β Functions as Gradients (Potential Flows)

U. Ellwanger (LPT Orsay)

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Motivation for Potential Flows:

- A-Theorem
- Relation to AdS/QFT correspondence (not considered here)
- Possible relations between coefficients of different β functions

Potential flows for dimensionless couplings g_i with β functions $\beta_i \equiv \mu \frac{\partial}{\partial \mu} g_i(\mu)$:

$$\eta^{ij}(g)\beta_j = \frac{\partial}{\partial g_i}\Phi(g)$$

$\eta^{ij}(g)$: Metric in the space of couplings, guarantees covariance under redefinitions $g_i \to g_i(g_j')$

$\Phi(g)$: "Potential"

Contract with β_i:

$$\beta_i \eta^{ij}(g)\beta_j = \beta_i \frac{\partial}{\partial g_i}\Phi(g) \equiv \mu \frac{d}{d\mu}\Phi(g)$$

(if $\Phi(g)$ has no explicit μ dependence)

$\longrightarrow$ If the symmetric part of η^{ij} is positive definite:

$$\mu \frac{d}{d\mu}\Phi(g) > 0 \leftrightarrow A - \text{Theorem } (A(g) \equiv \Phi(g))$$

Similar to Zamolodchikov's $C-$Theorem in $d = 2$, the RG flow is irreversible.
Nowadays: proven by Komorgodski-Schwimmer via unitarity of the dilaton S-Matrix

Does a potential flow lead to relations among coefficients of β functions via $(\partial_{g_i}\partial_{g_j} - \partial_{g_j}\partial_{g_i})\Phi(g) = 0$?

Yes if $\eta^{ij}(g)$ can be deduced independently from β_i, e.g. $\eta^{ij}(g) = c^{ij} + \mathcal{O}(g)$

U.E., 1812.06751:
But not in general: For any set of β_i a metric $\eta^{ij}(g)$ can be constructed such that a potential flow exists!

Moreover: Any potential $\Phi(g)$ which can be written as

$$\Phi(g) = \beta_i \frac{\partial}{\partial g_i} P(g)$$

in terms of an arbitrary prepotential $P(g)$ does the job!

Construction of $\eta^{ij}(g)$:

First: Consider the system of coupled RG equations $\mu\frac{\partial}{\partial\mu}g_i = \beta_i(g),\ i = 1 \dots n$, where $\beta_i(g)$ may depend on all n g_i.

To each (dim.-less) coupling g_i corresponds a dimensionful integration constant Λ_i. E.g. QCD:

$$\mu\frac{\partial}{\partial\mu}\alpha_{QCD} = b_1\alpha_{QCD}^2 + ... \qquad \rightarrow \qquad \alpha_{QCD} = \frac{1}{b_1 \ln\left(\frac{\Lambda_{QCD}}{\mu}\right) + \dots}$$

Trivial remark: $\Lambda_{QCD}\frac{\partial}{\partial\Lambda_{QCD}}|_\mu\alpha_{QCD} = -b_1\alpha_{QCD}^2$

Consider two couplings g_1, g_2 (a truncated scalar-Yukawa system) with

$$\mu \frac{\partial g_1}{\partial \mu} \equiv \beta_1 = a_1 g_1^2 + a_2 g_1 g_2, \quad \mu \frac{\partial g_2}{\partial \mu} \equiv \beta_2 = b_1 g_2^2$$

$$\to g_1(\mu, \Lambda_1, \Lambda_2) = \frac{a_2 - b_1}{a_1 b_1 \ln\left(\frac{\mu}{\Lambda_2}\right) + (a_2 - b_1)\left[\ln\left(\frac{\mu}{\Lambda_2}\right)\right]^{a_2/b_1} \cdot C_1\left(\frac{\Lambda_1}{\Lambda_2}\right)}$$

$$g_2(\mu, \Lambda_2) = \frac{1}{b_1 \ln\left(\frac{\Lambda_2}{\mu}\right)}$$

where $C_1\left(\frac{\Lambda_1}{\Lambda_2}\right)$ is arbitrary

$\rightarrow$ One can define

$$\Lambda_1 \frac{\partial g_1}{\partial \Lambda_1} \equiv -\hat{\beta}_1{}^1, \quad \Lambda_2 \frac{\partial g_1}{\partial \Lambda_2} \equiv -\hat{\beta}_1{}^2,$$

$$\Lambda_1 \frac{\partial g_2}{\partial \Lambda_1} \equiv -\hat{\beta}_2{}^1 \ (=0), \quad \Lambda_2 \frac{\partial g_2}{\partial \Lambda_2} \equiv -\hat{\beta}_2{}^2$$

where $\hat{\beta}_1{}^1 + \hat{\beta}_1{}^2 = \beta_1$, $\hat{\beta}_2{}^1 + \hat{\beta}_2{}^2 = \beta_2$ (the arbitrary function $C_1\left(\frac{\Lambda_1}{\Lambda_2}\right)$ drops out).

Next: Switch variables from

$$g_1(\Lambda_1, \Lambda_2),\; g_2(\Lambda_1, \Lambda_2) \to \Lambda_1(g_1, g_2),\; \Lambda_2(g_1, g_2)$$

or, better, to $\tau_i(g) = -\ln(\Lambda_i(g))$.

General case: n couplings g_a, β_a (not degenerate), scales Λ_i or τ_i

Define $\frac{\partial \tau_i}{\partial g_a}$; its inverse $\frac{\partial g_a}{\partial \tau_i}$ satisfies

$$\sum_i \frac{\partial g_a}{\partial \tau_i} = \sum_i \hat{\beta}_a{}^i = \beta_a$$

hence we obtain for any i

$$\sum_a \frac{\partial \tau_i}{\partial g_a} \beta_a = \sum_a \frac{\partial \tau_i}{\partial g_a} \sum_j \frac{\partial g_a}{\partial \tau_j} = \sum_j \delta^j{}_i = 1 \qquad \text{(Eq.1)}$$

Given all β_a: Solve Eq.(1) for n independent $\tau_i(g_a)$
$\longrightarrow \qquad \tau_i = \tau_0(g) + F_i(g)$ where
τ_0 solves the inhomogenous Eq.(1), this is the hard part
F_i solve the homogenous eq. $\sum_a \frac{\partial F_i}{\partial g_a} \beta_a = 0$

Next:

— Introduce an arbitrary prepotential $P(g)$ or $P(\tau(g))$

— Define a potential $\Phi(g) = \mu \frac{dP}{d\mu} = \frac{\partial P}{\partial g_a}\beta_a = \frac{\partial P}{\partial \tau_i}\frac{\partial \tau_i}{\partial g_b}\beta_b$

— Consider

$$\frac{\partial \Phi}{\partial g_a} = \left(\frac{\partial}{\partial g_a}\frac{\partial P}{\partial \tau_i}\right)\frac{\partial \tau_i}{\partial g_b}\beta_b + \underbrace{\frac{\partial P}{\partial \tau_i}\frac{\partial}{\partial g_a}\underbrace{\left(\frac{\partial \tau_i}{\partial g_b}\beta_b\right)}_{=1 \text{ from Eq.}(1)}}_{=0}$$

$$= \frac{\partial^2 P}{\partial \tau_i \partial \tau_j}\frac{\partial \tau_j}{\partial g_a}\frac{\partial \tau_i}{\partial g_b}\beta_b = \eta^{ab}\beta_b \text{ with } \eta^{ab} = \frac{\partial^2 P}{\partial \tau_i \partial \tau_j}\frac{\partial \tau_j}{\partial g_a}\frac{\partial \tau_i}{\partial g_b}$$

$\longrightarrow$ Potential flow with a metric η^{ab} which is manifestly symmetric and covariant under $g \to g(g')$, but depends on $P(g)$ and the solutions $\tau_i(g_a)$ or $g_a(\tau_i)$

Previous toy model: Take the simplest prepotential $P(g) = p_1 g_1 + p_2 g_2$

$\longrightarrow \Phi(g) = p_1 \beta_1 + p_2 \beta_2$

$\longrightarrow$ Metric

$$
\begin{aligned}
\eta^{11} &= \frac{p_1(2a_1 g_1 + a_2 g_2 - 2b_1 g_2)}{a_1 g_1^2 + a_2 g_1 g_2 - b_1 g_1 g_2} \\
\eta^{12} = \eta^{21} &= \frac{p_1 a_2 g_1}{a_1 g_1^2 + a_2 g_1 g_2 - b_1 g_1 g_2} \\
\eta^{22} &= \frac{-p_1 a_2 g_1^2}{g_2(a_1 g_1^2 + a_2 g_1 g_2 - b_1 g_1 g_2)} + \frac{2p_2}{g_2}
\end{aligned}
$$

— Not manifestly positive

— Singular for $g_a \to 0$

— More solvable examples with similar properties exist, e.g. a coupled system of three 2-loop gauge couplings (see Eur.Phys.J. C79 (2019) 198)

Lesson:

— Given n β functions β_a for n dim.-less couplings g_a: A gradient flow can always be constructed, but n solutions of Eq.(1) for the scales $\tau_i(g) = -\ln(\Lambda_i(g))$ have to be found.

— No relations among coefficients of different β functions emerge

Predictive Potential Flows

In 1974 Wallace and Zia observed in a multi-scalar theory with different couplings $\lambda^a_{ijkl}\phi^i\phi^j\phi^k\phi^l$, in dim. reg. + min. substraction (DRMS) to 3 loop order, a potential flow with a metric $\eta^{ab} = c\delta^{ab} + \dots$ which implies relations among the coefficients of β^a

In a series of papers Jack and Osborn (JO) studied the renormalizability of theories with local couplings $g_i(x)$ (scalar + Yukawa + gauge couplings) in a gravitational background, i.e. with a Lagrangian including terms like

$$\mathcal{L} \simeq \mathcal{F}_{ij}(g)\partial_\mu g_i \partial^\mu g_j \cdot R + \mathcal{G}_{ij}(g)\partial^\mu g_i \partial^\nu g_j \cdot (R_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}R) + \dots$$

In Nucl.Phys.B343 (1990) 647 they studied the β functions for the "coupling functions" $\mathcal{F}_{ij}(g), \mathcal{G}_{ij}(g)$ explicitely in perturbation theory in g_i in DRMS.

$\longrightarrow$ The absence of $\frac{1}{\varepsilon}$ poles in the β functions implies a potential flow, with η^{ij} symmetric up to 2 loop order, explaining the result of Wallace and Zia.

Osborn 1991: A potential flow follows from "Weyl consistency conditions", i.e. the fact that Weyl rescalings of local couplings $g_i(x)$ (anomalous if $\beta_i \neq 0$) must commute since these are Abelian $\leftrightarrow$ Wess-Zumino consistency conditions.

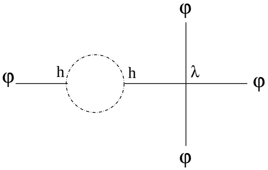
$\longrightarrow$ The same potential flow and relations between β functions follow, valid also for the Standard Model with Higgs^4 coupling λ, Higgs-top Yukawa coupling h_t and gauge couplings α_i:

$$\begin{array}{rcl} 1 \textit{ loop terms in } \beta_\lambda & \textit{related to} & 2 \textit{ loop terms in } \beta_{h_t} \\ & \textit{and to} & 3 \textit{ loop terms in } \beta_{\alpha_i} \\ 1 \textit{ loop terms in } \beta_{h_t} & \textit{related to} & 2 \textit{ loop terms in } \beta_{\alpha_i} \\ 2 \textit{ loop terms in } \beta_{\alpha_i} & \textit{related to} & 2 \textit{ loop terms in } \beta_{\alpha_j} \end{array}$$

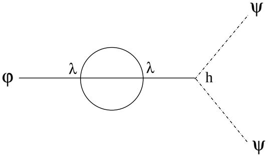
$\longrightarrow$ Have to reorganize the orders of perturbation theory?
(See e.g. Antipin et al., 1306.3234; Bond, Litim et al., 1710.07615)

Simple examples for $\mathcal{L} \sim \lambda\phi^4 + (h\phi\Psi\Psi + h.c.) + \ldots$

1) The only 1 loop contribution to $\beta_\lambda \sim \lambda h^2$ originates from



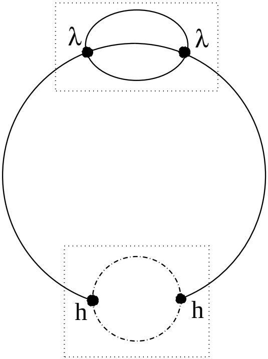
The only 2 loop contribution to $\beta_h \sim h\lambda^2$ originates from



One finds $\eta^{ab}\beta_b = \partial_a\Phi$ ($g_{a,b} \equiv \lambda, h$) with $\Phi \sim h^2\lambda^2 + \ldots$,
$\eta^{ab} \sim \delta^{ab}$ depending on the number of scalars/Fermions

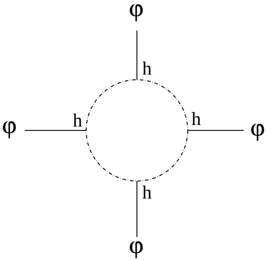
Origin

Absence of $\frac{1}{\varepsilon}$ poles in the β function for $\mathcal{A}_{ij}(g)$ in $\mathcal{L} \sim \mathcal{A}_{ij}(g)\Box g_i \Box g_j$ from the 4 loop vacuum diagram $\sim h^2\lambda^2$:

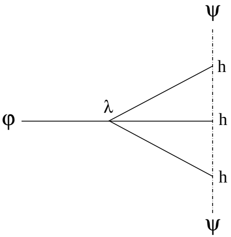


The fat dots denote local couplings $\lambda(x), h(x)$ which allow for inflow/outflow of momenta q; $\beta_{\mathcal{A}_{ij}}$ follows from $\mathcal{O}(q^4)$. (JO calculated $\beta_{\mathcal{A}_{ij}}$ in coordinate space.)

2) The only 1 loop contribution to $\beta_\lambda \sim h^4$ originates from



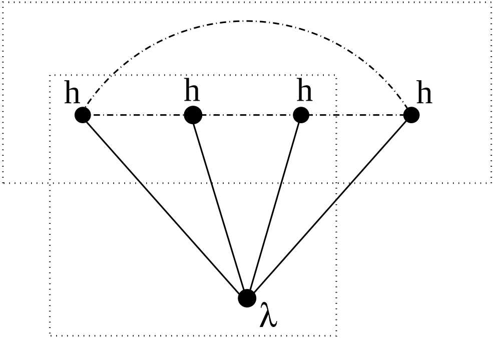
The only 2 loop contribution to $\beta_h \sim h^3\lambda$ originates from



One finds $\eta^{ab}\beta_b = \partial_a \Phi$ $(g_{a,b} \equiv \lambda, h)$ with $\Phi\ \sim h^4\lambda + \dots$, $\eta^{ab} \sim \delta^{ab}$ depending on the number of scalars/Fermions

Origin

Absence of $\frac{1}{\varepsilon}$ poles in $\beta_{\mathcal{A}_{ij}}$ from the 4 loop vacuum diagram $\sim h^4\lambda \sim q^4$:



Overlapping subdivergences are indicated by dotted boxes.

Lesson

- Relations between β functions β_i follow from vacuum diagrams with local couplings $g_i(x)$ including vertices $\sim \partial_\mu g_i, \Box g_i \dots$ which contribute to β functions like $\beta_{\mathcal{A}_{ij}}$.
- Local couplings $g_i(x) \equiv$ sources $J_i(x)$ for composite operators ϕ^4, $\phi\Psi\Psi$ etc.
  $\rightarrow$ Relations between β functions β_i follow from the consistent renormalization of correlation functions $\sim J_i^n(x)$ of composite operators. These play a central rôle for the AdS_5/QFT_4 correspondence where $J_i(x)$ are interpreted as fields on AdS_5.
- The origin of relations between different loop orders of different β functions is the fact that
  - gauge β functions follow from 2 point functions of gauge fields (+ Slavnov-Taylor identities),
  - Yukawa β functions follow from 3 point functions,
  - $\lambda\phi^4$ β functions follow from 4 point functions.
- These relations are missed in standard calculations of perturbative β functions!

Open Questions

- The above results are obtained in DRMS.
  Other regulators?
  E.g. a (Wilsonian-type) UV cutoff of propagators in terms of Schwinger parameters α:

$$\frac{1}{p^2+m^2} = \int_0^\infty e^{-\alpha(p^2+m^2)} \to \int_{1/\Lambda^2}^\infty e^{-\alpha(p^2+m^2)}$$

  Already used by BPHZ (in intermediate steps) for proof of renormalizability
  $\to$ Allows for a general proof of a potential flow?
- In DRMS dimensionful couplings like masses get only multiplicatively renormalized $\to$ no hierarchy problem in the absence of heavy fields.
  With a cutoff for Schwinger parameters the hierarchy problem $\Delta m^2 \sim \Lambda^2$ becomes manifest.
  $\to$ Constraints from a potential flow including masses within a Schwinger cutoff scheme?
  $\to$ Constraints from a consistent AdS/QFT correspondence?
  ...