FRG for Weyl Conformal Gravity

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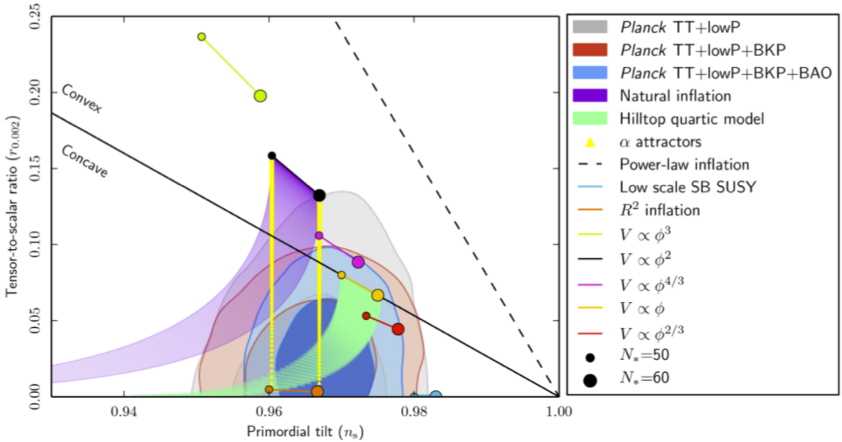
My stay will be in October, in INAF, Catania
under the supervision of prof. A. Bonanno.

Why conformal gravity

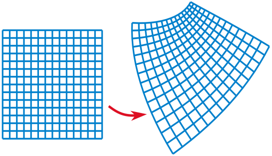


$$n_s = 1.00 - \varepsilon, \quad \varepsilon \ll 1$$

Why conformal gravity

Conformal symmetry and conformal transformations

- preserves only angles, distances are relative (gauge dependent)
- touches the differential structure of the spacetime
- leaves invariant only the causal structure of spacetime
- present for the classical matter system at very high energy or when particles are effectively massless
- can be additional symmetry of the gravitational system
- can be treated like gauge symmetry (can be gauged)



Advantages of conformal symmetry

Quantum conformality

- means that quantum fluctuations look the same at all energy scales (scale-invariance)
- additional symmetry on the quantum level
- constrains further quantum dynamics
- may be helpful in avoiding divergences
- leads to CFT's which describe FP's of RG
- gives fully idempotent quantum effective action Γ
- gives rise to very well-behaved quantum models, like $\mathcal{N} = 4$ SYM theory or $\mathcal{N} = 4$ conformal supergravity of Fradkin and Tseytlin '85
- may be instrumental in solving the issue of spacetime singularities
- is a starting point for various (non-conformal) deformations

Weyl gravity

Classical conformal Weyl gravity in $d = 4$

- is diffeomorphicaly invariant
- is conformally invariant
- defined in $d = 4$ by the action

$$S_{\rm conf} = \int d^4x \sqrt{|g|} \alpha_C C^2,$$

where

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - 2R_{[\mu[\rho}g_{\sigma]\nu]} + \frac{1}{3} g_{[\mu\rho}g_{\nu]\sigma} R$$

is a Weyl tensor (tensor of conformal curvature)

- uniqueness – defined by only one gravitational coupling constant: $\alpha_C > 0$

Weyl gravity

Advantages of conformal Weyl gravity in $d = 4$

- provides explanation for "almost" scale-invariant power spectrum of cosmological fluctuations
- describes accurately > 100 galactic rotation curves ("dark matter" problem)
- all vacuum solutions of Einsteinian gravitation are also exact vacuum solutions here (like Schwarzschild, Kerr spacetimes etc.)
- reproduces all precise tests of relativistic gravitation like Einstein theory in the vacuum
- resolves problem of classical singularities of GR (of Big Bang, or black hole) by allowing for conformal rescalings:

$$g_{\mu\nu}(x) \to g_{\mu\nu}(x)' = \Omega^2(x) g_{\mu\nu}(x)$$

Quantum Weyl gravity

Classical EM and YM (non-Abelian) gauge theories are conformally invariant in $d = 4$

Advantages of quantum conformal Weyl gravity in $d = 4$

- allows coupling of conformal matter in a conformal symmetry preserving way
- all the divergences from the matter sector (of the type C^2) are absorbed – conformal deWitt-Utiyama argument
- is a renormalizable quantum theory of gravitation
- is an asymptotically free theory in the UV regime (like QCD)
- reaches an UV FP of RG
- for $\mathcal{N} = 4$ super version is UV-finite (no divergences!)
- lets performance of well controlled quantum computations

Quantum Weyl gravity

Advantages of quantum conformal Weyl gravity in $d = 4$

- this approach to relativistic gravitation is fully Machian, gravitational field here is determined completely by the matter distribution everywhere else in the Universe
- for quantum conformality no conformal (gauged) anomaly on the quantum level, theory is with quantum scale-invariance, no beta functions, no RG flow, sits at the the FP of RG

Quantum Weyl gravity in $d = 4$ in Euclidean framework

- C^2 action is bounded from below, no conformal instability problem
- positive-definiteness of the partition function Z
- gives a consistent quantum statistical mechanics of 4-dimensional differential manifolds (quantum shape dynamics)

Quantum computations

Examples of quantum computations in Weyl Conformal Gravity

- within QFT of gravitational (and conformal) interactions
- within background-independent formalism (BFM)
- using only physical degrees of freedom (TT gravitons)

1. 1-loop partition function
2. functionally improved RG beta functions
3. quest for IR FP $\Longrightarrow$ windows for cosmology

York decomposition

York decomposition of gravitational fluctuations

- traceless

$$h_{\mu\nu} = \bar{h}_{\mu\nu} + \frac{1}{4} g_{\mu\nu} \phi$$

$$\bar{h}^{\mu}_{\mu} = 0, \quad \phi = h^{\mu}_{\mu}$$

- transverse and traceless (TT)

$$\bar{h}_{\mu\nu} = \bar{h}^{\perp}_{\mu\nu} + \nabla_\mu \eta^{\perp}_{\nu} + \nabla_\nu \eta^{\perp}_{\mu} + \nabla_\mu \nabla_\nu \sigma - \frac{1}{4} g_{\mu\nu} \Box \sigma$$

$$\nabla_\mu \bar{h}^{\perp}_{\mu\nu} = 0, \quad \nabla_\mu \eta^{\perp}_{\mu} = 0, \quad \bar{h}^{\perp\mu}_{\mu} = 0 \quad \text{with} \quad \bar{\phi} = \phi - \Box\sigma$$

- the physical field in conformal gravity is only TT graviton field $\bar{h}^{\perp}_{\mu\nu} = h^{TT}_{\mu\nu}$ due to diffeomorphism and conformal symmetry

Invariants in QG

Local curvature invariants in $d = 4$ of dimension four

- conformal invariant C^2 :

$$C^2 = R^2_{\mu\nu\rho\sigma} - 2R^2_{\mu\nu} + \frac{1}{3}R^2$$

$\delta_c(\sqrt{|g|}C^2) = 0$

- topological invariant GB :

$$\mathrm{GB} = R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2$$

$\delta\left(\int d^4x\sqrt{|g|}\mathrm{GB}\right) = 0$

- globally scale-invariant R^2:
  $\mathrm{dim}(R^2) = 4$

Background manifolds

- maximally symmetric spaces (MSS)

$$R_{\mu\nu\rho\sigma} = \frac{\Lambda}{3}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right) \quad \text{with} \quad \Lambda = \text{const}$$

  where also $R_{\mu\nu} = \Lambda g_{\mu\nu}$, $R = 4\Lambda$, $\mathrm{GB} = \frac{8}{3}\Lambda^2$ and $C_{\mu\nu\rho\sigma} = 0$, $C^2 = 0$

- Ricci-flat spaces

$$R_{\mu\nu} = 0,\ R = 0,\ R^2_{\mu\nu\rho\sigma} = C^2 = \mathrm{GB}$$

- Einstein spaces (ES)

$$R_{\mu\nu} = \Lambda g_{\mu\nu},\ R = d\Lambda,\ \Lambda = \text{const}$$

  where also $C^2 = R^2_{\mu\nu\rho\sigma} - \frac{8}{3}\Lambda^2$, $\mathrm{GB} = R^2_{\mu\nu\rho\sigma}$ and $\mathrm{GB} = C^2 + \frac{8}{3}\Lambda^2$

- they are all Bach-flat in $d = 4$ dimensions (classical solutions of conformal Weyl gravity)

Partition function on MSS

Derivation by Fradkin and Tseytlin '85

- second variation around MSS background in TT fluctuation fields

$$\delta^2 S = \int d^4x \sqrt{g}\, h_{\mu\nu}^{TT} \left(\Box - \frac{2}{3}\Lambda\right)\left(\Box - \frac{4}{3}\Lambda\right) h_{\mu\nu}^{TT}$$

- after taking Jacobian of the change of variables under PI and the FP determinant into account

$$Z^2_{1-\text{loop}} \sim \det{}^{-1}\left(\frac{\delta^2 S}{\delta h^2_{\mu\nu}}\right) = \frac{\det_{1T}\left(\Box + \Lambda\right)\det_0\left(\Box + \frac{4}{3}\Lambda\right)}{\det_{2TT}\left(\Box - \frac{2}{3}\Lambda\right)\det_{2TT}\left(\Box - \frac{4}{3}\Lambda\right)}$$

- correcting by contribution of zero modes

$$Z^2_{1-\text{loop}} = \frac{\det_1^2\left(\Box + \Lambda\right)\det_1\left(\Box + \frac{1}{3}\Lambda\right)\det_0\left(\Box + \frac{4}{3}\Lambda\right)}{\det_{2T}\left(\Box - \frac{2}{3}\Lambda\right)\det_{2T}\left(\Box - \frac{4}{3}\Lambda\right)\det_0\left(\Box + 2\Lambda\right)}$$

Partition function on Ricci-flat background

Derivation by Fradkin and Tseytlin '85

- second variation around Ricci-flat background in TT fluctuation fields

$$\delta^2 S = \int d^4x \sqrt{g}\, h_{\mu\nu}^{TT} \left(\Box - 2\hat{C}\right)^2 h_{\mu\nu}^{TT}$$

- after taking Jacobian of the change of variables under PI and the FP determinant into account

$$Z_{1-\mathrm{loop}}^2 \sim \det{}^{-1}\left(\frac{\delta^2 S}{\delta h_{\mu\nu}^2}\right) = \frac{\det_1^3 \Box \, \det_0^2 \Box}{\det_2^2 \left(\Box - 2\hat{C}\right)}$$

- there is no correcting contribution from zero modes

Functional RG flow

RG flow equation

$$\partial_t \Gamma_k = \mathrm{Tr}\left(\frac{\partial_t R_k \hat{\mathbf{1}}}{\hat{\Box} + R_k \hat{\mathbf{1}} + a\hat{\mathbf{1}}}\right)$$

Truncation ansatz

For the L.H.S. of the flow equation

$$k\frac{d}{dk}\Gamma_k^L = \beta_C C^2 + \beta_{\mathrm{GB}}\mathrm{GB}$$

at the one-loop level we have $\beta_R R^2 = 0$ in Weyl conformal gravity (Fradkin Tseytlin '85). For the R.H.S. $\Gamma_k^R = C^2$

Two beta functions β_C and β_{GB}

available by projecting the flow to MSS or Ricci-flat backgrounds (or ES in one stroke)

Flow on MSS

$$\partial_t \Gamma_k^L = \mathrm{Tr}_{2T}\left(\frac{(\partial_t R_k - \eta R_k)\,\hat{\mathbf{1}}}{\hat{\Box} + R_k\hat{\mathbf{1}} - \frac{2}{3}\Lambda\hat{\mathbf{1}}}\right) + \mathrm{Tr}_{2T}\left(\frac{(\partial_t R_k - \eta R_k)\,\hat{\mathbf{1}}}{\hat{\Box} + R_k\hat{\mathbf{1}} - \frac{4}{3}\Lambda\hat{\mathbf{1}}}\right) - 2\mathrm{Tr}_1\left(\frac{(\partial_t R_k - \eta R_k)\,\hat{\mathbf{1}}}{\hat{\Box} + R_k\hat{\mathbf{1}} + \Lambda\hat{\mathbf{1}}}\right) -$$

$$-\mathrm{Tr}_1\left(\frac{(\partial_t R_k - \eta R_k)\,\hat{\mathbf{1}}}{\hat{\Box} + R_k\hat{\mathbf{1}} + \frac{1}{3}\Lambda\hat{\mathbf{1}}}\right) + \mathrm{Tr}_0\left(\frac{(\partial_t R_k - \eta R_k)\,\hat{\mathbf{1}}}{\hat{\Box} + R_k\hat{\mathbf{1}} + 2\Lambda\hat{\mathbf{1}}}\right) - \mathrm{Tr}_0\left(\frac{(\partial_t R_k - \eta R_k)\,\hat{\mathbf{1}}}{\hat{\Box} + R_k\hat{\mathbf{1}} + \frac{4}{3}\Lambda\hat{\mathbf{1}}}\right)$$

from which we derive

$$\beta_{\mathrm{GB}} = \frac{1}{2}(2-\eta)\left[-\frac{21}{40}\left(1 - \frac{\frac{2}{3}\Lambda}{k^2}\right)^{-1} + \frac{9}{40}\left(1 - \frac{\frac{4}{3}\Lambda}{k^2}\right)^{-1} -\right.$$

$$\left. -\frac{179}{45}\left(1 + \frac{\Lambda}{k^2}\right)^{-1} - \frac{59}{90}\left(1 + \frac{\frac{1}{3}\Lambda}{k^2}\right)^{-1} + \frac{479}{360}\left(1 + \frac{2\Lambda}{k^2}\right)^{-1} - \frac{269}{360}\left(1 + \frac{\frac{4}{3}\Lambda}{k^2}\right)^{-1}\right]$$

- consistent with the one-loop results in Weyl gravity by Fradkin and Tseytlin, if $\eta = 0$ and $k \to \infty$, i.e. $\beta_{\mathrm{GB}} = -\frac{87}{20}$

Flow on Ricci-flat background

$$\partial_t \Gamma_k^L = 2\mathrm{Tr}_2\left(\frac{(\partial_t R_k - \eta R_k)\,\hat{\mathbf{1}}}{\hat{\Box} - 2\hat{C} + R_k \hat{\mathbf{1}}}\right) - 3\mathrm{Tr}_1\left(\frac{(\partial_t R_k - \eta R_k)\,\hat{\mathbf{1}}}{\hat{\Box} + R_k \hat{\mathbf{1}}}\right) - 2\mathrm{Tr}_0\left(\frac{(\partial_t R_k - \eta R_k)\,\hat{\mathbf{1}}}{\hat{\Box} + R_k \hat{\mathbf{1}}}\right)$$

from which we derive

$$\partial_t \Gamma_k^L = \frac{1}{2}(2-\eta)\int d^4x\sqrt{g}\left(2\frac{19}{18}C^2 - 3\left(-\frac{11}{180}\right)C^2 - 2\left(\frac{1}{180}\right)C^2\right) =$$

$$= (2-\eta)\int d^4x\sqrt{g}\left(\frac{411}{180}C^2\right)$$

$$\beta_C + \beta_{\mathrm{GB}} = \frac{1}{2}(2-\eta)\left(\frac{137}{60}\right)$$

- consistent with the one-loop results in Weyl gravity by Fradkin and Tseytlin, if $\eta = 0$, i.e. $\beta_C + \beta_{\mathrm{GB}} = \frac{137}{60} \quad \Longrightarrow \quad \beta_C = \frac{199}{30}$

Beta functions and fixed points

Beta functions

- the system for two beta functions can be solved algebraically
- anomalous dimension of the conformal graviton field

$$\eta = -\frac{1}{\omega_C}\partial_t \omega_C$$

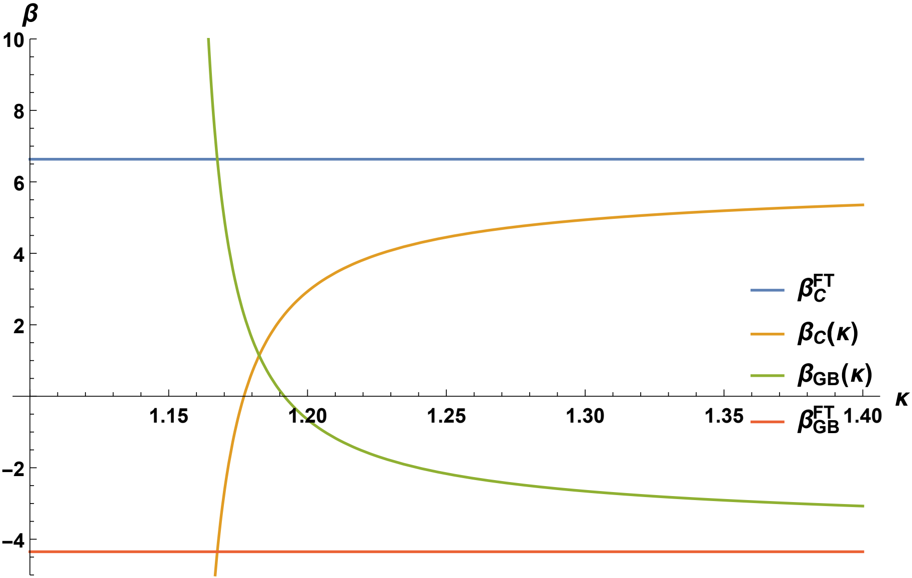
can be included for free

FP's

- UV: for $k \to +\infty$ asymptotic freedom in the coupling $\left(\sqrt{\omega_C}\right)^{-1}$ and $\left(\sqrt{-\omega_{\rm GB}}\right)^{-1}$, Gaussian FP
- IR: for $\beta_C = 0$ and $\beta_{\rm GB} = 0$ conditions we find respectively
  $\kappa_C \approx 1.17709 \quad \text{and} \quad \kappa_{\rm GB} \approx 1.19163$ for the case of $\Lambda > 0$
  $\kappa_C \approx 1.49722 \quad \text{and} \quad \kappa_{\rm GB} \approx 1.52128$ for the case of $\Lambda < 0$

  with the rescaling $\kappa = \frac{k}{\sqrt{|\Lambda|}}$; 2% confluence of FP's

Beta functions for $\Lambda > 0$



Conclusions

Final comments

- The IR FP is for finite values of energies (moreover Λ-dependent), but can be shifted into the deep IR, where formally $k = 0$
- The IR FP exists entirely due to threshold phenomena, and the inclusion of the anomalous dimension η (even with a supposed exact non-perturbative expression) does not change anything
- The FP is for any values of the couplings $\omega_{\rm GB}$ and ω_C. There is no any constraint from which it could be possible to find some special FP values of the couplings $\omega^*_{\rm GB}$ and ω^*_C. It is a 2-dimensional surface of FP's
- The IR FP describes crossover rather than a 2nd order PT

Physical features in cosmology

- UV FP – Big Bang
- IR FP – onset of inflation
- scale Λ – radius of the inflationary de Sitter background, when $\Lambda > 0$

Literature

1. "Effective action in quantum gravity", I.L. Buchbinder, S.D. Odintsov, I.L. Shapiro, Bristol, UK: IOP (1992) 413 p
2. "Conformal Supergravity", E.S. Fradkin, A.A. Tseytlin, Phys. Rept. 119 (1985) 233-362
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5. "Conformal Symmetry in Field Theory and in Quantum Gravity," L. Rachwał, Universe 4, no. 11, 125 (2018), **Best Paper Award**
6. "Infrared behavior of Conformal Gravity", P. Jizba, L. Rachwał, P. Urban (soon on arxive)

Grazie!

Thank you!

Unitarity issue

Possible solutions

- perturbative: Lee-Wick-Anselmi modification of Feynman prescription of avoiding poles of the propagator
- numerical: ghosts do not show up at low energies (below Planck scale) (Shapiro et al.)
- non-perturbative:
  - quantization within PT-symmetric form of QM (Bender, Mannheim),
  - perturbation around wrong vacuum (instabilities like with tachyons), another gravitational vacuum (MSS) should be used instead,
  - unitarity is safe due to non-trivial UV FP (asymptotic safety),
  - conformal supergravity (Fradkin, Tseytlin) is so special that the problem is solved thanks to supersymmetry and conformal symmetries (in $\mathcal{N} = 4$ SYM there is no problem with unitarity),
  - in true quantum CFT there are no scattering amplitudes