Spectral functions with the Functional Renormalization Group

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Functional and Renormalization Group Methods
in Quantum and Statistical Physics

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CRC-TR 211

Outline

I) Introduction and motivation

II) Theoretical setup

- Functional Renormalization Group (FRG)
- Effective model for Quantum Chromodynamics (QCD)
- Analytic continuation procedure

III) Results

- Quark spectral function
- (Pseudo-)scalar meson spectral functions and transport coefficients
- (Axial-)vector meson spectral functions and dilepton rates

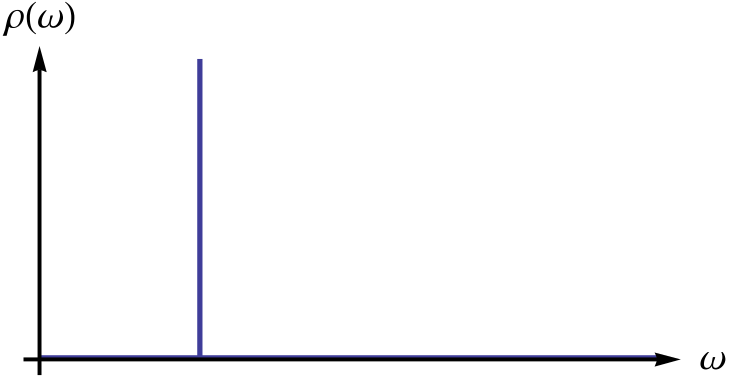
IV) Summary and outlook

I) Introduction and motivation

[courtesy L. Holicki]

What is a spectral function?

$$\rho(\omega) = -\frac{1}{\pi}\mathrm{Im}D^R(\omega)$$



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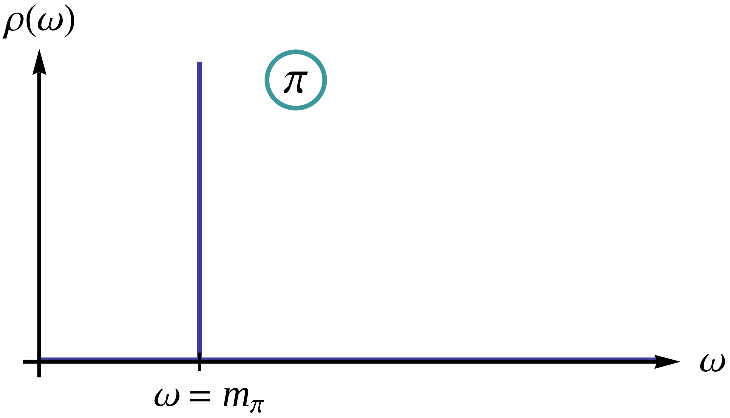

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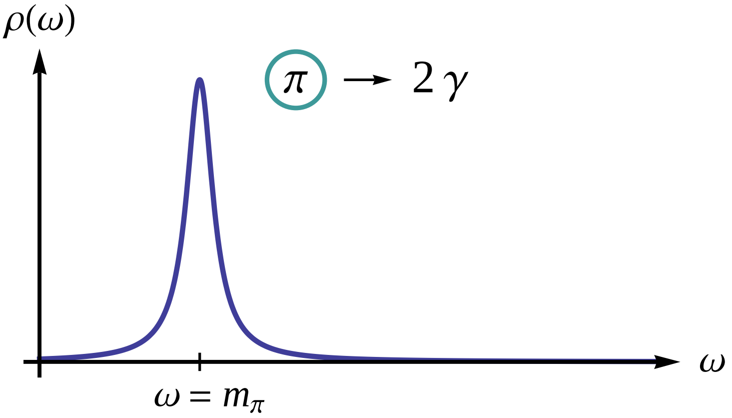

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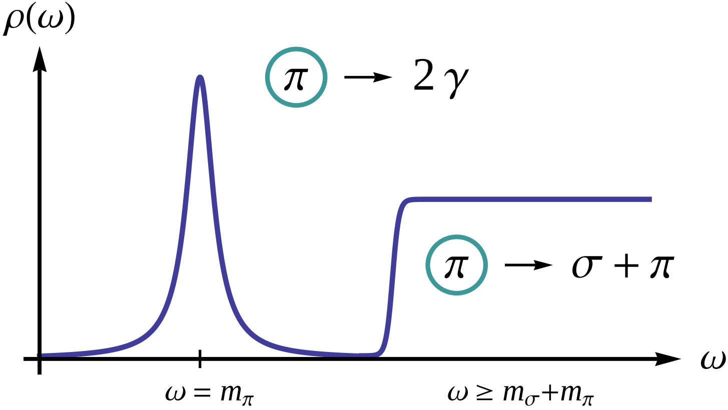


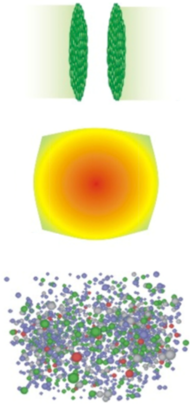
Why are spectral functions interesting?

Spectral functions determine both
real-time and imaginary-time propagators,

- $D^R(\omega) = -\int d\omega' \frac{\rho(\omega')}{\omega' - \omega - i\varepsilon}$
- $D^A(\omega) = -\int d\omega' \frac{\rho(\omega')}{\omega' - \omega + i\varepsilon}$
- $D^E(p_0) = \int d\omega' \frac{\rho(\omega')}{\omega' + ip_0}$

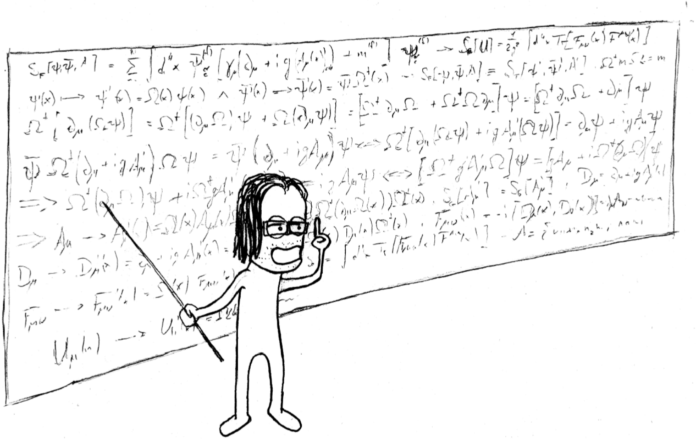
and thus allow access to many observables,
e.g. transport coefficients like the shear viscosity:

- $\eta = \frac{1}{24} \lim_{\omega \to 0} \lim_{|\vec{p}| \to 0} \frac{1}{\omega} \int d^4x \; e^{ipx} \left\langle \left[T_{ij}(x), T^{ij}(0) \right] \right\rangle$



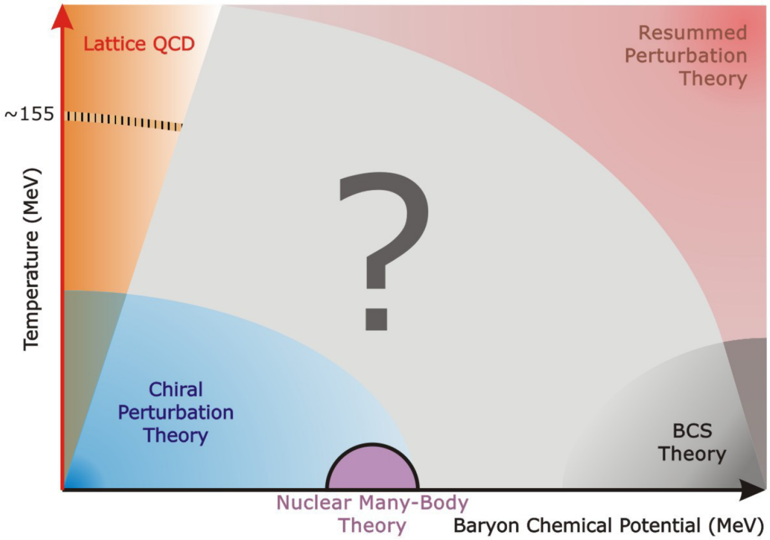
[B. Mueller, arXiv: 1309.7616]

II) Theoretical setup



[courtesy L. Holicki]

Methods to study the QCD phase diagram



Consistent theoretical framework

The Functional Renormalization Group (FRG) is a non-perturbative framework that is able to compute the thermodynamical and spectral properties of QCD matter on the **same footing!**

Advantages:

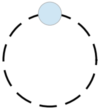
- thermodynamic consistency: direct connection between chiral symmetry restoration and spectral properties:
- preservation of symmetries and their breaking patterns
- properly deals with phase transitions since it includes both **thermal** and **quantum** fluctuations
- no sign problem
- well-defined and straightforward analytic continuation procedure

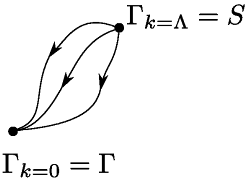
Functional Renormalization Group

Flow equation for the effective average action Γ_k:

$$\partial_k \Gamma_k = \frac{1}{2} \mathrm{STr} \left(\partial_k R_k \left[\Gamma_k^{(2)} + R_k \right]^{-1} \right)$$

[C. Wetterich, Phys. Lett. **B 301** (1993) 90]

$\partial_k \Gamma_k \;=\; \frac{1}{2}$ 



[wikipedia.org/wiki/Functional_renormalization_group]

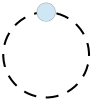
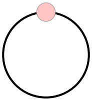
- Γ_k interpolates between bare action S at $k = \Lambda$ and effective action Γ at $k = 0$
- regulator R_k acts as a mass term and suppresses fluctuations with momenta smaller than k
- the use of 3D regulators allows for a simple analytic continuation procedure

Quark-meson model

Ansatz for the scale-dependent effective average action:

$$\Gamma_k[\overline{\psi}, \psi, \phi] = \int d^4x \left\{ \overline{\psi} \left(\partial\!\!\!/ + h(\sigma + i\vec{\tau}\vec{\pi}\gamma_5) - \mu\gamma_0 \right) \psi + \frac{1}{2}(\partial_\mu \phi)^2 + U_k(\phi^2) - c\sigma \right\}$$

- effective low-energy model for QCD with two flavors
- describes spontaneous and explicit chiral symmetry breaking
- flow equation for the effective average action:

$$\partial_k \Gamma_k = \frac{1}{2}$$  $-$ 

Flow of the effective potential at $\mu = 0$ and $T = 0$

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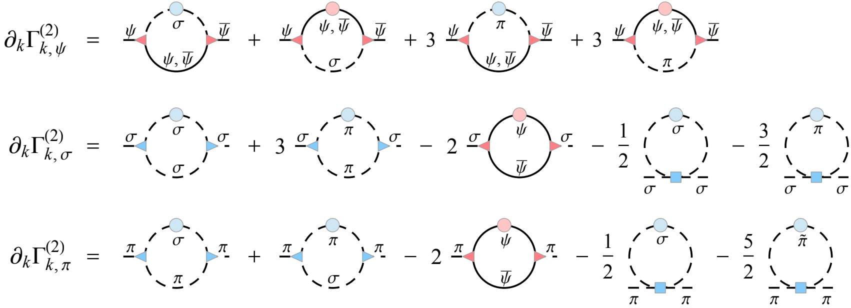
Flow equations for two-point functions

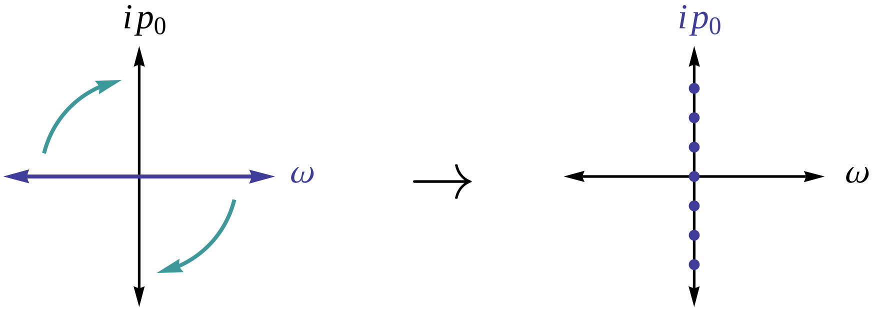


- quark-meson vertices are given by $\Gamma^{(3)}_{\bar{\psi}\psi\sigma} = h$, $\Gamma^{(3)}_{\bar{\psi}\psi\vec{\pi}} = ih\gamma^5\vec{\tau}$
- mesonic vertices from scale-dependent effective potential: $U^{(3)}_{k,\phi_i\phi_j\phi_m}$, $U^{(4)}_{k,\phi_i\phi_j\phi_m\phi_n}$
- one-loop structure and 3D regulators allow for a simple analytic continuation!

[R.-A. Tripolt, L. von Smekal, and J. Wambach, Phys. Rev. **D 90**, 074031 (2014)]

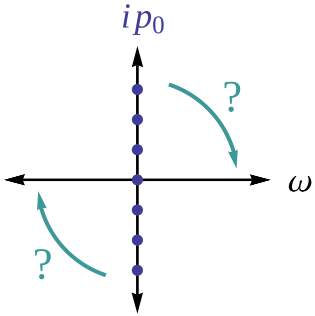
The analytic continuation problem

Calculations at finite temperature are often performed using imaginary energies:



The analytic continuation problem

Analytic continuation problem: How to get back to real energies?



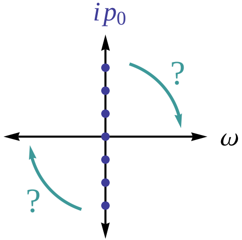
Two-step analytic continuation procedure

1) Use periodicity w.r.t. imaginary energy $ip_0 = i2n\pi T$:

$$n_{B,F}(E + ip_0) \to n_{B,F}(E)$$

2) Substitute p_0 by continuous real frequency ω:

$$\Gamma^{(2),R}(\omega, \vec{p}) = -\lim_{\epsilon \to 0} \Gamma^{(2),E}(ip_0 \to -\omega - i\epsilon, \vec{p})$$



Spectral function is then given by

$$\rho(\omega, \vec{p}) = -\frac{1}{\pi} \mathrm{Im} \frac{1}{\Gamma^{(2),R}(\omega, \vec{p})}$$

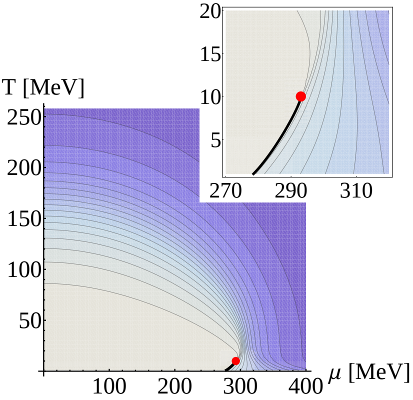
[R.-A. T., N. Strodthoff, L. v. Smekal, and J. Wambach, Phys. Rev. **D 89**, 034010 (2014)]
[J. M. Pawlowski, N. Strodthoff, Phys. Rev. **D 92**, 094009 (2015)]
[N. Landsman and C. v. Weert, Physics Reports 145, 3&4 (1987) 141]

III) Results

[courtesy L. Holicki]

Phase diagram of the quark-meson model

- chiral order parameter σ_0 decreases towards higher T and μ
- a crossover is observed at $T \approx 175$ MeV and $\mu = 0$
- critical endpoint (CEP) at $\mu \approx 292$ MeV and $T \approx 10$ MeV
- we will study spectral functions along $\mu = 0$ and $T \approx 10$ MeV



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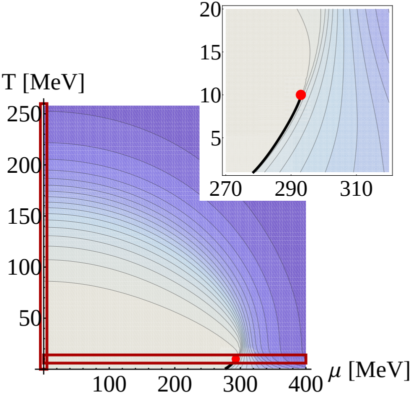

[R.-A. T., N. Strodthoff, L. v. Smekal, and J. Wambach, Phys. Rev. **D 89**, 034010 (2014)]

Flow of quark spectral function at $\mu = T = 0$

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[R.-A. T., J. Weyrich, L. v. Smekal, and J. Wambach, Phys.Rev. D98 (2018) no.9, 094002]

Flow of σ and π spectral function at $\mu = T = 0$

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σ and π spectral function for $T > 0$ at $\mu = 0$

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σ and π spectral function for $\mu > 0$ at T_c

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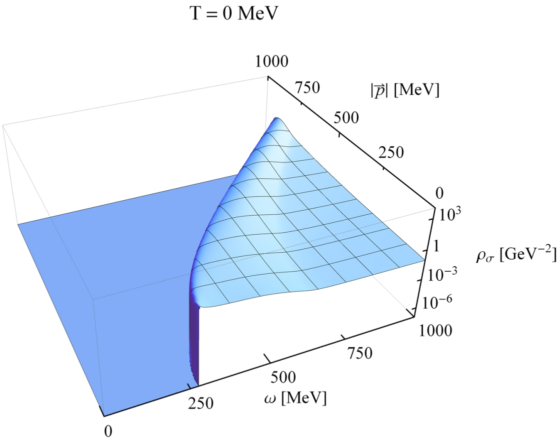
σ spectral function vs. ω and $\vec{p}$ at $\mu = T = 0$

- time-like region ($\omega > \vec{p}$) is Lorentz-boosted to higher energies
- space-like region ($\omega < \vec{p}$) is non-zero at finite T due to space-like processes



σ spectral function vs. ω and $\vec{p}$ for $T > 0$, $\mu = 0$

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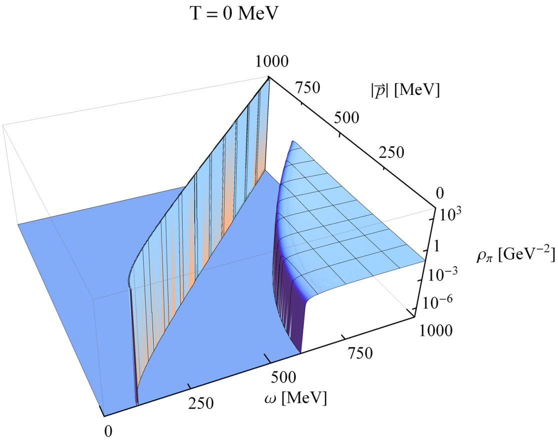
π spectral function vs. ω and $\vec{p}$ at $\mu = T = 0$

- time-like region ($\omega > \vec{p}$) is Lorentz-boosted to higher energies
- capture process $\pi^* + \pi \to \sigma$ is suppressed at large $\vec{p}$
- space-like region ($\omega < \vec{p}$) is non-zero at finite T due to space-like processes



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Shear viscosity

Applying the Green-Kubo formula for the shear viscosity

$$\eta = \frac{1}{24} \lim_{\omega \to 0} \lim_{|\vec{p}| \to 0} \frac{1}{\omega} \int d^4x \; e^{ipx} \left\langle \left[T_{ij}(x), T^{ij}(0) \right] \right\rangle$$

to the quark-meson model with energy-momentum tensor

$$T^{ij}(x) = \frac{i}{2} \left(\overline{\psi} \gamma^i \partial^j \psi - \partial^j \, \overline{\psi} \gamma^i \psi \right) + \partial^j \sigma \partial^i \sigma + \partial^j \vec{\pi} \partial^i \vec{\pi}$$

gives

$$\eta_{\sigma,\pi} \propto \int \frac{d\omega}{2\pi} \int \frac{d^3p}{(2\pi)^3} \; p_x^2 \, p_y^2 \, n_B'(\omega) \, \rho_{\sigma,\pi}^2(\omega, \vec{p})$$

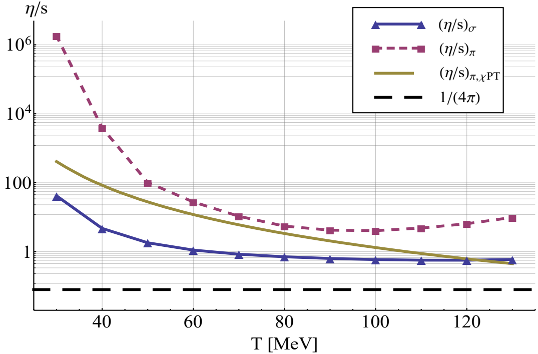
[R.-A. Tripolt, L. von Smekal, and J. Wambach, Int.J.Mod.Phys. E26 (2017) no.01n02, 1740028]

Shear viscosity over entropy density η/s at $\mu = 0$

- $\eta_{\pi,\chi\mathrm{PT}}$: result from chiral perturbation theory

[Lang, Kaiser, Weise, EPJ **A 48**, 109 (2012)]

- large shear viscosity at low temperatures due to small width of pion peak
$\rightarrow 4\pi$ processes missing
- η/s is always larger than the AdS/CFT limiting value of $\eta/s \geq 1/4\pi$

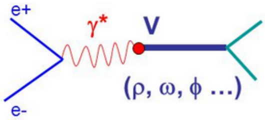


[R.-A. Tripolt, L. von Smekal, and J. Wambach, Int.J.Mod.Phys. E26 (2017) no.01n02, 1740028]

Vector Meson Spectral Functions

Important in heavy-ion collisions:

- Vector mesons mix with photons and can decay directly into lepton pairs
- Lifetime $\tau_\rho \approx 1.3$ fm/c, smaller than lifetime of fireball (≈ 10 fm/c)
- dileptons allow to measure in-medium vector-meson spectral functions

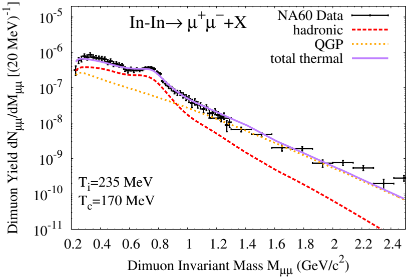


Dilepton rate:

$$\frac{dN_{ll}}{d^4x d^4q} \sim \mathrm{Im}\Pi^{\mu\nu}_{\mathrm{em}}(M, q; \mu, T)$$

For low-energy regime $M \leq 1$ GeV (VMD):

$$\mathrm{Im}\Pi^{\mu\nu}_{\mathrm{em}} \sim \mathrm{Im}D^{\mu\nu}_{\rho} + \frac{1}{9}\mathrm{Im}D^{\mu\nu}_{\omega} + \frac{2}{9}\mathrm{Im}D^{\mu\nu}_{\phi}$$



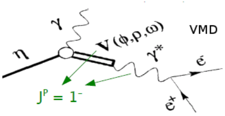
[Rapp, van Hees, Phys.Lett. B **753** (2016) 586-590]

Modeling vector mesons

- Sakurai (1960): vector mesons as gauge bosons of $SU(2)$ gauge symmetry
  - $\rightarrow$ Electromagnetic-hadronic interaction via exchange of vector mesons
  - $\rightarrow$ Current Field Identity (CFI):

$$j^{\mu}_{\mathrm{em}} = \frac{m_\rho^2}{g_\rho}\rho^\mu + \frac{m_\omega^2}{g_\omega}\omega^\mu + \frac{m_\phi^2}{g_\phi}\phi^\mu$$

$\Rightarrow$ **Vector Meson Dominance (VMD)**



[Berghaeuser, www.staff.uni-giessen.de (2016)]

- Lee and Nieh (1960s): Gauged linear sigma model with $SU(2)_L \times SU(2)_R$

$\Rightarrow$ $\boldsymbol{\rho}$ **meson** and chiral partner $\boldsymbol{a_1}$ **meson** as gauge bosons

Gauged linear-sigma model with quarks

- $SU(2)_L \times SU(2)_R$: corresponds to chiral symmetry of two-flavor QCD
- Additional gauge symmetry $U(1)$ to include photon field

Ansatz for the effective average action $\Gamma_k \equiv \Gamma_k[\sigma, \pi, \rho, a_1, \psi, \bar{\psi}, A_\mu]$:

$$\Gamma_k = \int d^4x \Big\{ \bar{\psi}\left(\not{D} - \mu\gamma_0 + h_S\left(\sigma + \mathrm{i}\vec{\tau}\vec{\pi}\gamma_5\right) + \mathrm{i}h_V\left(\gamma_\mu\vec{\tau}\vec{\rho}^\mu + \gamma_\mu\gamma_5\vec{\tau}\vec{a}_1^\mu\right)\right)\psi + U_k(\phi^2)$$

$$- c\sigma + \frac{1}{2}\left|(D_\mu - \mathrm{i}gV_\mu)\Phi\right|^2 + \frac{1}{8}\mathrm{Tr}\left(V_{\mu\nu}V^{\mu\nu}\right) + \frac{1}{4}m_{V,k}^2\mathrm{Tr}\left(V_\mu V^\mu\right)\Big\}$$

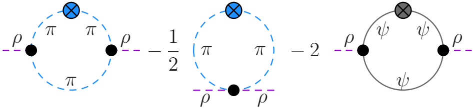
with

$$V_{\mu\nu} = D_\mu V_\nu - D_\nu V_\mu - \mathrm{i}g\left[V_\mu, V_\nu\right], \quad D_\mu\psi = \left(\partial_\mu - \mathrm{i}eA_\mu Q\right)\psi,$$

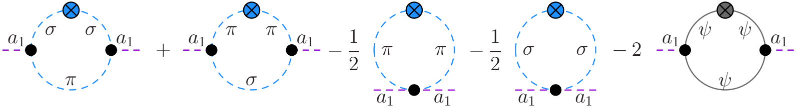
$$D_\mu V_\mu = \partial_\mu V_\nu - ieA_\mu[T_3, V_\nu], \quad \phi \equiv (\vec{\pi}, \sigma), \quad V_\mu \equiv \vec{\rho}_\mu\vec{T} + \vec{a}_{1,\mu}\vec{T}^5$$

Flow equations for ρ and a_1 2-point functions

$\partial_k \Gamma^{(2)}_{\rho,k} =$



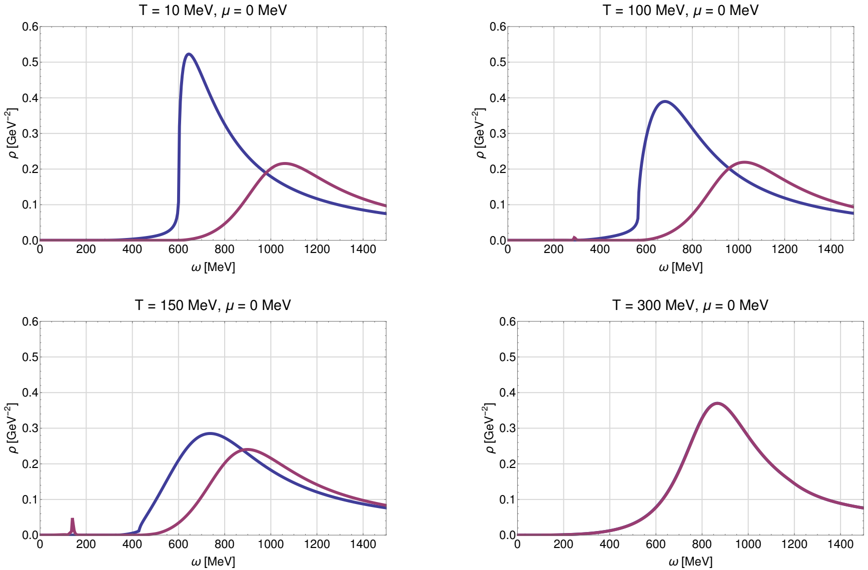
$\partial_k \Gamma^{(2)}_{a_1,k} =$



- neglect vector mesons inside the loops
- vertices extracted from ansatz for the effective average action Γ_k
- tadpole diagrams give ω-independent contributions

[C. Jung, F. Rennecke, R.-A. T., L. von Smekal, and J. Wambach, Phys. Rev. **D 95**, 036020 (2017)]

T-dependence of ρ and a_1 spectral functions



[C. Jung, F. Rennecke, R.-A. T., L. von Smekal, and J. Wambach, Phys. Rev. **D 95**, 036020 (2017)]

T-dependence of ρ and a_1 spectral functions

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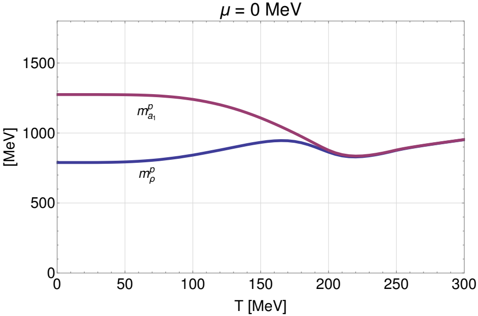
T-dependence of ρ and a_1 pole masses

- pole masses in the vacuum:

$$m_\rho^p = 789 \text{ MeV}, \quad m_{a_1}^p = 1275 \text{ MeV}$$

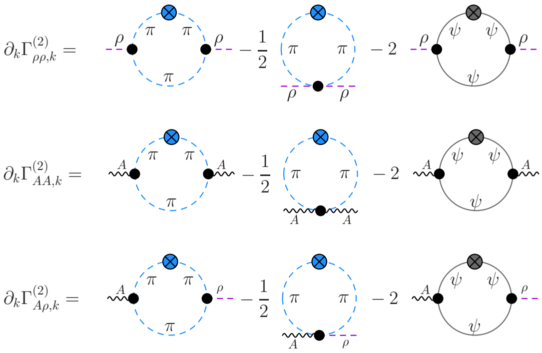
- degeneration of ρ and a_1 spectral functions in chirally symmetric phase
- broadening of spectral functions with increasing T
- pole masses do not vary much, no dropping ρ mass

$\Rightarrow$ consistent with broadening/melting-ρ-scenario



[C. Jung, F. Rennecke, R.-A. T., L. von Smekal, and J. Wambach, Phys. Rev. **D 95**, 036020 (2017)]

Electromagnetic (EM) spectral function



$$\begin{pmatrix} \Gamma^{(2)}_{AA} & \Gamma^{(2)}_{A\rho} \\ \Gamma^{(2)}_{\rho A} & \Gamma^{(2)}_{\rho\rho} \end{pmatrix} \xrightarrow{\text{diagonalize}} \begin{pmatrix} \tilde{\Gamma}^{(2)}_{AA} & 0 \\ 0 & \tilde{\Gamma}^{(2)}_{\rho\rho} \end{pmatrix}, \quad \tilde{\Gamma}^{(2)}_{AA} = \overbrace{\Gamma^{(2)}_{AA} - \frac{\Gamma^{(2)}_{A\rho}\Gamma^{(2)}_{\rho A}}{\Gamma^{(2)}_{\rho\rho}}}^{\mathcal{O}(e^2)} + \mathcal{O}(e^4)$$

[R.-A. T., C. Jung, N. Tanji, L. von Smekal, and J. Wambach, Nucl.Phys. A982 (2019) 775-778]

Calculation of dilepton rates

- We use the Weldon formula for the thermal dilepton rate:

$$\frac{d^8 N_{l\bar{l}}}{d^4 x d^4 q} = \frac{\alpha}{12\pi^3}\left(1 + \frac{2m^2}{q^2}\right)\left(1 - \frac{4m^2}{q^2}\right)^{1/2} q^2 (2\rho_T + \rho_L) n_B(q_0)$$

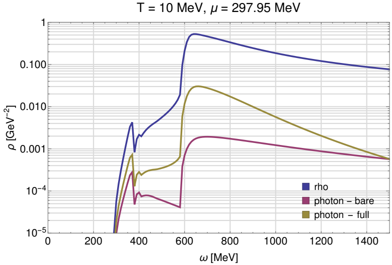
- in the following we assume $m = 0$ and set the external spatial momentum to zero, such that $\rho_T = \rho_L = \rho_{\tilde{A}\tilde{A}}$

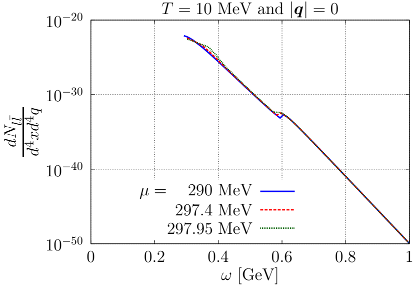
[H. A. Weldon, Phys. Rev. D42, 2384 (1990)]

[R.-A. T., C. Jung, N. Tanji, L. von Smekal, and J. Wambach, Nucl.Phys. A982 (2019) 775-778]

EM spectral function and dilepton rates

- U(1)-gauging of the linear-sigma model: rho mixes with the photon $\rightarrow$ diagonalization
- we use the Weldon formula to calculate the dilepton rate
- signatures for the CEP will show up when implementing fluctuating vector mesons and self-consistent spectral functions!





[R.-A. T., C. Jung, N. Tanji, L. von Smekal, and J. Wambach, Nucl.Phys. A982 (2019) 775-778]

Summary and Outlook

- the Functional Renormalization Group is a powerful non-perturbative framework to study the properties of strong-interaction matter at finite temperature and chemical potential
- analytically continued flow equations allow for the calculation of in-medium spectral functions and transport coefficients in the context of chiral symmetry restoration

Outlook:

- study spectral functions of quarks and baryons at finite density and temperature
- calculate realistic dilepton rates and identify signatures of phase transitions
- improve the calculation of transport coefficients like the shear viscosity and the electrical conductivity