Geometry of Bounded Critical Phenomena

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The people

- Andrea Trombettoni (CNR-IOM, Trieste)

Thanks to math people

- María del Mar González (UAM Madrid)
- Robin C. Graham (Washington U, Seattle)

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- Nicolò Defenu (U Heidelberg)

The paper

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Geometry of bounded critical phenomena
Giacomo Gori, Andrea Trombettoni

... joint mathematical work is coming soon

Outline

(Bounded) Criticality and Conformal invariance

- Symmetries at criticality. . .
- . . . in a bounded system

Uniformisation

- Yamabe Equation
- Anomalous dimension incluson
- Fractional Yamabe Equation
- Conformally covariant formulation

Interesting technicalities

- Conformal fractional laplacian. . .
- . . . in bounded domains
- Fractional Yamabe in a slab

Results

- Monte Carlo simulations
- One-point correlations
- Two-point correlations

Conclusions & Perspectives

Statistical Models at Criticality

Ising model

$$\mathcal{H}(\{s_i\}) = -\sum_{<i,j>} s_i s_j$$

- $s_i = \pm 1$ and $i = (i_1, i_2, \dots, i_d)$, $j = (j_1, j_2, \dots, j_d) \in \mathbb{Z}^d$
- $< i, j >$ nearest neighbors
- weight of a configuration $\{s_i\}$: $p(\{s_i\}) = \frac{1}{\mathcal{Z}} e^{-\beta \mathcal{H}(\{s_i\})}$

Averages of observables $\mathcal{O}(\{s_i\})$ are calculated as

$$\langle \mathcal{O} \rangle = \sum_{\{s_i\}} \mathcal{O}(\{s_i\}) e^{-\beta \mathcal{H}(\{s_i\})}$$

Symmetries in the continuum description

In the continuum limit (lattice spacing $\ll$ correlation length ξ)
$s_i \Rightarrow \phi(x)$ with $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ for infinite system.
We highly expect good properties under:

- Translations $\mathcal{T}$, $x \to x + a$ (d)

$$\langle \phi(x)\phi(y) \rangle = f(x - y)$$

A little less trivial

- Rotations $\mathcal{R}$, $x \to R\,x$ with $R^T R = \mathbb{1}$ ($\frac{d(d-1)}{2}$)

$$\langle \phi(x)\phi(y) \rangle = f(|x - y|)$$

for spinless operators. In general (spinful operators) we have covariance

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At the critical point ξ diverges. Continuum description becomes THE description. The system forgets about the lattice spacing and becomes *scale invariant*,

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- Scaling $\mathcal{S}$, $x \to \lambda x$ (1)
$$\langle\phi(0)\phi(\lambda x)\rangle = \lambda^{-\Delta_\phi}\langle\phi(0)\phi(x)\rangle$$

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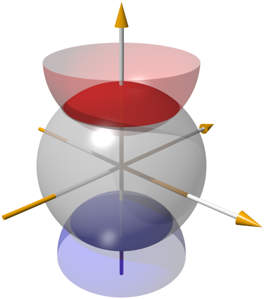
this is the defining property of the critical point (RG approach). BONUS SYMMETRY!: what if we manage to construct transformations that locally is a scaling?

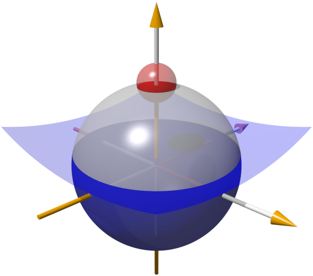
- Inversion $\mathcal{I}$, $x \to \frac{x}{|x|^2}$
- Special conformal transformations (SCT) $\mathcal{K} = \mathcal{I}\mathcal{T}\mathcal{I}$, ($d$)

The conformal symmetry group $\{\mathcal{T}, \mathcal{R}, \mathcal{S}, \mathcal{K}\}$ $(1 + d + \frac{d(d-1)}{2} + d = \frac{(d+1)(d+2)}{2})$

How do SCT look like?



$$\xrightarrow{\mathcal{K}}$$



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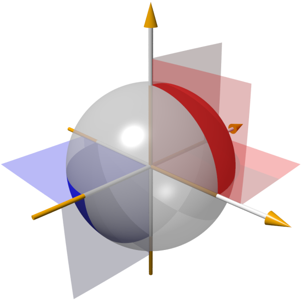
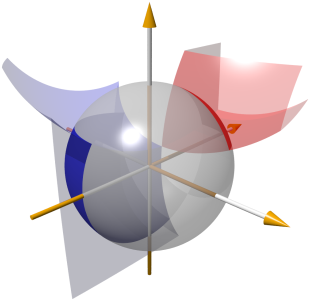


$$\xRightarrow{\mathcal{K}}$$



What can be gained from conformal symmetry?

Structure of correlators

$$\langle \phi(x)\phi(y)\rangle = \frac{1}{|x-y|^{\Delta_\phi}}$$

$$\langle \phi_1(x)\phi_2(y)\phi_3(z)\rangle = \frac{C_{123}}{|x-y|^{\Delta_1+\Delta_2-\Delta_3}|y-z|^{\Delta_2+\Delta_3-\Delta_1}|z-x|^{\Delta_3+\Delta_1-\Delta_2}}$$

If we build a Field Theory with such a symmetry we obtain very nice objects, CFT. Recently (CFT + unitarity):

- Conformal Bootstrap: good basis of operators yield exceptionally good results for Ising Critical Exponents $\Delta_\phi = 0.5181489(10)$[1].

[1] *Precision Islands in the Ising and O(N) Models*, Kos, Filip; Poland, David; Simmons-Duffin, David; Vichi, Alessandro; Journal of High Energy Physics. 2016 (8) 36.

The d=2 case

Every holomorphic function $z \to f(z)$ with $z = x_1 + i\, x_2$ is locally a scaling.
Infinite dimensional symmetry group!

- Any $z \to z^n$ with $n \in \mathbb{Z}$ will do

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However: for example $z \to z^2$ map is not one-to-one in $\mathbb{R}^2$.
The ones mapping $\mathbb{R}^2$ into itself in a one-to-one fashion are just the Möbius transformations (6)

$$f(z) = \frac{az + b}{cz + d} \quad \text{with } ad - cb \neq 0$$

Criticality in bounded domains

Take a (simply connected domain) Ω in $\mathbb{R}^d$. Put a critical system in it with "respectful" boundary conditions

- Extremely useful setting for experiments
- Structure of correlators (smaller symmetry group)
- SCT look more natural

Classical results[2]. In semiinfinite system $x_d > 0$ defining $|x-y|^2_{d-1} = \sum_{i=1}^{d-1}(x_i - y_i)^2$:

$$\langle \phi(x) \rangle \propto x_d^{-\Delta_\phi}$$

$$\langle \phi(x)\phi(y) \rangle \propto x_d^{-\Delta_\phi} y_d^{-\Delta_\phi} F\left(\frac{|x-y|^2_{d-1} + x_d^2 + y_d^2}{2x_d y_d}\right)$$

For the 2d case

- Extremely powerful setting (2dBCFT) a wealth of results known[3]

[2] *Conformal invariance and surface critical behavior*, Cardy, John L.; Nuclear Physics B **240**, 514 (1984).

[3] *Boundary conformal field theory* Cardy, John L.; `arXiv:hep-th/0411189`, in Encyclopedia of Mathematical Physics, Elsevier (2006).

Uniformisation

A system at criticality in a bounded domain will try to modify its (flat euclidean) metric in order to be "as uniform as possible".
Otherwise stated it will try to forget the boundaries.
Looking for a geometric description We start with $ds^2 = g_{ij}dx^i dx^j$

$$g_{ij} = \delta_{ij}$$

Allowed modifications of the metric: conformal changes

$$g_{ij} = \frac{\delta_{ij}}{\gamma(x)^2}$$

the two metric are said to belong to the same conformal class.

Riemannian geometry

(General Relativity) reminder

$$
\begin{aligned}
\Gamma^i_{jk} &= \frac{1}{2} g^{il} \left(\partial_k g_{lj} + \partial_j g_{lk} - \partial_l g_{jk}\right) \\
\mathrm{Ric}_{ij} &= \partial_l \Gamma^l_{ji} - \partial_j \Gamma^l_{li} + \Gamma^l_{l\lambda} \Gamma^\lambda_{ji} - \Gamma^l_{j\lambda} \Gamma^\lambda_{li} \\
R_g &= \mathrm{Ric}_{ij} g^{ji}
\end{aligned}
$$

where

- Γ^i_{jk} Schwarz-Christoffel symbols
- Ric Ricci tensor curvature
- R_g Ricci scalar curvature

Implementing uniformisation

Intrinsic geometric quantity should be constant. Find $\gamma(x)$ such that

$$R_{\delta/\gamma(x)^2} = \kappa$$

Yamabe problem[4] stated in the '60s. Still of interest to people doing differential geometry.
Possible values for κ:

- $\kappa > 0$ "spherical" ($\mathbb{S}^d$ special example). Problem: no boundary
- $\kappa = 0$ "flat" the one we started with
- $\kappa < 0$ "hyperbolic" ($\mathbb{H}^d$ special example). Looks promising: (infinitely distant) boundary.

For our case Yamabe equation reads:

$$1 - |\nabla\gamma(x)|^2 + \frac{2}{d}\gamma(x)\Delta\gamma(x) = 0$$

[4] *On a deformation of Riemannian structures on compact manifolds*, Yamabe, Hidehiko; Osaka Journal of Mathematics **12**, 21 (1960).

Properties & Simple solutions of Yamabe Equation

- $\gamma(x) = 0$ if $x \in \partial\Omega$
- $\gamma(x) \simeq \mathrm{dist}(x, \partial\Omega)$ if x is close to $\partial\Omega$
- Scaling $\gamma^{\lambda\Omega}(\lambda x) = \lambda\gamma^{\Omega}(x)$
- Semi-infinite system $x_d > 0$

$$\gamma(x) = x_d$$

- Inside balls $x_d > 0$

$$\gamma(x) = \frac{1 - |x|^2}{2}$$

In these two examples actually a model of $\mathbb{H}^d$ is built that is a fully homogeneous space with all sectional curvatures negative and constant

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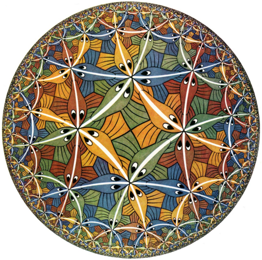
Uniformisation in 2d

Unit disk: $|x| < 1$



$$\gamma(x) = \frac{1-|x|^2}{2}$$

Unit strip: $-1 < x_1 < 1$

$$\gamma(x) = \frac{2}{\pi} \cos \frac{\pi x_1}{2}$$

Can't tell one point from another: Complete uniformisation.

Conjectures*

Given the scaling properties of γ (it has dimensions of a length) we are lead to:

Conjecture* for one-point correlators (C1pt*)

$$\langle \phi(x) \rangle = const. \times \gamma(x)^{-\Delta_\phi},$$

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Conjecture* for two-point correlators (C2pt*)

$$\langle \phi(x)\phi(y) \rangle = \gamma(x)^{-\Delta_\phi} \gamma(y)^{-\Delta_\phi} F(\mathfrak{D}_{\delta/\gamma^2}(x,y)).$$

$\mathfrak{D}_{\delta/\gamma^2}(x,y)$ is the distance between points x and y calculated with the metric δ/γ^2. These Conjectures* will be soon corrected to account for anomalous dimensions.

Yamabe equation

Yamabe equation

$$1-|\nabla\gamma(x)|^2+\frac{2}{d}\gamma(x)\Delta\gamma(x)=0$$

can also be cast in the form (nonlinear eigenvalue problem)

$$(-\Delta)\gamma(x)^{-\frac{d-2}{2}}=-\frac{d(d-2)}{4}\gamma(x)^{-\frac{d+2}{2}}.$$

we recognize scaling dimensions of the magnetization $\frac{d-2}{2}$ and the conjugate magnetic field $-\frac{d+2}{2}$ for a free field theory. In real life we have $\Delta_\phi\neq\frac{d-2}{2}=0.5181\ldots$ (Ising 3D).

Gaussian reasoning

Action of a theory at most quadratic in the fields

$$\mathcal{S}_Q[\phi(x)] = -\int \mathrm{d}^d x \frac{1}{2}\phi(x)(A\phi)(x) - \int \mathrm{d}^d x b(x)\phi(x),$$

Relation between applied field and magnetization
$A\langle\phi(x)\rangle_Q = b(x)$.
Assuming $A = (-\Delta)$ and that both field and magnetization can be related to a γ_Q we get Yamabe equation:

$$(-\Delta)\gamma_Q(x)^{-\frac{d-2}{2}} = const. \times \gamma_Q(x)^{-\frac{d+2}{2}}$$

How to account for anomalous dimensions?

$$(-\Delta)^{\frac{d}{2}-\Delta_\phi}\left[\gamma_{(\Delta_\phi)}(x)\right]^{-\Delta_\phi} = const. \times \left[\gamma_{(\Delta_\phi)}(x)\right]^{-d+\Delta_\phi},$$

Fractional Yamabe problem [5]

[5] *Fractional Laplacian in conformal geometry* Chang, Sun-Yung Alice; González, María del Mar; Advances in Mathematics **226**, 1410 (2011).

Conjectures

Conjecture for one-point correlators (C1pt)

$$\langle \phi(x) \rangle = const. \times \gamma_{(\Delta_\phi)}(x)^{-\Delta_\phi},$$

Conjecture for two-point correlators (C2pt)

$$\langle \phi(x)\phi(y) \rangle = \gamma_{(\Delta_\phi)}(x)^{-\Delta_\phi} \gamma_{(\Delta_\phi)}(y)^{-\Delta_\phi} F(\mathfrak{D}_{\delta/\gamma^2_{(\Delta_\phi)}}(x,y)).$$

Where $\gamma_{(\Delta_\phi)}(x)$ is the solution of FYE in Ω.

$$(-\Delta)^{\frac{d}{2}-\Delta_\phi} \left[\gamma_{(\Delta_\phi)}(x)\right]^{-\Delta_\phi} = const. \times \left[\gamma_{(\Delta_\phi)}(x)\right]^{-d+\Delta_\phi},$$

Conformally covariant operator

Conformal change: $g \to g' = g/w^2$ for a "good" operator A we want $A_{g'}(w^p \varphi) = w^q A_g(\varphi)$.
Conformal laplacian:

$$\mathcal{L}_g^{(1)} = (-\Delta_g) + \frac{d-2}{4(d-1)} R_g$$

with $p = \frac{d-2}{2}$, $q = \frac{d+2}{2}$, and $(-\Delta_g)$ Laplace-Beltrami operator.
If we manage to construct $\mathcal{L}_g^{(\frac{d}{2}-\Delta_\phi)}$ FYE reads[6]:

$$\mathcal{L}_{\delta/w^2}^{(\frac{d}{2}-\Delta_\phi)} \left(\frac{\gamma_{(\Delta_\phi)}(x)}{w(x)} \right)^{-\Delta_\phi} = \frac{\Upsilon(\Delta_\phi)}{\Upsilon(d-\Delta_\phi)} \left(\frac{\gamma_{(\Delta_\phi)}(x)}{w(x)} \right)^{-d+\Delta_\phi} .$$

Choosing $w(x) = \gamma_{(\Delta_\phi)}(x)$:

$$\mathcal{L}_{\delta/\gamma^2_{(\Delta_\phi)}}^{(\frac{d}{2}-\Delta_\phi)}(\mathbb{1}) = \frac{\Upsilon(\Delta_\phi)}{\Upsilon(d-\Delta_\phi)}(\mathbb{1})$$

[6] $\Upsilon(x) = \Gamma(1-x)\cos(\pi x/2)$

What's it all about?

For domains where complete uniformisation can be achieved:

- Semi-infinite systems ($x_d > 0$)
- Balls ($|x| < 1$)
- Any domain in 2d

we recover what is known from the group approach.

$$\mathfrak{D}_{\delta/x_d^2}(x, y) = \operatorname{arccosh}\left[\frac{|x-y|_{d-1}^2 + x_d^2 + y_d^2}{2x_d y_d}\right]$$

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New predictions:

any domain in $d > 2$ differing semi-infinite and balls.
Our work will be aimed to the slab domain $-1 < x_1 = x < 1$ in $d = 3$. Ideal geometry for confronting with experiments!

Fractional laplacian (in $\mathbb{R}^d$)

Given a function $f(x)$ compute $(-\Delta)^s f = g$

- Singular integral $g(x) = const. \int d^d x \frac{f(x)-f(y)}{|x-y|^{d+2s}}$
- Fourier transform $\hat{g}(k) = const. \hat{f}(k)|k|^{2s}$
- Inverse operator $f(x) = const. \int d^d x \frac{g(x)}{|x-y|^{d-2s}}$
- Heat semigroup $g(x) = const. \int_0^\infty dt \frac{(e^{t\Delta}-\mathbb{1})f(x)}{t^{1+s}}$
- ...
- ...
- PDE Extension (wait for next slide)

They are all (under conditions) equivalent [7]

[7] *Ten equivalent definitions of the fractional Laplace operator*, M. Kwaśnicki, Fract. Calc. Appl. Anal., **20**, (2017), p. 7-51

The square root of the laplacian

Set up this PDE in $d+1$ dimensions $(\mathbf{x}, y)$
$(-\Delta)^{1/2} f = g$

$$\begin{cases} \Delta_{\mathbf{x}} u(\mathbf{x}, y) + \partial_{yy} u(\mathbf{x}, y) = 0, & \text{for } y > 0 \\ u(\mathbf{x}, 0) = f(x) \end{cases}$$

Impose Dirichlet BC, read off Neumann BC

$$g(\mathbf{x}) = (-\Delta)^{1/2} f(\mathbf{x}) = -\left. \partial_y u(\mathbf{x}, y) \right|_{y=0}$$

The fractional laplacian: scattering theory[8]

- Extension the space $\mathbb{R}^d$ to $\mathbb{R}^{d+1}$, $(\mathbf{x}, y)$: when $y \to 0$ we retrieve the base space $\mathbb{R}^d$.
- Near $y = 0$, $g_+ \sim \frac{dx^2}{y^2}$.
- g_+ solution of Einstein equations in vacuum $\mathrm{Ric}_{g_+} + dg = 0$

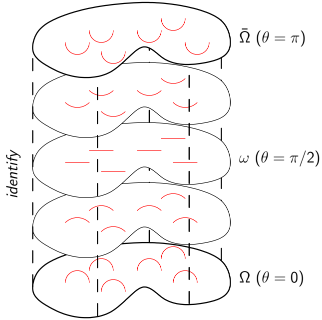
$$\begin{cases} (-\Delta_{g_+})U = \Delta_\phi(d - \Delta_\phi)U \\ U = y^{\Delta_\phi} F_I + y^{d-\Delta_\phi} F_O \end{cases}$$

F_I and F_O are regular on $y = 0$ and $F|_{y=0} = f$. The fractional laplacian is $(-\Delta)^{d/2-\Delta_\phi} f = const. \times F_O|_{y=0}$

- If you change to another variable $y \to \upsilon$ eigenfunction stays the same $U = y^{\Delta_\phi} F_I + y^{d-\Delta_\phi} F_O = \upsilon^{\Delta_\phi} F_I' + \upsilon^{d-\Delta_\phi} F_O'$
- $y = \left.\frac{dy}{d\upsilon}\right|_\Omega \upsilon = w(x)^{-1} \upsilon$

[8] *Scattering matrix in conformal geometry* Graham, C. Robin; Zworski, Maciej; Inventiones mathematicae **152**, 89 (2003).

The extension space for bounded domains



Canonical form of the metric[9]:

$$g_+ = (\sin\theta)^{-2}(d\theta^2 + g_\theta) = (\sin\theta)^{-2}(d\theta^2 + dx^2/x_d^2)$$

[9] *Formal theory of cornered asymptotically hyperbolic Einstein metrics*, McKeown, E. Stephen; arXiv:1708.02390 (2017).

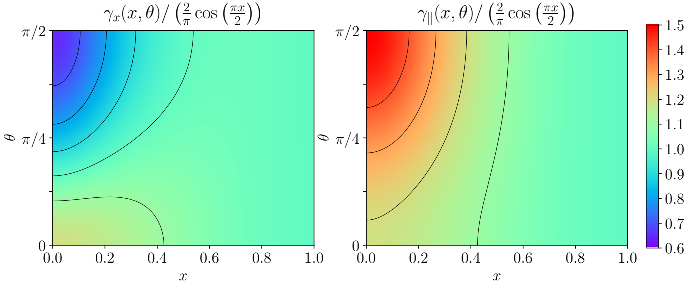
EE solution for the slab geometry

Solutions of Einstein equations

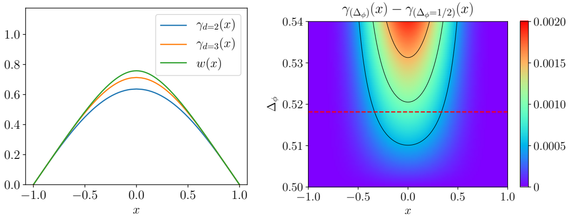
$$\mathrm{Ric}_{g_+} = -d\, g_+$$

can be chosen in a diagonal form (for any $d > 2$):

$$g_+ = (\sin\theta)^{-2}\left[d\theta^2 + dx_1^2/\gamma_x(x_1,\theta)^2 + (dx_2^2 + dx_3^2)/\gamma_{\|}(x_1,\theta)^2\right]$$



Fractional Yamabe solution



Ansatz for solution

$$U = \sin(\theta)^{\Delta_\phi} \left(\sum_{i=0,\, i\,\mathrm{even}}^{N_\theta} \mathcal{F}_i(x) \sin(\theta)^{2i} + \sum_{i=1,\, i\,\mathrm{odd}}^{N_\theta} \mathcal{F}_i(x) \sin(\theta)^{2s+i-1} \right)$$

eigenvalue problem solved by Chebyshev spectral methods yields a matrix operator $\mathcal{J}$ (inverse of the fractional laplacian).
Nonlinear eigenvalue converted in a minimization problem for
$\mathcal{E}[\rho] = \mathcal{J}[\rho] - \rho^{\frac{\Delta_\phi}{d-\Delta_\phi}} = 0$ where $\rho = (\gamma/w)^{-d+\Delta_\phi}$.

Numerical Experiments

Improved Blume-Capel model[10] (Ising universality class)

$$\mathcal{H} = -\beta \sum_{<i,j>} s_i s_j + D \sum_i s_i^2 \qquad s_i = \pm 1, 0$$

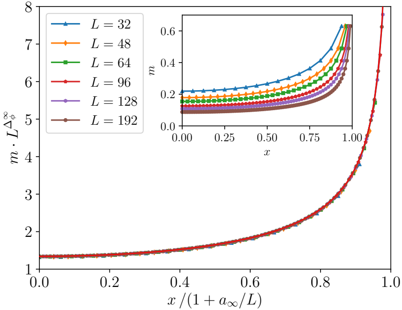
- Best available values for the system to be at the critical point
- In a slab domain of transverse dimension $0 < i_1 < L$ with fixed boundary conditions $s_i = +1$ if $i_1 = 0$ or $i_1 = L$.
- Large transverse dimension i_2, $i_3 = 1, \ldots, 6L$. Periodic boundary conditions.
- Cluster + local moves, $\sim 10^7$ samples, largest size $L = 192$.

Recorded observables:

- One-point function: $\langle s_i \rangle$
- Two-point function: $\langle s_i s_j \rangle$

[10] *Finite size scaling study of lattice models in the three-dimensional Ising universality class*, M. Hasenbush, Phys. Rev. B **82**, 174433 (2010).

One-point functions I

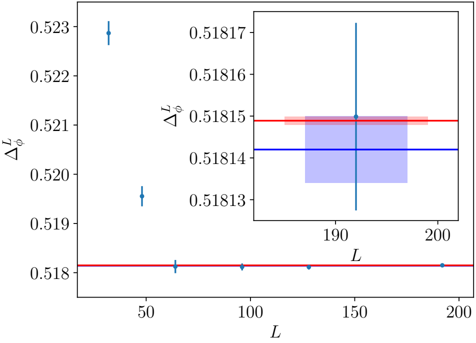


From **C1pt** magnetization profile

$$m_i = \alpha L^{-\Delta_\phi} \left[\gamma_{(\Delta_\phi)} \left(\frac{Lx}{L + a} \right) \right]^{-\Delta_\phi}$$

- Fitting parameters: α, a (extrapolation length), Δ_ϕ
- Adaptive windowing in order to have a 0.95 p-value

One-point functions II



Δ_ϕ as a function of L (blue), Conformal Bootstrap (red)

Reference	Method	Δ_ϕ
Hasenbusch (2010)	MC	0.518135(50)
Ferrenberg et al. (2018)	MC	0.51801(35)
Sheer El-Showk et al. (2014)	Conformal Bootstrap	0.518154(15)
Kos et al. (2016)	Conformal Bootstrap	0.5181489(10)
This work	Critical Geometry	0.518142(8)

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Using $\eta = 2\Delta_\phi - (d-2)$:

$$\begin{aligned}\eta^{MC} &= 0.036270(25)\\ \eta^{CB} &= 0.0362978(20)\\ \eta^{DE} &= 0.0358(6)\\ \eta^{CG} &= 0.036284(16)\end{aligned}$$

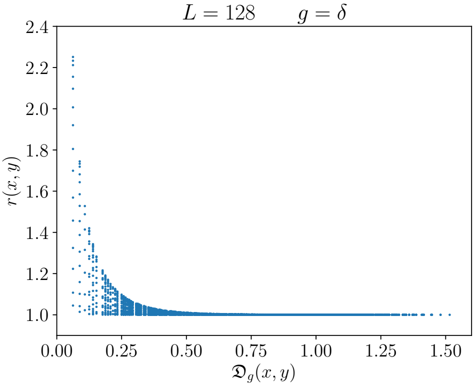
Two-point functions I

Correlation ratio

$$r(x,y) = \frac{\langle \phi(x)\phi(y)\rangle}{\langle \phi(x)\rangle\langle \phi(y)\rangle}$$

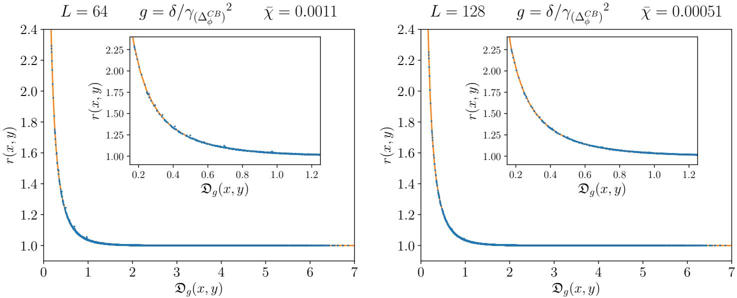
According to **C2pt** it should depend just on $\mathfrak{D}_g(x,y)$. r evaluated for 7672 independent couples of points.

Bad choice (euclidean distance) $\Rightarrow$ no collapse



Two-point functions II

FYE distance $\gamma_{(\Delta_\phi^{CB})}$:



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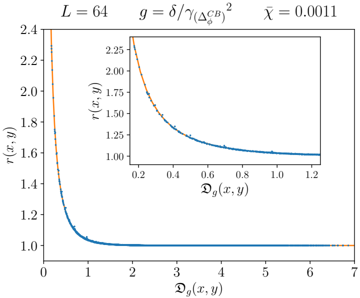
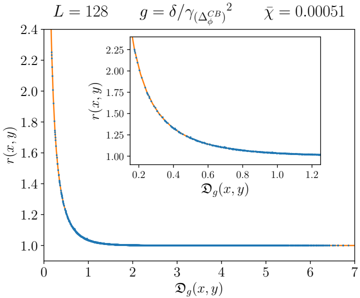
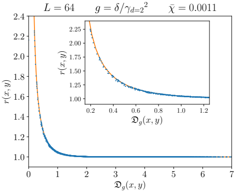
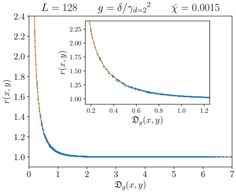




Strip distance $\gamma = (2/\pi)\cos \pi x/2$:





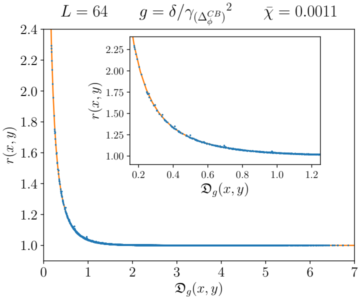
Fit with: $f(x) = 1 + \sum_{i=1}^{3} a_i e^{-b_i x}$

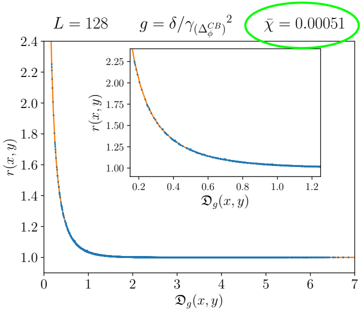
Figure of merit

$$\bar{\chi} = \sqrt{\frac{[r - f(\mathfrak{D}_g)]^2}{n_{\mathrm{d.o.f.}}}}$$

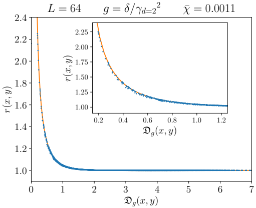
Two-point functions II

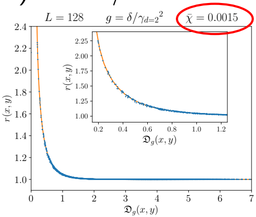
FYE distance $\gamma_{(\Delta_\phi^{CB})}$:





Strip distance $\gamma = (2/\pi)\cos \pi x/2$:





Fit with:
$f(x) = 1 + \sum_{i=1}^{3} a_i e^{-b_i x}$
Figure of merit

$$\bar{\chi} = \sqrt{\frac{[r - f(\mathfrak{D}_g)]^2}{n_{\mathrm{d.o.f.}}}}$$

Conclusions

- Clean separation between geometry/kinematics and dynamics/interaction
- Solution of geometry/kinematics

  $\Rightarrow$ Prediction of critical magnetization profile
  $\Rightarrow$ Precise estimate of $\Delta_\phi = 0.518142(8)$
  $\Rightarrow$ Structure of two point functions verified

Perspectives

To do list

- More operators (ε energy operator)
- More models (3D O(2), percolation 3-4-5D)
- More boundary conditions (ordinary, special)

Bolder directions

- Compatibility conditions $\Leftrightarrow$ OPE?
- Algebraic counterpart to "critical Geometry"?

Thank you