# Criticality and Phase Diagram of (Classical and) Quantum Long-Range Spin Systems

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***Functional and Renormalization-Group Methods – Italian Meeting (FRGIM 2019), ECT* - Trento, 16 September 2019***

# Outline

→ FRG for long-range O(N) models

→ Classical Long-Range Spin Systems

→ Anisotropic Long-Range Spin Systems

→ Quantum Long-Range Spin Systems

# Motivations

→ **Long-lasting interest in properties of long-range systems:**

ensemble inequivalence, non-equilibrium stationary states, inhomogeneous ground-states...

→ **Recent realizations of quantum long-range models in AMO systems, including <u>tunable</u> range of interactions**

→ **Analogies & differences between classical and quantum long-range interactions:** What classical effects survive in quantum long-range systems and what are changed?

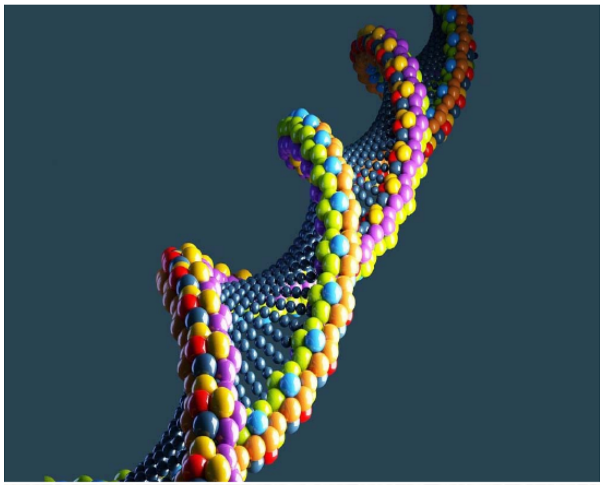
# Outline

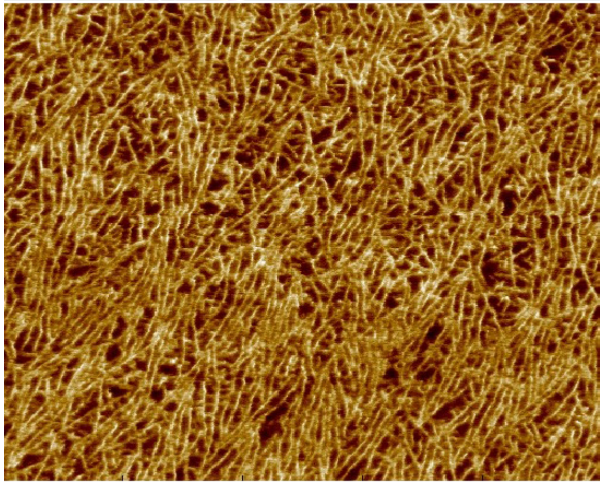
→ **Classical Long-Range Systems**

→ **Anisotropic Long-Range Systems**

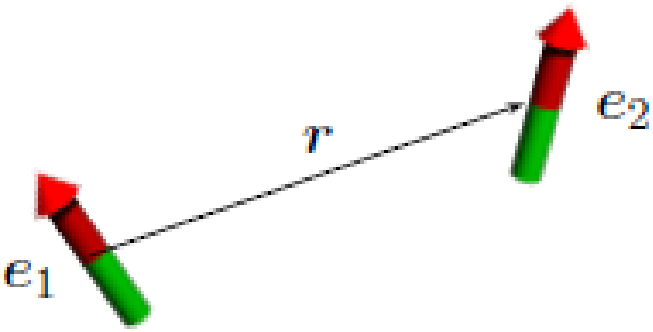
→ **Quantum Long-Range Systems**

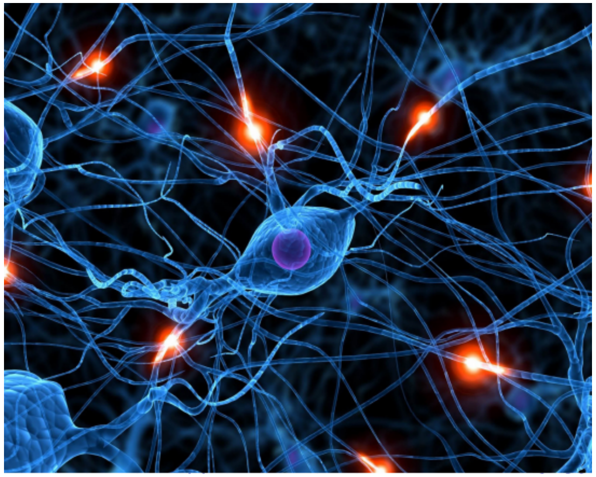
# Long-Range Interacting Systems





$$\frac{1}{r^{d+\sigma}}$$



# Spin systems

**Why spin systems**

- spin systems are the testbed of statistical mechanics.
- Various Monte Carlo (MC) and perturbative results available.
- Diverse interesting physical problems in a single formalism.

# Classical spin systems

## Lattice Hamiltonian

$$H = -\frac{J}{2}\sum_{ij}\frac{1}{|i-j|^{d+\sigma}}\boldsymbol{S}_i\boldsymbol{S}_j$$

## Mean Field Propagator

$$G(q)^{-1} = J(q) = \int d^d x\, J(i-j)e^{iq\cdot(i-j)}$$

## Leading momentum term

$$\lim_{q\to 0} G^{-1}(q) \propto q^{\sigma} \quad \text{if} \quad \sigma \le 2$$

$$\lim_{q\to 0} G^{-1}(q) \propto q^{2} \quad \text{if} \quad \sigma > 2$$

# Long-range interactions in d dimensions

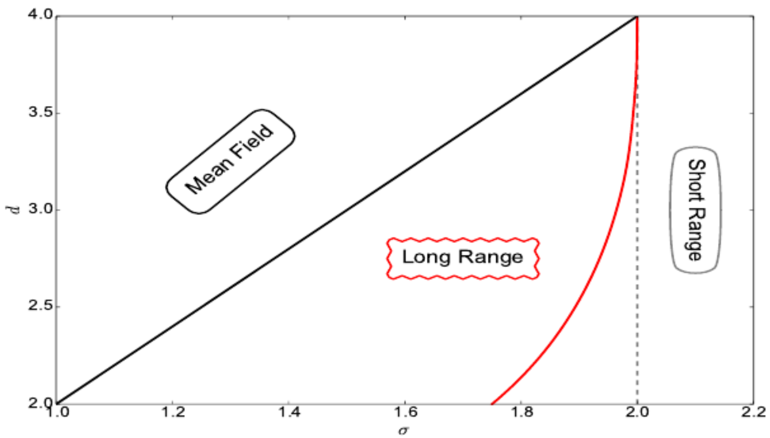
## Sak's Results

The anomalous dimension cannot be less than $\eta_{\text{SR}}$,

$$\eta = 2 - \sigma \quad \sigma < \sigma^*$$
$$\eta = \eta_{\text{SR}} \quad \sigma > \sigma^*$$

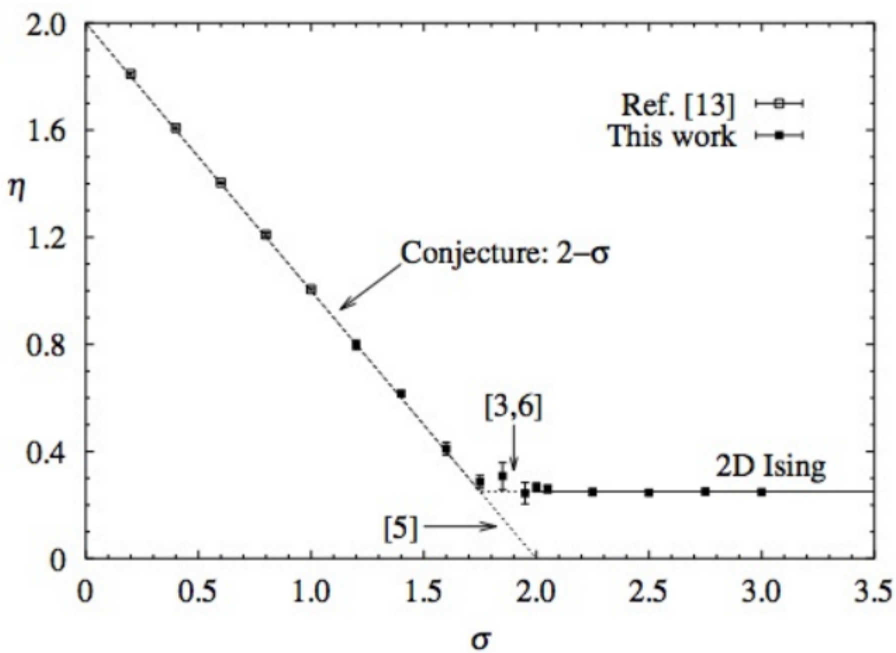
where $\sigma* = 2 - \eta_{\text{SR}}$. **No discontinuity** is present.

## System regimes




[J. Sak, PRB (1973)]

# Is the Sak result correct?

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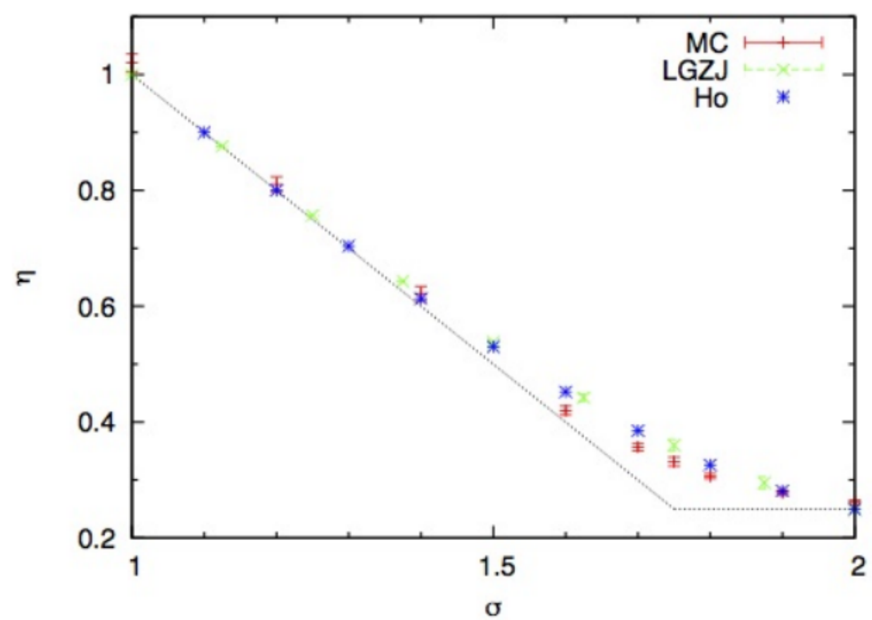



Figure: MC 2002 [E. Luijte and H. W. Blote, PRL (2002)]

Figure: MC 2013 [M. Picco, ArXiv 2013]

Last MC results are in Sak favour [T. Horita, H. Suwa, and S. Todo, PRE (2017)]
→ see as well later [recent review: N. Defenu, A. Codello, S. Ruffo, and A. Trombettoni, arXiv:1908.05158]

# Effective dimension

**Ginzburg-Landau Free Energy**

$$\Phi_{\rm SR} = \int d^{d_{\rm SR}}x \left\{ -Z_k \psi \Delta \psi + \mu \psi^2 + g\psi^4 \right\} + \cdots$$

$$\Phi_{\rm LR} = \int d^{d_{\rm LR}}x \left\{ -Z_k \psi \Delta^{\frac{\sigma}{2}} \psi - Z_{2,k} \psi \Delta \psi + \mu \psi^2 + g\psi^4 \right\} + \cdots$$

**Effective dimension results**

$$Z_k = Z_{2,k} = 1 \rightarrow d_{\rm SR} = \frac{2 d_{\rm LR}}{\sigma}$$

$$Z_{2,k} = 1 \rightarrow d_{\rm SR} = \frac{(2 - \eta_{\rm SR}) d_{\rm LR}}{\sigma}$$

The functional RG equations are the same of short-range system in an *effective* dimension

# Effective dimension (II)

$$D'_{\text{eff}} = \frac{[2 - \eta_{\text{SR}}(D'_{\text{eff}})]d}{\sigma}$$

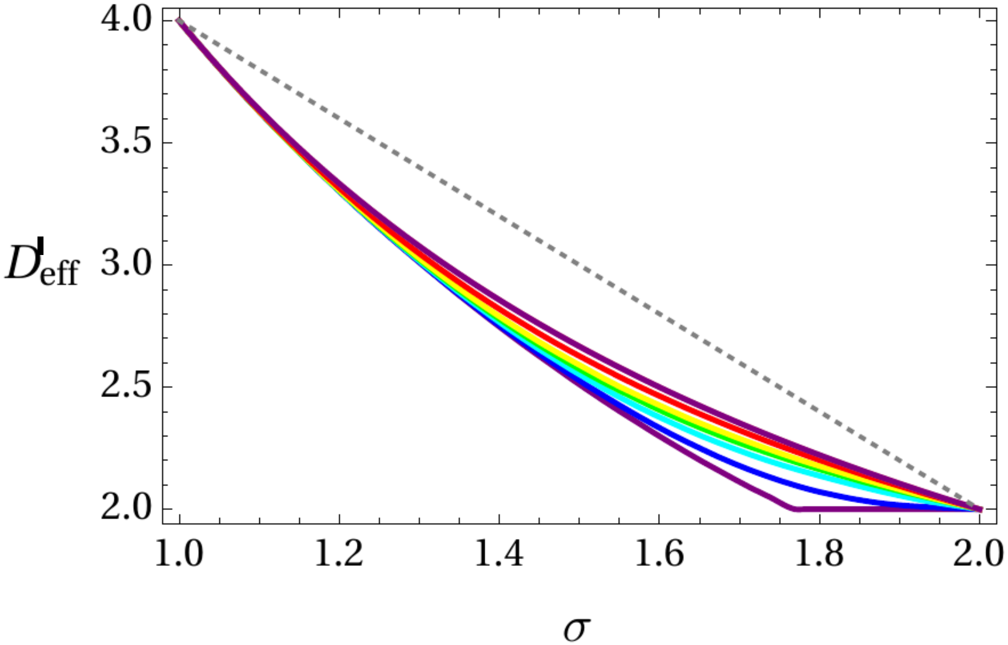



FIG. 6. (Color online) Different results for the effective dimension $D'_{\text{eff}}$ of a LR O($N$) model in $d = 2$: from left to right $N = 1,2,3,5,100$ using the data from [27,28]. The $N = 100$ case already practically overlaps with $D_{\text{eff}} = 2d/\sigma$ valid in the large-$N$ limit. The gray dashed line is the proposal made in [16].

# An important check...

$$\Gamma_k[\phi] = \int d^d x \Big\{ Z_{\sigma,k} \partial_\mu^{\frac{\sigma}{2}} \phi_i \partial_\mu^{\frac{\sigma}{2}} \phi_i + Z_{2,k} \partial_\mu \phi_i \partial_\mu \phi_i + U_k(\rho) \Big\},$$

$$R_k(q) = Z_\sigma (k^\sigma - q^\sigma) \theta(k^\sigma - q^\sigma) + Z_2 (k^2 - q^2) \theta(k^2 - q^2)$$

$$J_\sigma = \frac{Z_\sigma}{Z_2}$$

# An important check...(continued)

$$\partial_t \bar{J}_\sigma = (\sigma - 2)\bar{J}_\sigma + \eta_2 \bar{J}_\sigma\,,$$

$$\eta_2 = \frac{(2+\sigma \bar{J}_\sigma)^2 \bar{\rho}_0 \bar{U}_k''(\bar{\rho}_0)^2}{(1+\bar{J}_\sigma)^2[1+\bar{J}_\sigma + 2\bar{\rho}_0 \bar{U}_k''(\bar{\rho}_0)]^2}\,,$$

$$\partial_t \bar{U}_k(\bar{\rho}) = -d\bar{U}_k(\bar{\rho}) + (d-2+\eta_2)\bar{\rho}\,\bar{U}_k'(\bar{\rho}) + (N-1)\frac{1-\frac{\eta_2}{d+2}+\frac{\sigma}{2}\bar{J}_\sigma}{1+\bar{J}_\sigma+\bar{U}_k'(\bar{\rho})} + \frac{1-\frac{\eta_2}{d+2}+\frac{\sigma}{2}\bar{J}_\sigma}{1+\bar{J}_\sigma+\bar{U}_k'(\bar{\rho})+2\bar{\rho}\,\bar{U}_k''(\bar{\rho})}\,.$$

[N. Defenu, A. Codello, S. Ruffo, and A. Trombettoni, arXiv:1908.05158]

# FRG main results

**I Approximation Level: No anomalous dimension**

$d_{\mathrm{SR}} = \frac{2 d_{\mathrm{LR}}}{\sigma}$: Exact $N \to \infty$, Correct $\sigma$ ranges, $\sigma^* = 2$

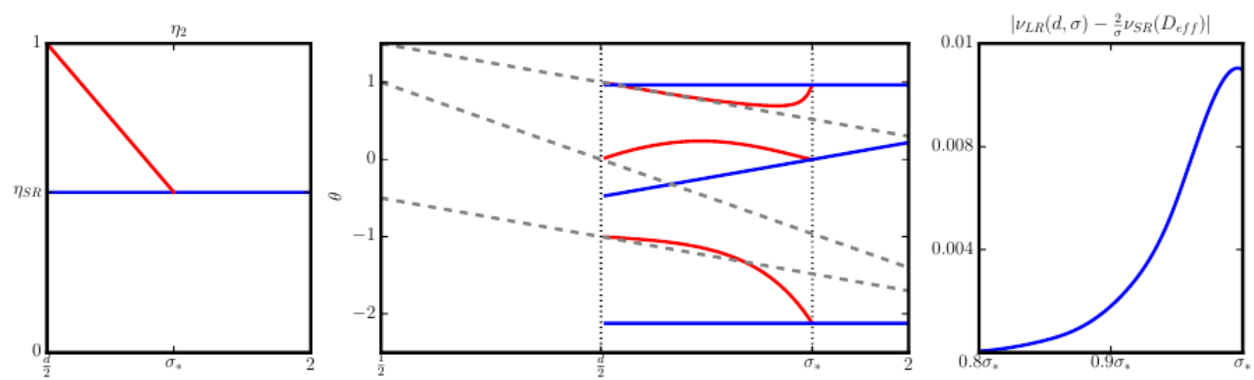
**II Approximation Level: Pure Long range case**

$d_{\mathrm{SR}} = \frac{(2-\eta_{\mathrm{SR}}) d_{\mathrm{LR}}}{\sigma}$: Exact $N \to \infty$, Correct $\sigma$ ranges, $\sigma^* = 2 - \eta_{\mathrm{SR}}$

**III Approximation Level: Mixed theory space**

Competition between Short and Long range fixed points: ~~$d_{\mathrm{SR}}$~~

**Fixed Points Solutions and Stability**



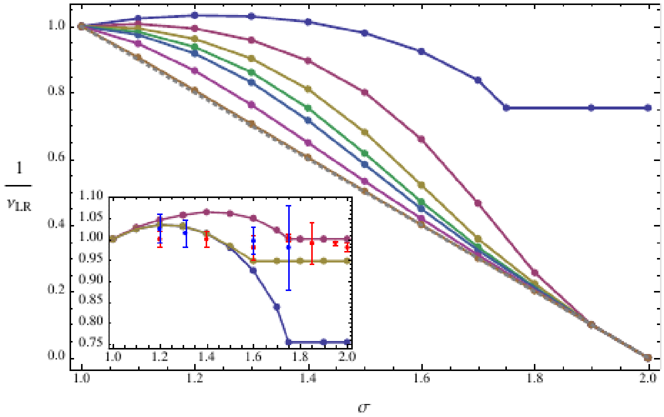

Functional RG reproduces the Sak result! However: do the effective dimension is *exact*?most probably (perhaps...) no

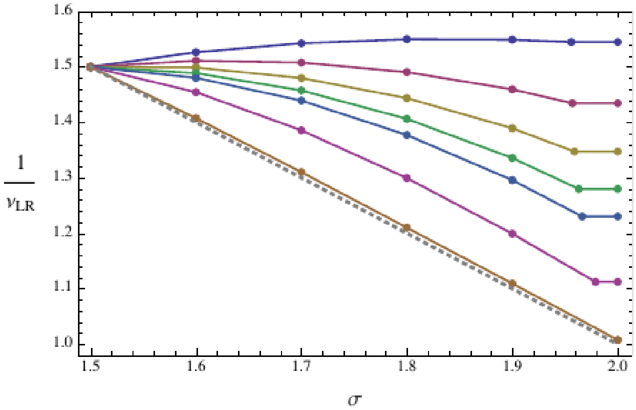
# FRG results (II)

**Short Range Corrections**

Short Range corrections spoil dimensional equivalence. Small everywhere but at $\sigma \simeq \sigma_*$.

**Correlation Length Exponent**




[N. Defenu, A. Trombettoni, and A. Codello, PRE (2015)]

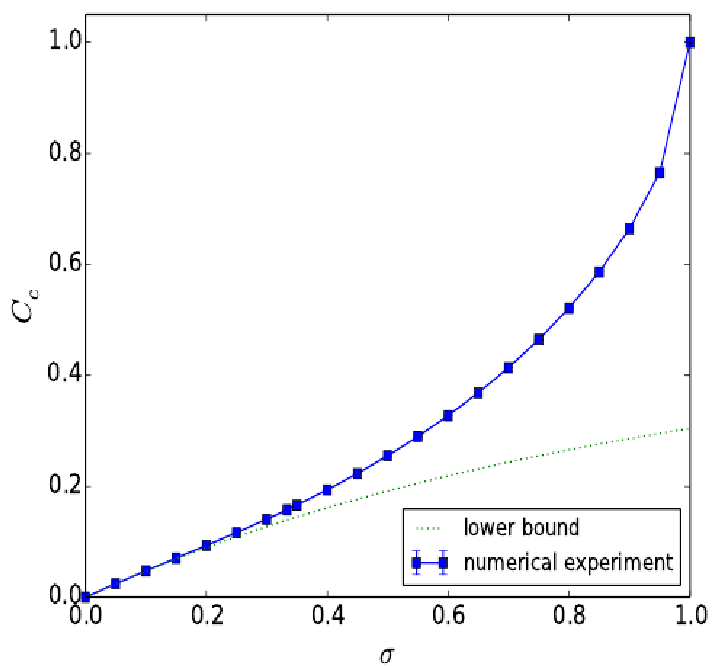
# Long-range percolation (I)

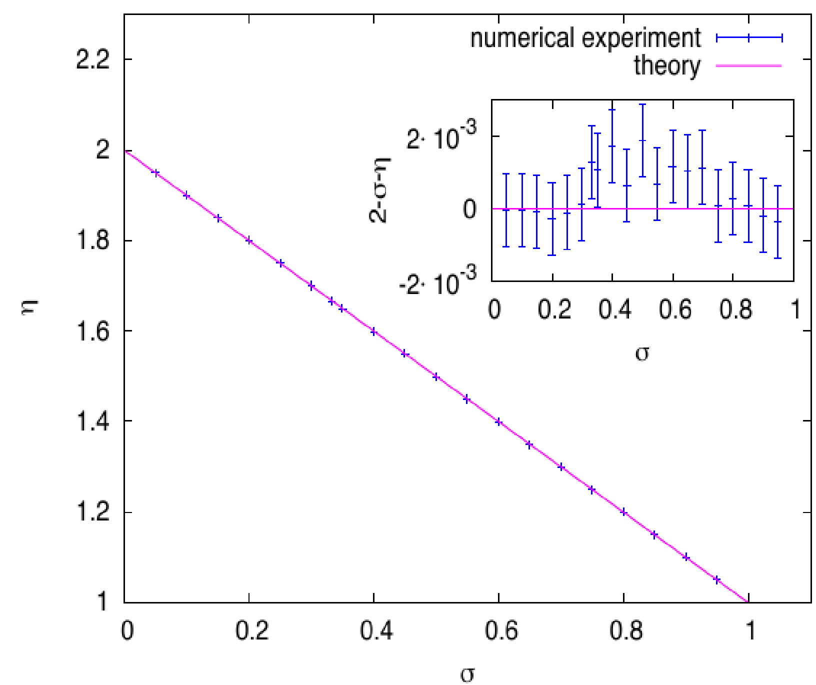
## Bond LR Percolation

Bond percolation on a one dimensional $N$ sites lattice

$$p_{i,j} = p_{|i-j|} = \frac{C}{|i-j|^{1+\sigma}}.$$

## Critical coupling and anomalous dimension




[G. Gori, M. Michelangeli, N. Defenu and A. Trombettoni, PRE (2017)]

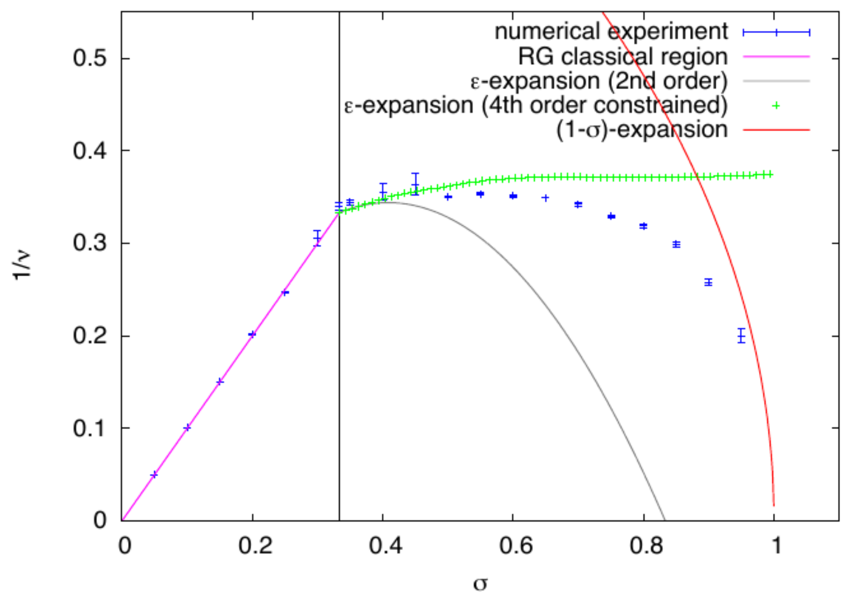
# Long-range percolation (I)

## Bond LR Percolation

Perturbative expansion

$$\nu_{SR} = \frac{2\beta + \gamma}{d_{SR}} = \frac{1}{2} + \frac{5}{84}\varepsilon + \frac{589}{37044}\varepsilon^2 + O(\varepsilon^3).$$

## Numerical results




We conclude that the Sak result is confirmed and that effective dimension works reasonably well [G. Gori, M. Michelangeli, N. Defenu and A. Trombettoni, PRE (2017)]

# Outline

→ Classical Long-Range Systems

→ **Anisotropic Long-Range Systems**

→ Quantum Long-Range Systems

# Outline

→ **Classical Long-Range Systems**

→ **Classical Anisotropic Long-Range Systems: are there genuinely anisotropic fixed points?**

→ **Quantum Long-Range Systems**

# Anisotropic long-range spin systems

## Lattice Hamiltonian

$$H = -\sum_{i \neq j} \frac{J_\parallel}{2} \frac{\boldsymbol{S}_i \boldsymbol{S}_j}{r_{\parallel,ij}^{d_1+\sigma}} \delta(\boldsymbol{r}_{\perp,ij}) - \sum_{i \neq j} \frac{J_\perp}{2} \frac{\boldsymbol{S}_i \boldsymbol{S}_j}{r_{\perp,ij}^{d_2+\tau}} \delta(\boldsymbol{r}_{\parallel,ij}).$$

## Mean Field Propagator

$$\lim_{q \to 0} G(q)^{-1} = \lim_{q \to 0} J(q) = Z_\parallel q_\parallel^\sigma + Z_\perp q_\perp^\tau + \mu + O(q^2)$$

## Effective field theory

$$\int d^d x \left\{ -\frac{Z_\parallel}{2} \phi(x) \Delta^{\sigma/2} \phi(x) - \frac{Z_\parallel}{2} \phi(x) \Delta^{\tau/2} \phi(x) + \ldots + U(\phi(x)) \right\}$$

# Critical beahviour

## Asymptotic propagators

$$G(q_1, q_2) \approx q_1^{-\sigma+\delta\eta_\sigma} G(1, q_2 q_1^{-\theta}) \approx q_2^{-\tau+\delta\eta_\tau} G(q_1 q_2^{-\frac{1}{\theta}}, 1)$$

## Correlation Lengths

$$\xi_{\parallel} \approx |T - T_c|^{-\nu_1} \qquad \xi_{\perp} \approx |T - T_c|^{-\nu_2},$$

## Anisotropy index

$$\frac{\sigma - \delta\eta_\sigma}{\tau - \delta\eta_\tau} = \frac{\nu_2}{\nu_1} = \theta.$$

## Mean field Results

$$\delta\eta_\sigma = \delta\eta_\tau = 0, \quad \nu_1 = \sigma^{-1}, \quad \nu_2 = \tau^{-1}.$$

# Phase diagram and anomalous dimension

**Region I: Anisotropic pure LR system.**

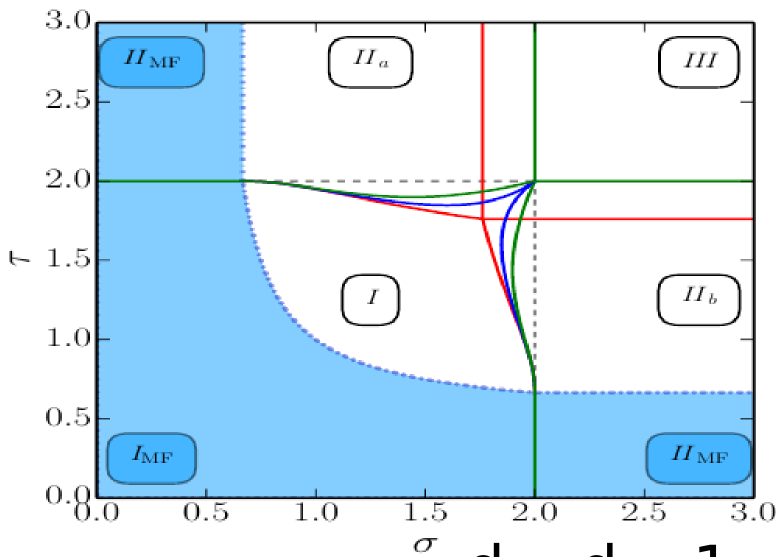
$D_{eff} = d_1 + \theta d_2$: Exact for $N \to \infty$. $\sigma^* = 2 - \eta(\tau)$ and $\tau^* = 2 - \eta(\sigma)$

**Region II: Anisotropic mixed S-LR region.**

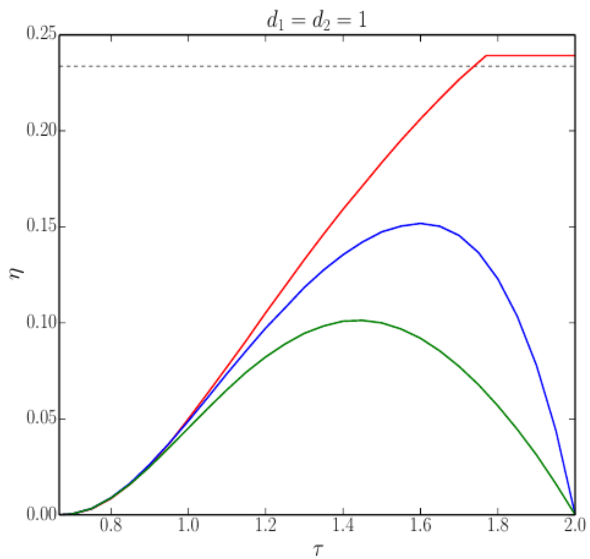
$D_{eff} = d_1 + \theta d_2$: Exact for $N \to \infty$. $\tau^* = 2 - \eta_{SR}$. $\theta = \frac{2-\eta(\tau)}{\tau}$.

**Region III: Isotropic SR case.**

Universality class of an isotropic SR system in dimension $d = d_1 + d_2$.




$d_1 = d_2 = 1$




[N. Defenu, A. Trombettoni, and S. Ruffo, PRB (2016)]

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→ Classical Long-Range Systems

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→ **Quantum Long-Range Systems**

# Quantum Ising and rotor O(N) models

$$H_{\mathrm{I}} = -\sum_{ij} \frac{J_{ij}}{2} \sigma_i^z \sigma_j^z - \mathcal{H} \sum_i \sigma_i^x$$

Quantum Ising

$$H_{\mathrm{R}} = -\sum_{ij} \frac{J_{ij}}{2} \hat{\boldsymbol{n}}_i \cdot \hat{\boldsymbol{n}}_j + \frac{\lambda}{2} \sum_i \mathcal{L}_i^2$$

Quantum O(N) rotor

$$J_{ij} = \frac{J}{r_{ij}^{d+\sigma}}$$

# Field theory formalism

$$S[\varphi] = \int d\tau \int d^d x \{K \partial_\tau \varphi_i \partial_\tau \varphi_i - Z \varphi_i \Delta^{\frac{\sigma}{2}} \varphi - Z_2 \varphi_i \Delta \varphi + U(\rho)\}$$

$$\rho = \sum_i \frac{\varphi_i^2}{2} \quad i \in \{1, N\}$$

Critical exponents:

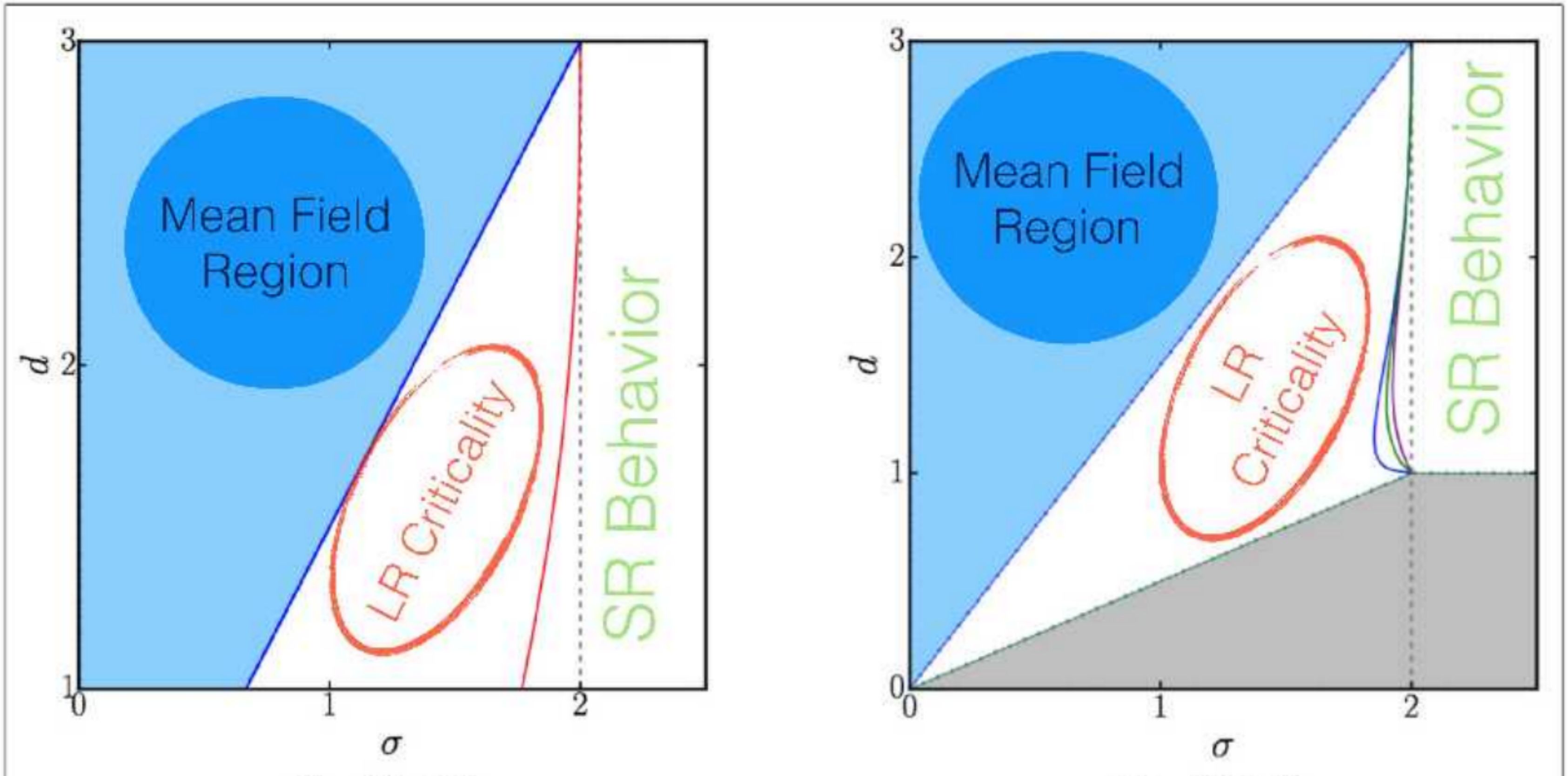
$$\xi \propto (\lambda - \lambda_c)^{-\nu} \qquad \lim_{q \to 0} G^{-1}(0,q) \propto q^{2-\eta} \qquad \Delta \propto (\lambda - \lambda_c)^{-z\nu}$$

How are computed in the functional RG formalism:

$$\frac{\partial \log K_k}{\partial \log k} = -\eta_\omega \qquad \frac{\partial \log Z_k}{\partial \log k} = -\delta\eta \qquad \frac{\partial \log Z_{2,k}}{\partial \log k} = -\eta$$

$$z = \frac{2-\eta}{2-\eta_\omega}$$

# Phase diagram




(a) $N < 2$ (b) $N \geq 2$

Mean field exponents

$$\eta = 2 - \sigma,$$

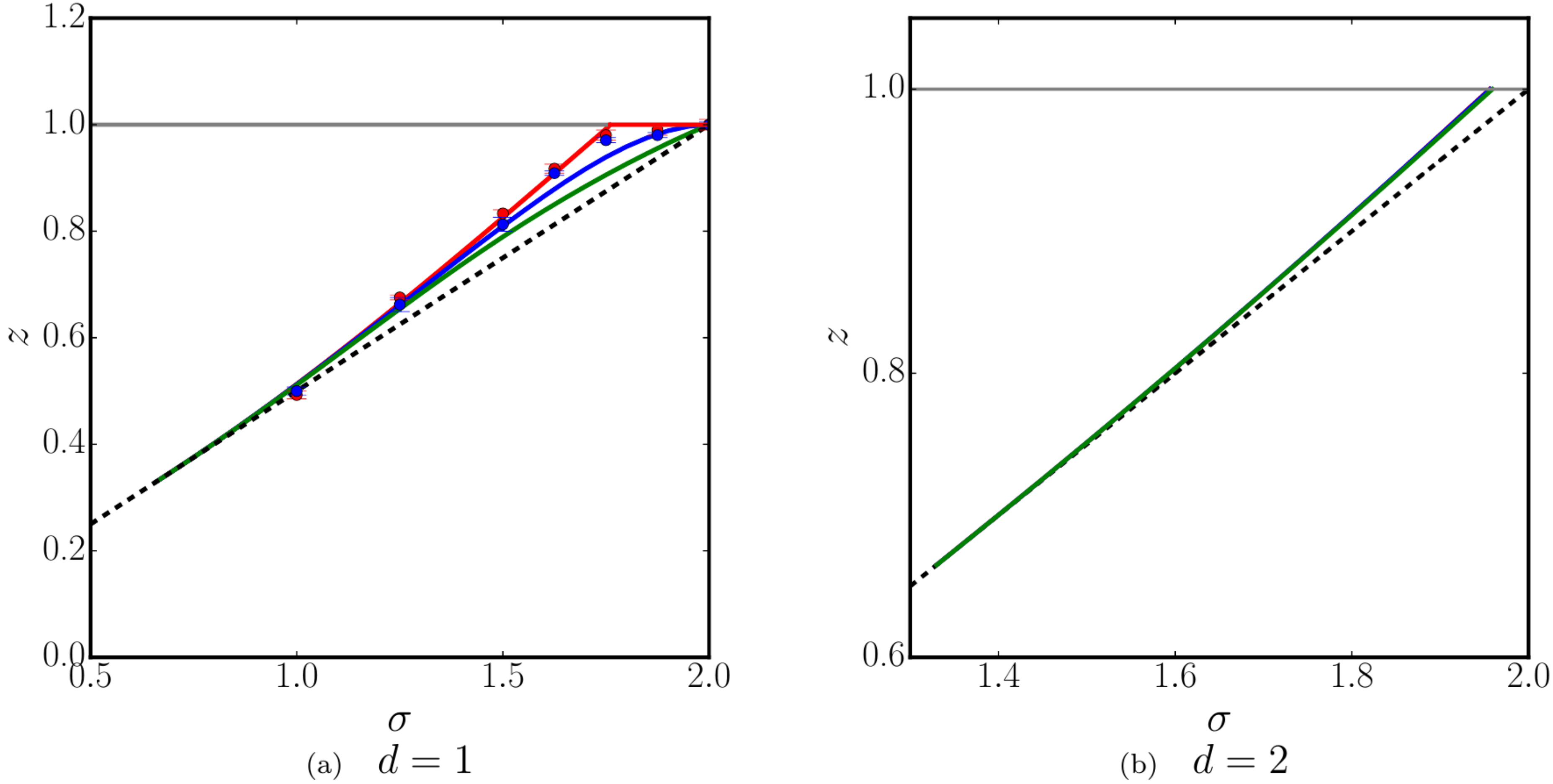
$$z = \frac{\sigma}{2},$$

$$\nu = \sigma^{-1}$$

Boundary region

$$\sigma_* = 2 - \eta_{\mathrm{SR}}$$

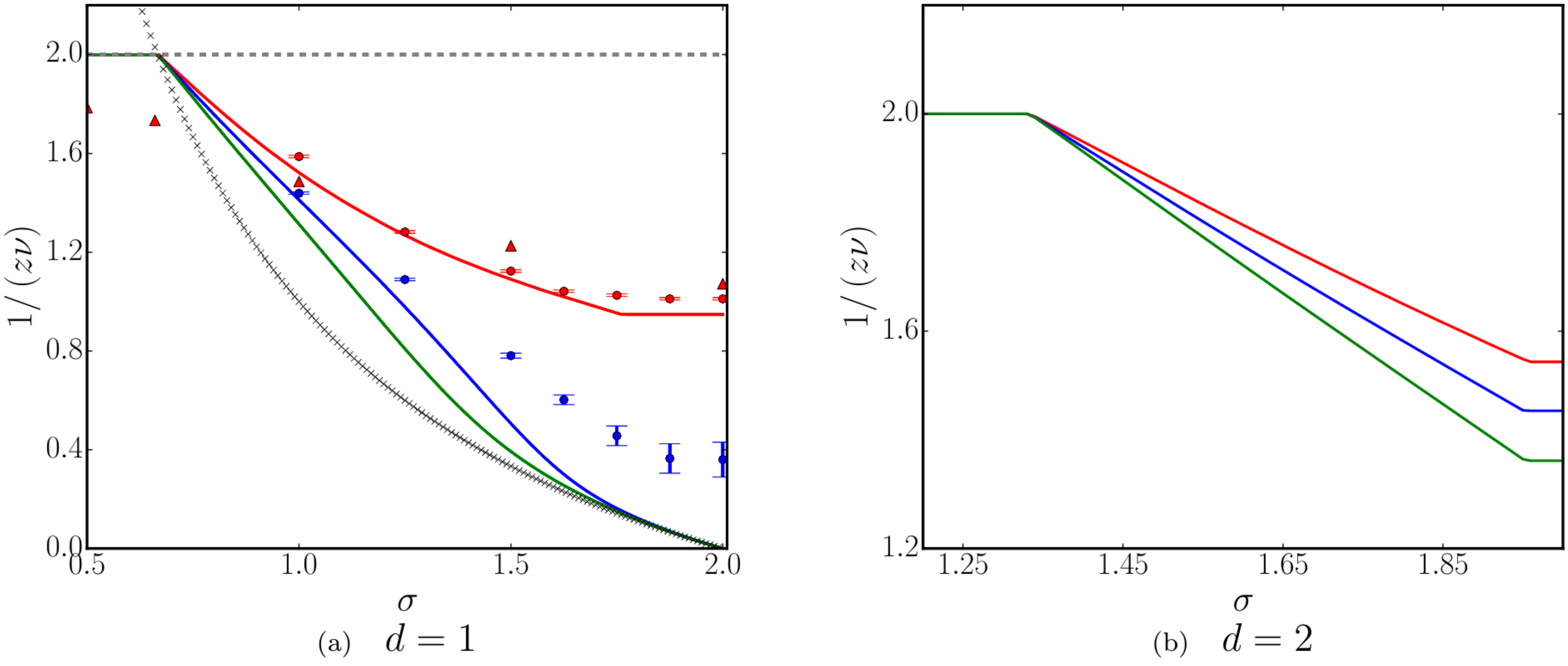
# Dynamical critical exponent




Monte Carlo results taken from I.B. Sperstad, E.B. Stiansen and A. Sudbo, PRB (2012)

Discrepancies for N=2 → BKT behaviour?

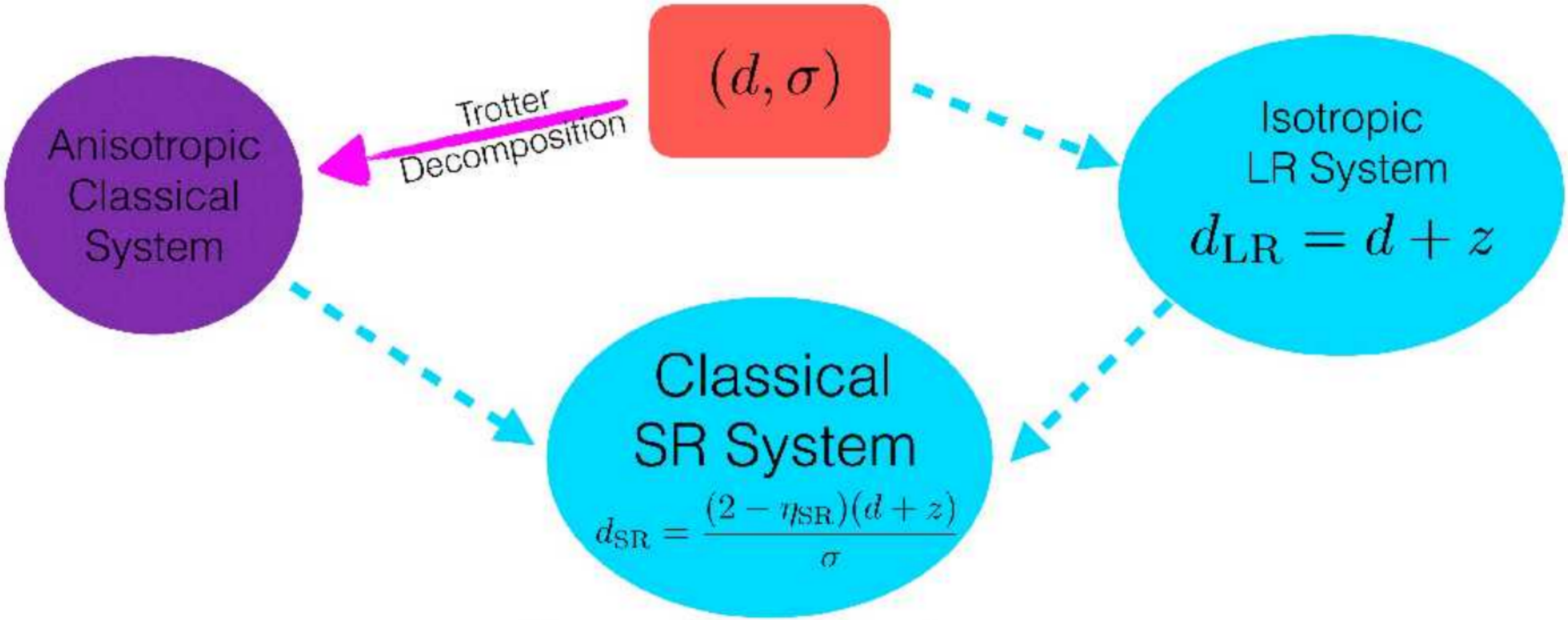
# Correlation length exponent




N=1 → red
N=2 → blue
N=3 → green

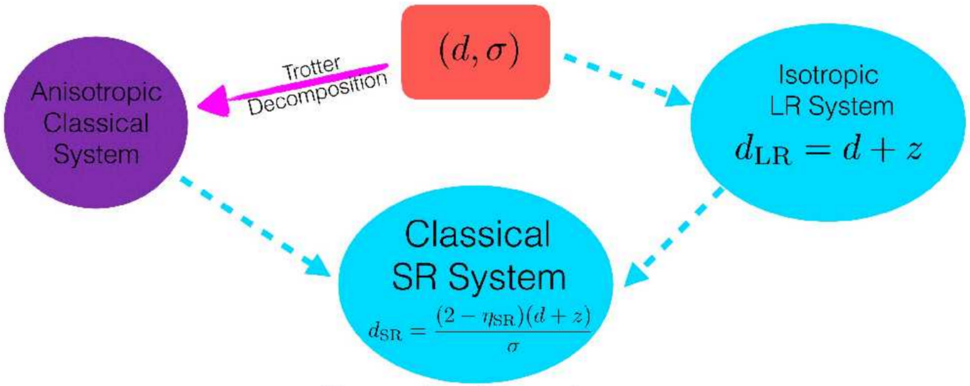
[N. Defenu, A. Trombettoni, and S. Ruffo, PRB (2017); N. Defenu, A. Codello, S. Ruffo, and A. Trombettoni, arXiv:1908.05158]

# Summary



$(d, \sigma)$
Trotter Decomposition
Anisotropic Classical System
Isotropic LR System
$d_{\mathrm{LR}} = d + z$
Classical SR System
$d_{\mathrm{SR}} = \frac{(2-\eta_{\mathrm{SR}})(d+z)}{\sigma}$

# Summary




→ Functional RG allows for an unified treatment of critical properties of classical and quantum long-range systems

→ Many things remain to be done:

→ need for a systematic comparison with numerical results

→ from FRG side: improve the results!

→ study of BKT physics in long-range quantum spin systems

→ applications of functional RG to non-equilibrium properties in presence of long-range couplings

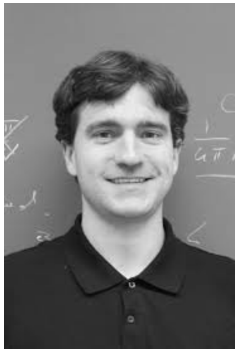
→ application to physical systems with tunable range [see next talk by Tommaso Macri']

→ ...

# Acknoledgements

**N. Defenu** **G. Gori** **A. Codello** **S. Ruffo**






Thank you!