Conformal Bootstrap and Continuous Phase Transitions in 3D

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CFTs

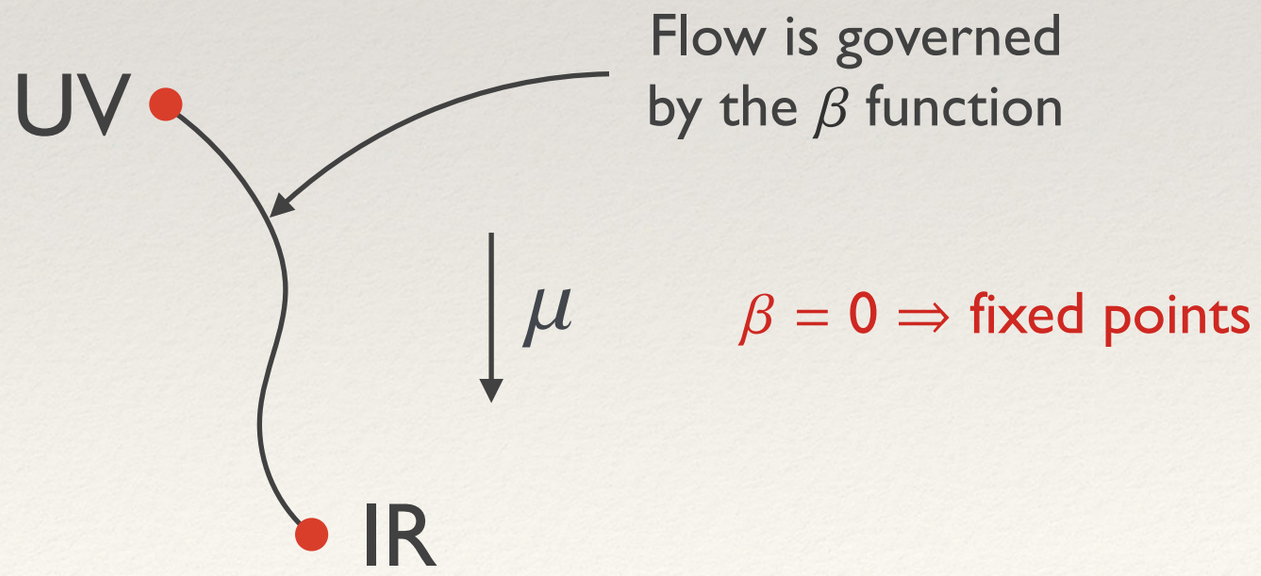
QFT provides an immensely powerful framework for the description of the subatomic world, condensed matter systems, etc.

CFTs appear in the study of

- Renormalization group
- AdS/CFT and thus gravity
- String theory
- Critical phenomena

How do we study conformal field theories (CFTs)?

One way is to view CFTs as endpoints of renormalization group flows.



Conformal Bootstrap

Bootstrap: Solve a theory simply by imposing self-consistency conditions/constraints!

First suggested in the 1960s by Chew for the S-matrix, and by Polyakov in the 1970s for CFTs.

The first successful numerical implementation of the method for CFTs appeared in 2008. (Rattazzi, Rychkov, Tonni & Vichi)

The numerical conformal bootstrap:

- Is non-perturbative.
- Is not specific to any theory (does not need a Lagrangian).
- Can be used in any spacetime dimension.
- Exploits the power of conformal symmetry.
- Has errors that are under control.

3D Ising Model

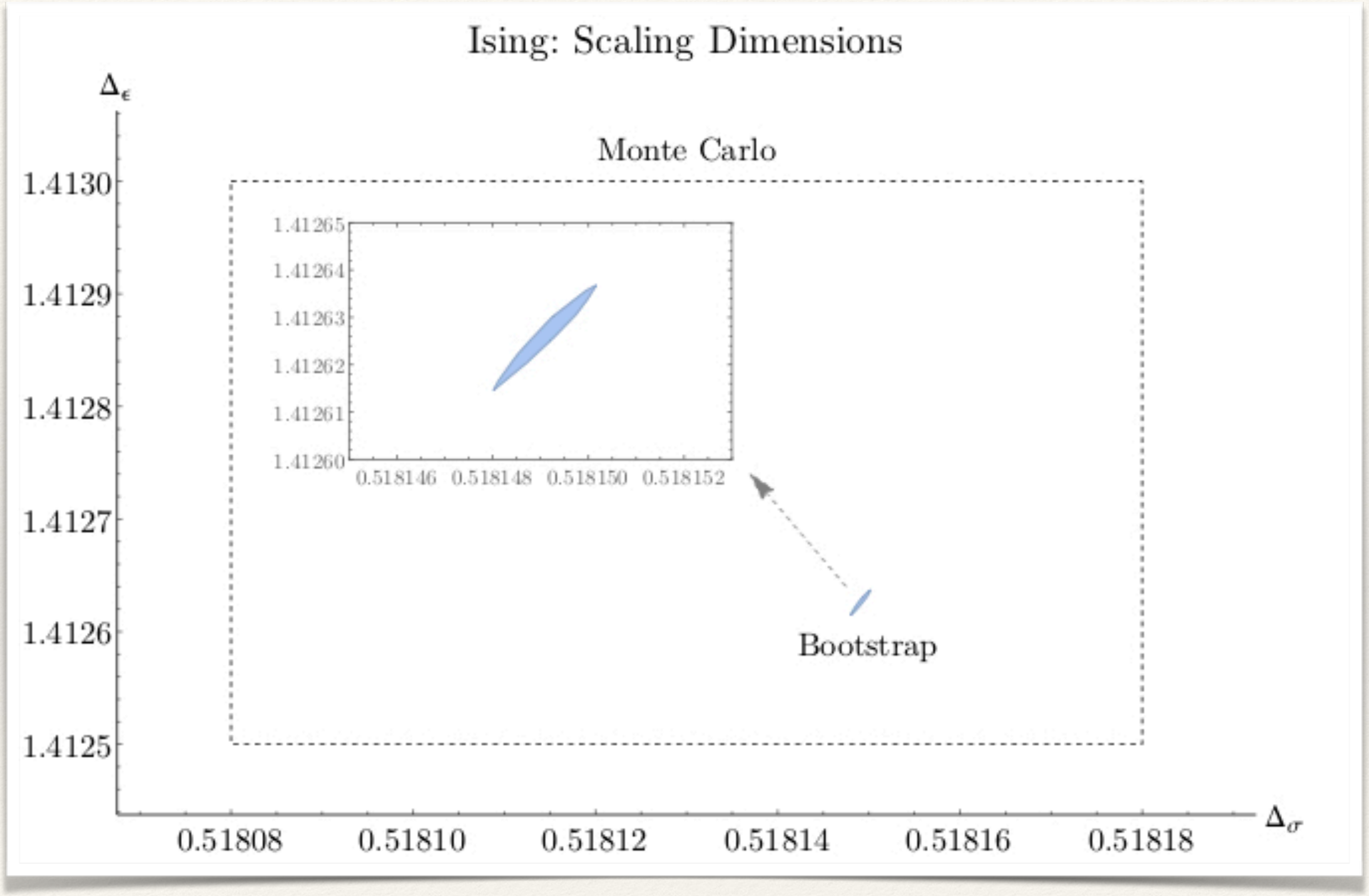
For the 3D Ising model everyone agrees:

- ε expansion
- FRG
- Monte Carlo
- …
- Experiments

Any new computational approach to critical phenomena better reproduce the Ising results.

The conformal bootstrap reproduces these results and in fact has succeeded in giving the most precise determination of the critical exponents to date!

3d Ising Model



(Kos, Poland, Simmons-Duffin & Vichi, 2016)

Outline

The bootstrap philosophy

The cubic bootstrap

Structural phase transitions

The MN bootstrap

Summary

Operators in CFTs

In CFTs operators can be grouped into primaries and descendants:

$$K_\mu(O(0)) = 0 \qquad \text{or} \qquad K_\mu(O(0)) \neq 0.$$

Descendants are derivatives of primaries.

Correlation functions of primary operators are severely constrained:

$$\langle O(x)O(0)\rangle = \frac{c_O}{(x^2)^{\Delta_O}}, \qquad \langle O_1(x)O_2(0)\rangle = 0,$$

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)\rangle = \frac{C_{123}}{({x_{12}}^2)^{\frac{1}{2}(\Delta_1+\Delta_2-\Delta_3)}({x_{23}}^2)^{\frac{1}{2}(\Delta_2+\Delta_3-\Delta_1)}({x_{13}}^2)^{\frac{1}{2}(\Delta_1+\Delta_3-\Delta_2)}}.$$

Four-point Functions in CFTs

These constraints arise from the fact that with two or three points in space one cannot write down any conformally invariant quantities.

This is not so with four points:

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \qquad \text{and} \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}.$$

The four-point function of primary operators in CFTs is given in terms of an arbitrary function of u and v:

$$\langle O_1(x_1) O_2(x_2) O_3(x_3) O_4(x_4) \rangle = \left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{1}{2}(\Delta_1 - \Delta_2)} \left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{1}{2}(\Delta_3 - \Delta_4)} \frac{g(u,v)}{(x_{12}^2)^{\frac{1}{2}(\Delta_1+\Delta_2)} (x_{34}^2)^{\frac{1}{2}(\Delta_3+\Delta_4)}}.$$

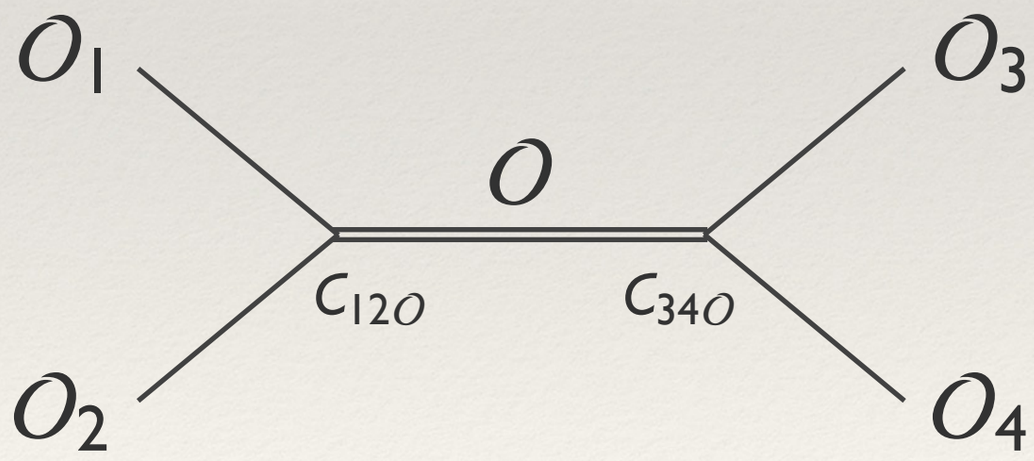
Conformal Blocks

In remarkable work Dolan and Osborn computed this function $g(u, v)$ for scalar external operators in 2001.

Using the operator product expansion (OPE),

$$\mathcal{O}_1(x) \times \mathcal{O}_2(0) = \sum_{\mathcal{O}} \frac{C_{12\mathcal{O}}}{(x^2)^{\frac{1}{2}(\Delta_1+\Delta_2-\Delta_{\mathcal{O}})}} \mathcal{O}(0)$$

one can view the four point function as a sum over contributions of the form



The exchanged operators can be primaries or descendants.

Conformal Blocks

Dolan and Osborn managed to resum the contributions of the descendants of any given primary, and determine the dependence of the four-point function on u and v.

As a result,

Conformal blocks

$$g(u,v) = \sum_{\mathcal{O}} C_{12\mathcal{O}} C_{34\mathcal{O}}\, g_{\mathcal{O}}(u,v).$$

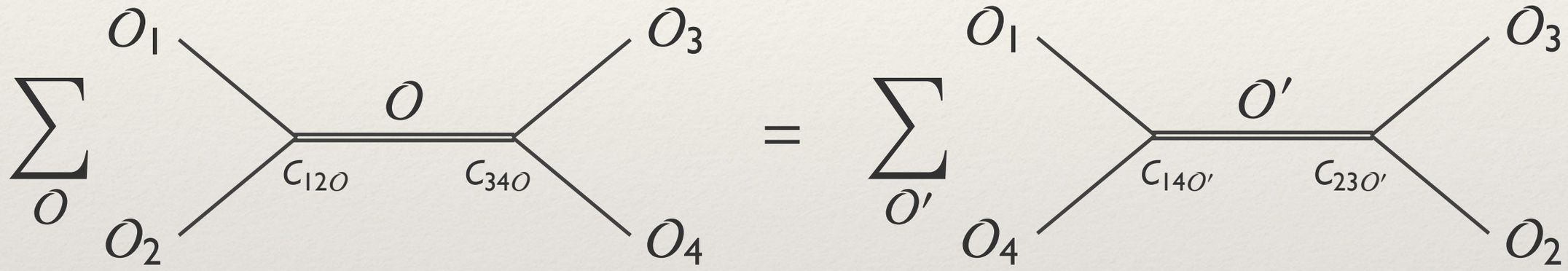
The conformal blocks depend on the scaling dimension and the spin of the exchanged primary operators.

The four-point function follows from knowing all possible exchanged primary operators.

Crossing Symmetry

We are now ready to set up the constraint used in the conformal bootstrap program.

It must be that



$$(\text{kin. fac.}) \times \sum_{\mathcal{O}} C_{12\mathcal{O}} C_{34\mathcal{O}}\, g_{\mathcal{O}}(u,v) = (\text{kin. fac.}) \times \sum_{\mathcal{O}'} C_{14\mathcal{O}'} C_{23\mathcal{O}'}\, g_{\mathcal{O}'}(v,u)$$

This simple constraint provides the basis of all bootstrap analyses.

How To Extract Information

Let's take the simple case of a CFT containing a scalar operator ϕ.

Its four-point function can be expressed in two ways:

$$\langle \overbrace{\phi(x_1)\phi(x_2)}\,\overbrace{\phi(x_3)\phi(x_4)}\rangle = \frac{1}{(x_{12}^2 x_{34}^2)^{\Delta_\phi}} \sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}} g_{\mathcal{O}}(u,v),$$

$$\langle \overbrace{\phi(x_1)\,\overbrace{\phi(x_2)}\,\phi(x_3)}\phi(x_4)\rangle = \frac{1}{(x_{14}^2 x_{23}^2)^{\Delta_\phi}} \sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}} g_{\mathcal{O}}(v,u).$$

Equality of the right-hand sides gives

$$\sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}} \mathcal{F}_{\mathcal{O}}(u,v) = 0, \quad \mathcal{F}_{\mathcal{O}}(u,v) = u^{-\Delta_\phi} g_{\mathcal{O}}(u,v) - v^{-\Delta_\phi} g_{\mathcal{O}}(v,u).$$

How To Extract Information: OPE Coefficients

$$\sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}} \mathcal{F}_{\mathcal{O}}(u,v) = 0, \quad \mathcal{F}_{\mathcal{O}}(u,v) = u^{-\Delta_\phi} g_{\mathcal{O}}(u,v) - v^{-\Delta_\phi} g_{\mathcal{O}}(v,u).$$

If our theory is unitary then

$$C^2_{\phi\phi\mathcal{O}} > 0.$$

Let us separate the contributions of the identity operator and another operator $\mathcal{O}_0$ of interest:

$$C^2_{\phi\phi\mathcal{O}_0} \mathcal{F}_{\mathcal{O}_0} = -\mathcal{F}_1 - \sum_{\mathcal{O}\neq 1,\mathcal{O}_0} C^2_{\phi\phi\mathcal{O}} \mathcal{F}_{\mathcal{O}}.$$

We can act on this equation with a linear functional α:

$$C^2_{\phi\phi\mathcal{O}_0} \alpha(\mathcal{F}_{\mathcal{O}_0}) = -\alpha(\mathcal{F}_1) - \sum_{\mathcal{O}\neq 1,\mathcal{O}_0} C^2_{\phi\phi\mathcal{O}} \alpha(\mathcal{F}_{\mathcal{O}}).$$

How To Extract Information: OPE Coefficients

$$C^2_{\phi\phi\mathcal{O}_0}\alpha(\mathcal{F}_{\mathcal{O}_0}) = -\alpha(\mathcal{F}_1) - \sum_{\mathcal{O}\neq 1,\mathcal{O}_0} C^2_{\phi\phi\mathcal{O}}\alpha(\mathcal{F}_{\mathcal{O}}).$$

We can now demand

$$\alpha(\mathcal{F}_{\mathcal{O}_0}) = 1 \qquad \text{and} \qquad \alpha(\mathcal{F}_{\mathcal{O}}) \geq 0.$$

Then,

$$C^2_{\phi\phi\mathcal{O}_0} = -\alpha(\mathcal{F}_1) - \sum_{\mathcal{O}\neq 1,\mathcal{O}_0} (\text{positive} \times \text{positive}) \leq -\alpha(\mathcal{F}_1).$$

If we now minimize $-\alpha(\mathcal{F}_1)$ we have an upper bound on $C_{\phi\phi\mathcal{O}_0}$!

This is an optimization problem, solved numerically on the computer.

With this method we can obtain rigorous bounds on the interaction strength of CFTs!

More Information: Operator Dimensions

There is another way to write the crossing constraint:

$$-\mathcal{F}_{\mathbb{1}} = \sum_{\mathcal{O}\neq\mathbb{1}} C^2_{\phi\phi\mathcal{O}}\mathcal{F}_{\mathcal{O}}.$$

If we act with α as before,

$$-\alpha(\mathcal{F}_{\mathbb{1}}) = \sum_{\mathcal{O}\neq\mathbb{1}} C^2_{\phi\phi\mathcal{O}}\alpha(\mathcal{F}_{\mathcal{O}}),$$

make some assumptions about one or more of the $\mathcal{O}$'s and demand

$$\alpha(\mathcal{F}_{\mathbb{1}}) = 1 \qquad \text{and} \qquad \alpha(\mathcal{F}_{\mathcal{O}}) \geq 0,$$

then if such functional α exists we have a contradiction:

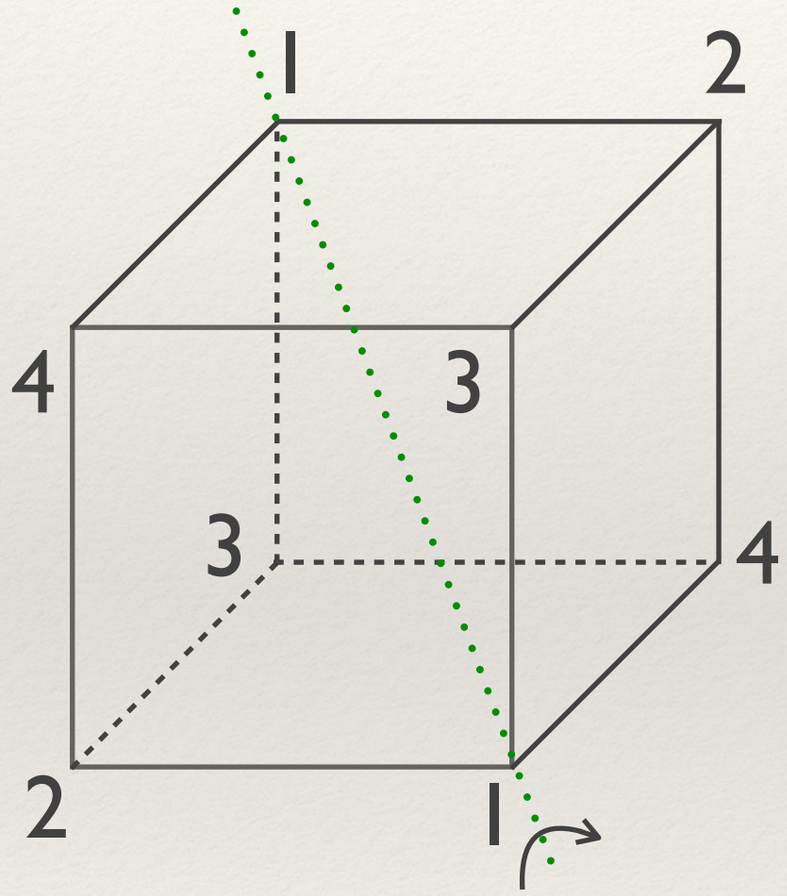
$$-1 \overset{!}{=} \text{positive}.$$

This allows us to exclude CFTs with certain operator spectra.

Cubic CFTs in 3D

Cubic CFTs have a discrete global symmetry,

$$O_h = \mathbb{Z}_2^3 \rtimes S_3 \simeq S_4 \times \mathbb{Z}_2 \,.$$



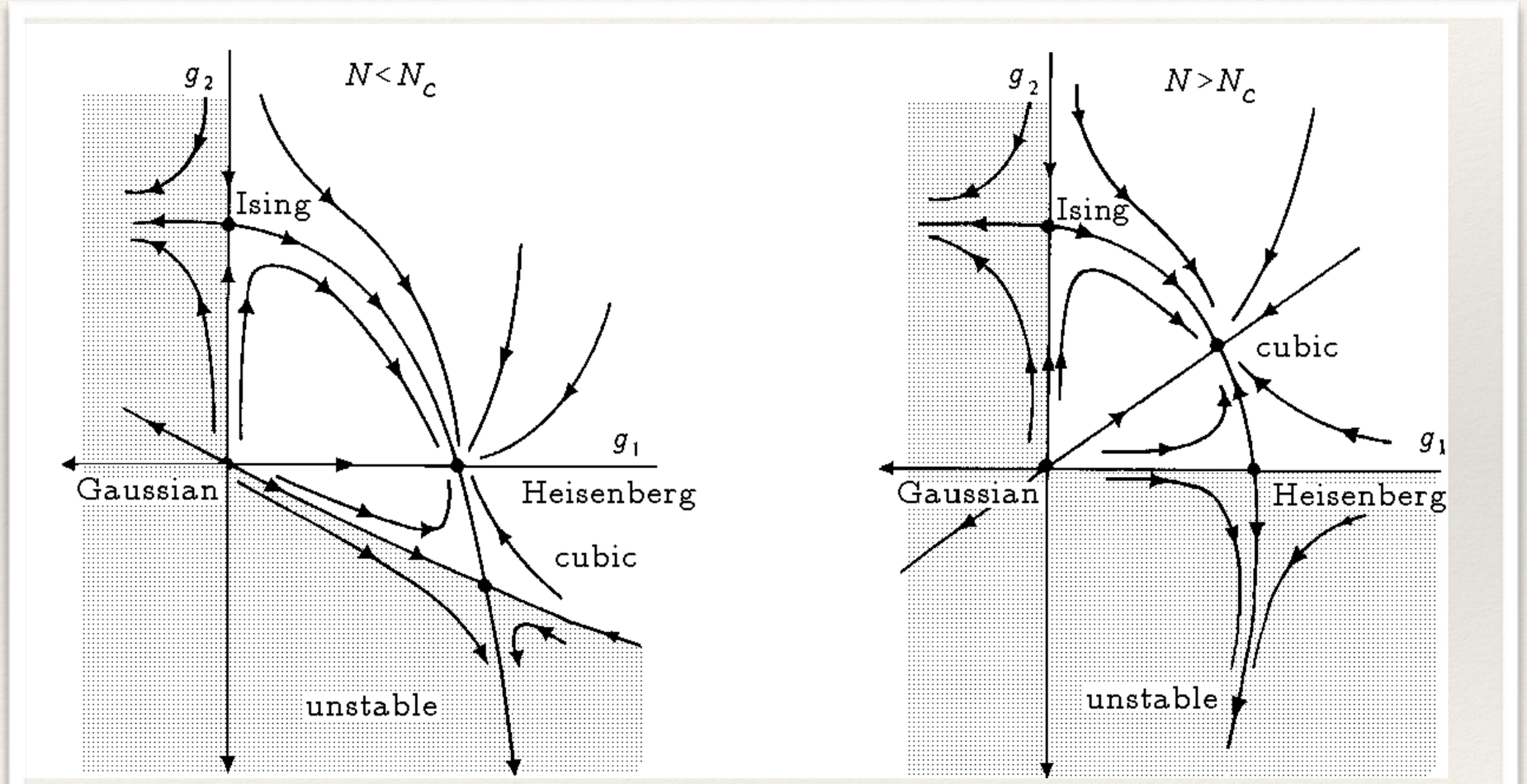
$$\vec{\phi} = (\phi_1, \phi_2, \phi_3)$$

$$V = \frac{1}{8}\lambda(\phi^2)^2 + \frac{1}{24}g\sum_{i=1}^{3}\phi_i^4$$

Relevant whenever we have cubic lattices, e.g. cubic magnets like Fe or Ni.

Cubic CFTs in 3d

Using the ε expansion one can find a fixed point with cubic symmetry, but the critical exponents are almost identical to those of the Heisenberg model.



(Kleinert & Schulte-Frohlinde, 1995)

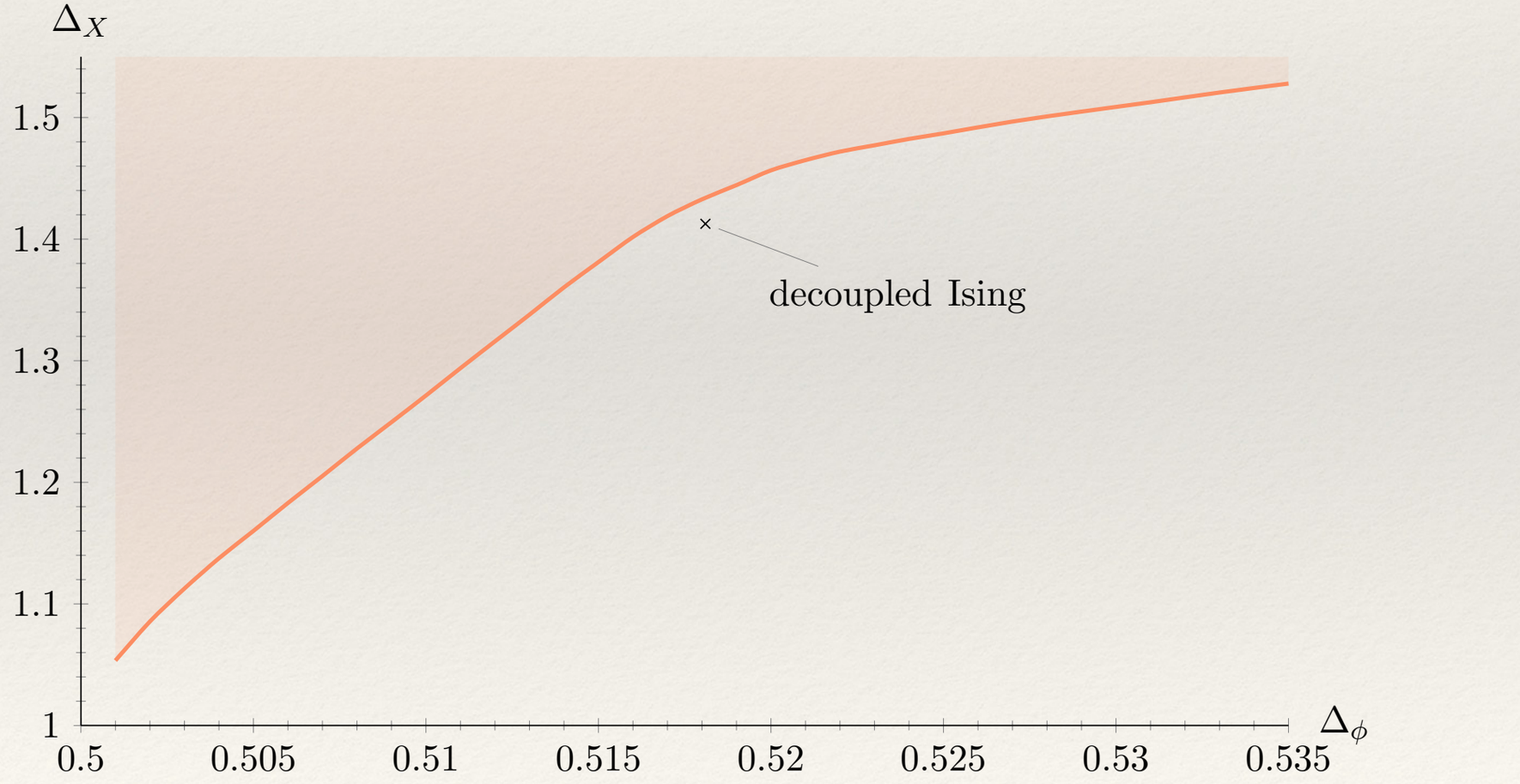
Bootstrapping Cubic CFTs in 3d

We will look at the four-point function

$$\langle \phi_i(x_1)\phi_j(x_2)\phi_k(x_3)\phi_l(x_4)\rangle .$$

The OPE follows from group theory:

$$\phi_i \times \phi_j \sim \delta_{ij} S + X_{(ij)} + Y_{(ij)} + A_{[ij]}$$



(AS, 2018)

Bootstrapping Cubic CFTs in 3d

Let us now consider the system of four-point functions

$$\langle \phi_i(x_1)\phi_j(x_2)\phi_k(x_3)\phi_l(x_4)\rangle,$$
$$\langle \phi_i(x_1)X_{jk}(x_2)\phi_l(x_3)X_{mn}(x_4)\rangle,$$
$$\langle X_{ij}(x_1)X_{kl}(x_2)X_{mn}(x_3)X_{pq}(x_4)\rangle.$$

We will assume that the dimension of X lies on the bound.

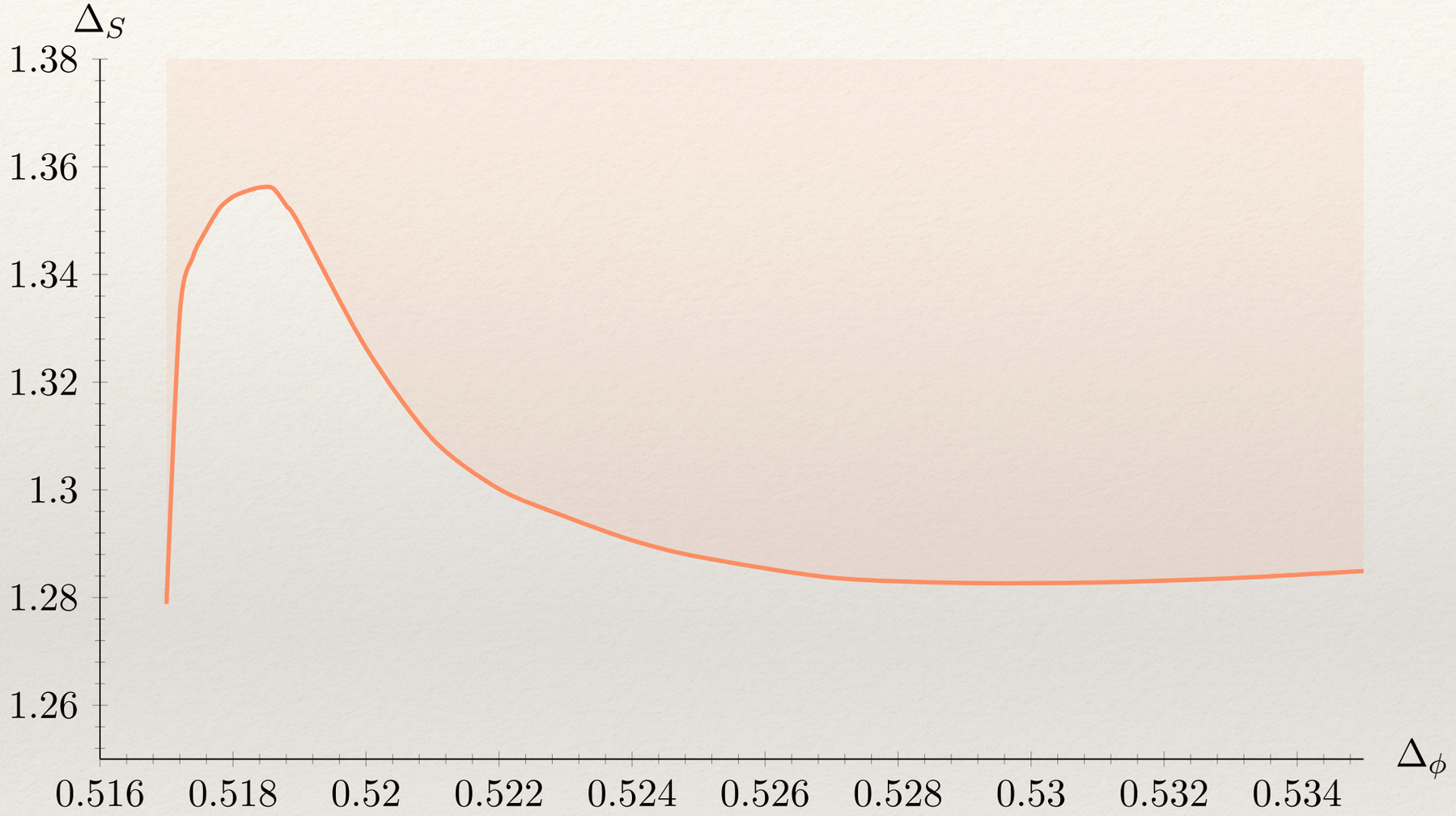
This is now a much bigger numerical optimization problem.

In order to make progress we need to make assumptions.

It turns out that it is important to limit the number of relevant operators.

Bootstrapping Cubic CFTs in 3d

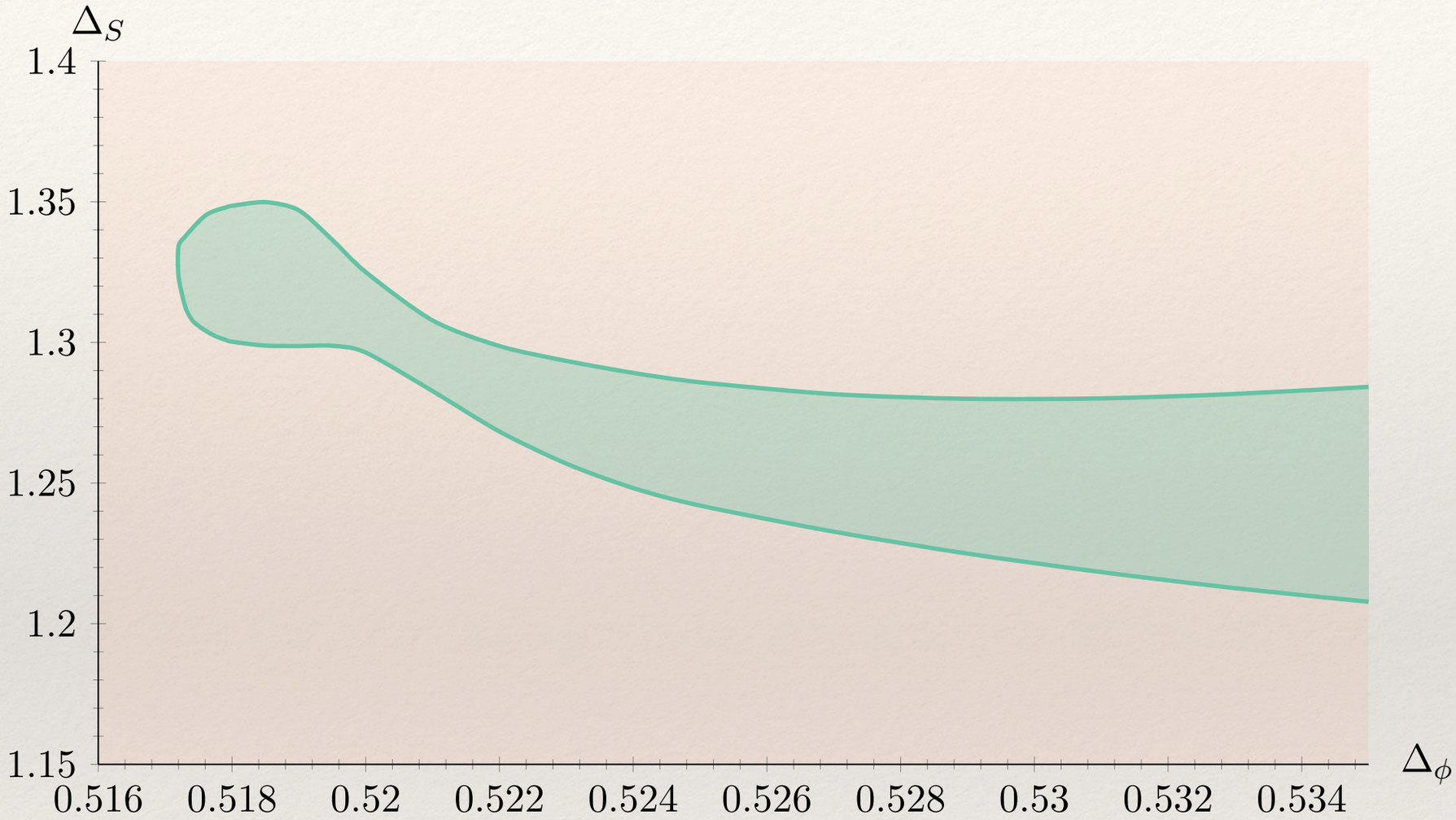
$$\Delta_{X'} > 3$$



(Kousvos & AS, 2018)

Bootstrapping Cubic CFTs in 3d

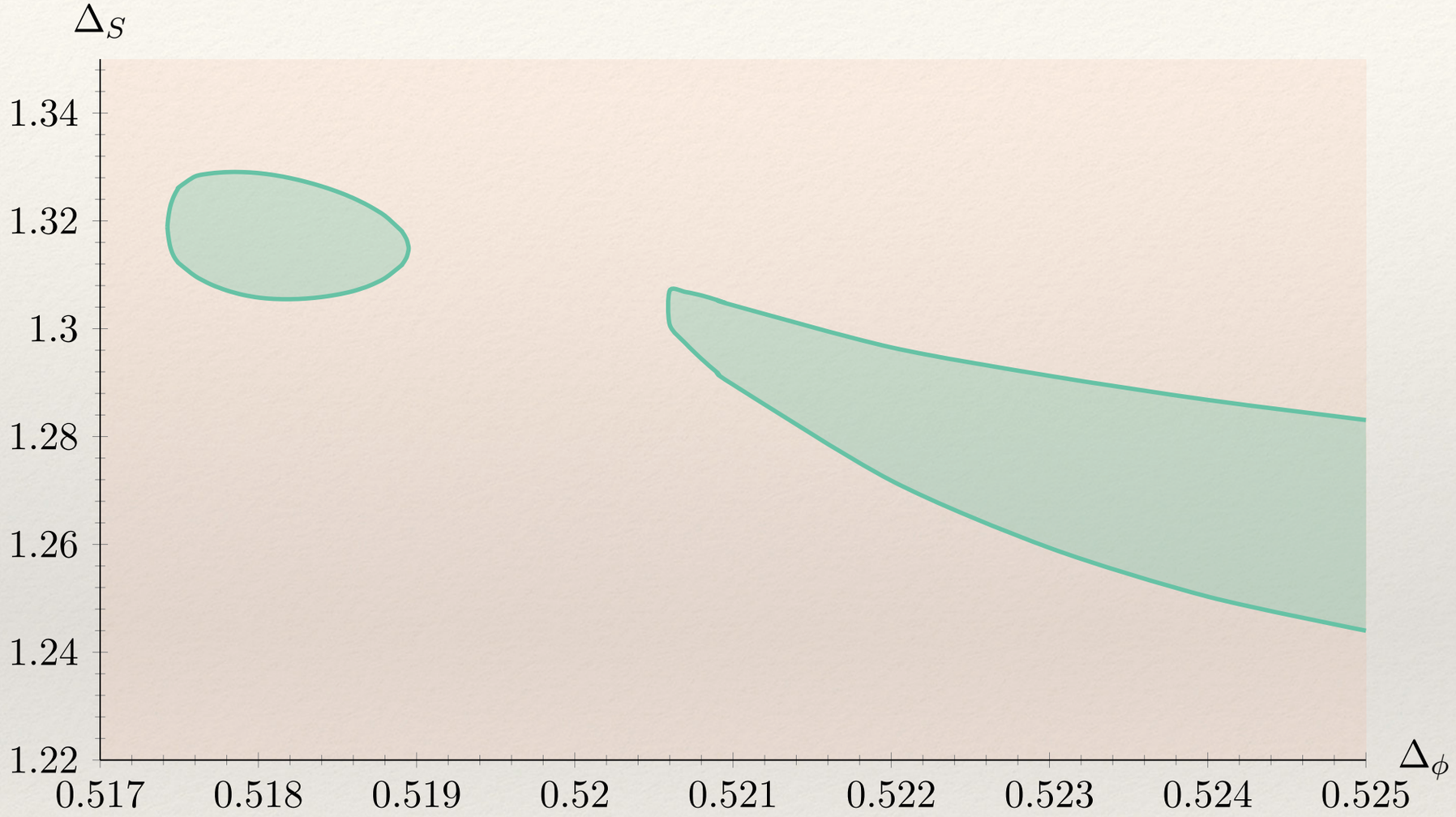
$$\Delta_{X'}, \Delta_{S'} > 3$$



(Kousvos & AS, 2018)

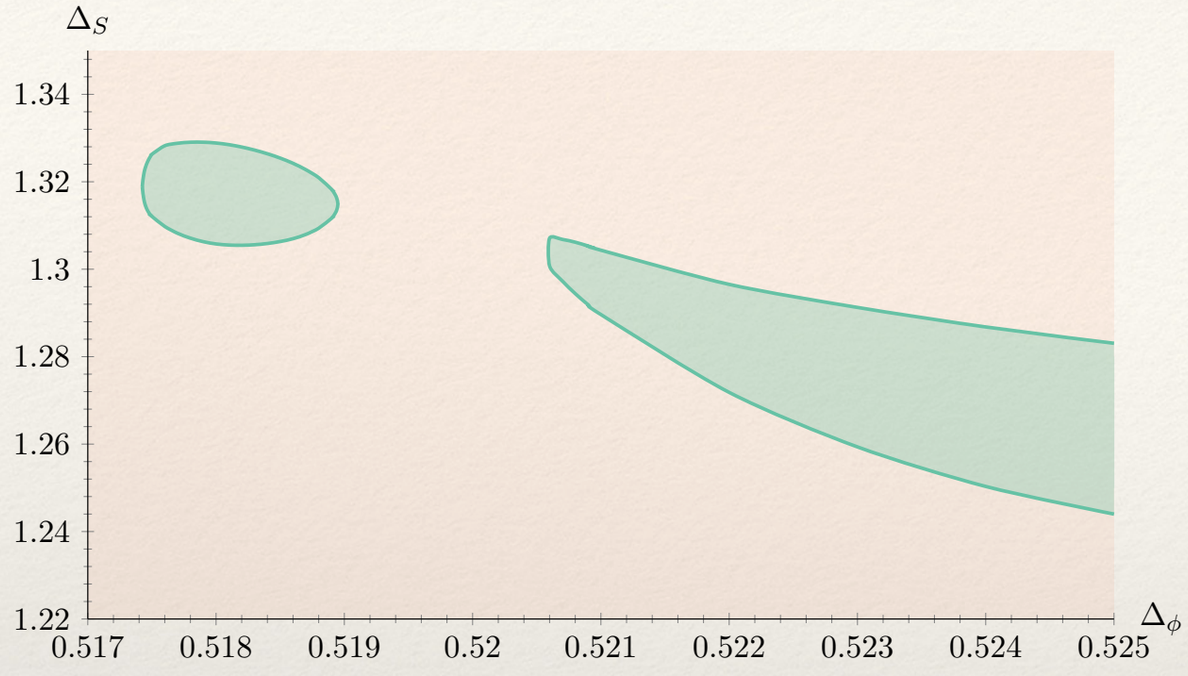
Bootstrapping Cubic CFTs in 3d

$$\Delta_{X'} > 3, \quad \Delta_{S'} > 3.8$$



(Kousvos & AS, 2018)

Bootstrapping Cubic CFTs in 3d



The presence of the island indicates that we found a very special solution to the crossing equation.

The critical exponents obtained differ significantly from those of the ε expansion:

$$\beta \approx 0.308 \pm 0.002\,, \quad \nu \approx 0.594 \pm 0.004,$$
$$\beta^{(\varepsilon)} \approx 0.368\,, \quad \nu^{(\varepsilon)} \approx 0.709.$$

Does the ε expansion fail, or have we found a new CFT?

Structural Phase Transitions

In many crystals the crystallographic structure changes at some critical temperature.

This happens usually in a discontinuous way, but there are cases where the transition is continuous.

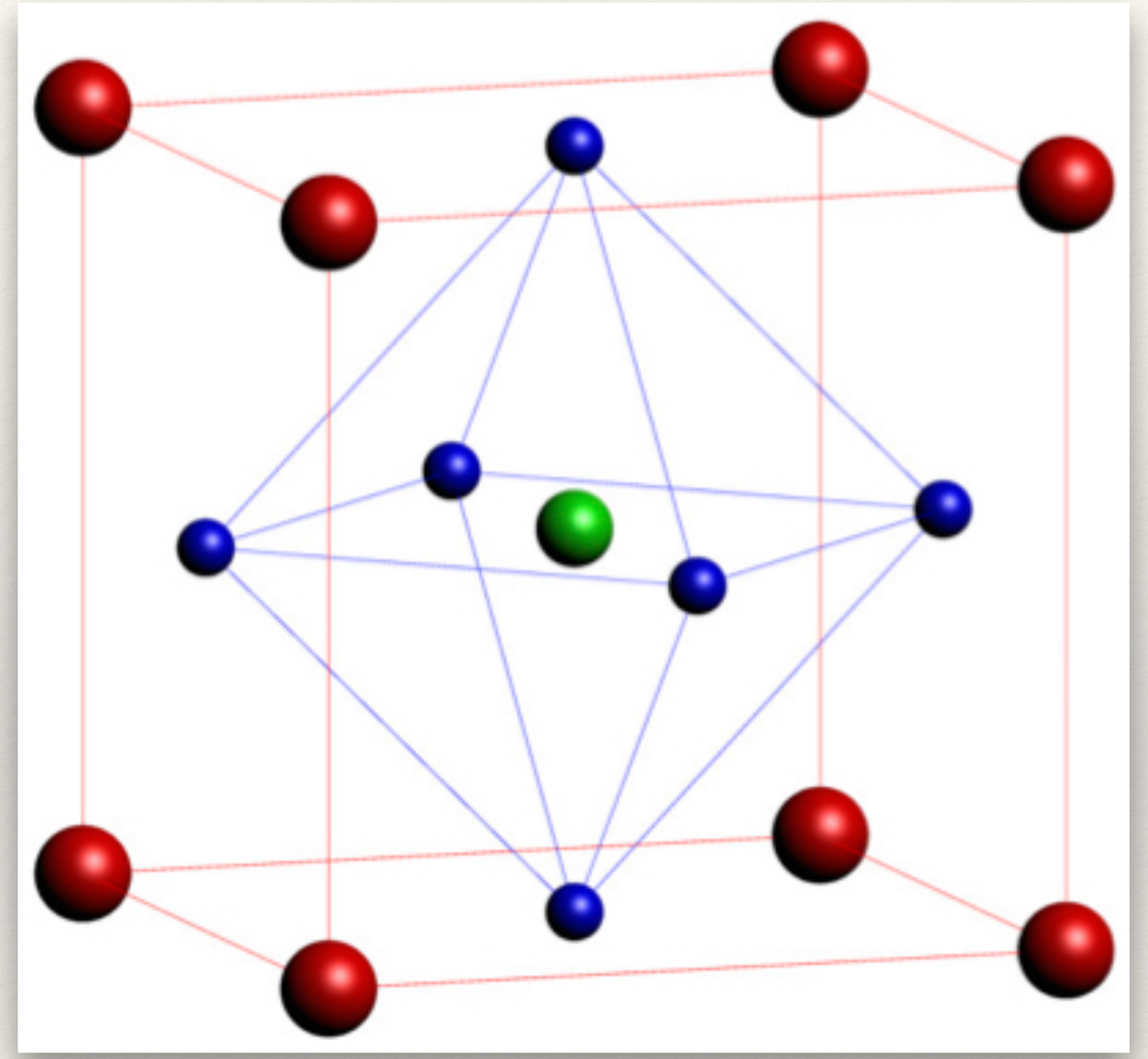
In fact, Landau developed his theory of phase transitions in the late 1930s motivated by continuous structural phase transitions.

In structural phase transitions the symmetry of the two phases is different, contrary to the case of liquid/gas transitions.

The thermodynamic state of the system in either phase becomes the common state at the transition point (no equilibrium of two distinct phases).

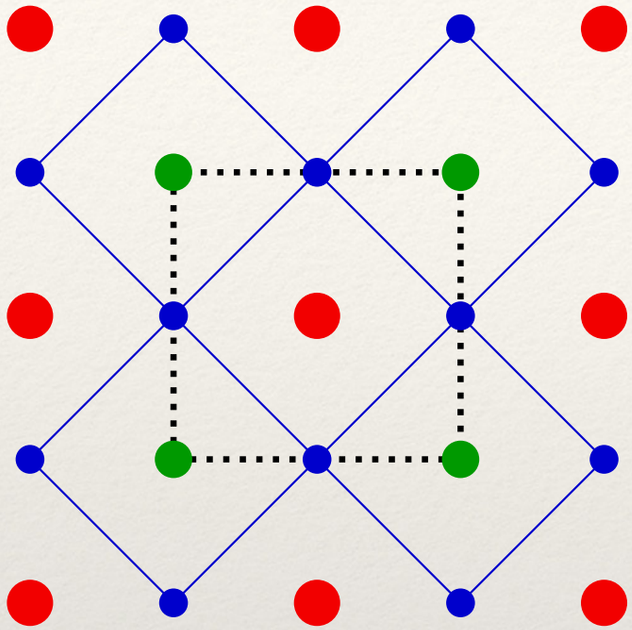
One phase must have higher symmetry than the other. The smaller symmetry group is a subgroup of the bigger one.

Perovskites

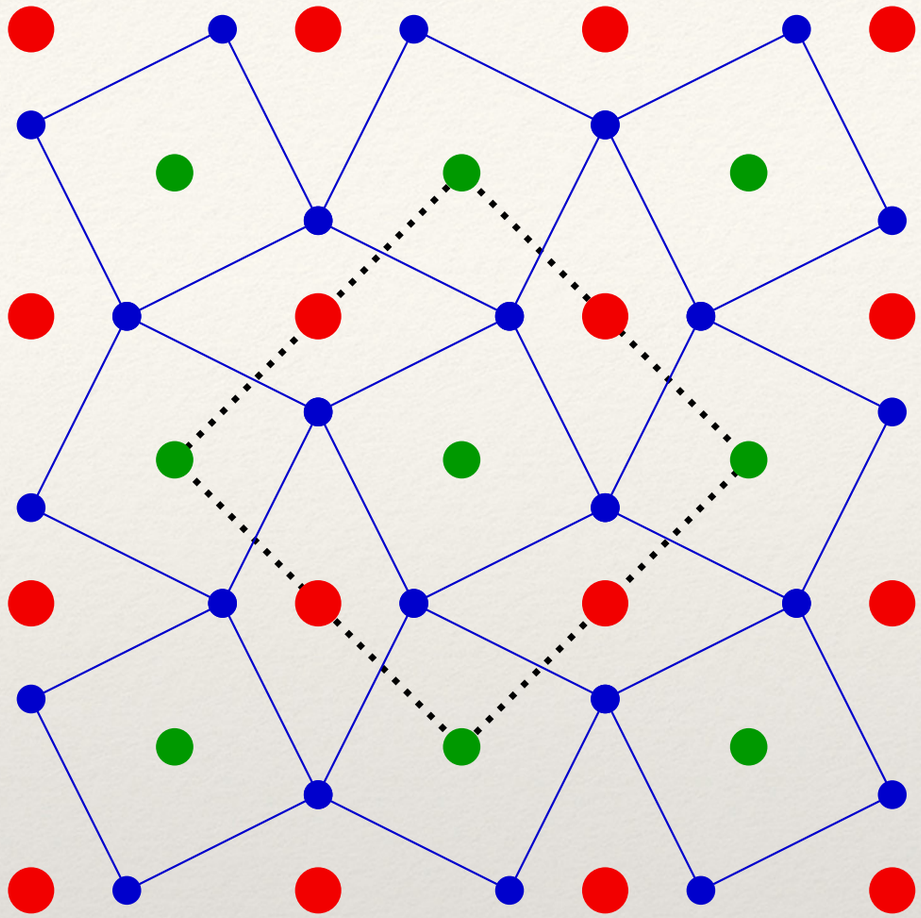


ABX_3

$SrTiO_3$



(a) $T > T_c$



(b) $T < T_c$

$T_c \approx 105$ K

Symmetry breaking:

$O_h = \mathbb{Z}_2^3 \rtimes S_3 \simeq S_4 \times \mathbb{Z}_2 \longrightarrow D_{4h} \simeq D_4 \times \mathbb{Z}_2$

$SrTiO_3$

Measurements:

$$\beta = 0.33 \pm 0.02\,, \quad \nu = 0.63 \pm 0.07$$

(Müller & Berlinger, 1971) (von Waldkirch, Müller, Berlinger & Thomas, 1972)

Bootstrap:

$$\beta = 0.308 \pm 0.002\,, \quad \nu = 0.594 \pm 0.004$$

The same critical exponent β has been measured in the ferromagnetic phase transition of EuS. (Heller & Benedek, 1965)

Clear goal: get better experimental results for the critical exponents.

MN Theories

Symmetry group: $MN_{m,n} = O(m)^n \rtimes S_n$

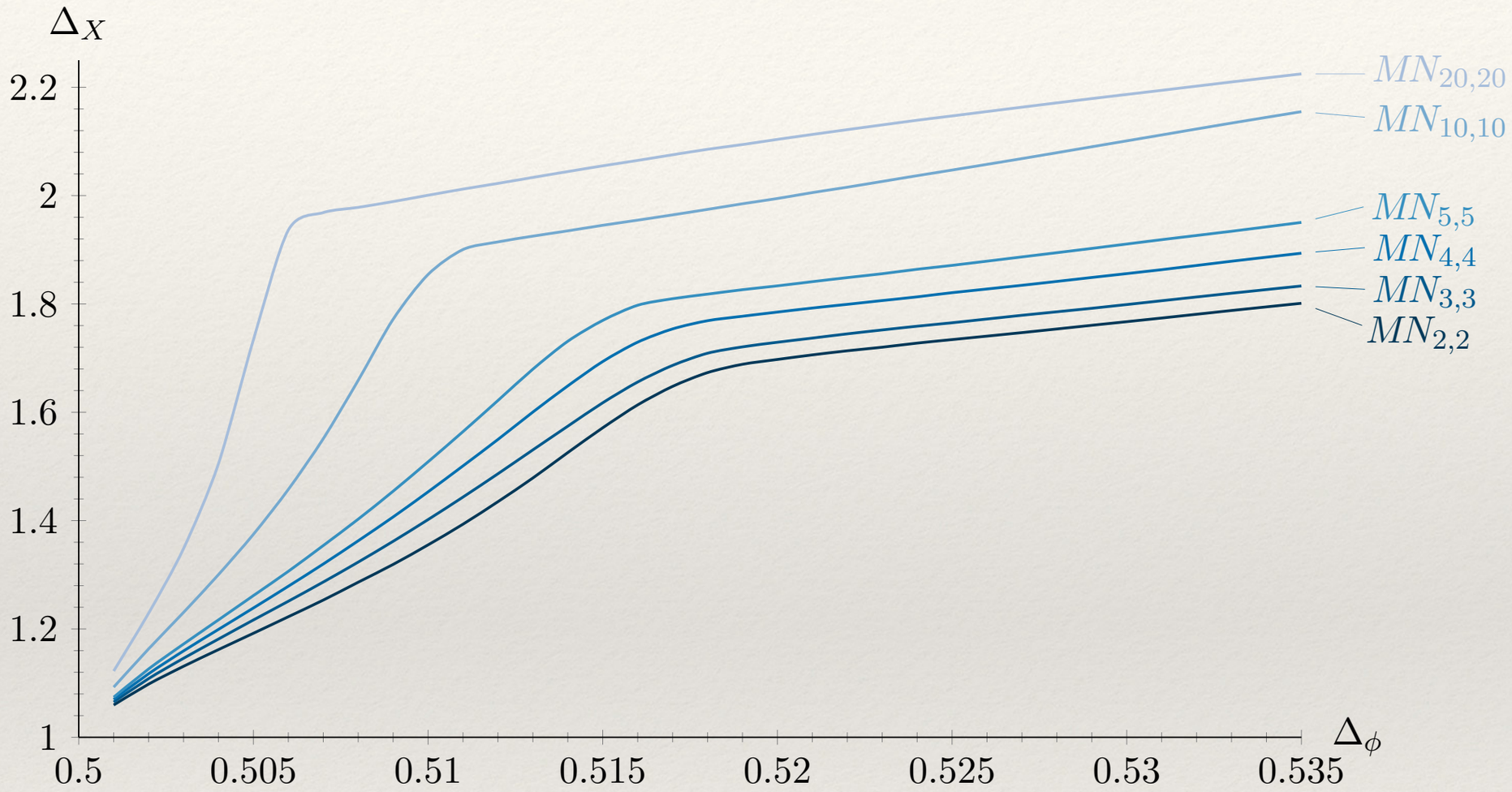
In the ε expansion we have a two-coupling theory:

$$V = \lambda(\phi^2)^2 + g\left[(\phi_1^2 + \cdots + \phi_m^2)^2 + \cdots + (\phi_{m(n-1)+1}^2 + \cdots + \phi_{mn}^2)^2\right]$$

$O(2)^2 \rtimes S_2$: Describes NbO_2, Ho, Dy, Tb

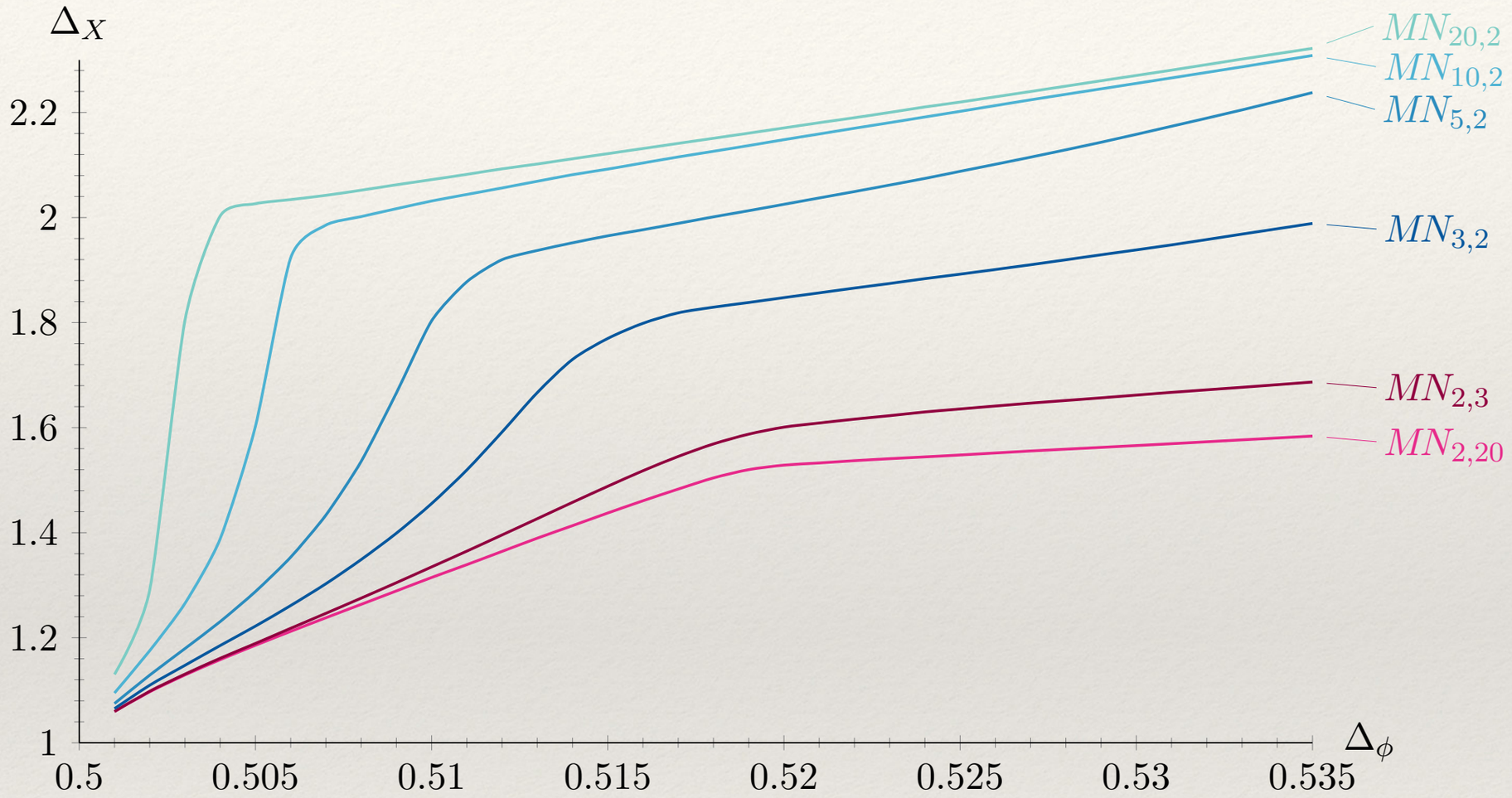
$O(2)^3 \rtimes S_3$: Describes K_2IrCl_6, TbD_2, Nd

MN Theories



(AS, 2019)

MN Theories



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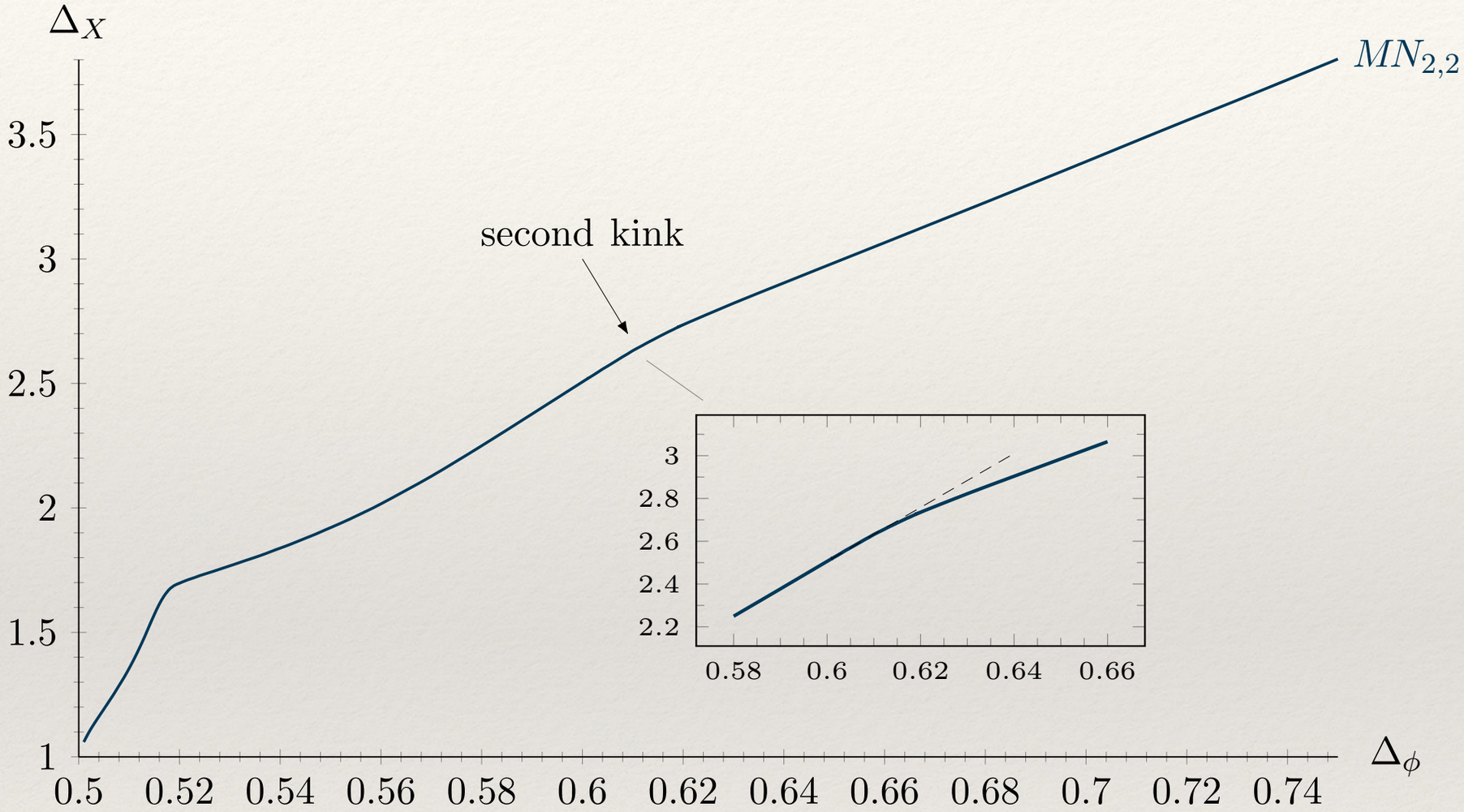
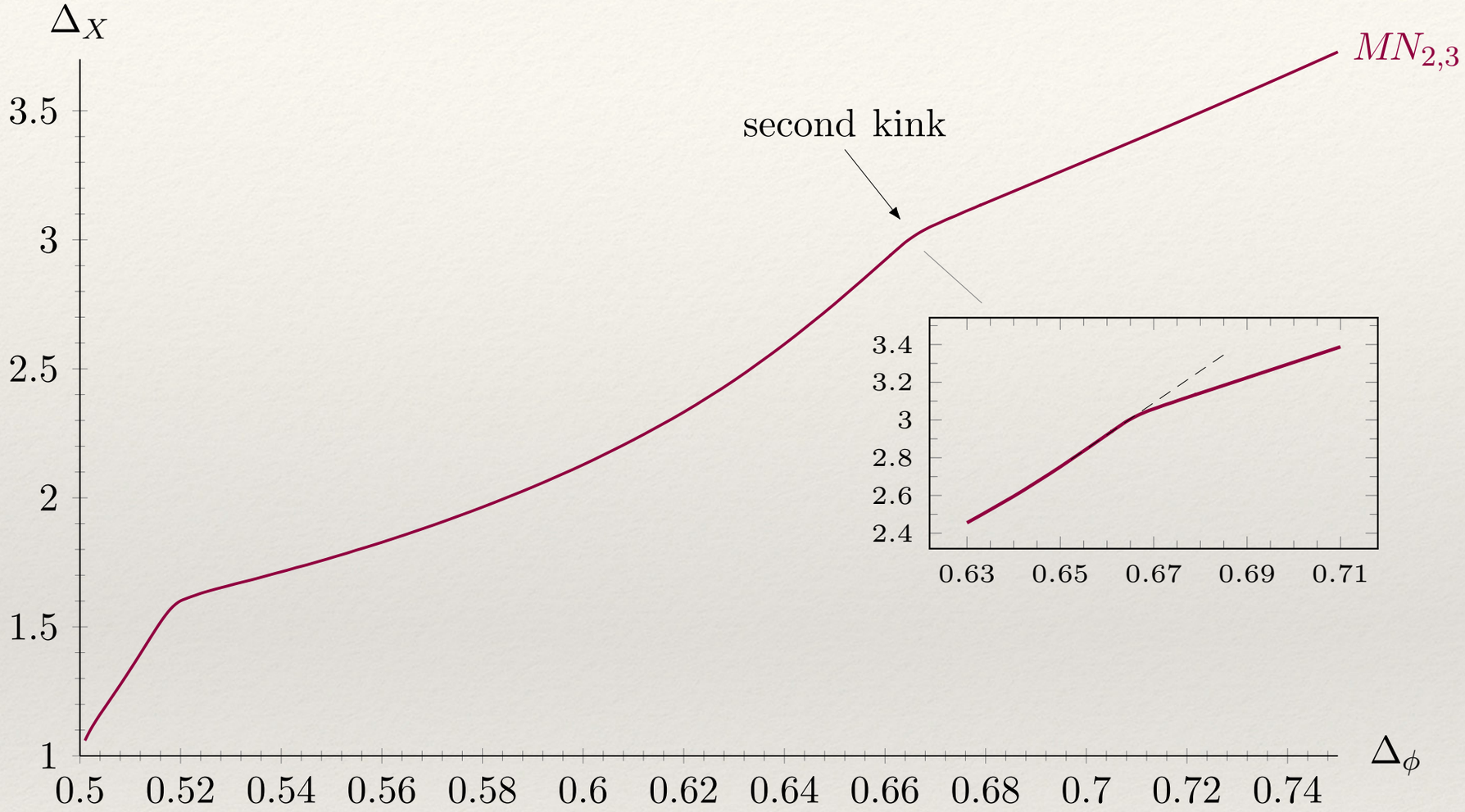


(AS, 2019)

MN Theories



(AS, 2019)

$$O(2)^2 \rtimes S_2$$

Experiments:

NbO_2	Ho and Dy	Tb
$\beta = 0.40^{+0.04}_{-0.07}$	$\beta = 0.39(4)$	$\beta = 0.23(4)$
	$\nu = 0.57(4)$	$\nu = 0.53(4)$

Perturbative methods:

$$\beta \approx 0.370\,, \quad \nu \approx 0.715$$

Bootstrap:

$$\beta = 0.293(3) \qquad \beta = 0.355(5)$$

$$\nu = 0.566(6) \qquad \nu = 0.576(8)$$

$$O(2)^3 \rtimes S_3$$

Experiments:

Nd

$$\beta = 0.36(2)$$

Perturbative methods:

$$\beta \approx 0.363\,, \quad \nu \approx 0.702$$

Bootstrap:

$$\beta = 0.301(3) \qquad \beta = 0.394(5)$$

$$\nu = 0.581(6) \qquad \nu = 0.590(8)$$

Summary

The numerical conformal bootstrap provides a widely-applicable and robust method for the study of CFTs.

There is a host of experimental results pertaining to structural and other types of phase transitions that have no theoretical explanation yet.

The conformal bootstrap has suggested the existence of previously undiscovered CFTs with potential relevance to structural and other phase transitions.

Are there other new nonperturbative universality classes relevant for physical systems?

Thank you!