RG improved four-point functions

Two examples in $d = 3$ and $d = 4$

ALBERTO TONERO

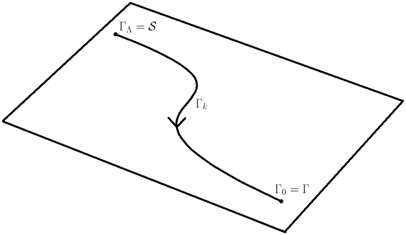
Carleton University, Ottawa

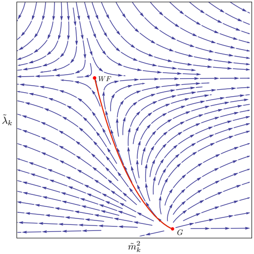
FRGIM - ECT* (Trento)
September 19, 2019

Department of Physics

Goal

- Compute 1PI correlation functions from an effective action Γ_0 that is obtained by integrating its RG flow along a critical trajectory that connects an IR to an UV fixed point





- Show that good IR/UV behavior of couplings in terms of RG scale k is reflected into good IR/UV behavior of correlation functions as function of external momenta p_i

The method I

One-loop Euclidean effective action

$$\Gamma[\varphi] = S[\varphi] + \frac{1}{2}\mathrm{Tr}\log S^{(2)}[\varphi]$$

Regularized effective action $\left(S_\Lambda^{(2)}[\varphi] = \frac{\delta^2 S_\Lambda[\varphi]}{\delta\varphi\delta\varphi}\right)$

$$\Gamma_k = S_\Lambda + \frac{1}{2}\mathrm{Tr}_\Lambda \log\left[S_\Lambda^{(2)} + R_k\right]$$

Λ: UV cut-off regulator

R_k: IR regulator

Scale derivative $(\partial_t = k\partial_k)$

$$\partial_t \Gamma_k = \frac{1}{2}\mathrm{Tr}\frac{\partial_t R_k}{S_\Lambda^{(2)} + R_k}$$

The method II

RG improvement

$$S_\Lambda^{(2)} \to S_k^{(2)}$$

Flow equation

$$\partial_t \Gamma_k = \frac{1}{2}\mathrm{Tr}\frac{\partial_t R_k}{S_k^{(2)} + R_k}$$

One-loop RG improved effective action

$$\Gamma_0 = \Gamma_\Lambda - \frac{1}{2}\int_0^\Lambda \frac{dk}{k}\,\mathrm{Tr}\frac{\partial_t R_k}{S_k^{(2)} + R_k}$$

RG improved 1PI correlation functions

$$\langle \varphi(x_1)\varphi(x_2)\ldots\varphi(x_n)\rangle_{\text{1PI}} = \left.\frac{\delta\Gamma_0[\varphi]}{\delta\varphi(x_1)\ldots\delta\varphi(x_n)}\right|_{\varphi=0}$$

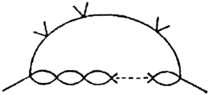
Example I: $\lambda\varphi^4$ in $d = 3$

[A. Codello and AT, PRD 94, 025015 (2016)]

Motivations

- Possible appearance of IR divergences for $p \to 0$ in the massless case in one-loop four-point correlation function $\sim \frac{g^2}{|p|^{4-d}}$

- Iterations n times inside UV convergent diagrams will produce IR divergences if $n(4-d) > d$



[G. Parisi, *Statistical Field Theory*; C. Itzykson and J. M. Drouffe, *Statistical Field Theory*]

The model

- The Euclidean action

$$S_\Lambda = \int d^d x \left\{ \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{m_\Lambda^2}{2}\varphi^2 + \frac{\lambda_\Lambda}{4!}\varphi^4 \right\}$$

- Hessian

$$S_k^{(2)} = -\Box + m_k^2 + \frac{\lambda_k}{2}\varphi^2$$

- RG flow

$$\partial_t\Gamma_k = \frac{1}{2}\mathrm{Tr}\frac{\partial_t R_k}{S_k^{(2)} + R_k}$$

The four-point structure function F_k

Scale dependent effective action

$$\Gamma_k = \int d^d x \left\{ \frac{1}{2} \varphi [-\Box + \Sigma_k(-\Box) + m_k^2] \varphi + \frac{1}{4!} \varphi^2 F_k(-\Box) \varphi^2 \right\} + \mathcal{O}(\varphi^6)$$

- 1PI four–point correlation function:

$$\langle \varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\rangle_{\text{1PI}} = \frac{1}{6} \delta_{xx_1} \delta_{xx_2} F_0(-\Box) \delta_{xx_3} \delta_{xx_4} + \text{ permutations}$$

Flow equations

RG equations ($X = -\Box$)

$$
\begin{aligned}
\partial_t \Sigma_k(X) &= 0 \\
\partial_t m_k^2 &= -\frac{1}{2}\frac{\lambda_k}{(4\pi)^{d/2}} Q_{\frac{d}{2}-1}\left[h_k\right] \\
\partial_t F_k(X) &= \frac{3}{2}\frac{\lambda_k^2}{(4\pi)^{d/2}} \int_0^1 d\xi \, Q_{\frac{d}{2}-2}\left[h_k^{X\xi(1-\xi)}\right]
\end{aligned}
$$

For $X \to 0$

$$
\partial_t \lambda_k = \frac{3}{2}\frac{\lambda_k^2}{(4\pi)^{d/2}} Q_{\frac{d}{2}-2}\left[h_k\right]
$$

Functional trace evaluated using non-local heat-kernel expansion

Beta functions in $d = 3$

Dimensionless couplings $\tilde{m}_k^2 = m_k^2/k^2$ and $\tilde{\lambda}_k = \lambda_k/k$

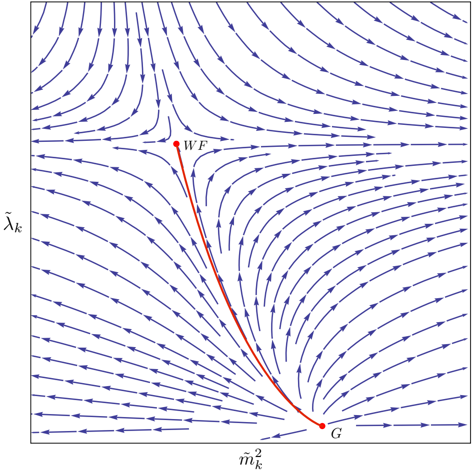
$$\begin{aligned} \partial_t \tilde{m}_k^2 &= -2\tilde{m}_k^2 - \frac{\tilde{\lambda}_k}{4\pi^2} \frac{1}{1+\tilde{m}_k^2} \\ \partial_t \tilde{\lambda}_k &= -\tilde{\lambda}_k + \frac{3\tilde{\lambda}_k^2}{8\pi^2} \frac{1}{1+\tilde{m}_k^2} \end{aligned}$$

Fixed points (FPs)

$\tilde{m}_*^2 = -\frac{1}{3}$ $\tilde{\lambda}_* = \frac{16\pi^2}{9}$ (IR) (Wilson-Fisher)

$\tilde{m}_*^2 = 0$ $\tilde{\lambda}_* = 0$ (UV) (Gaussian)

Phase diagram in $d = 3$



Critical trajectory connecting the Gaussian and the Wilson-Fisher FPs

Four point structure function in $d = 3$

Four-point function $F_0(X) = F_\Lambda(X) - \Delta F(X)$

$$\Delta F(X) = \frac{3}{8\pi^2}\frac{1}{\sqrt{X}}\int_0^\infty \frac{dk}{k}\lambda_k^2 \frac{k^2}{k^2+m_k^2}\, g\left(\frac{X}{k^2}\right)$$

- One-loop perturbative result massless case ($\lambda_k \to \lambda_0$, $m_k^2 \to 0$)

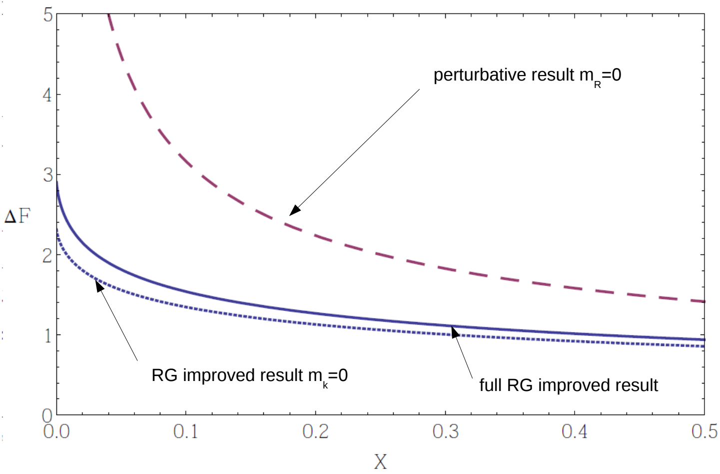
$$\Delta F^{1L}(X) = \frac{3\lambda_0^2}{16}\frac{1}{\sqrt{X}}$$

- One-loop improved result massless case ($\lambda_k \to \frac{8\pi^2 k C_1}{3C_1+k}$, $m_k^2 \to 0$)

$$\Delta F^{I}(X) = \frac{24 C_1^2 \pi^2}{\sqrt{X}}\int_0^\infty dk\, \frac{k}{(3C_1+k)^2}\, g\left(\frac{X}{k^2}\right)$$

RG improved structure function

Comparison between perturbative and improved results



Comments

- The path integral has been performed by weighting the momentum modes according to their RG relevance, i.e. according to the value of the running couplings at that scale

- We obtain a consistent IR finite $(q \to 0)$ four-point function for scalar theories in $d = 3$ in the massless case

Example II: $SU(2)$ NLSM in $d = 4$

[old work in progress]

$SU(2)$ NLSM RG flow in $d = 4$

Classical Euclidean action $[\partial\varphi \cdot \partial\varphi = h_{\alpha\beta}(\varphi)\partial_\mu\varphi^\alpha\partial^\mu\varphi^\beta]$

$$\mathcal{S}^E = \frac{1}{2f^2}\int d^4x\,\partial\varphi\cdot\partial\varphi$$

One-loop RG flow using NLHK ($\Box = \partial^2$)

$$\begin{aligned}\partial_t\Gamma_k^E[\varphi] &= \frac{1}{2}\frac{1}{(4\pi)^2}k^2\int d^4x\,\partial\varphi\cdot\partial\varphi \\ &+ \frac{1}{2}\frac{1}{(4\pi)^2}\frac{1}{16}\int d^4x\,\partial\varphi\cdot\partial\varphi\,F'_{\mathcal{M}}(-\Box/k^2)\,\partial\varphi\cdot\partial\varphi \\ &+ (3-\text{combinations})\end{aligned}$$

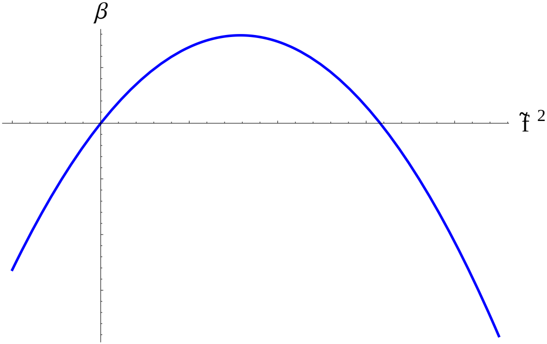
Effective action

$$\Gamma_k^E[\varphi] = \frac{1}{2f_k^2}\int d^4x\,\partial\varphi\cdot\partial\varphi + \dots$$

RG flow of f

Beta function *[A. Codello and R. Percacci, PLB 672 (2009) 280]*

$$\partial_t f_k^2 = -\frac{k^2 f_k^4}{(4\pi)^2} \qquad \Rightarrow \qquad f_k^2 = \frac{32\pi^2}{k^2 + C}$$



$$\tilde{f}^2 = k^2 f^2 \qquad\qquad \beta_{\tilde{f}^2} = 2\tilde{f}^2 - \frac{\tilde{f}^4}{(4\pi)^2}$$

Effective action at one-loop I

(i) integrate the flow at one-loop

$$
\begin{aligned}
\Gamma^E_\Lambda[\varphi] - \Gamma^E_0[\varphi] &= \frac{1}{2}\frac{1}{(4\pi)^2}\frac{\Lambda^2}{2}\int d^4x\,\partial\varphi\cdot\partial\varphi \\
&+ \frac{1}{2}\frac{1}{(4\pi)^2}\frac{1}{16}\int d^4x\,\partial\varphi\cdot\partial\varphi\left(1-\frac{1}{2}\log\frac{-\Box}{\Lambda^2}\right)\partial\varphi\cdot\partial\varphi \\
&+ (3-\text{combinations})
\end{aligned}
$$

(ii) fix the action at $k = \Lambda$

$$
\Gamma^E_\Lambda[\varphi] = \frac{1}{2f_\Lambda^2}\int d^4x\,h_{\alpha\beta}\partial_\mu\varphi^\alpha\partial^\mu\varphi^\beta - \frac{l_{2,\Lambda}}{2}\int d^4x\,\partial\varphi\cdot\partial\varphi\partial\varphi\cdot\partial\varphi + \ldots
$$

(iii) renormalize to eliminate the divergences

$$
\frac{1}{f_\Lambda^2} \to \frac{1}{f_0^2} + \frac{\Lambda^2}{32\pi^2} \qquad l_{2,\Lambda} \to l_{2,0}(\mu) + \frac{1}{(4\pi)^2}\frac{1}{48}\log\frac{\mu^2}{\Lambda^2} \qquad \ldots
$$

Effective action at one-loop II

(iv) go to canonically normalized fields

$$\varphi^\alpha \to f_0 \pi^\alpha$$

$$\begin{aligned}\Gamma_0^E[\pi] &= \frac{1}{2}\int d^4x\, \partial\pi\cdot\partial\pi - \frac{l_{2,0} f_0^4}{2}\int d^4x\, \partial\pi\cdot\partial\pi\partial\pi\cdot\partial\pi \\ &+ \frac{1}{2}\frac{1}{(4\pi)^2}\frac{f_0^4}{16}\int d^4x\, \partial\pi\cdot\partial\pi\left(1-\frac{1}{2}\log\frac{-\Box}{\mu^2}\right)\partial\pi\cdot\partial\pi \\ &+ \dots\end{aligned}$$

(v) perform the analytic continuation of the action to Minkowski space

$$\Gamma_0^E[\pi] \to \Gamma_0^M[\pi]$$

[A. Codello, R. Percacci, L. Rachwal and AT, Eur.Phys.J. C76 (2016) no.4, 226]

RG improved effective action I

(i) go to canonically normalized fields $f_k^2 = \frac{32\pi^2}{k^2+C}$

$$\partial\varphi \cdot \partial\varphi \to f^2 \partial\pi \cdot \partial\pi$$

(ii) perform the RG improvement

$$f^2 \partial\pi \cdot \partial\pi \to f_k^2 \partial\pi \cdot \partial\pi \qquad h_{\alpha\beta}(\varphi) \to h_{\alpha\beta}(f_0\pi)$$

and substitute into the RG flow

$$\begin{aligned} \partial_t \Gamma_k^E[\pi] &= \frac{1}{2}\frac{1}{(4\pi)^2} k^2 f_k^2 \int d^4x\, \partial\pi \cdot \partial\pi \\ &+ \frac{1}{2}\frac{1}{(4\pi)^2}\frac{f_k^4}{16} \int d^4x\, \partial\pi \cdot \partial\pi\, F'_{\mathcal{M}}(-\Box/k^2)\, \partial\pi \cdot \partial\pi + \ldots \end{aligned}$$

(iii) integrate the flow

$$\frac{1}{32\pi^2} \int \frac{dk}{k} k^2 f_k^2 = \frac{1}{2} \log \frac{\Lambda^2}{C} \qquad \frac{F^I_{\mathcal{M}}(-\Box/C)}{C^2} \equiv \frac{1}{512\pi^2} \int \frac{dk}{k} f_k^4 F'_{\mathcal{M}}$$

RG improved effective action II

(iv) renormalize to get the following improved action

$$\begin{aligned}\Gamma^E_{0,I}[\pi] &= \frac{1}{2}\int d^4x\,\partial\pi\cdot\partial\pi \\ &+ \int d^4x\,\partial\pi\cdot\partial\pi\,\frac{F^I_{\mathcal{M}}(-\Box/C)}{C^2}\,\partial\pi\cdot\partial\pi+\ldots\end{aligned}$$

(v) perform the analytic continuation of the action to Minkowski space

$$\Gamma^E_{0,I}[\pi]\to\Gamma^M_{0,I}[\pi]$$

Scattering amplitude I

The pion scattering

$$\pi^i(p_1)\pi^k(p_2) \to \pi^l(p_3)\pi^m(p_4)$$

Amplitude

$$A(ik \to lm) = A(s,t,u)\delta_{ik}\delta_{lm} + A(t,s,u)\delta_{il}\delta_{km} + A(u,t,s)\delta_{im}\delta_{kl}$$

- Using $\Gamma_0^M[\pi]$ we get the well known result

$$\begin{aligned} A(s,t,u) &= \frac{f_0^2}{4}s + \frac{l_{1,0}(\mu)}{2} f_0^4(t^2+u^2) + l_{2,0}(\mu) f_0^4 s^2 \\ &+ \frac{f_0^4}{512\pi^2}\left(\frac{10s^2+13(t^2+u^2)}{9}\right) \\ &- \frac{1}{3}\frac{f_0^4}{512\pi^2}\left(2s^2\log\frac{-s}{\mu^2} + t(t-u)\log\frac{-t}{\mu^2} + u(u-t)\log\frac{-u}{\mu^2}\right) \end{aligned}$$

[J. Gasser and H. Leutwyler, Annals Phys. 158 (1984) 142]

Scattering amplitude II

- Using $\Gamma_{0,I}^{M}[\pi]$ we get instead

$$
\begin{aligned}
A^I(s,t,u) &= \frac{8\pi^2}{C}s + 2\frac{s^2}{C^2}F_{\mathcal{M}}^I(-s/C) + \frac{t^2}{C^2}F_{\mathcal{M}}^I(-t/C) + \frac{u^2}{C^2}F_{\mathcal{M}}^I(-u/C) \\
&+ \frac{u^2-s^2}{C^2}F_{\Omega}^I(-t/C) + \frac{t^2-s^2}{C^2}F_{\Omega}^I(-u/C)
\end{aligned}
$$

Asymptotic behavior for fixed scattering angle x and large s

$$
\begin{aligned}
\mathrm{Re}\,A &\simeq s^2 \log \frac{s}{\mu^2} \\
\mathrm{Re}\,A^I &\simeq \log \frac{s}{C}
\end{aligned}
$$

with $C = \frac{32\pi^2}{f^2}$

- Asymptotically safe pions ?

Conclusions

- We performed the path integral by weighting the momentum modes according to their RG relevance, i.e. according to the value of the running couplings at that scale

- *Including the non-trivial information of the phase diagram of the theory it is possible to cure both UV and IR behaviors when appropriate FPs are present*

- We obtain a consistent IR and UV finite four-point functions as function of p for those scalar theories in $d = 3$ and $d = 4$

Thank you!

Carleton
UNIVERSITY

Back up

Heat-kernel traces

Flow equation using Laplace transform

$$\partial_t \Gamma_k = \frac{1}{2} \mathrm{Tr}\, h_k(-\Box + U, \omega) = \frac{1}{2} \int_0^\infty ds\, \tilde{h}_k(s,\omega) \mathrm{Tr}\, e^{-s(-\Box + U)}$$

Non-local heat kernel expansion

$$\mathrm{Tr}\, e^{-s(-\Box+U)} \quad = \quad \frac{1}{(4\pi s)^{d/2}} \int d^d x \sqrt{g} \left\{ \mathbf{1} - sU + U f_U(-s\Box) U + O(U^3) \right\}$$

Structure functions

$$f_U(x) = \frac{1}{2} \int_0^1 d\xi\, e^{-x\xi(1-\xi)}$$

Q-functionals

$$Q_n[h_k^a] = \int_0^\infty ds s^{-n} e^{-sa} \tilde{h}_k(s,\omega)$$

where $h_k^a(z,\omega) \equiv h_k(z+a,\omega)$

Q-functionals in $d = 3$

We have

$$\int_0^1 d\xi \, Q_{-\frac{1}{2}} \left[h_k^{X\xi(1-\xi)} \right] \quad = \quad \frac{2}{\sqrt{\pi}} \frac{k^2}{k^2 + m_k^2} \frac{1}{\sqrt{X}} g(X/k^2)$$

where

$$g(X/k^2) = \left[\log \frac{2k + \sqrt{X}}{2k - \sqrt{X}} \theta(4k^2 - X) + \log \frac{\sqrt{X} + 2k}{\sqrt{X} - 2k} \theta(X - 4k^2) \right]$$

Pion scattering amplitude

Action

$$\begin{aligned}\Gamma^M[\pi] &= \frac{1}{2}\int d^4x\,\partial_\mu\pi^\alpha\partial^\mu\pi^\alpha + B_0\int d^4x\,\left[(\partial_\mu\pi^\alpha\pi^\alpha)^2 - (\partial_\mu\pi^\alpha\partial^\mu\pi^\alpha)(\pi^\beta\pi^\beta)\right]\\ &+ B_1\int d^4x\,\partial_\mu\pi^a\partial_\nu\pi^a\partial^\mu\pi^b\partial^\nu\pi^b + B_2\int d^4x\,\partial_\mu\pi^a\partial^\mu\pi^a\partial_\nu\pi^b\partial^\nu\pi^b\\ &+ \int d^4x\,\partial_\mu\pi^\alpha\partial^\mu\pi_\alpha\hat{F}_{\mathcal{M}}(\Box)\partial_\nu\pi^\beta\partial^\nu\pi_\beta + \int d^4x\,\partial_\mu\pi^\alpha\partial^\mu\pi_\beta\hat{F}_{\mathcal{M}}(\Box)\partial_\nu\pi^\beta\partial^\nu\pi_\alpha\\ &+ \int d^4x\,\partial^\mu\pi^\beta\partial_\nu\pi_\alpha\hat{F}_{\Omega}(\Box)\partial_\mu\pi^\alpha\partial^\nu\pi_\beta - \int d^4x\,\partial^\mu\pi_\alpha\partial_\nu\pi^\beta\hat{F}_{\Omega}(\Box)\partial_\mu\pi^\alpha\partial^\nu\pi_\beta + \mathcal{O}(\end{aligned}$$

Amplitude $A(s,t,u)$

$$\begin{aligned}A(s,t,u) &= 6B_0 s + B_1(t^2+u^2) + 2B_2 s^2 + 2s^2\hat{F}_{\mathcal{M}}(-s) + t^2\hat{F}_{\mathcal{M}}(-t) + u^2\hat{F}_{\mathcal{M}}(-u)\\ &+ (u^2-s^2)\hat{F}_{\Omega}(-t) + (t^2-s^2)\hat{F}_{\Omega}(-u)\end{aligned}$$

RG improved amplitude terms

We have the following terms

$$B_0 = \frac{4\pi^2}{3C} \quad , \quad B_1 = 0 \quad , \quad B_2 = 0$$

$$\hat{F}_{\mathcal{M}}(-s) = F^I_{\mathcal{M}}(s) = \frac{\pi^2}{C^2} \frac{\sqrt{-s}\sqrt{4C-s}\log(\frac{C}{-s}) + (2C-s)\log(\frac{\sqrt{-s}+\sqrt{4C-s}}{-\sqrt{-s}+\sqrt{4C-s}})}{\sqrt{-s}\sqrt{4C-s}}$$

$$\begin{aligned}\hat{F}_{\Omega}(-t) &= F^I_{\Omega}(t) \\ &= \frac{\pi^2}{3C^2} \frac{(-8C^2 - 2Ct + t^2)\log\left(\frac{\sqrt{-t}+\sqrt{4C-t}}{-\sqrt{-t}+\sqrt{4C-t}}\right) + \sqrt{-t}\sqrt{4C-t}(4C + t\log(\frac{-t}{C}}{-t\sqrt{-t}\sqrt{4C-t}}\end{aligned}$$