient can be determined considering $V = \frac{1}{2}\lambda_2\varphi^2 + \frac{1}{3!}\lambda_3\varphi^3$ from $\beta_0 = a\lambda_2^2 + b\lambda_2\lambda_3^2$ which comes from the sunset diagram

$$\text{sunset} = -\frac{1}{12}\frac{\lambda_3^2}{(4\pi)^d}\text{Vol}\int_0^\infty \frac{ds_1\, ds_2\, ds_3}{(s_1 s_2 + s_1 s_3 + s_2 s_3)^{\frac{d}{2}}} e^{-\lambda_2(s_1+s_2+s_3)} \to \frac{1}{4\epsilon}\frac{\lambda_2\lambda_3^2}{(4\pi)^4}\text{Vol}$$

This corresponds to a two loop vacuum renormalization $\beta_0^{II} = -\frac{1}{2}\frac{\lambda_2\lambda_3^2}{(4\pi)^4}$ and thus $b = -\frac{1}{2}\frac{1}{(4\pi)^4}$

Universality Classes

$(4\pi)^{\frac{d_c}{2}L}\,\beta_V$	d_c	$L=1$	$L=2$	$L=3$
`Sine-Gordon`	2	$-V^{(2)}$	0	0
`Tetracritical`	$\frac{8}{3}$	0	0	$\frac{1}{8}\Gamma(\frac{1}{3})^3(V^{(4)})^2$
`Tricritical`	3	0	$\frac{1}{3}\Gamma(\frac{1}{2})^2(V^{(3)})^2$	0
`Tri-Lee-Yang`	$\frac{10}{3}$	0	0	$\frac{\Gamma(\frac{1}{2})^4\Gamma(\frac{2}{3})}{\Gamma(\frac{4}{3})^2}[\frac{1}{9}V^{(2)}(V^{(4)})^2-\frac{1}{2}(V^{(3)})^2V^{(4)}]$
`Ising`	4	$\frac{1}{2}(V^{(2)})^2$	$-\frac{1}{2}V^{(2)}(V^{(3)})^2$	$\frac{1}{16}(V^{(2)})^2(V^{(4)})^2+\frac{7}{4}V^{(2)}(V^{(3)})^2V^{(4)}-\frac{1-4\zeta_3}{8}(V^{(3)})^4$
`Lee-Yang`	6	$-\frac{1}{6}(V^{(2)})^3$	$-\frac{23}{144}(V^{(2)})^3(V^{(3)})^2$	$-\frac{1595+432\zeta_3}{5184}(V^{(2)})^3(V^{(3)})^4$

$(4\pi)^{\frac{d_c}{2}L}\beta_Z$	d_c	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$
Sine-Gordon	2	0	0	0	0	0
Tetracritical	$\frac{8}{3}$	0	0	0	0	$-\frac{\Gamma(\frac{1}{3})^6}{1120}(V^{(8)})^4$
Tricritical	3	0	0	0	$-\frac{\Gamma(\frac{1}{2})^4}{45}(V^{(6)})^4$	0
Tri-Lee-Yang	$\frac{10}{3}$	0	0	$-\frac{3}{40}\Gamma(\frac{2}{3})^3(V^{(5)})^4$	0	0
Ising	4	0	$-\frac{1}{6}(V^{(4)})^2$	0	$\frac{1}{8}(V^{(4)})^4$	$\cdots$
Lee-Yang	6	$-\frac{1}{6}(V^{(3)})^2$	$-\frac{13}{216}(V^{(3)})^4$	$-\frac{5195-2592\zeta_3}{31104}(V^{(3)})^6$	$\cdots$	$\cdots$

Beta functions from the beta functionals

- Ising dimensionful beta functionals

$$\beta_V = \frac{1}{(4\pi)^2}\frac{1}{2}(V^{(2)})^2 - \frac{1}{(4\pi)^4}\frac{1}{2}V^{(2)}(V^{(3)})^2 + \cdots$$
$$\beta_Z = -\frac{1}{(4\pi)^4}\frac{1}{6}(V^{(4)})^2 + \cdots$$

- Dimensionless variables: $v(\varphi) \equiv (4\pi)^2\mu^{-(4-\epsilon)}V(\varphi\mu^{(2-\epsilon+\eta)/2})$ and $z(\varphi) = \mu^\eta Z(\varphi\mu^{(2-\epsilon+\eta)/2})$

$$\beta_v = -4v + \varphi v' + \epsilon\left(v - \frac{1}{2}\varphi v'\right) + \frac{1}{2}\eta\varphi v' + \frac{1}{2}(v'')^2 - \frac{1}{2}v''(v''')^2$$
$$\beta_z = \eta z + \varphi z' - \frac{\epsilon}{2}\varphi z' + \frac{1}{2}\eta\varphi z' - \frac{1}{6}(v^{(4)})^2$$

Note: this is the step where we introduce ϵ!

- Expand the potential in Taylor series ($\mathbb{Z}_2$ even and odd operators)

$$v(\varphi) = \sum_{n=1}^{\infty}\frac{\lambda_n}{n!}\varphi^n$$

then the beta functional β_V becomes *generating function of beta functions*

- The anomalous dimension obtains from the normalization $z(0) = 1$ and is $\eta = \frac{\lambda_4^2}{6}$.

Beta functions

- Beta function system

$$
\begin{aligned}
\beta_1 &= -\left(3-\frac{\epsilon}{2}\right)\lambda_1+\lambda_2\lambda_3-\frac{1}{2}\lambda_3^3-\lambda_2\lambda_3\lambda_4+\frac{1}{12}\lambda_1\lambda_4^2\\
\beta_2 &= -2\lambda_2+\lambda_4\lambda_2+\lambda_3^2-\frac{5}{2}\lambda_3^2\lambda_4-\frac{5}{6}\lambda_2\lambda_4^2-\lambda_2\lambda_3\lambda_5\\
\beta_3 &= -\left(1+\frac{\epsilon}{2}\right)\lambda_3+3\lambda_4\lambda_3-\frac{23}{4}\lambda_3\lambda_4^2+\lambda_2\lambda_5-\frac{7}{2}\lambda_3^2\lambda_5-3\lambda_2\lambda_4\lambda_5-\lambda_2\lambda_3\lambda_6\\
\beta_4 &= -\epsilon\lambda_4+3\lambda_4^2-\frac{17}{3}\lambda_4^3+4\lambda_3\lambda_5-3\lambda_2\lambda_5^2+\lambda_2\lambda_6-\frac{9}{2}\lambda_3^2\lambda_6-22\lambda_3\lambda_4\lambda_5\\
&\quad -4\lambda_2\lambda_4\lambda_6-\lambda_2\lambda_3\lambda_7\\
\beta_5 &= \ldots
\end{aligned}
$$

- $\frac{1}{4!}\lambda_4\phi^4$ two loop beta function recovered
- In dimensional regularization the fixed point is very simple:

$$\beta_i = 0 \qquad \Rightarrow \qquad \lambda_i^* = \left(\frac{\epsilon}{3}+\frac{17\epsilon^2}{81}\right)\delta_{i,4}$$

- Expand around the fixed point: $\lambda_i = \lambda_i^* + \delta\lambda_i$ with $\Lambda_i \equiv i!\,\delta\lambda_i$

$$
\begin{aligned}
\partial_t \Lambda_1 &= -\left(3 - \frac{\epsilon}{2} - \frac{\epsilon^2}{108}\right)\Lambda_1 + 12\left(1 - \frac{\epsilon}{3}\right)\Lambda_2\Lambda_3 + \frac{4}{3}\epsilon\,\Lambda_1\Lambda_4 + \ldots \\
\partial_t \Lambda_2 &= -\left(2 - \frac{\epsilon}{3} - \frac{19\epsilon^2}{162}\right)\Lambda_2 + 24\left(1 - \frac{5}{9}\epsilon\right)\Lambda_2\Lambda_4 + 18\left(1 - \frac{5}{6}\epsilon\right)\Lambda_3^2 + \ldots \\
\partial_t \Lambda_3 &= -\left(1 - \frac{\epsilon}{2} + \frac{\epsilon^2}{108}\right)\Lambda_3 + 72\left(1 - \frac{23}{18}\epsilon\right)\Lambda_3\Lambda_4 + \ldots \\
\partial_t \Lambda_4 &= -\left(-\epsilon + \frac{17\epsilon^2}{27}\right)\Lambda_4 + 72\left(1 - \frac{17}{9}\epsilon\right)\Lambda_4^2 + \ldots
\end{aligned}
$$

- CFT data encoded in the beta functions:

1. RG eigenvalues θ_i Related to the scaling dimension $\Delta_i = d - \theta_i$ of the composite operators ϕ^i
2. `Gaussian` OPE coefficients Obtainable from Wick's theorem at $d = d_c$
3. RG OPE coefficients ${c^a}_{bc}$ Note: $O(\epsilon^2)$ terms receive contributions from NNLO and are omitted

CFT data in the ϵ-expansion

- General form of beta functions around a FP in CFT perturbation theory

$$\partial_t \Lambda^a = -(d-\Delta_a)\Lambda^a + \sum_{b,c} C^a{}_{bc}\, \Lambda^b \Lambda^c + O(\Lambda^3)$$

transforms under a change of variables $\tilde{\Lambda}_i = \Lambda_i + \cdots$ as:

1. The spectrum is universal

$$\tilde{\Delta}_a = \Delta_a$$

and up to $O(\epsilon^2)$ is scheme independent

2. The $c^i{}_{jk}$ are not universal away from d_c

$$\tilde{c}^c{}_{ab} = c^c{}_{ab} + \frac{1}{2}\left(\theta_c - \theta_a - \theta_b\right) \frac{\partial^2 \Lambda_c}{\partial \tilde{\Lambda}_p \partial \tilde{\Lambda}_q} \frac{\partial \tilde{\Lambda}_p}{\partial \Lambda_a} \frac{\partial \tilde{\Lambda}_q}{\partial \Lambda_b}$$

but are scheme independent up to order $O(\epsilon)$

- LO and NLO beta functionals are scheme independent (i.e. a, b, c are scheme independent)

$$\beta_V = a(V^{(2)})^2 + bV^{(2)}(V^{(3)})^2 + \cdots \qquad \beta_Z = c(V^{(4)})^2 + \cdots$$

- $\Rightarrow$ θ_i are scheme independent up to order $O(\epsilon^2)$
- $\Rightarrow$ $c^i{}_{jk}$ are scheme independent up to order $O(\epsilon)$ (since NLO terms enter at order $O(\epsilon^2)$)
- **Question**: How do we compute the non-perturbative $C^i{}_{jk}$ with the RG? (Pagani+Sonoda)

SDE+CFT

- FPRG gives a coherent description of multicritical models in the ϵ-expansion (Codello+Safari+Vacca+Zanusso)[arXiv:1705.05558]
- RG OPE coefficients are scheme independent at order $O(\epsilon)$ ...
- **... can we compare with other analytical approaches?**
- $\Rightarrow$ **use SDE+CFT to study multicritical models**
  (Codello+Safari+Vacca+Zanusso)[arXiv:1703.04830;1809.05071]
  We generalize the arguments made for `Tricritical` ($m = 3$) in (Nii)(Hasegawa+Nakayama), for `Ising` ($m = 4$) in (Rychkov+Tan)(Nakayama)(Nii) and for `Lee-Yang` ($m = 6$) in (Basu+Krishnan)(Nii), and assume that for each value of m the multicritical models at the critical point are CFTs for any dimension $2 \leq d \leq d_c$
- **SDE+CFT is an elegant method for the computation of CFT data at LO in the ϵ-expansion**
- The key idea is that all the CFT data must interpolate with that of the `Gaussian` theory in the limit $\epsilon \to 0$.
- Achieve consistency between conformal symmetry and the equations of motion through the use of Schwinger-Dyson equations

RG OPE $\cap$ CFT OPE

- Ising

  Relevant:

$$c^1{}_{23} = 6 - 2\epsilon \qquad c^2{}_{33} = 18 - 15\epsilon$$

  Marginal:

$$c^1{}_{14} = \frac{2}{3}\epsilon$$

- Tricritical

  Relevant:

$$c^1{}_{25} = 6\epsilon \qquad c^1{}_{34} = 24 - \frac{72}{5}\epsilon \qquad c^2{}_{35} = 60 - 90\epsilon$$

$$c^2{}_{44} = 96 - \frac{18}{5}(32 + 3\pi^2)\epsilon \qquad c^3{}_{45} = 240 - 6(98 + 9\pi^2)\epsilon \qquad c^4{}_{55} = 600 - 15(167 + 18\pi^2)\epsilon$$

  Marginal:

$$c^1{}_{16} = \frac{6}{5}\epsilon \qquad c^2{}_{26} = \frac{192}{5}\epsilon$$

- Tetracritical

  Relevant:

$$c^1{}_{27} = \frac{36}{5}\epsilon \qquad c^1{}_{36} = \frac{972}{35}\epsilon \qquad c^1{}_{45} = 120 - \frac{432}{7}\epsilon \qquad \cdots$$

  Marginal:

$$c^1{}_{18} = \frac{72}{35}\epsilon \qquad c^2{}_{28} = \frac{432}{7}\epsilon \qquad c^3{}_{38} = \frac{60696}{35}\epsilon$$

- **OPE coefficients computed with RG and SDE+CFT agree in all cases considered!**

Lee-Yang $[d_c = 6 \; ; \; L_{\mathrm{LO}} = 1]$

- **Only non-trivial single scalar UC in $d = 4$... but is non-unitary** $\lambda_3^* \sim \sqrt{-\epsilon}$
- Beta functionals (LO+NLO)

$$\beta_V = -\frac{1}{6}\frac{(V^{(2)})^3}{(4\pi)^3} - \frac{23}{144}\frac{(V^{(2)})^3(V^{(3)})^2}{(4\pi)^6} \qquad \beta_Z = -\frac{1}{6}\frac{(V^{(3)})^2}{(4\pi)^3} - \frac{13}{216}\frac{(V^{(3)})^4}{(4\pi)^6}$$

- Spectrum (LO+NLO)

$$\gamma_i = \frac{1}{18} i(6i-7)\epsilon - \frac{1}{2916} i(414i^2 - 1371i + 1043)\epsilon^2 - \left(\frac{\epsilon}{3} + \frac{47}{486}\epsilon^2\right)\delta_{i,3}$$

- RG OPE coefficients

$$c^k{}_{ij} = \sqrt{\frac{2}{3}}\sqrt{-\epsilon}\, i(i-1)j(j-1)\delta_{i+j,k+3} + \sqrt{-6\epsilon}\,\delta_{i,3}\delta_{j,3}\delta_{k,3} + \sqrt{\frac{2}{3}}\sqrt{-\epsilon}\,(i\delta_{j,3}\delta_{i,k} + j\delta_{i,3}\delta_{j,k})$$

- First two scheme independent RG OPE coefficients agree with SDE+CFT

$$c^1{}_{2\,2} = -4\sqrt{\frac{2}{3}}\sqrt{-\epsilon} \qquad c^1{}_{1\,3} = \sqrt{\frac{2}{3}}\sqrt{-\epsilon}$$

Tri-Lee-Yang $[d_c = \frac{10}{3}\ ;\ L_{\rm LO} = 3]$

- **New UC never considered in perturbation theory!**
- Together with Ising and Lee-Yang only non-trivial UC in $d = 3$
- Only requires $\epsilon = \frac{1}{3}$ to reach the nearest physical dimensions. Example of well behaved ϵ-expansion?
- Beta functionals (LO)

$$\beta_V = \frac{1}{(4\pi)^5} \frac{\Gamma(\frac{1}{2})^4 \Gamma(\frac{2}{3})}{\Gamma(\frac{4}{3})^2} \left\{ \frac{1}{9} V^{(2)} (V^{(4)})^2 - \frac{1}{2} (V^{(3)})^2 V^{(4)} \right\}$$

$$\beta_Z = -\frac{1}{(4\pi)^5} \frac{3\Gamma(\frac{2}{3})^3}{40} (V^{(5)})^2$$

- Spectrum (LO)

$$\gamma_i = \frac{\epsilon}{153} \left(\frac{52}{5} i - \frac{139}{12} i^2 - \frac{1}{2} i^3 + \frac{19}{12} i^4 - \delta_{i,5} \right) \qquad \frac{\gamma_2}{\gamma_1} = 42 + O(\epsilon)$$

- CFT data at LO agrees with SDE+CFT (these ratios do not depend on $g(\epsilon)$)

$$\frac{{c^1}_{15}}{\sqrt{\gamma_1}} = 4\sqrt{15} + O(\epsilon) \qquad \frac{{c^1}_{24}}{\sqrt{\gamma_1}} = 32\sqrt{15} + O(\epsilon) \qquad \frac{{c^1}_{33}}{\sqrt{\gamma_1}} = -108\sqrt{15} + O(\epsilon)$$

- For more details and numerical estimates (Codello+Safari+Vacca+Zanusso)[arXiv:1706.06887]

Sine-Gordon $[d_c = 2 \; ; \; L_{\rm LO} = 1]$

- $d_c = 2$ corresponds to the Sine-Gordon universality class. To *all orders* the beta functionals are

$$\beta_V = -\frac{1}{4\pi} V'' \qquad\qquad \beta_Z = 0$$

- Flow of the dimensionless potential in $d = 2$ is

$$\beta_v = -2v(\varphi) - \frac{1}{4\pi} v''(\varphi)$$

- Because the field is canonically dimensionless in $d = 2$ fluctuations do not generate a nonzero anomalous dimension
- Interestingly, the fixed point solution of the Sine-Gordon UC can be obtained by direct integration. Using $v''(0) = \sigma$ as boundary condition we obtain

$$v(\varphi) = -\frac{\sigma}{8\pi} \cos(\sqrt{8\pi}\varphi)$$

in which we can recognize the well-known Coleman phase $\sqrt{8\pi}$

Relation with Non-Perturbative RG (NPRG)

- Exact RG flow equation for the generator of the irreducible 1PI diagrams (Wetterich) (Morris) (Polchinski)

$$\partial_t \Gamma_\mu = \frac{1}{2}\mathrm{Tr}\left(\Gamma_\mu^{(2)} + R_\mu\right)\partial_t R_\mu$$

- We can compute the flow of the effective potential (LPA)

$$\beta_V = \partial_t V = c_d \frac{\mu^{d+2}}{\mu^2 + V''} \qquad c_d = \frac{1}{(4\pi)^{d/2}\Gamma(d/2+1)}$$

- Let us expand β_V in powers of V''

$$\beta_V = c_d \left\{ \mu^d - \mu^{d-2} V'' + \mu^{d-4}(V'')^2 - \mu^{d-6}(V'')^3 + \ldots \right\}.$$

- Terms independent of μ correspond to the $\frac{1}{\epsilon}$ poles of dimensional regularization

$$\beta_V = -c_2 V'' = -\frac{1}{4\pi} V'' \qquad d = 2$$

$$\beta_V = c_4 (V'')^2 = \frac{1}{2(4\pi)^2}(V'')^2 \qquad d = 4$$

$$\beta_V = -c_6 (V'')^3 = -\frac{1}{6(4\pi)^3}(V'')^3 \qquad d = 6$$

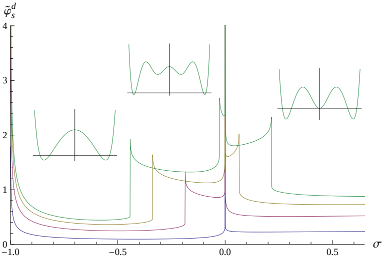
- **We see `Ising`, `Lee-Yang` and `Sine-Gordon` but what about all other with $L_{\mathrm{LO}} \geq 2$?**

Multicritical models in the LPA

- Scaling solutions are given by the ODE

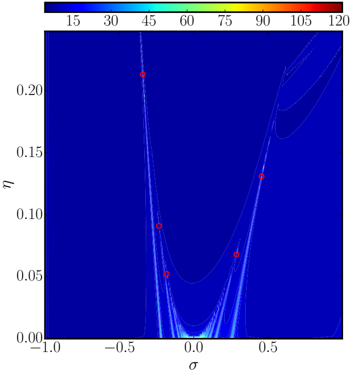
$$0 = -dv + \left(\frac{d}{2} - 1 + \frac{\eta}{2}\right)\varphi v' + c_d \frac{1 - \frac{\eta}{d+2}}{1 + v''}$$

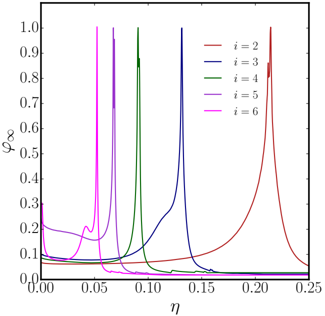
  with initial condition $v'(0) = 0$ and $v''(0) = \sigma$ that encode the $\mathbb{Z}_2$ symmetry
- Spike plot (Morris) (Codello)



Spike plot for unitary models in $2 \leq d < \infty$: `Ising`, `Tricritical`, `Tetracritical`, ...

- `SineGordon`: when $d = 2$ and $\eta = 0$ scaling solutions are periodic functions with Coleman phase $\sqrt{8\pi}$





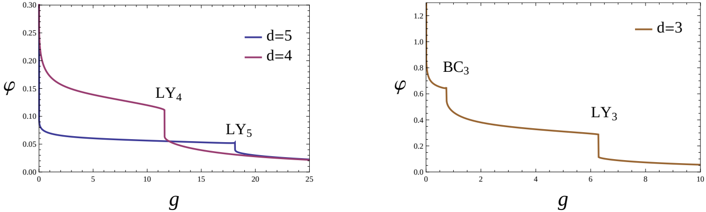
$d = 2$ unitary models are seen using $O(\partial^2)$ derivative expansion (Morris) (Defenu+Codello)[arXiv:1711.01809]

- $\mathcal{PT}$ symmetry $v(\varphi) \to v(-\varphi)^*$
- Scaling solutions for complex potential $v(\varphi) = ih(\varphi)$

$$0 = -dh + \left(\frac{d}{2} - 1 + \frac{\eta}{2}\right)\varphi h' + c_d h'' \frac{1 - \frac{\eta}{d+2}}{1 + (h'')^2}$$

with initial conditions $h(0) = 0$ and $h'''(0) = g$

- Spike plot (Zambelli+Zanusso)



Spike plot for non-unitary models in $d = 5, 4, 3$

- **We see all UCs: the LPA really captures information from every loop order!**

- Multicomponent world (Osborn+Stergiou)

  "hardly a *terra incognita*, nevertheless there is as yet no *mappa mundi*"

- FPRG is the most agile tool for charting of this vast landscape of theories
- N component scalar order parameter

$$S = \int \mathrm{d}^d x \left\{ \frac{1}{2} \partial\phi_i \cdot \partial\phi_i + V(\phi_1, ..., \phi_N) \right\}$$

- **The magic of the functional constraint: (almost) no new computation needed!**
- The universal LO and NLO coefficients are the $N = 1$ in $d_c = 4, 3, \frac{8}{3}, ...$ (unitary family)

$$(V^{(2)})^2 \to V_{ij} V_{ij} \qquad V^{(2)}(V^{(3)})^2 \to V_{ij} V_{iab} V_{jab}$$

- The universal LO coefficients are the $N = 1$ in $d_c = 6, \frac{10}{3}, ...$ (non-unitary family). For the NLO coefficients we have only a constraint

$$(V^{(2)})^3 (V^{(3)})^2 \to \alpha V_{ik} V_{kl} V_{lj} V_{iab} V_{jab} + \beta V_{ik} V_{kj} V_{lm} V_{ija} V_{lmb} + \gamma V_{ia} V_{jb} V_{kc} V_{abc} V_{ijk}$$

  with $\alpha + \beta + \gamma = 1$

- **RG information and group information factorize at LO (and at NLO for unitary family)!**

- Maximal symmetry of kinetic term and one invariant $\rho = \frac{1}{2}\varphi_i\varphi_i$

$$S = \int \mathrm{d}^d x \left\{ \frac{1}{2}\partial\phi_i \cdot \partial\phi_i + U(\rho) \right\}$$

- LO and NLO beta functional in $d_c = 4$ are

$$\beta_V = \frac{1}{2}\frac{1}{(4\pi)^2} V_{ij}V_{ij} - \frac{1}{2}\frac{1}{(4\pi)^2} V_{ij}V_{iab}V_{jab}$$

- The only invariant is $\rho = \frac{1}{2}\varphi_i\varphi_i$ and the potential is a function of it $U(\rho) \equiv V(\varphi_i)$
- The LO and NLO beta functional is

$$\begin{aligned}\beta_U &= (N-1)\left\{ \frac{1}{2}(U')^2 - 3\rho(U'')^2 \left(U' + \frac{2}{3}\rho U'' \right) \right\} \\ &\quad + \frac{1}{2}\left(U' + 2\rho U''\right)^2 - 9\rho\left(U' + 2\rho U''\right)\left(U'' + \frac{2}{3}\rho U''' \right)^2\end{aligned}$$

with $\eta = \frac{1}{6N} V_{ijkl}V_{ijkl} = 3N(N+2)(U'')^2 + \ldots$

- Reproduces all O(N) universal beta functions and critical exponents at LO and NLO
- LO β_U agrees with expansion of $O(N)$ LPA

Tricritical-O(N) $[d_c = 3\ ;\ L_{\rm LO} = 2]$

- General beta functional in $d_c = 3$ at LO is

$$\beta_V = \frac{1}{3}\frac{1}{(4\pi)^3} V_{ijk} V_{ijk} + \ldots$$

The Triciritcal-O(N) LO beta functional turns out to be

$$\beta_U = 2(N-1)\rho(U'')^2 + 6\rho\left(U'' + \frac{2}{3}\rho U'''\right)^2 + \ldots$$

- $O(N)$ LPA is able to "see" all multicritical scaling solutions (Codello+D'Odorico+Defenu)
- Non trivial large N limit (Osborn+Stergiou) (Delamotte+Shunsuke) (Katsis+Tetradis)
- For a way to construct $d_c = 6$ $O(N)$ models see (Klebanov+Giombi) (Percacci+Vacca)
- Higher multicritical $O(N)$ models $d_c = \frac{8}{3}, \ldots$ are all analyzable by FPRG and NPRG (Codello+Delamotte+Defenu+Shunsuke in preparation)

- Permutation group S_{N+1} (symmetry of hyper-tetrahedron) in $d_c = 6$

$$S = \int \mathrm{d}^d x \left\{ \frac{1}{2} \partial\phi_i \cdot \partial\phi_i + U(\rho, \tau, \sigma, ...) \right\}$$

- **Random Cluster Model**: $\texttt{Potts}_1 = \texttt{Percolation}$ and $\texttt{Potts}_0 = \texttt{SpanningForest}$
- Number of invariants $= N \Rightarrow$ functional analysis possible only at fixed N
- General LO beta functional in $d_c = 6$

$$\beta_V = -\frac{1}{3} \frac{1}{(4\pi)^3} V_{ij} V_{jk} V_{ki} + ... \qquad \beta_{Z_{ij}} = -\frac{1}{6} \frac{1}{(4\pi)^3} V_{iab} V_{jab} + ...$$

- $N = 2$ LO beta functional ($\texttt{Potts}_3$) is ($\rho = \phi_1^2 + \phi_2^2$, $\tau = \frac{3}{\sqrt{2}} \phi_2 (\phi_2^2 - 3\phi_1^2)$)

$$\begin{aligned} \beta_U \quad = \quad & 9 \left(3U_{\tau\tau} \left(2 \left(\rho^3 - 6\tau^2 \right) U_{\rho\rho} - 3\rho\tau U_\tau \right) - 6 \left(\rho^3 - 6\tau^2 \right) U_{\rho\tau}^2 \right. \\ & + U_\rho \left(3\rho^2 U_{\tau\tau} + 8\rho U_{\rho\rho} + 24\tau U_{\rho\tau} \right) - 12 U_\tau \left(\rho^2 U_{\rho\tau} + 2\tau U_{\rho\rho} \right) - 6\rho U_\tau^2 + 4U_\rho^2 \Big) \\ & - \frac{3}{4} \left(3\rho^2 U_{\tau\tau} + 8 \left(\rho U_{\rho\rho} + U_\rho + 3\tau U_{\rho\tau} \right) \right)^2 \end{aligned}$$

- Reproduces all $\texttt{Potts}_{N+1}$ universal beta functions and critical exponents at LO and NLO
- LO β_U agrees with expansion of LPA analysis (Ben Ali Zinati+Codello)[arXiv:1707.03410]

- Permutation group S_{N+1} in $d_c = \frac{10}{3}$ (Codello+Safari+Vacca+Zanusso in preparation)

$$S = \int \mathrm{d}^d x \left\{ \frac{1}{2}\partial\phi_i \cdot \partial\phi_i + \frac{1}{5!}\left(\lambda_{5,1}\delta_{(i_1 i_2}Q^{(3)}_{i_3 i_4 i_5)} + \lambda_{5,2}Q^{(5)}_{i_1 i_2 i_3 i_4 i_5}\right)\phi_{i_1}\dots\phi_{i_5}\right\}$$

arbitrary N invariants $Q^{(k)}_{i_1\cdots i_k} = \sum_\alpha e^\alpha_{i_1}\cdots e^\alpha_{i_k}$

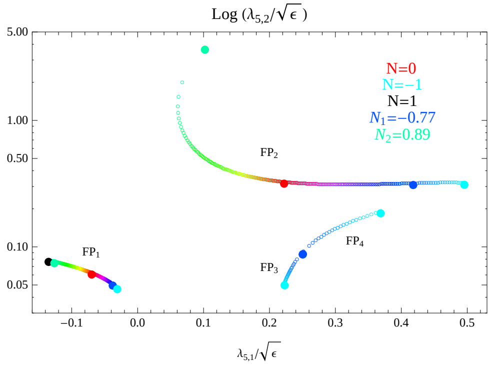
- General LO beta functionals are

$$\beta_V = \frac{1}{3}V_{ijkl}V_{ijkm}V_{lm} - \frac{3}{2}V_{ijk}V_{ilm}V_{jklm} \qquad\qquad \beta_{Z_{ij}} = -\frac{1}{30}V_{iklmn}V_{jklmn}$$

- Universal beta functions

$$\begin{aligned}
\beta_{5,1} &= -\frac{3\epsilon}{2}\lambda_{5,1} - \frac{3}{200}(17N^2+833N-3510)\lambda_{5,1}^3 + \frac{10}{3}(2N^2+25N-25)\lambda_{5,2}^3 \\
&\quad -\frac{1}{5}(97N^2-532N+432)\lambda_{5,1}^2\lambda_{5,2} - \frac{1}{4}(25N^3-181N^2+277N-673)\lambda_{5,1}\lambda_{5,2}^2 \\
\beta_{5,2} &= -\frac{3\epsilon}{2}\lambda_{5,2} - \frac{3}{25}(5N+144)\lambda_{5,1}^3 - \frac{1}{12}(459N^3-919N^2+1459N-919)\lambda_{5,2}^3 \\
&\quad -\frac{1}{40}(25N^2+3533N-4038)\lambda_{5,1}^2\lambda_{5,2} - \frac{1}{2}(217N^2-373N+300)\lambda_{5,1}\lambda_{5,2}^2
\end{aligned}$$

Multiciritical Potts models: preliminary results



- No real fixed point in the $N \geq 2$. Consistent with the annihilation scenario
- Multicritical fixed points for $N = 1$ (multicriticial `Percolation`) and $N = 0$ (multicritical `SpanningForest`)
- $N = 1$ is probably related to `CorrelatedPercolation` (percolation of Ising model clusters)
- NPRG analysis is work in progress

Platonic Field Theory

	Polytope	Schläfli	$\mathcal{G}$
N=2	n-Polygon	$\{n\}$	$\mathbb{D}_n$
N=3	Tetrahedron	$\{3,3\}$	S_4
	Octahedron	$\{3,4\}$	$S_4 \times \mathbb{Z}_2$
	Cube	$\{4,3\}$	$S_4 \times \mathbb{Z}_2$
	Icosahedron	$\{3,5\}$	$A_5 \times \mathbb{Z}_2$
	Dodecahedron	$\{5,3\}$	$A_5 \times \mathbb{Z}_2$
N=4	5-cell	$\{3,3,3\}$	S_5
	16-cell	$\{3,3,4\}$	$(\mathbb{Z}_2)^4 \rtimes S_4$
	8-cell	$\{4,3,3\}$	$(\mathbb{Z}_2)^4 \rtimes S_4$
	24-cell	$\{3,4,3\}$	F_4
	120-cell	$\{3,3,5\}$	H_4
	600-cell	$\{5,3,3\}$	H_4

Platonic solids and their symmetry groups

- Explore scalar QFTs with the internal symmetries of Polygons ($N = 2$), Platonic solids ($N = 3$) and Hyper-Platonic solids ($N = 4$)
(Ben Ali Zinati+Codello+Gori)[arXiv:1902.05328]
- New UC in $d = 3$ with good critical exponents: Pentagon. Simulation is progress!
- Many PFTs with $d_c < 3$ can lead to new (non-rational) CFTs in $d = 2$

Platonic Field Theory

	Polytope		d_c	Fixed Points
N=2		Triangle	6	$\texttt{Potts}_3$
		Square	4	2×Ising, O(2)
		Pentagon	10/3	Pentagon
		Hexagon	3	Tri-O(2)
		Heptagon	14/5	Heptagon
		Octagon	8/3	Tetra-O(2)
	⋮	⋮	⋮	⋮
N=3		Tetrahedron	6	No real FP
			4	3×Ising, O(3), $\texttt{Cubic}_3$
		Octahedron	4	3×Ising, O(3), $\texttt{Cubic}_3$
			3	3×Tri-Ising, Tri-O(3), ϕ^6-$\texttt{Cubic}_3$
		Icosahedron	3	Tri-O(3)
			8/3	Tetra-O(3)
			5/2	Penta-O(3), $\texttt{Ico}_{1\leq i\leq 2}$
N=4		5-cell	6	No real FP
			4	O(4), $\texttt{Quartic-Potts}_5$
			10/3	No real FP
		16-cell	4	4×Ising, O(4)
			3	4×Tri-Ising, Tri-O(4), ϕ^6-$\texttt{Cubic}_4$
			8/3	4×Tetra-Ising, Tetra-O(4), ϕ^8-$\texttt{Cubic}_4$
		24-cell	3	Tri-O(4)
			8/3	Tetra-O(4)
			5/2	Penta-O(4), $\texttt{24-cell}_1$
			12/5	Hexa-O(4), $\texttt{24-cell}_{1\leq i\leq 2}$
		600-cell	12/5	Hexa-O(4)
			⋮	⋮
			20/9	Deca-O(4)
			⋮	⋮
			15/7	Triaconta-O(4)

	Universality Class	d_c	η	ν
$N=1$	Ising	4	$\frac{1}{54}\epsilon^2$	$\frac{1}{2}+\frac{1}{12}\epsilon+\frac{7}{162}\epsilon^2$
	Tri-Ising	3	$\frac{1}{500}\epsilon^2$	$\frac{1}{2}+\frac{1}{125}\epsilon^2$
	Tetra-Ising	$\frac{8}{3}$	$\frac{9}{85750}\epsilon^2$	$\frac{1}{2}+\frac{27}{68600}\epsilon^2$
$N=2$	$\texttt{Potts}_3$	6	$\frac{1}{3}\epsilon$	$\frac{1}{2}-\frac{5}{12}\epsilon$
	O(2)	4	$\frac{1}{50}\epsilon^2$	$\frac{1}{2}+\frac{1}{10}\epsilon+\frac{11}{200}\epsilon^2$
	Pentagon	$\frac{10}{3}$	$\frac{3}{5}\epsilon$	$\frac{1}{2}+\frac{3}{20}\epsilon$
	Tri-O(2)	3	$\frac{1}{392}\epsilon^2$	$\frac{1}{2}+\frac{1}{98}\epsilon^2$
	Heptagon	$\frac{14}{5}$	$\frac{10}{7}\epsilon$	$\frac{1}{2}+\frac{5}{14}\epsilon$
	Tetra-O(2)	$\frac{8}{3}$	$\frac{9}{59858}\epsilon^2$	$\frac{1}{2}+\frac{135}{239432}\epsilon^2$
$N=3$	O(3)	4	$\frac{5}{242}\epsilon^2$	$\frac{1}{2}+\frac{5}{44}\epsilon+\frac{345}{5324}\epsilon^2$
	$\texttt{Cubic}_3$	4	$\frac{5}{243}\epsilon^2$	$\frac{1}{2}+\frac{1}{9}\epsilon+\frac{599}{8748}\epsilon^2$
	Tri-O(3)	3	$\frac{35}{11532}\epsilon^2$	$\frac{1}{2}+\frac{35}{2883}\epsilon^2$
	ϕ^6-$\texttt{Cubic}_3$	3	$0.00261529\,\epsilon^2$	$\frac{1}{2}+0.0104612\,\epsilon^2$
	Tetra-O(3)	$\frac{8}{3}$	$\frac{945}{4798802}\epsilon^2$	$\frac{1}{2}+\frac{14175}{19195208}\epsilon^2$
$N=4$	O(4)	4	$\frac{1}{48}\epsilon^2$	$\frac{1}{2}+\frac{1}{8}\epsilon+\frac{7}{96}\epsilon^2$
	$\texttt{Quartic-Potts}_5$	4	$\frac{55}{2646}\epsilon^2$	$\frac{1}{2}+\frac{5}{42}\epsilon+\frac{22465}{222264}\epsilon^2$
	Tri-O(4)	3	$\frac{1}{289}\epsilon^2$	$\frac{1}{2}+\frac{4}{289}\epsilon^2$
	ϕ^6-$\texttt{Cubic}_4$	3	$0.00322216\,\epsilon^2$	$\frac{1}{2}+0.0128886\,\epsilon^2$
	Tetra-O(4)	$\frac{8}{3}$	$\frac{9}{36980}\epsilon^2$	$\frac{1}{2}+\frac{27}{29584}\epsilon^2$
	ϕ^8-$\texttt{Cubic}_4$	$\frac{8}{3}$	$0.000196765\,\epsilon^2$	$\frac{1}{2}+0.000737867\,\epsilon^2$

Fixed points and critical exponents of PFTs

Towards a classification of UCs for general N

- **We have all the beta functions for multicomponent scalars in $d = 3$ and $d = 4$**
- Within the ϵ-expansion **LO analysis is enough to classify fixed points** (Osborn+Stergiou)
- **Example: $N = 2$ in $d_c = 4$** the most general potential has 5 marginal couplings

$$V(\phi_1, \phi_2) = \lambda_1 \phi_1^4 + \lambda_2 \phi_2 \phi_1^3 + \lambda_3 \phi_2^2 \phi_1^2 + \lambda_4 \phi_2^3 \phi_1 + \lambda_5 \phi_2^4$$

- From the LO beta functional in $d_c = 4$

$$\beta_V = \frac{1}{2} V_{ij} V_{ij}$$

We derive the LO beta functions

$$\begin{aligned}
\beta_1 &= -\epsilon\lambda_1 + 72\lambda_1^2 + 9\lambda_2^2 + 2\lambda_3^2 \\
\beta_2 &= -\epsilon\lambda_2 + 72\lambda_1\lambda_2 + 24\lambda_3\lambda_2 + 12\lambda_3\lambda_4 \\
\beta_3 &= -\epsilon\lambda_3 + 18\lambda_2^2 + 18\lambda_4\lambda_2 + 16\lambda_3^2 + 18\lambda_4^2 + 24\lambda_1\lambda_3 + 24\lambda_3\lambda_5 \\
\beta_4 &= -\epsilon\lambda_4 + 12\lambda_2\lambda_3 + 24\lambda_4\lambda_3 + 72\lambda_4\lambda_5 \\
\beta_5 &= -\epsilon\lambda_5 + 2\lambda_3^2 + 9\lambda_4^2 + 72\lambda_5^2
\end{aligned}$$

General $N = 2$ case in $d_c = 4$

- The Landau-Ginzburg potentials turn out to be ($\epsilon \times \ldots$)

0	Gauss
$\frac{1}{72}\phi_2^4$	Ising $\times$ Gauss
$\frac{1}{288}\left(\phi_1 - \phi_2\right)^4$	Ising $\times$ Gauss
$\frac{1}{288}\left(\phi_1 + \phi_2\right)^4$	Ising $\times$ Gauss
$\frac{1}{144}\left(\phi_1^4 + 6\phi_2^2\phi_1^2 + \phi_2^4\right)$	Ising $\times$ Ising
$\frac{1}{288}\left(3\phi_1^2 + 2\phi_2\phi_1 + \phi_2^2\right)\left(\phi_1^2 - 2\phi_2\phi_1 + 3\phi_2^2\right)$	Ising $\times$ Ising
$\frac{1}{288}\left(3\phi_1^2 - 2\phi_2\phi_1 + \phi_2^2\right)\left(\phi_1^2 + 2\phi_2\phi_1 + 3\phi_2^2\right)$	Ising $\times$ Ising
$\frac{1}{80}\left(\phi_1^2 + \phi_2^2\right)^2$	O(2)
$\frac{1}{72}\phi_1^4$	Ising $\times$ Gauss
$\frac{1}{72}\left(\phi_1^4 + \phi_2^4\right)$	Ising $\times$ Ising

- Field redefinitions remove redundancy (Osborn+Stergiou) ($\rightarrow$ Safari's talk)
- Same strategy can be followed for all d_c, in particular for those that lead to non-trivial UCs in $d = 3$ ($d_c = 6, 4, \frac{10}{3}$) and in $d = 4$ ($d_c = 6$)
- $\Rightarrow$ **we can systematically span all physically relevant UCs**
- Classification of UCs within the ϵ-expansion is reduced to group theory problem

Conclusion and Outlook

- **FPRG**: functional constraint very efficient in re-organizing perturbative RG
- Allows the computation of *some* CFT data in the ϵ-expansion
- General setting for the classification of universality classes in the ϵ-expansion
- **Question I**: *Can we classify all* $N = 3, 4, \ldots$ *universality classes in* $d = 3$*?*
- **Generalization I**: Add fermions and gauge fields [Yukawa: (Jena Group) (Herbut's Group)]
- **Generalization II**: Higher derivative theories $S = \int \frac{1}{2}\phi\Box^k\phi + V(\phi)$ (Vacca+Safari)
- **Question II**: *Are there non-trivial unitary fixed points in* $d = 4$*?*
- **Generalization III**: Quantum Gravity in the ϵ-expansion (ϵQG) in $d_c = 2, 4, 6, \ldots$
- **Question III**: *Asymptotic Safety in* $d = 4$ *by "pulling down" the* $d_c = 6$ *theory?*