Towards the gauge beta function at $\mathcal{O}(1/N_f^2)$ and $\mathcal{O}(1/N_f^3)$

Manuel Reichert

Functional and Renormalization Group methods, Trento, 19. September 2019

CP^3-Origins, SDU Odense, Denmark

Nicola Dondi, Gerald Dunne, MR, Francesco Saninno: arXiv:1903.02568
Nicola Dondi, MR, Francesco Saninno: in preparation

CP^3 Origins
Cosmology & Particle Physics

CP3

Which matter systems are asymptotically safe in $d = 4$?

- Gauge-Yukawa theories at large N_f & N_c (perturbatively) [Litim, Sannino '14]
- How far does this extend to small N_c?
- Test gauge theories at large N_f non-perturbatively

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Standard QCD picture:

- Small N_f: asymptotic freedom & confinement in the IR
- Medium N_f: asymptotic freedom & IR Banks-Zaks fixed point
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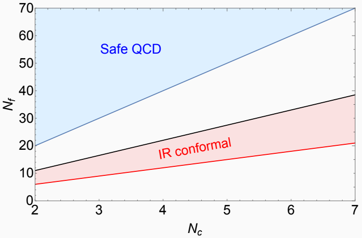

[Antipin, Sannino '17]

Gauge theories at large N_f

Usual expansion of beta function in gauge coupling

$$\beta(g) = \sum_{k=0}^{\infty} g^{3+2k} \beta^{(k)}$$

Introduce 't Hooft like coupling

$$K = \frac{g^2 N_f S_2(R)}{4\pi^2} \sim \mathcal{O}(1)$$

Allows to reorganise beta function

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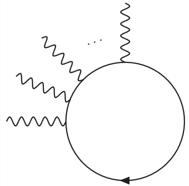
Advantages of $1/N_f$ expansion:

- Perturbative in the expansion parameter $1/N_f$
- Non-perturbative in the 't Hooft gauge coupling
- Model building in SM extensions [Mann et al. '17; Pelaggi et al. '18; Molinaro et al. '18; ...]

Topologies of diagrams and counting of vertices

Due to the rescaled gauge coupling $g \sim \sqrt{K/N_f}$



$$\sim g^n N_f = \mathcal{O}\left(\frac{1}{N_f^{n/2-1}}\right)$$

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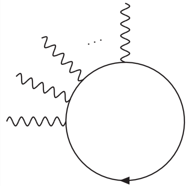

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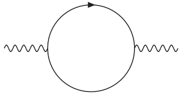
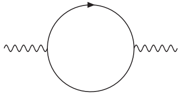
Bubble chain



$\ldots$



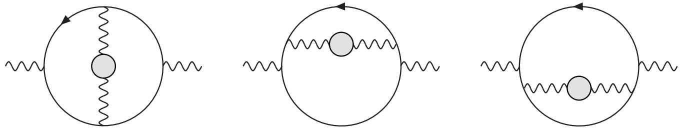
$$= \mathcal{O}(1)$$

Each gauge line is always fully dressed with fermion loops

QED beta function at $1/N_f$

[Palanques-Mestre, Pascual '84]

Feynman diagrams at $1/N_f$:



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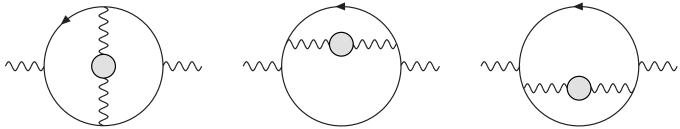

Resummation formula for $1/\epsilon$ pole

$$\sum_{n=1}^{\infty} \left(-\frac{2K_0}{3}\right)^n \left.\frac{H(n+1,\epsilon)}{(n+1)\epsilon}\right|_{\frac{1}{\epsilon}} = -\frac{2}{3}\int_0^K \mathrm{d}x \left(1-\frac{x}{K}\right) H(0,\frac{2}{3}x)$$

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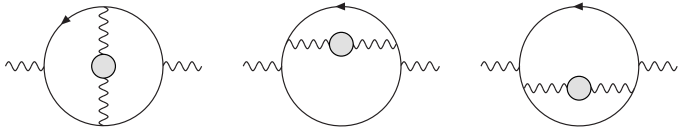

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$$\beta_{\mathrm{QED}}(K) = \frac{2}{3}K^2 + \frac{K^2}{2N_f}\int_0^K \mathrm{d}x\, F_{\mathrm{QED}}(x) + \mathcal{O}\left(\frac{1}{N_f^2}\right)$$

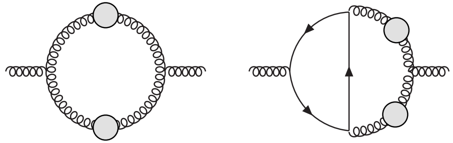
$$F_{\mathrm{QED}}(x) = -\frac{(x+3)(x-\frac{9}{2})(x-\frac{3}{2})\sin\left(\frac{\pi x}{3}\right)\Gamma\left(\frac{5}{2}-\frac{x}{3}\right)}{27\cdot 2^{\frac{2x}{3}-5}\pi^{\frac{3}{2}}(x-3)x\,\Gamma\left(3-\frac{x}{3}\right)}$$

Radius of convergence: $K_* = \frac{15}{2}$

QCD beta function at $1/N_f$

[Gracey '96; Holdom '10]

New Feynman diagrams at $1/N_f$:



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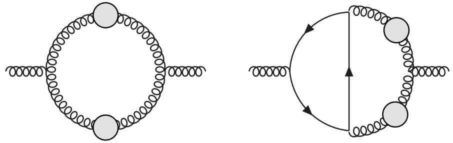

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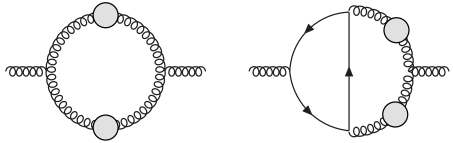

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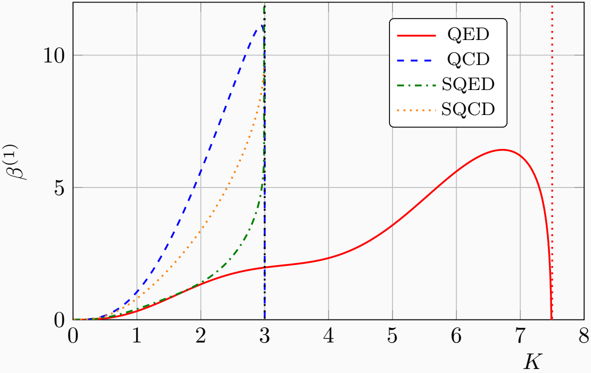
$$\beta_{\text{QCD}}(K) = \frac{2K^2}{3}\left(1 - \frac{11}{4N_f}\frac{C_2(G)}{S_2(R)}\right) + \frac{K^2}{2N_f}\int_0^K \mathrm{d}x\, F_{\text{QCD}}(x) + \mathcal{O}\left(\frac{1}{N_f^2}\right)$$

$$F_{\text{QCD}}(x) = \frac{2^{1-\frac{2x}{3}}\sin\left(\frac{\pi x}{3}\right)\Gamma\left(\frac{5}{2}-\frac{x}{3}\right)}{27\pi^{3/2}(x-3)^2 x\,\Gamma\left(3-\frac{x}{3}\right)}\left[\frac{C_2(G)}{S_2(R)}(4x^4 - 42x^3 + 288x^2 - 1161x) - 4\frac{d(G)}{d(R)}(x-3)(x+3)(2x-9)(2x-3)\right].$$

Radius of convergence: $K_* = 3$

Beta functions of (S)QED and (S)QCD

$$\beta(K) = \beta^{(0)}(K) + \frac{\beta^{(1)}(K)}{N_f} + \dots$$



UV fixed point for QED & QCD

Landau pole for SQED & SQCD

How physical are these fixed points?

- The fermion mass anomalous dimension goes to zero in QCD and to infinity in QED [Antipin, Sannino '17]
- Hints for FP in QCD at medium N_f from resummations with Meijer G-functions [Antipin, Maiezza, Vasquez '18]
- Lattice studies inconclusive so far [Leino, Rindlisbacher, Rummukainen, Sannino, Tuominen '19]
- Poles might be resummable within the $1/N_f$ series [Alanne, Blasi, Dondi '19]

How to go beyond $1/N_f$

- The next orders in the $1/N_f$ expansion would test the physical nature of the FP
- No known resummation formula for two bubble-chains, needed for $1/N_f^2$ and higher orders
- Can we extract the location of the pole, the residuum, etc., with a finite amount of coefficients?

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Two methods:

- Large-order behaviour of expansion coefficients
- Padé approximants

Large-order behaviour: Darboux's Theorem

The nearby singularity determines the large order growth of the expansion coefficients a_n. E.g. for expansion around $z = 0$

- pole of order p at z_0 ($f(z) \sim \phi(z)(1 - z/z_0)^p + \text{finite}$)

$$a_n \sim \frac{1}{z_0^n} \binom{n+p-1}{n} \phi(z_0) + \dots$$

- logarithmic branch cut at z_0 ($f(z) \sim \phi(z) \ln(1 - z/z_0) + \text{finite}$)

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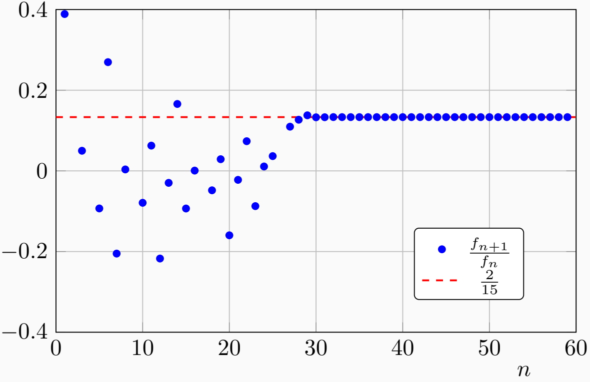
Expectation for QED $F_{\text{QED}} = \sum_n f_n x^n$

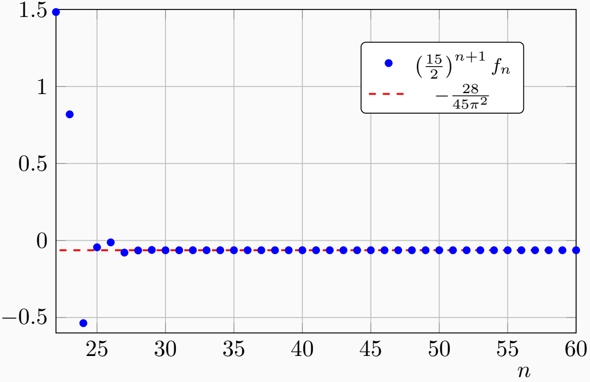
$$f_n \sim \left[R_0 \left(\frac{2}{15}\right)^n + R_1 \left(\frac{2}{21}\right)^n + R_2 \left(\frac{2}{27}\right)^n + \dots \right]$$

Large order behaviour of F_{QED}



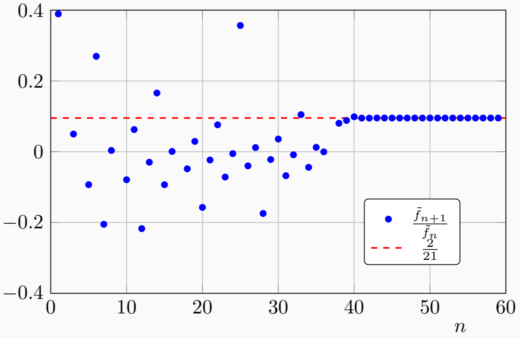
Ratio test $\frac{f_{n+1}}{f_n}$ reveals location of the first pole

Large order behaviour of F_{QED}



With the knowledge of the pole the residuum is computable

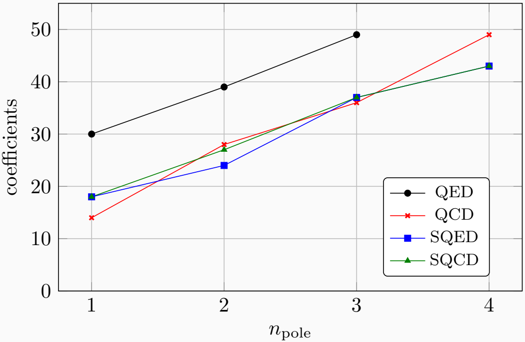
Large order behaviour of F_{QED}



Subtracting the first pole reveals the second pole

$$\tilde{f}_n = f_n + \frac{28}{45\pi^2}\left(\frac{15}{2}\right)^{-n-1}$$

How many coefficients are needed?



"Closer" to the origin $\rightarrow$ less coefficients are needed

Padé methods

Analytic continuation of truncated Taylor series by ration of two polynomials

$$F_{\mathrm{QED}}(x) \approx \sum_{n=0}^{M} f_n\, x^n \quad \longrightarrow \quad \mathcal{P}^{[R,S]}(x) = \frac{P_R(x)}{Q_S(x)}$$

with $R + S = M$.

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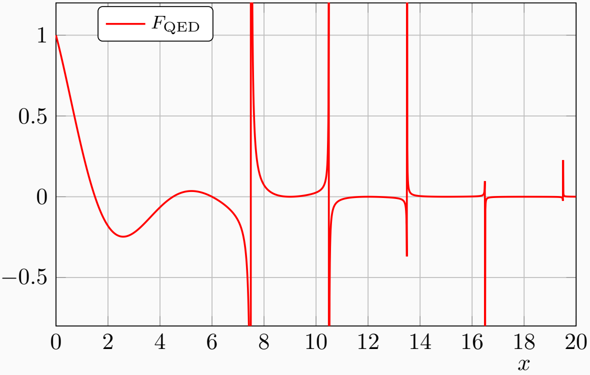
Rewriting of resummed $F_{\mathrm{QED}}(x)$

$$F_{\mathrm{QED}}(x) \sim \frac{\Gamma(1+\frac{x}{3})}{\Gamma(\frac{1}{2}+\frac{x}{3})} \frac{\sin^2\left(\frac{\pi x}{3}\right)}{\cos\left(\frac{\pi x}{3}\right)}$$

Padé approximant with $2R \approx S$ should lead to best results.

Padé approximants of F_{QED}



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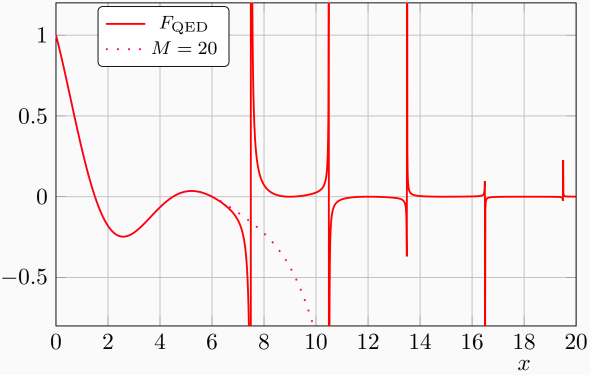

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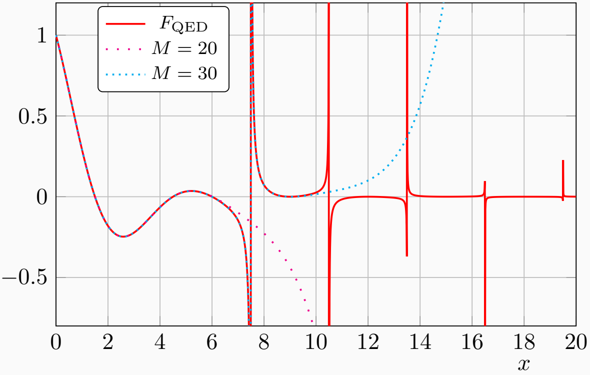

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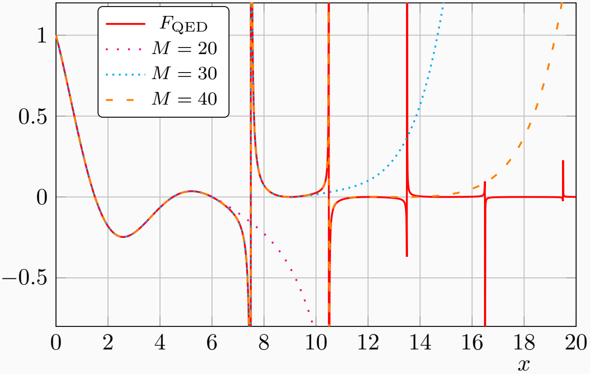

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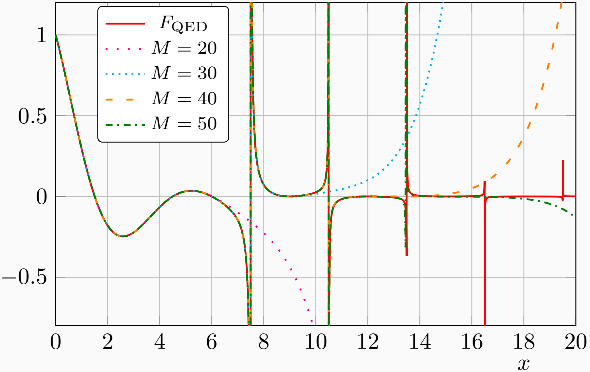

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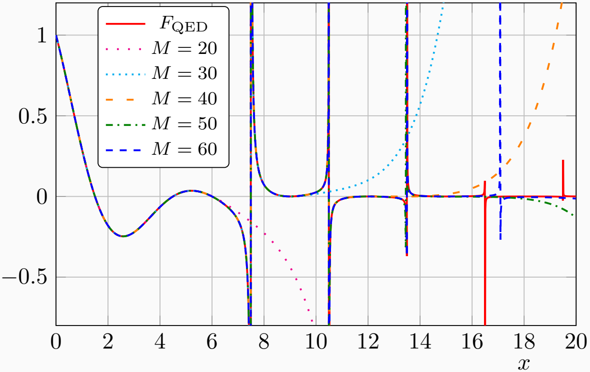

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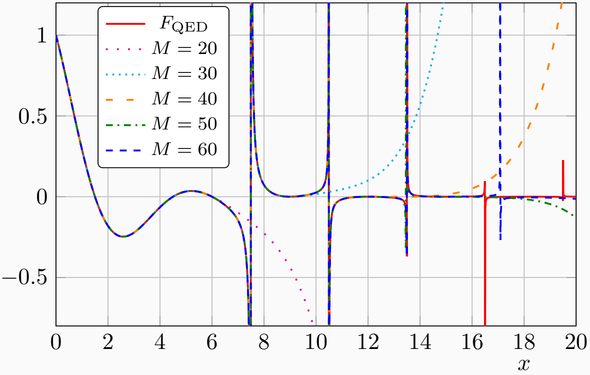
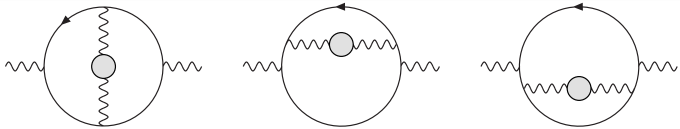


- Need ~ 30 coefficients to resolve first singularity (similar to large order growth analysis)
- Can resolve function beyond the first singularity

QED beta function

$1/N_f$:



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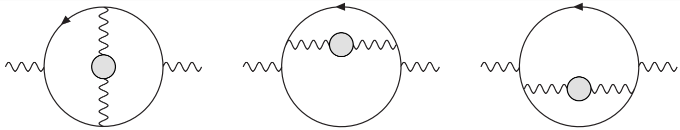
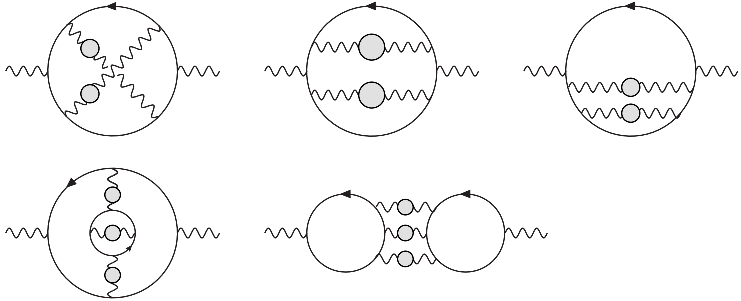


$1/N_f^2$ (subset):



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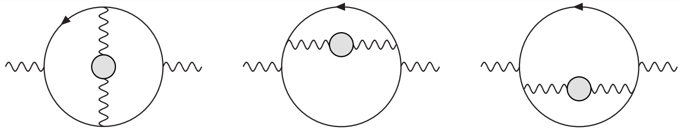

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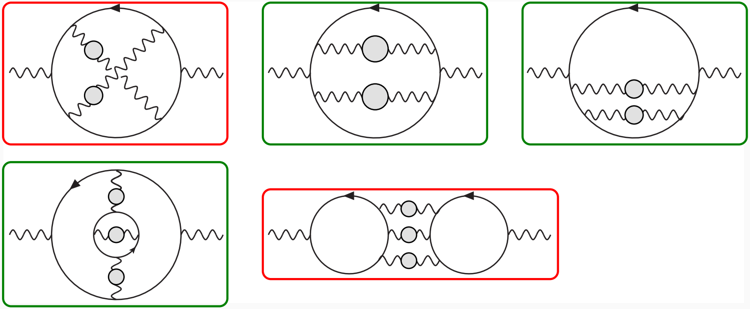
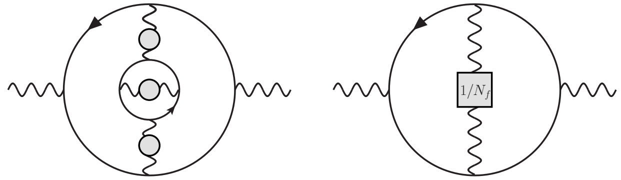


Master integral known / not know

Beyond $1/N_f$: nested diagrams

Nested sub-part of beta function: gauge & RG scale independent



Computation up to K^{44}

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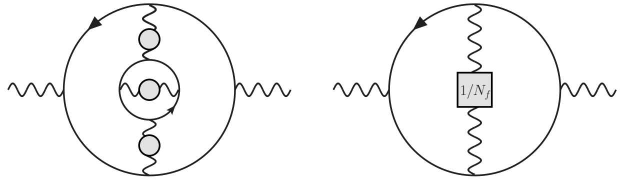
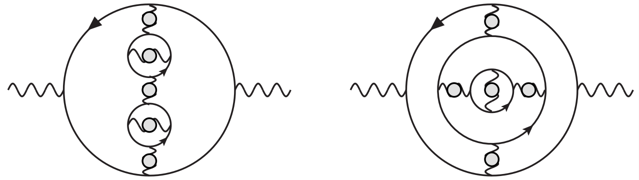


Computation up to K^{44}

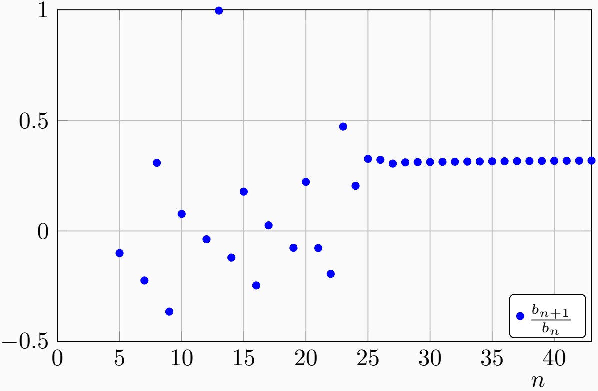
At $\mathcal{O}(1/N_f^3)$



Computation up to K^{32}

Ratio test at $\mathcal{O}(1/N_f^2)$

$$\beta^{(2)}_{\text{nested}} = \sum_n b_n K^n$$



New finite radius of convergence

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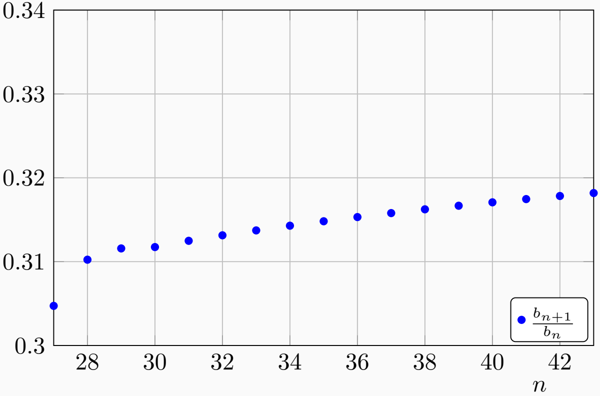

New finite radius of convergence
but extreme slow convergence

Richardson extrapolation

Enhance the convergence of the series

$$a_n = s + \frac{A}{n} + \frac{B}{n^2} + \frac{C}{n^3} + \dots$$

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First Richardson ($B = C = \ldots = 0$)

$$\mathrm{R}^{(1)} a_n \equiv s = (n+1) a_{n+1} - n a_n$$

Richardson extrapolation

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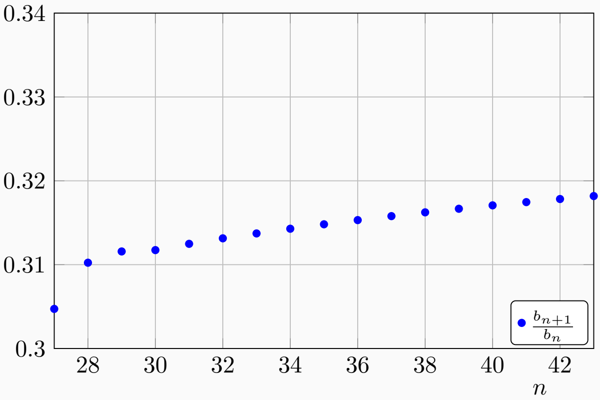
Second Richardson ($C = \dots = 0$)

$$\mathrm{R}^{(2)} a_n \equiv s = \frac{1}{2} \left((n+2)^2 a_{n+2} - 2(n+1)^2 a_{n+1} + n^2 a_n \right)$$

Ratio test at $\mathcal{O}(1/N_f^2)$



Bare series: $K^* = 3.14$

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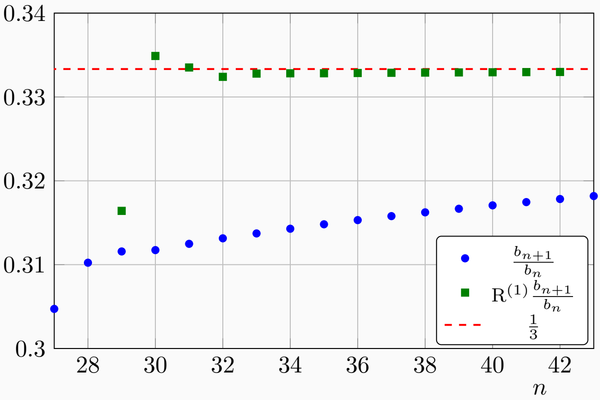
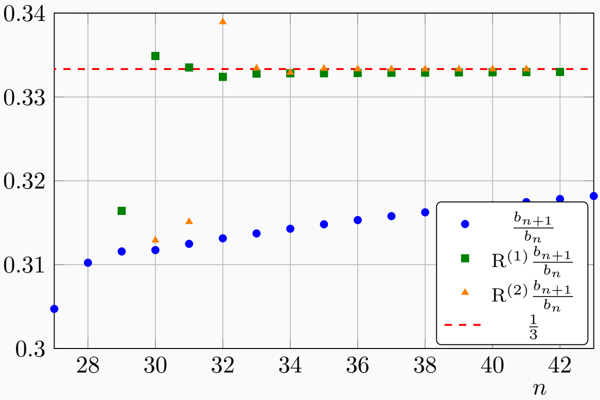


Bare series: $K^* = 3.14$

First Richardson: $K^* = 3.003$

Ratio test at $\mathcal{O}(1/N_f^2)$

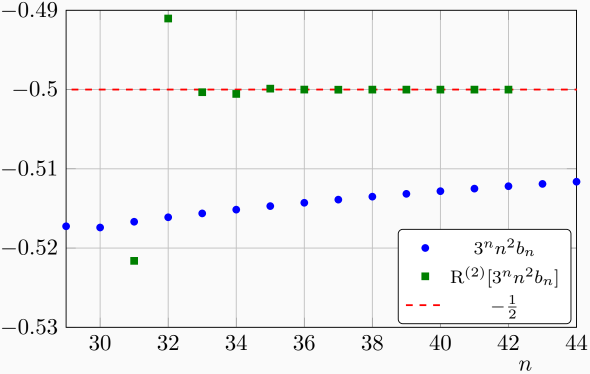


Bare series: $K^* = 3.14$

First Richardson: $K^* = 3.003$

Second Richardson: $K^* = 3.00008$

Residue at $\mathcal{O}(1/N_f^2)$



Bare series: $3^n n^2 b_n = -0.512$

Second Richardson: $3^n n^2 b_n = -0.500007$

Subleading behaviour

$$\tilde{b}_n = b_n + \frac{1}{2}\frac{1}{3^n}\frac{1}{n^2}$$

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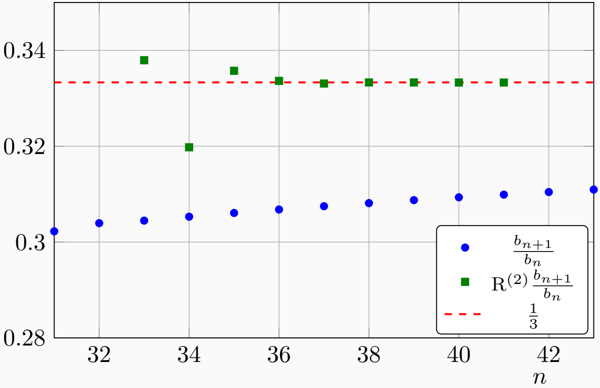
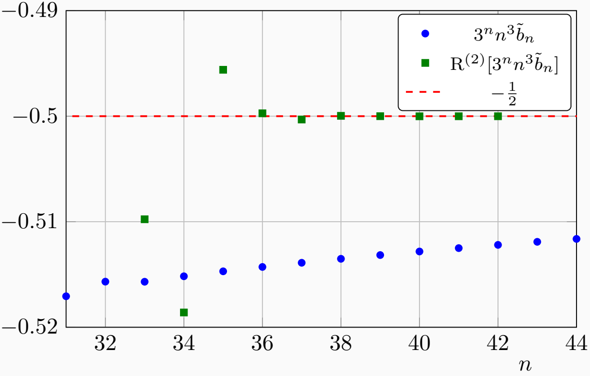


Bare series: $K^* = 3.215$

Second Richardson: $K^* = 3.0003$

Subleading behaviour

$$\tilde{b}_n = b_n + \frac{1}{2}\frac{1}{3^n}\frac{1}{n^2}$$



Bare series: $3^n n^3 \tilde{b}_n = -0.512$

Second Richardson: $3^n n^3 \tilde{b}_n = -0.500007$

Large-order behaviour

Large-order behaviour

$$
\begin{aligned}
b_n &\sim -\frac{1}{2}\frac{1}{3^n}\left(\frac{1}{n^2}+\frac{1}{n^3}+\dots\right)+\mathcal{O}\left(\frac{1}{(x>3)^n}\right)\\
&= -\frac{1}{2}\frac{1}{3^n}\frac{1}{n(n-1)}
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\sum_{n=4}^{\infty} b_n K^n \sim \frac{1}{6}(K-3)\ln\left(1-\frac{K}{3}\right)+\text{finite}
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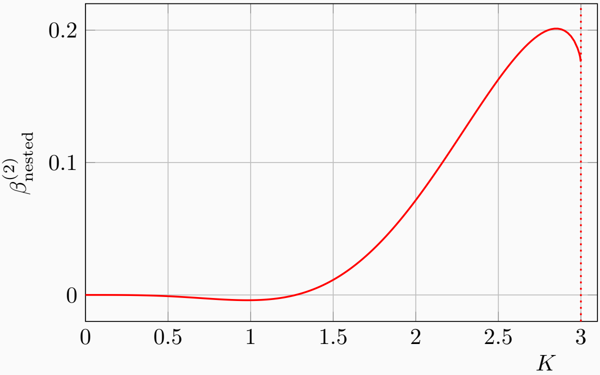
Logarithmic branch cut but no pole!

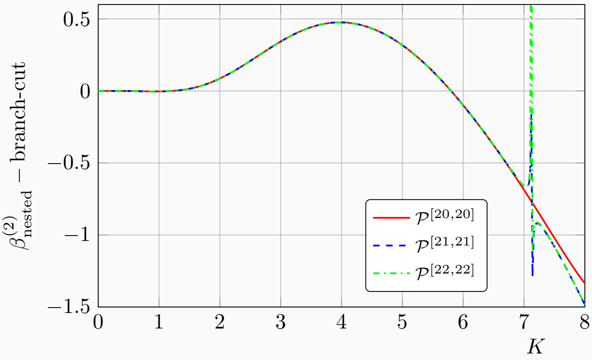
Nested beta function at $1/N_f^2$



"Exact" nested beta function up to $K = 3$

Beta function unphysical beyond $K = 3$ or magic cancellation needed

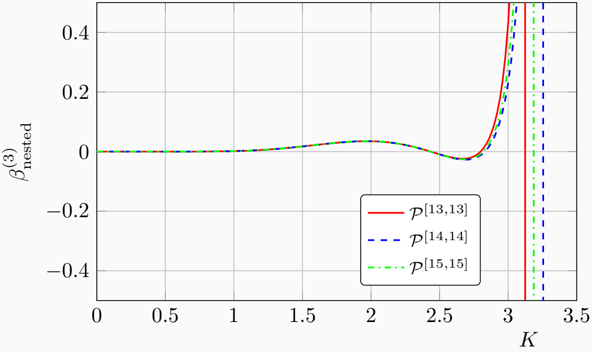
Nested beta function at $1/N_f^2$ beyond the first branch cut



No singularity before $K = 15/2$

Positive pole at $K = 15/2$?

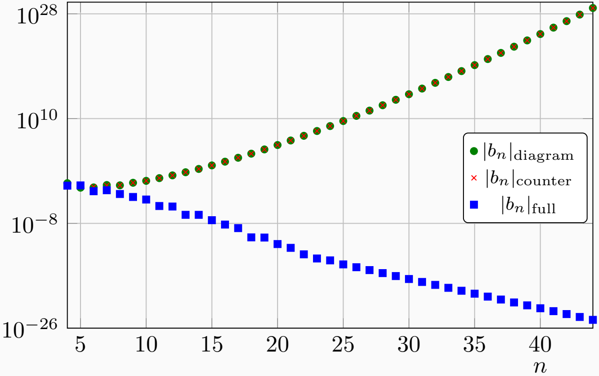
Nested beta function at $1/N_f^3$



No singularity before $K = 3$

Branch cut at $K = 3$?

Factorially divergent diagrams



Two factorially divergent contributions but the sum goes to zero

Is there are smarter way of computing it?

Summary and outlook

- Which matter theories in $d = 4$ are asymptotically safe?
- Non-perturbative resummation applied to gauge theories at large N_f
- Large-order behaviour & Padé methods constitute powerful tools
- First partial result beyond $\mathcal{O}(1/N_f)$ for QED:
  New logarithmic branchcut at $K^* = 3$ without pole
- ToDo: Remaining diagrams (Master integrals?) & QCD

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Thank you for your attention