Higher Derivative Gravity in the Functional RG approach

Kevin Falls, SISSA

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Introduction

- Can we make sense of gravity as a local quantum field theory?
- RG point of view: Continuum limits exist at fixed points of the RG flow.
- Predictions: Number of relevant directions = number free parameters.

Introduction

- Einstein's Theory:

$$S = \frac{1}{16\pi G_N} \int d^4x \sqrt{g}(2\Lambda - R)$$

- Perturbatively non-renormalisable'.

- The dimensionless coupling $\tilde{G} = Gk^2 \to 0$ for $k \to 0$: GR is perturbative around an IR fixed point $\tilde{G} = 0$.

- The question of what happens when $k \to \infty$ cannot be answered with perturbation theory.

- Choices: Give up on Einstein's theory or give up on perturbation theory.

Higher derivative gravity and asymptotic freedom in $D = 4$ dimensions

- Higher-derivative gravity

$$S = \int d^4x \sqrt{|g|} \Big[-Z_N(R - 2\Lambda) + \frac{1}{2\lambda} C^2 - \frac{\omega}{3\lambda} R^2 - \frac{1}{\rho} E \Big]$$

- Perturbatively renormalisable in an expansion around $\lambda = 0$ (Stelle '77).
- Asymptotic freedom (Avramidi and Barvinsky '85):

$$\beta_\lambda = -\frac{133}{160\pi^2} \lambda^2$$

- Open problem: Is the higher derivative theory unitary? → Talk by Piva

Before going non-perturbative...

- First approach to asymptotic safety: start in $D = 2 + \varepsilon$ dimensions
- One loop beta function

$$\beta_{\tilde{G}} = \varepsilon \tilde{G} - b \tilde{G}^2$$

- IR fixed point: $\tilde{G} = 0$
- UV fixed point $\tilde{G}_* = \varepsilon / b$.

Asymptotic Safety

- Non-perturbative investigations made possible by FRG.

- Beyond perturbation theory canonical power counting no longer dictates which couplings are relevant

- The beta functions define the RG flow which is a vector field on the theory space spanned by the couplings $\tilde{\lambda}_i = k^{-d_i}\lambda_i$.

$$S = \sum_i \lambda_i \mathcal{O}_i \tag{1}$$

- We do not know which terms are relevant a priori: We have to calculate the scaling exponents

Asymptotic Safety

- Asymptotic Safety: UV complete and predictive theory corresponds to a fixed point of the RG flow with finite number of relevant (i.e. IR repulsive) directions.

$$\beta_i(\tilde{\lambda}_*) = 0\,, \qquad M_{ij} = \frac{\partial \beta_i}{\partial \tilde{\lambda}_j}\bigg|_{\tilde{\lambda}=\tilde{\lambda}_*} \tag{2}$$

- Number of relevant direction = number of negative eigenvalues of M.
- Since the number of relevant directions is equal to the number of free parameters the theory will be predictive.

Asymptotic Safety

- Relevant (irrelevant) directions flow away (towards) from the fixed point into the IR.

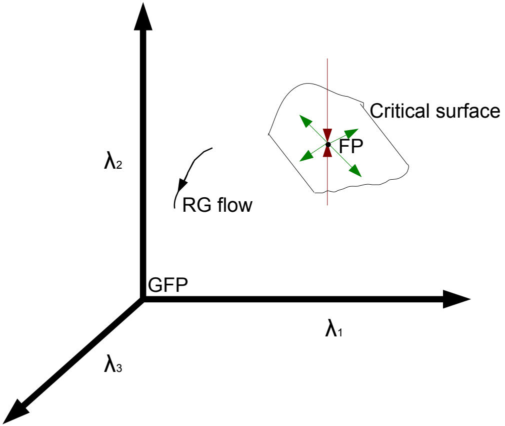


Figure: UV fixed point and critical surface. Arrows point from the UV to the IR.

Einstein-Hilbert approximation

- Einstein-Hilbert truncation

$$\Gamma_k = \int d^4x \sqrt{\det g} \frac{1}{16\pi G}(-R + 2\Lambda) + S_{\mathrm{gf}} + S_{\mathrm{gh}}$$

- Two running couplings Newton's constant and the cosmological constant.
- Evaluate the trace using the heat kernel expansion at $g = \bar{g}$

$$\frac{1}{2}\mathrm{Tr}\frac{1}{\Gamma_k^{(2)}[g,\bar{g}] + \mathcal{R}_k[\bar{g}]} k\partial_k \mathcal{R}_k[\bar{g}] = \int d^4x \sqrt{\det g}\Big(C_0(R,\Lambda,G) + \frac{R}{6}C_1(R,\Lambda,G)\Big) + \ldots$$

involves an expansion in curvature.

Einstein-Hilbert approximation

- Action is given by the Einstein-Hilbert action with a cosmological constant (Reuter '96, Souma '99)

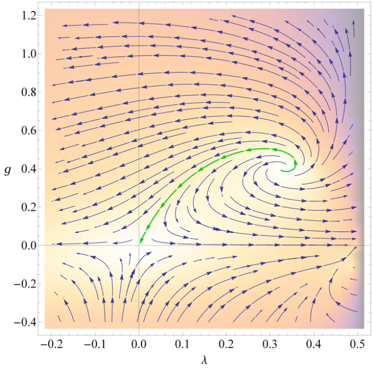


Figure: UV fixed point with RG trajectory leading to the Gaussian fixed point.

Extended background field approximations

- Actions of the $f(R)$ type (Codello, Percacci, Rahmede; Machado, Saueressig). Both polynomial (KF, Litim, Nikolakopoulos, Rahmede, Schröder) and functional approximations (Benedetti, Caravelli; Morris, Dietz; Demmel, Saueressig, Zanusso).
- Curvature squared R^2, C^2 (Benedetti, Machado, Saueressig).
- Polynomials of R and $R_{\mu\nu}R^{\mu\nu}$ (KF, Litim, King, Nikolakopoulos, Rahmede).
- Flow of the two-loop counter term C^3 (Gies, Knorr, Lippoldt, Saueressig).
- Flows on foliated spacetimes (Biemans, Platania, Saueressig).

Beyond the background field approximation

- The action is a functional of $\Gamma[\bar{g}, g] = \Gamma[\bar{g}, \bar{g} + h]$.
- Need to resolve the dependence on both metrics or equivalently on the background and the fluctuation.
- Two philosophies "bi-metric" $\Gamma[\bar{g}, g]$ (Reuter, Becker,..) and "vertex expansion" $\Gamma[\bar{g}, g] = \sum_n \Gamma^{(n)}[\bar{g}, \bar{g}] \frac{1}{n!} h^n$.

Vertex expansion

- Vertex expansion $\bar{g}_{\mu\nu} = \delta_{\mu\nu}$ and $g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$. Expand in Feynman Diagrams.
- Flow for n-vertex depends on $n+2$-vertex.
- Flows of propagator and three-vertex

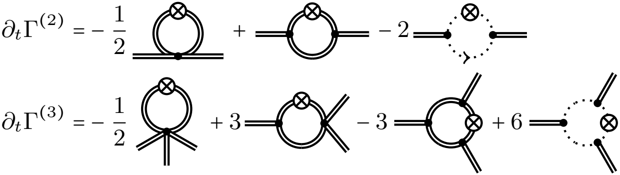


Figure: Flow of the graviton two- and three-point function.

- State of the art: Flows for the graviton four point function (Denz, Pawlowski, Reichert 2016). Closes the flow of the two point function.

Effective average action: $F(R)$ approximation

- $F(R)$ approximation

$$\Gamma_k = \int d^4x \sqrt{\det g} F(R) + \text{gauge fixing} + \text{ghosts}$$

project onto spacetimes of constant curvature

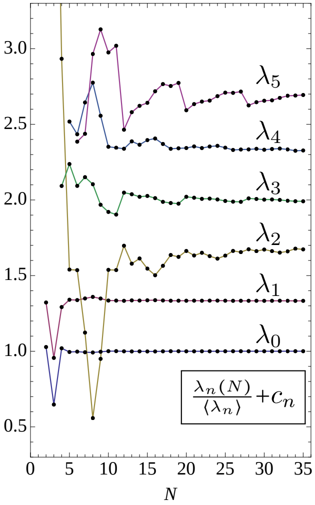
$$R_{\mu\nu\rho\sigma} = \frac{R}{d(d-1)} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho})$$

- Fixed point equation: Third order partial differential equation for $F(R)$ (Codello, Percacci and Rahmede 2007, Machado and Saueressig 2007)
- Bootstrap hypothesis (KF, D Litim, K Nikolakopoulos, C Rahmede 2013): canonical scaling $\vartheta_{G,n} = 2n - 4$ remains to be a good ordering principle beyond perturbation theory

$$F(R) = \frac{1}{16\pi} \sum_{n=0}^{N-1} \lambda_n (R/k^2)^n$$

Effective average action: $F(R)$ approximation

KF, D Litim, K Nikolakopoulos, C Rahmede 2013, 2014



Bootstrap

- Polynomial $F(R)$ approximation up to $N = 35$ (KF, D Litim, K Nikolakopoulos, C Rahmede 2013, 2014)
- Near gaussian exponents.

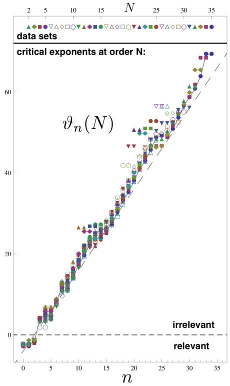


Figure: Critical exponents

Effective average action: $F(R)$ approximation

- First $F(R)$ flow equations which were written down possess a fixed point in polynomial approximations.
- Fixed point found in approximations up to order $N = 35$.
- Convergence of the fixed points and critical exponents.
- Three relevant directions.
- However due to the choice of regulator the equations have poles at finite curvature which prevent global solutions (Benedetti and Caravelli 2012, Dietz and Morris 2012) for $0 \le R \le \infty$.
- No exact de-Sitter solutions $RF'(R) - 2F(R) = 0$ are found.

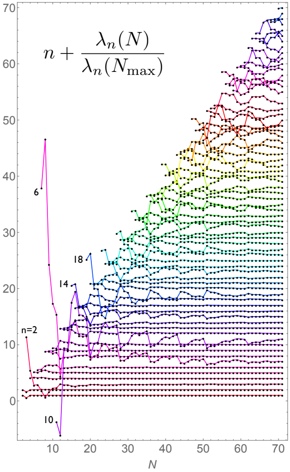
Effective average action: $F(R)$ approximation

- A new renormalisation group flow for $f(R)$ quantum gravity which ensures that technical poles at positive Ricci curvature are absent (KF, D Litim, J, Schröder):

$$\frac{df''}{dR} = \frac{(3-R)^2 f'' + (3-2R)f' + 2f}{\left(\frac{181}{1680}R^4 + \frac{29}{15}R^3 + \frac{91}{10}R^2 - 54\right)R} \times$$
$$\times \Bigg[48\pi\left(2f - Rf'\right) - \frac{211R^2 - 810R - 1080}{90}$$
$$- \frac{\left(\frac{311}{756}R^3 - \frac{1}{3}R^2 - 90R + 240\right)f' - \left(\frac{311}{756}R^3 - \frac{1}{6}R^2 - 30R + 60\right)Rf''}{3f - (R-3)f'}$$
$$- \frac{\left(\frac{37}{756}R^3 + \frac{29}{15}R^2 + 18R + 48\right)f' - \left(\frac{37}{756}R^4 + \frac{29}{10}R^3 + \frac{121}{5}R^2 + 12R - 216\right)f''}{(3-R)^2 f'' + (3-2R)f' + 2f} \Bigg].$$

- Possibility of global solutions for $-\infty \leq R \leq \infty$ (three fixed singularities vs three initial conditions).
- Polynomial studies up to order $N = 71$.

- Polynomial $F(R)$ approximation up to $N = 71$ (KF, D Litim, J Schröder)



- Polynomial $F(R)$ approximation up to $N = 71$ (KF, D Litim, J, Schröder 2018)

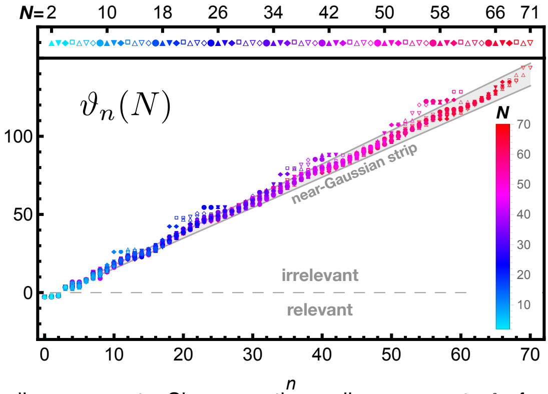


Figure: Universal scaling exponents. Shown are the scaling exponents ϑ_n for all approximation orders N. The inset and the top legend explain the colour-coding of approximation orders N. The gray-shaded area indicates that most scaling exponents, with increasing n, take values narrowly above, yet increasingly close to classical values within the near-Gaussian strip.

- The choice $\bar{\Gamma} = \int \sqrt{g} F_k(R)$ is not a unique way to close the equations even we project onto a maximally symmetric spacetime.
- We can distinguish a set of operators

$$\bar{\Gamma} = \int \sqrt{g} \sum_n \lambda_n O_n[g]$$

  if

$$\mathcal{O}_n\big|_{\text{sphere}} \propto \sqrt{g} R^n \, .$$

- Allowing operators which differ from powers of R we can test the effect of different tensor structures e.g. $R_{\mu\nu} R^{\mu\nu}$.
- Our choice (KF, C. King, D. Litim, K. Nikolakopoulos, C. Rahmede, Phys. Rev. D 97, 086006 (2018))

$$\bar{\Gamma} = \int \sqrt{g} \left[F(R_{\mu\nu} R^{\mu\nu}) + R \cdot Z(R_{\mu\nu} R^{\mu\nu}) \right]$$

  is as *ad hoc* as $F(R)$.
- Main physical difference is that we have a higher derivative hessian for the transverse traceless modes $\implies$ more degrees of freedom.

convergence of the couplings

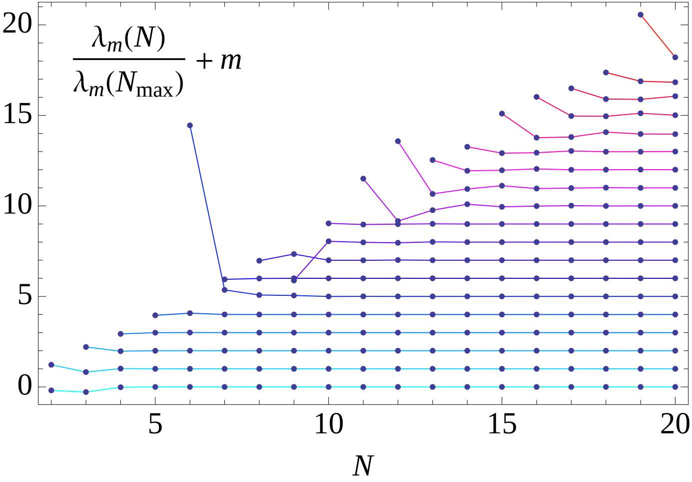


Figure: Couplings λ_m ($m = 0, \cdots, 18$ from bottom to top) with increasing approximation order N, normalised to their value at the highest order in the approximation $N_{\max} = 21$. The shift by m is added for better display. We observe a fast convergence with approximation order.

convergence: critical exponents

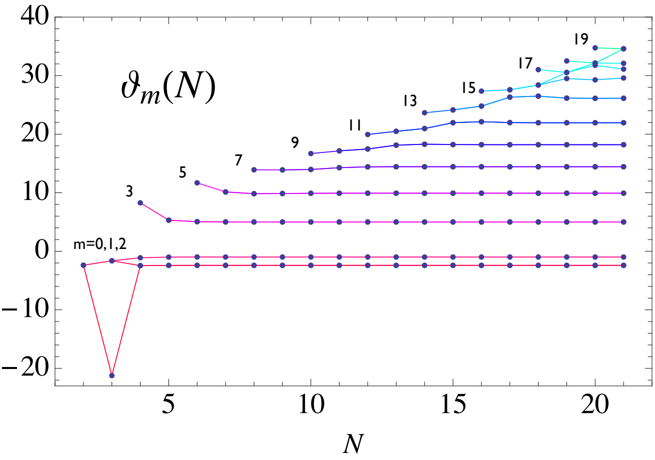


Figure: The universal eigenvalues $\vartheta_m(N)$ (real part if complex) for specific values of m as functions of the order of the polynomial approximation N.

bootstrap

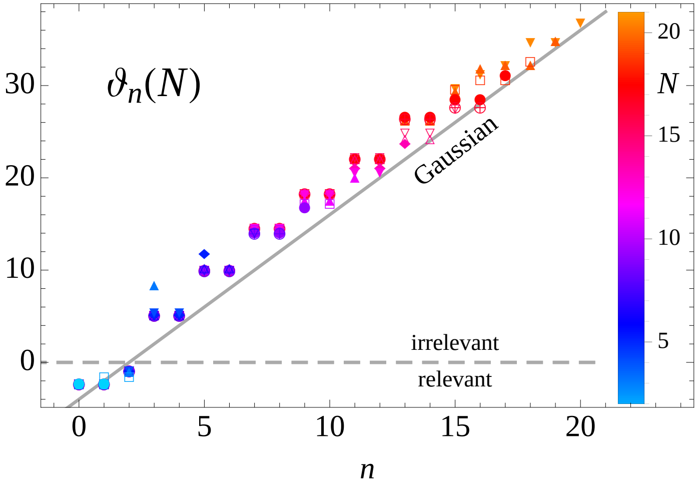


Figure: The universal scaling exponents $\vartheta_n(N)$ (real parts if complex) are shown for all approximation orders N, in comparison with classical exponents (straight line). We observe three relevant eigenvalues. All other eigenvalues are increasingly irrelevant, approaching Gaussian values from above.

- (anti)-de Sitter solutions in the $f(R)$

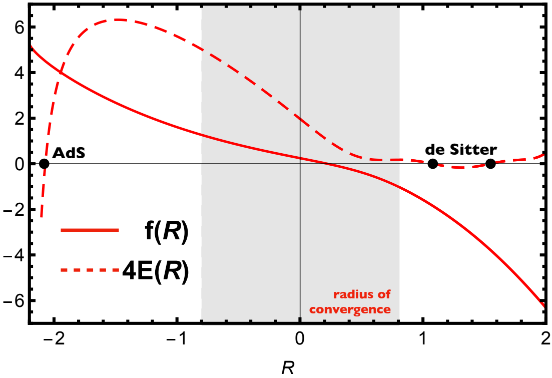


Figure: Fixed point action and quantum equations of motion. Shown are the functions $f(R)$ (full line) and $4E(R) = 8f(R) - 4Rf'(R)$ (dashed line) within their full domains of validity $|R| < R_+$. We observe that $f'(R) < 0$ throughout. The quantum equations of motion offers two de Sitter and an anti de Sitter solution (black dots), just outside the polynomial radius of convergence $E(R) > 0$ throughout.

(anti)-de Sitter solutions

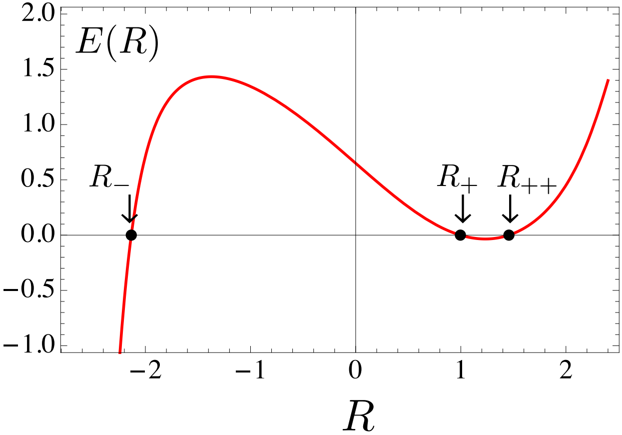


Figure: Shown is the equation of motion $E = \frac{1}{2} f\left(\frac{R^2}{4}\right) - \frac{R^2}{8} f'\left(\frac{R^2}{4}\right) + \frac{1}{4} R\, z\left(\frac{R^2}{4}\right) - \frac{R^3}{8} z'\left(\frac{R^2}{4}\right)$ zeros correspond to (anti)-de Sitter solutions.

Curvature squared gravity

- Action

$$\Gamma_k = \int d^4x\sqrt{|g|}\Big[- Z_N(R - 2\Lambda) + \frac{1}{2\lambda}C^2 - \frac{\omega}{3\lambda}R^2 - \frac{1}{\rho}E\Big] + \Gamma_{\rm gf} + \Gamma_{\rm gh}$$

- To resolve each independent operator one has to use a general background.
- To simplify the calculation one can go to Einstein-space and compute the running of two independent couplings out of three

$$R_{\mu\nu} = \frac{1}{4}g_{\mu\nu}R$$

  but then $E = C^2 + \frac{1}{6}R^2$. Can only resolve $g_{4a} = -\frac{\omega}{3\lambda} - \frac{1}{6}\rho$ and $g_{4b} = \frac{1}{2\lambda} - \frac{1}{\rho}$
- Fixed point (D. Benedetti, P. F. Machado and F. Saueressig 2009)

$$\tilde{\Lambda}^* = 0.218\,, \;\; \tilde{G}^* = 1.96\,, \;\; g^*_{4a} = 0.008\,, g^*_{4b} = -0.0050$$

- Three relevant directions

$$\theta_0 = 2.51\,, \;\; \theta_1 = 1.69\,, \;\; \theta_2 = 8.40\,, \;\; \theta_3 = -2.11$$

Curvature squared gravity

- Action

$$\Gamma_k = \int d^4x\sqrt{|g|}\Big[- Z_N(R - 2\Lambda) + \frac{1}{2\lambda}C^2 - \frac{\omega}{3\lambda}R^2 - \frac{1}{\rho}E\Big] + \Gamma_{\rm gf} + \Gamma_{\rm gh}$$

- **New calculation on a general background (KF, Ohta, Percacci)**

- To resolve each independent operator one has to use a general background.

- Utilise off-diagonal heat kernel technology (Groh, Saueressig, Zanusso 2011)

- Because $\int d^4x\sqrt{|g|}E$ the beta functions are independent of $1/\rho$. Consequently ρ will always be zero at any fixed point since $\beta_\rho \propto \rho^2$.

Hessian of higher derivative gravity

- The rhs of the flow involves the second functional derivative of the action
- We choose the minimal gauge: Four derivative terms $K\Delta^2$ where $\Delta = -\nabla^2$
- The Hessian is of the form

$$\Gamma_k^{(2)} = K_k(\Delta^2 + V_k^{\mu\nu}\nabla_\mu\nabla_\nu + U_k) \tag{3}$$

$$V_k^{\mu\nu} = V_{k,0}^{\mu\nu} + V_{k,1}^{\mu\nu} \tag{4}$$

where $V_{k,0}^{\mu\nu} \sim Z_N$ and $V_{k,1}^{\mu\nu} \sim R$

- $V_k^{\mu\nu}\nabla_\mu\nabla_\nu$ also contains 'minimal terms'

$$V_k^{\mu\nu}\nabla_\mu\nabla_\nu = K_{N,k}\Delta + V_k^{'\mu\nu}\nabla_\mu\nabla_\nu \tag{5}$$

Cutoff schemes

- We have considered two schemes
- In the first scheme the cutoff takes its structure from the fourth derivative terms

$$\mathcal{R} = K_k R_k(\Delta^2) \tag{6}$$

- In the second scheme we also include the second derivative terms

$$\mathcal{R} = K_k R_k(\Delta^2) + K_{N,k} R_k(\Delta) \tag{7}$$

We use the optimised cutoff

$$R(z^n) = (-z^n + k^{2n})\Theta(-z^n + k^{2n}) \tag{8}$$

Expansion of the propagator

- We need to invert the two point function $\Gamma_k^{(2)} + \mathcal{R}_k = F(\Delta) + U + V_k'^{\mu\nu}\nabla_\mu\nabla_\nu$
- This is technically challenging due non-minimal derivative terms in $V_k'^{\mu\nu}\nabla_\mu\nabla_\nu$.
- For now we additionally expand in Z_N as well as curvature.
- Our first approximation involved treating $R \sim Z_N \sim \Lambda$
- This is too server an approximation: λ has no non-trivial point; the only FP is at $\lambda_* = 0$.

Curvature squared gravity (Preliminary results!)

- Several fixed points.
- IR gaussian fixed point where the dimensionful Newton coupling becomes constant

$$\tilde{G} \sim k^2 G_N \to 0\,, \quad \beta_\lambda \sim 0.0109366\lambda^2\,, \quad \omega_* = -0.613866 \tag{9}$$

- This is the classical GR regime.
- Well known asymptotically free UV fixed point when $\lambda \to 0$ but $\tilde{G} \to G_*$

$$\beta_\lambda = -\frac{133}{160\pi^2}\lambda^2 \tag{10}$$

$$\lambda_* = 0\,, \quad \rho_* = 0\,, \quad \omega_* = -0.0228639\,, \quad G_* = 2.24308 \tag{11}$$

Curvature squared gravity (Preliminary results!)

- Four fully interacting UV fixed points.
- However, flowing away from the fixed points only one connects with the IR fixed point
- First Scheme:

$$\tilde{G}_* = 1.4013\,, \;\; \lambda_* = 31.393\,, \;\; \omega_* = 1.0088$$

Three relevant directions (excluding ρ)

$$\theta = \{2.8585,\; 2.6953,\; 0.5386,\; -1.2640\}$$

- Second Scheme:

$$\tilde{G}_* = 1.4386\,, \;\; \lambda_* = 33.050\,, \;\; \omega_* = 1.0038$$

Three relevant directions (excluding ρ)

$$\theta = \{2.7812,\; 2.7075,\; 0.5579,\; -1.3633\}$$

Curvature squared gravity (Preliminary results!)

- Flowing away from the UV fixed point we connect to the IR fixed point

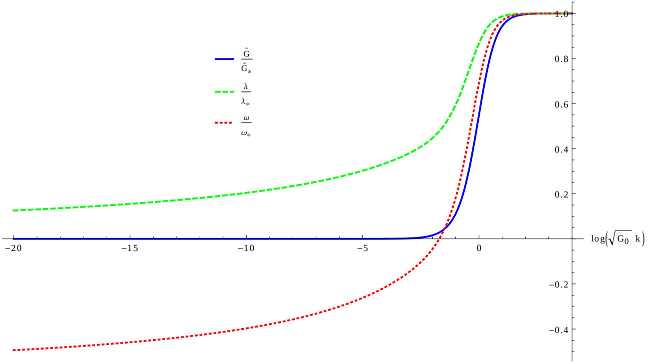


Figure: The cutoff dependence of $\tilde{G}$, λ and ω

Curvature squared gravity (Preliminary results!)

- Flow away from the fully interacting UV fixed point.

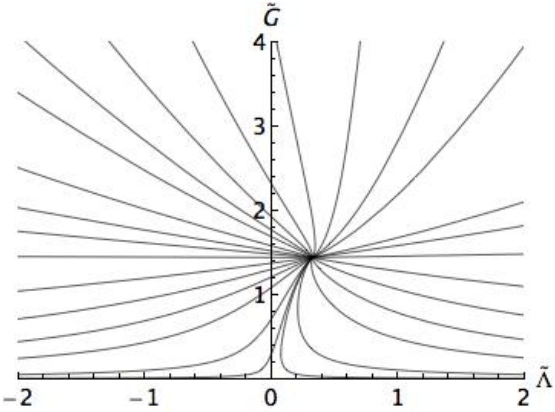


Figure: Flow in the $\tilde{\Lambda}$-$\tilde{G}$ plane.

Conclusion and outlook

- Asymptotic safety offers a possible mechanism to renormalise gravity if a fixed point with a finite number of relevant directions
- Many studies support the existence of a UV fixed point in quantum gravity and in gravity-matter systems.
- More sophisticated techniques needed to extend approximations.
- Increasing the number of canonically irrelevant operators in approximation does not increase the number of relevant directions.
- Working on a general background such that we can distinguish all operators is technically challenging.
- Preliminary results are encouraging: fixed point three relevant directions which connects to the IR.