# Convergence of the derivative expansion

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## Wilson's idea: a scale by scale summation over fluctuations

Traditional formulation of field theory= integral problem:

$$\mathcal{Z} = \int D\phi(x) e^{-H[\phi(x)] + \int d^d x\, B(x)\phi(x)}$$

with for instance

$$H = \int d^d x \left( \frac{1}{2}(\nabla\phi(x))^2 + \frac{1}{2} m_0^2 \phi(x)^2 + \frac{g_0}{4!}\phi^4(x) \right)$$

While perturbation theory consists in expanding in power series $\exp\left(-\frac{g_0}{4!}\int d^d x\, \phi^4(x)\right)$.

Wilson's idea: transform it into a differential problem. This is the "block-spin" idea:
The fluctuations are integrated over scale by scale and not in one shot.

## Wilson's RG: general idea

Build an interpolation between

$H$= hamiltonian or action defined at $k = \Lambda \sim a^{-1}$
no fluctuations are taken into account

and

the Gibbs free energy $\Gamma$ at scale $k = L^{-1} \to 0$
all fluctuations have been integrated over

Gibbs free energy = Legendre transform of the Helmoltz free energy $F = \log \mathcal{Z}$ = 1PI generating functional.
$\Rightarrow$ introduce a new scale $k$ that will vary between $\Lambda$ and 0 such that when $k$ is decreased more and more fluctuations are integrated over.

## Integration over the rapid modes:

$$\mathcal{Z}[B] = \int D\varphi \ \exp\Big\{ - H[\varphi] + \int_x B(x)\varphi(x)\Big\}$$

hyp.: the system is close to criticality ($\xi \gg a \sim \Lambda^{-1} \Rightarrow m_R \ll \Lambda$)

- $\phi(\vec{x}) = \langle \varphi(\vec{x}) \rangle$
- $\Gamma[\phi] + \ln \mathcal{Z}[B] = \int_x B_x \phi_x$

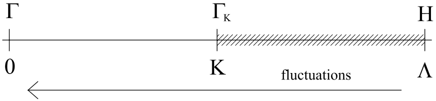
## Idea: deform the model

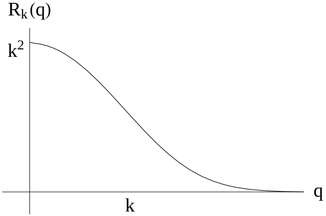
integrate over the rapid modes only $\rightarrow$ freeze the slow modes $\rightarrow$ make them non-critical $\rightarrow$ give them a large mass.

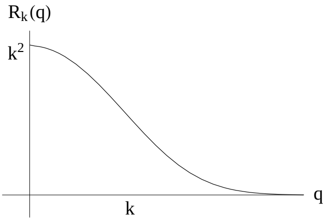
build a one-parameter family of models, indexed by a scale $k$:

$$\mathcal{Z}[B] \to \mathcal{Z}_k[B] = \int D\varphi \;\exp\left\{ -H[\varphi] - \Delta H_k[\varphi] + \int_x B(x)\varphi(x)\right\}$$

$$\Delta H_k[\varphi] = \frac{1}{2}\int_q R_k(q)\,\varphi_q\varphi_{-q}$$






• when $k = \Lambda$ all fluctuations are frozen $\Rightarrow$ mean field is exact:

$$\Rightarrow \quad \Gamma_{k=\Lambda}[\phi] = H[\phi]$$

• when $k = 0$ all fluctuations are integrated out and the original model is retrieved

$$\forall q, \quad R_{k=0}(q) = 0, \qquad \Rightarrow \quad \mathcal{Z}_{k=0}[J] = \mathcal{Z}[J] \text{ and } \Gamma_{k=0} = \Gamma$$

then $\Gamma_k[\phi]$ interpolates between the microphysics at $k = \Lambda$ and the macrophysics at $k = 0$.

The flow equation for $\Gamma_k[\phi]$ writes:

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} \int_q \partial_k R_k(q) G[q;\phi]$$

where $G[q;\phi]$ is the full 2-point function (full propagator):
$G[q;\phi] = (\Gamma_k^{(2)} + R_k)^{-1}$ with $\Gamma_k^{(2)}[q;\phi] = \frac{\delta^2 \Gamma_k[\phi]}{\delta\phi(q)\delta\phi(-q)}$

Some properties of the flow equation:
– differential formulation of field theory
– involves only one integral
– the initial condition is the (microscopic) bare theory
– good properties of decoupling of the massive and rapid modes
– starting point of non-perturbative approximation schemes (not linked to an expansion in a small coupling constant)

– BUT it is a tremendously complicated equation: functional, non linear and integral...
$\Rightarrow$ it leads to very few exact results and requires approximation.

# Sector at vanishing momentum : Derivative Expansion (DE)

$$\Gamma_k[\phi] = \int_x \left\{ U_k(\phi) + \frac{1}{2} Z_k(\phi)(\nabla\phi(x))^2 + O(\nabla^4) \right\}$$

flow of $\Gamma_k \Rightarrow$ flow of functions: $U_k(\phi), Z_k(\phi), \ldots$

The DE consists in keeping all $\Gamma_k^{(n)}$ correlation functions and expanding in their momenta (more precisely in $\frac{p_i}{k}$).

Most celebrated: Local Potential Approximation (LPA):

$$\Gamma_k^{\mathrm{LPA}} = \int d^d x \left( U_k(\phi) + \frac{1}{2}(\nabla\phi)^2 \right)$$

- bare momentum dependence of $\Gamma_k^{(2)}(p)$;
- zero momentum approximation for all other correlation functions;

$\rightarrow$ exact for the flow of $U_k$ when $N \rightarrow \infty$, 1-loop exact in $d = 4 - \epsilon$, 1-loop exact for O($N$) in $d = 2 + \epsilon$ (actually: the LPA').

## Nonuniversal results obtained with the LPA: $T_c$ in $d = 3$

$$\Gamma_k^{\mathrm{LPA}} = \int d^d x \left( U_k(M) + \frac{1}{2}(\nabla M)^2 \right)$$

$\rightarrow$ requires a reformulation of the NPRG formalism on the lattice

(T. Machado, N. Dupuis: Phys. Rev. E 82, 041128 (2010) )

$$\partial_k U_k(M) = \frac{k^{d+1}}{k^2 + U_k''(M)}$$

| | $T_c^{\mathrm{MF}}/J$ | $T_c^{\mathrm{exact}}/J$ | $T_c^{\mathrm{NPRG}}/J$ |
|---|---|---|---|
| Ising 3D | 6 | 4.51 | 4.48 |
| XY 3D | 3 | 2.20 | 2.18 |
| Heisenberg 3D | 2 | 1.44 | 1.42 |

## Universal results obtained with the LPA: $\nu$ and $\eta$ in $d = 3$

$\nu = 0.651$ and $\eta = 0$

"exact" results

$\nu = 0.630$ and $\eta = 0.0363$

Accuracy on both universal and nonuniversal results = 3 – 4 %

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Other completely nontrivial results:

- convexity of the effective potential $U(\phi)$ in the broken phase,
- phase diagrams in reaction-diffusion systems, ...

BUT the successes of the LPA do not prove the convergence of the DE!

## Beyond LPA for the O($N$) models

$$\Gamma_k[\phi] = \int_x \left\{ U_k(\phi) + \frac{1}{2} Z_k(\phi)(\nabla\phi(x))^2 + O(\nabla^4) \right\}$$

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- Only preliminary results for Ising at order $\nabla^4$ of the DE

L. Canet, B. D., D. Mouhanna, J. Vidal, Phys.Rev. B68, 2003

- And O($N$) models studied in detail at order $\nabla^2$ only in 2014!

P. Jakubczyk, N. Dupuis, B. D., Phys. Rev. E 90, 2014

## Two crucial questions: the convergence and the accuracy of the DE

Two strategies:

- produce analytical arguments about the convergence of the DE,
- obtain numbers at order $\nabla^6$.

... and study in detail the impact of the choice of regulator $R_k(q)$ on the results:

- in the exact theory: physical quantities do not depend on $R_k$,

- whereas a spurious dependence shows up once approximations are performed.

## The DE of the $\phi^4$ theory ($k = 0$) in the massive case in $d = 3$

Renormalized mass at $p = 0 \leftrightarrow$ inverse correlation length: $m = \xi^{-1}$

Massive theory in QFT = Non critical theory in Stat. Mech.
$\Rightarrow$ It is non singular.

$\rightarrow$ When $m > 0$, the momentum expansion of $\Gamma^{(2)}(p, m, \phi = 0)$ exists and has a finite radius of convergence:

$$\frac{\Gamma^{(2)}(p,m)}{\Gamma^{(2)}(0,m)} = \frac{\Gamma^{(2)}_{k=0}(p,m)}{\Gamma^{(2)}_{k=0}(0,m)} = 1 + \frac{p^2}{m^2} + \sum_{n=2}^{\infty} c_n \left(\frac{p^2}{m^2}\right)^n .$$

The $c_n$ are universal close to criticality.

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The series is alternating and has a finite radius of convergence:

- in the symmetric phase: $\mathcal{R} = 9$: $c_{n+1}/c_n \sim -1/9$ for $n \to \infty$,
- in the broken phase: $\mathcal{R} = 4$: $c_{n+1}/c_n \sim -1/4$ for $n \to \infty$.

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because the singularity nearest to the origin in the complex plane of $p^2$ is at $|p|^2 = 9m^2$ (symm. phase) or $|p|^2 = 4m^2$ (broken phase).

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Because the Minkowskian (that is, Wick rotated) version of the theory has a 3- (2-) particle cut, that is, a branch cut at $p = 3m$ ($p = 2m$) in the symmetric (broken) phase (Källén-Lehmann decomposition).

Symm phase: $c_2 = -4 \times 10^{-4}$, $c_3 = 0.9 \times 10^{-5}$,
broken phase: $c_2 \simeq -10^{-2}$, $c_3 \simeq 4 \times 10^{-3}$.

## The DE of the NPRG in the critical case: the $d = 3$ Ising ($\phi^4$) model

For $k > 0$, a critical theory is made noncritical by $R_k(q)$
$\Rightarrow R_k(q)$ gives an effective mass $m_{\text{eff}}$ to the theory.

We expect the regularized critical theory to be roughly “in between” the theory in its symmetric and broken phase:

**Expected radius of conv.: $4 \leq \mathcal{R} \leq 9$.**

# Order 6 of the DE

$$
\begin{aligned}
\Gamma_k[\phi] \;\; &= \int d^dx \Big[ U_k(\phi) + \tfrac{1}{2} Z_k(\phi)(\partial\phi)^2 \\
&+ \tfrac{1}{2} W_k^a(\phi)(\partial_\mu\partial_\nu\phi)^2 + \tfrac{1}{2}\phi W_k^b(\phi)(\partial^2\phi)(\partial\phi)^2 \\
&+ \tfrac{1}{2} W_k^c(\phi)\left((\partial\phi)^2\right)^2 + \tfrac{1}{2} X_k^a(\phi)(\partial_\mu\partial_\nu\partial_\rho\phi)^2 \\
&+ \tfrac{1}{2}\phi \tilde{X}_k^b(\phi)(\partial_\mu\partial_\nu\phi)(\partial_\nu\partial_\rho\phi)(\partial_\mu\partial_\rho\phi) \\
&+ \tfrac{1}{2}\phi \tilde{X}_k^c(\phi)\left(\partial^2\phi\right)^3 + \tfrac{1}{2} \tilde{X}_k^d(\phi)\left(\partial^2\phi\right)^2(\partial\phi)^2 \\
&+ \tfrac{1}{2} \tilde{X}_k^e(\phi)(\partial\phi)^2(\partial_\mu\phi)(\partial^2\partial_\mu\phi) + \tfrac{1}{2} \tilde{X}_k^f(\phi)(\partial\phi)^2(\partial_\mu\partial_\nu\phi)^2 \\
&+ \tfrac{1}{2}\phi \tilde{X}_k^g(\phi)\left(\partial^2\phi\right)\left((\partial\phi)^2\right)^2 + \tfrac{1}{96} \tilde{X}_k^h(\phi)\left((\partial\phi)^2\right)^3 \Big].
\end{aligned}
$$

To reach the fixed point (FP): need to work with dimensionless and renormalized functions: $\tilde{x} = kx$, $\tilde{\phi}(\tilde{x}) = \sqrt{Z_k^0}\, k^{(2-d)/2}\, \phi(x)$

$Z_k(\phi) = Z_k^0\, z_k(\tilde{\phi})$, $W_k^c(\phi) = (Z_k^0)^2 k^{-d} w_k^c(\tilde{\phi})$

At criticality and for $s = 6$, the analogue of

$$\frac{\Gamma^{(2)}(p,m)}{\Gamma^{(2)}(0,m)} = \frac{\Gamma^{(2)}_{k=0}(p,m)}{\Gamma^{(2)}_{k=0}(0,m)} = 1 + \frac{p^2}{m^2} + \sum_{n=2}^{\infty} c_n \left(\frac{p^2}{m^2}\right)^n$$

is

$$\frac{\Gamma^{(2)}_k(p,\phi) + R_k(0)}{\Gamma^{(2)}_k(0,\phi) + R_k(0)} = 1 + \frac{Z_k p^2 + W^a_k p^4 + X^a_k p^6}{U''_k + R_k(0)}$$

$$\xrightarrow[k\to 0]{} 1 + \frac{p^2}{m^2_{\text{eff}}} + \frac{w^*_a v^{*\prime\prime}}{z^{*2}} \frac{p^4}{m^4_{\text{eff}}} + \frac{x^*_a v^{*\prime\prime 2}}{z^{*3}} \frac{p^6}{m^6_{\text{eff}}}$$

with $m^2_{\text{eff}} = k^2 v^{*\prime\prime}/z^*$ and $v^{*\prime\prime} = u^{*\prime\prime} + R_k(0)/Z^0_k k^2$.

$c_2$ is analogous to $w^*_a v^{*\prime\prime}/z^{*2}$
$c_3$ is analogous to $x^*_a v^{*\prime\prime 2}/z^{*3}$.

## Procedure and goal

1. Choose a regulator $R_k(q)$ and compute at order $s = 0, \cdots, 6$ the FP functions: $u^*, \cdots, x_h^*$;
2. Compute physical quantities, e.g. critical exponents, and study their dependence on $R_k$ and their accuracy/convergence with $s$;
3. Compute $m_{\text{eff}}^2 = k^2 v^{*\prime\prime}/z^*$, $w_a^* v^{*\prime\prime}/z^{*2}$ and $x_a^* v^{*\prime\prime 2}/z^{*3}$ to see whether they behave respectively as the mass generated by the regulator and the analogues of $c_2$ and $c_3$.
4. Obtain criteria defining what a good regulator is;
5. Conclude about the convergence of the DE.

# The choice of $R_k(q)$

- DE = Taylor expansion of all $\Gamma_k^{(n)}(\{\mathbf{p}_i\})$ in powers of $\mathbf{p}_i \cdot \mathbf{p}_j / k^2$

$\Rightarrow$ valid provided $\mathbf{p}_i \cdot \mathbf{p}_j / k^2 < \mathcal{R}$ with $\mathcal{R} \simeq 4 - 9$;
$\Rightarrow$ whenever a $\Gamma_k^{(n)}$ is replaced in a flow equation by its DE, the momentum region beyond $\mathcal{R}$ must be efficiently cut off.

- Good news: all flow equations involve $\partial_t R_k(q^2)$ because

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} \int_q \partial_k R_k(q) G[q; \phi]$$

$\Rightarrow \partial_t R_k(q^2)$ must almost vanish for $|\mathbf{q}| \gtrsim k$.

- $R_k(q^2)$ should behave as $k^2$ for $|\mathbf{q}| < k$ to freeze the slow modes.

- $\partial_t R_k(q^2)$ and $\partial^n_{q^2} R_k(q^2)$ appear in the flow for $n \leq s/2$: they should decrease monotonically to avoid a "bump" at $q^2 = q_0^2 > 0$ where the DE is not accurate.

# The choice of $R_k(q)$

We have used three families of regulators

$$W_k(q^2) = \alpha Z_k^0 k^2 \, y/(\exp(y) - 1) \tag{1a}$$

$$\Theta_k^n(q^2) = \alpha Z_k^0 k^2 \, (1-y)^n \theta(1-y) \quad n \in \mathbb{N} \tag{1b}$$

$$E_k(q^2) = \alpha Z_k^0 k^2 \, \exp(-y) \tag{1c}$$

where $y = q^2/k^2$ and $\alpha$ is varied between 0.1 and 10.

Physical quantities, e.g. critical exponents, depend on $\alpha$ at any order $s$ of the DE $\Rightarrow$ one source of arbitrariness that needs to be fixed.

Is it the only one?

## A second source of arbitrariness

- $\eta_k = -k\partial_k \ln(Z_k^0)$ becomes the anomalous dimension $\eta$ at the FP.

- $Z_k^0$ defined by $Z_k(\phi) = Z_k^0\ z_k(\tilde{\phi})$  ($z_k(\tilde{\phi})$ reaches a FP value).

- The absolute normalization of both $Z_k^0$ and $z_k(\tilde{\phi})$ is defined by a "renormalization condition", e.g. $z_k(\tilde{\phi}_0) = 1$. In the exact theory, no physical quantity depends on $\tilde{\phi}_0$.

- At any order $s$ of the DE, the critical exponents depend on $\tilde{\phi}_0 \Rightarrow$ another source of arbitrariness that needs to be fixed.

- However, the variations of $\tilde{\phi}_0$ can be compensated by the variations of $\alpha \Rightarrow$ there is only one arbitrariness and not two: possible to fix $\tilde{\phi}_0$ wherever we want (as long as there is a FP) and to study only the variations of $\alpha$. (reparametrization invariance is not lost within the DE).

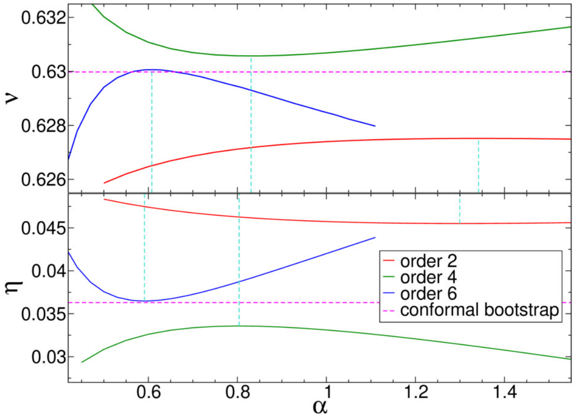


Figure: Exponent values $\nu(\alpha)$ and $\eta(\alpha)$ at different orders of the DE for the exponential regulator. Vertical lines indicate $\alpha_{\text{opt}}$. LPA ($s = 0$) results do not appear within the narrow ranges of values chosen here.

## Principle of minimal sensitivity

The dependence of the exponents on $\alpha$ is such that:

- There is an extremum for both $\nu(\alpha)$ and $\eta(\alpha)$, $\forall s \in [0, 6]$.

- The concavity of the curves $\nu(\alpha)$ and $\eta(\alpha)$ alternates at each order (coming from the alternating nature of the DE itself).

- The extremum is chosen as THE optimal value because it is the point where the exponent depends the least on $\alpha$ (principle of minimal sensitivity).

- At a given order $s$, $\alpha^{\nu}_{\text{opt}} \neq \alpha^{\eta}_{\text{opt}}$ but they are closer and closer as $s$ increases.

- The concavity becomes larger for both exponents as $s$ increases $\Rightarrow$ PMS Is crucial to select "the best" value of the exponents.

| derivative expansion | $\nu$ | $\eta$ |
|---|---|---|
| $s = 0$ (LPA) | 0.651(1) | 0 |
| $s = 2$ | 0.6278(3) | 0.0449 (6) |
| $s = 4$ | 0.63039(18) | 0.0343(7) |
| $s = 6$ | 0.63012(5) | 0.0361 (3) |
| $s \to \infty$ | 0.6300(2) | 0.0358(6) |
| conformal bootstrap | 0.629971(4) | 0.0362978(20) |

error bars for $s = 0, \cdots, 6 =$ dispersion of results at order $s$ from regulator to regulator.

The $s \to \infty$ extrapolation is based on:
$\nu(s) = \nu_\infty + a_\nu \beta^{-s/2} + b_\nu (-1)^{s/2} \beta^{-s/2}$ with $\beta \in [4, 9]$.

## Back to $m_{\text{eff}}$, $c_2$ and $c_3$

We have:

$$\frac{\Gamma_k^{(2)}(p,\phi)+R_k(0)}{\Gamma_k^{(2)}(0,\phi)+R_k(0)} = 1 + \frac{Z_k p^2 + W_k^a p^4 + X_k^a p^6}{U_k'' + R_k(0)}$$

$$\xrightarrow[k\to 0]{} 1 + \frac{p^2}{m_{\text{eff}}^2} + \frac{w_a^* v^{*\prime\prime}}{z^{*2}}\frac{p^4}{m_{\text{eff}}^4} + \frac{x_a^* v^{*\prime\prime 2}}{z^{*3}}\frac{p^6}{m_{\text{eff}}^6}$$

with $m_{\text{eff}}^2 = k^2 v^{*\prime\prime}/z^*$ and $v^{*\prime\prime} = u^{*\prime\prime} + R_k(0)/Z_k^0 k^2$.

$c_2$ (resp. $c_3$) is analogous to $w_a^* v^{*\prime\prime}/z^{*2}$ (resp. $x_a^* v^{*\prime\prime 2}/z^{*3}$).

$\rightarrow$ if $m_{\text{eff}}^2$ is generated by the regulator, then $m_{\text{eff}}^2 \propto R_k(0) = \alpha k^2$,

$\rightarrow$ $c_3/c_2$ must typically be in [4, 9].

## Numerical results for $m_{\text{eff}}^2$ and $c_3/c_2$

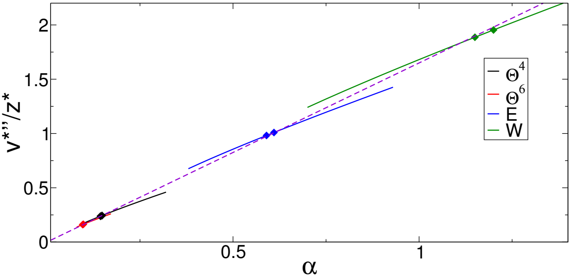


Figure: Squared dimensionless mass generated by the regulator $\tilde{m}^2_{\mathrm{eff}}(\phi_{\mathrm{min}}) = v^{*\prime\prime}/z^*|_{\phi_{min}}$ computed at the minimum of the potential.

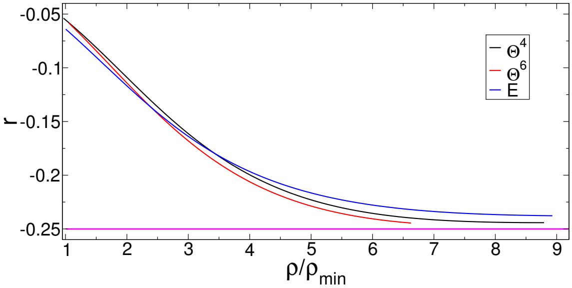


Figure: The ratio $r = x_a^* u^{*\prime\prime}/(w_a^* z^*)$ as a function of $\tilde{\rho} = \tilde{\phi}^2/2$. The line $r = 0.25$ is a guide for the eyes.

## A simple model for $\Gamma_k^{(2)}(p)$

At criticality:

when $p \gg k$, $\Gamma_k^{(2)}(p,\phi=0) \simeq \Gamma_{k=0}^{(2)}(p,\phi=0) \propto p^{2-\eta}$.

when $p \ll k$, $\Gamma_k^{(2)}(p,\phi=0) = \left(U_k'' + Z_k p^2 + W_k^a p^4 + X_k^a p^6\right)_{\phi=0}$

A simple way of matching these two expressions for $p \sim k$:

$$\Gamma_k^{(2)}(p,\phi=0) \simeq Ap^2(p^2+bk^2)^{-\eta/2} + m_k^2$$

with $A$ and $b$ two constants and $m_{k=0}=0$.

Expand in powers of $p^2/k^2 \Rightarrow$ an alternating series:
- with a negative coefficient for $p^4$ and a positive one for $p^6$
- all coefficients of the series from $p^4$ are proportional to $\eta$

$\Rightarrow$ they are naturally small!

# Conclusion

The DE is an alternating series. It has a finite radius of convergence. For the $\phi^4$ theory it is either 9 or 4.

$R_k(q)$ must almost vanish beyond typically $q^2 = 4k^2$, to cut efficiently the region $q^2 > 4k^2$ in the flow equations: no problem in replacing all $\Gamma_k^{(n)}$ by their Taylor expansion in the flow equations.

The PMS plays a crucial because the dependence on $\alpha$ increases with the order of the DE.

$\eta$ is NOT the small expansion parameter of the DE. However, all coefficients of the DE, starting from order $p^4$, are proportional to $\eta$ and are therefore naturally small $\Rightarrow$ fast convergence.

The analysis above can be used to select optimal regulators. Wilson-Polchinski version of the RG does not converge well.

## Preliminary results for $N = 2$

Conf. Bootstrap
$\nu = 0.6719(11), \ \eta = 0.0385(7)$

Monte-Carlo
$\nu = 0.6717(1), \ \eta = 0.0381(2)$

Experiments (space shuttle)
$\nu = 0.6709(1)$

DE
$\nu = 0.6727(5)$