Numerical Toy Models of Fast and Slow Neutrino Oscillations

1. Formalism
2. Setup of Self-Interaction, Matter, Source Terms and Boundary Conditions
3. Some Results
4. General Tendencies, role of Matter Term
5. Conclusions/Outlook

Deutsche Forschungsgemeinschaft
PROJEKTTRÄGER FÜR DAS Bundesministerium für Bildung und Forschung

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Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG

Formalism

For N_f flavours N_fxN_f density matrices are defined as (Wigner distributions)

$$\rho_{ij}(\mathbf{r},\mathbf{p}) \equiv \int d^3\mathbf{r}'\, e^{-i\mathbf{p}\cdot\mathbf{r}'} \left\langle a_j^\dagger(\mathbf{r}-\mathbf{r}'/2)a_i(\mathbf{r}+\mathbf{r}'/2)\right\rangle = \int \frac{d^3\boldsymbol{\Delta}}{(2\pi)^3}\, e^{i\boldsymbol{\Delta}\cdot\mathbf{r}} \left\langle a_j^\dagger(\mathbf{p}-\boldsymbol{\Delta}/2)a_i(\mathbf{p}+\boldsymbol{\Delta}/2)\right\rangle ,$$

and analogously for anti-neutrinos. The equations of motion are Liouville equations with vacuum terms and refractive terms from a background medium and from self-interactions:

$$\partial_t \rho(\mathbf{r},\mathbf{p}) + \mathbf{v}(\mathbf{r},\mathbf{p})\cdot\nabla_{\mathbf{r}}\rho(\mathbf{r},\mathbf{p}) = -\,i\left[\Omega^0_{\mathbf{p}} + \Omega_m(\mathbf{r}) + \Omega^{\rm S}(\mathbf{r},\mathbf{p}), \rho_{\mathbf{p}}\right] ,$$

where $\Omega_p{}^0$ is the vacuum term, Ω_m is the matter term, and

$$\Omega^{\rm S}(\mathbf{r},\mathbf{p}) = \mu(\mathbf{r})\sum_{\mathbf{q}\neq\mathbf{p}}(1-\mathbf{v}_{\mathbf{p}}\cdot\mathbf{v}_{\mathbf{q}})\Big\{G_{\rm S}[\rho(\mathbf{r},\mathbf{q})-\bar{\rho}(\mathbf{r},\mathbf{q})]G_{\rm S} + G_{\rm S}{\rm Tr}\left[(\rho(\mathbf{r},\mathbf{q})-\bar{\rho}(\mathbf{r},\mathbf{q}))G_{\rm S}\right]\Big\} ,$$

where in general G_S=diag(1,…,1) for active neutrinos.

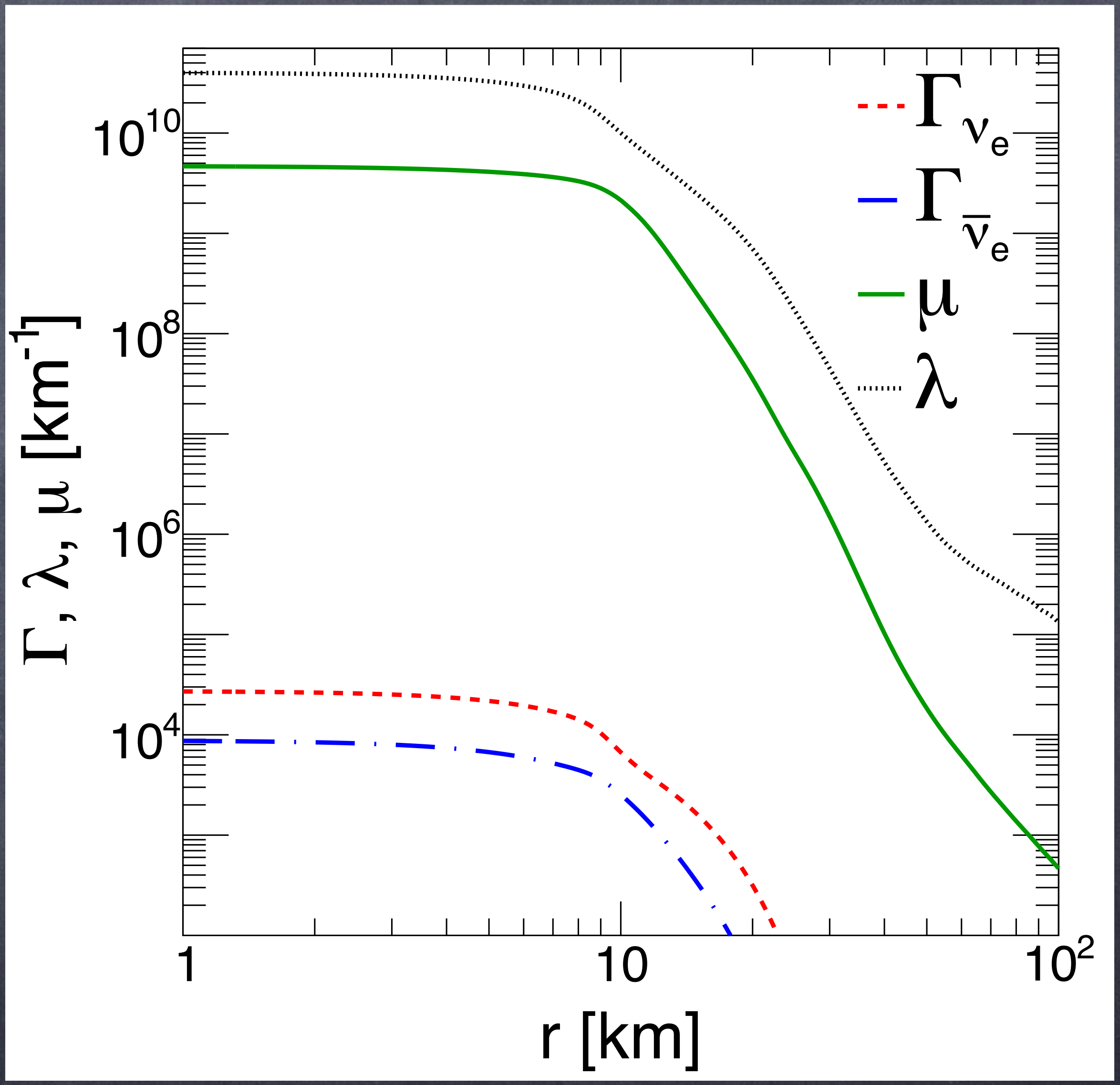
In general the Liouville term is

$$\frac{1}{2}\left\{ \nabla_{\mathbf{r}}\rho(\mathbf{r},\mathbf{p}), \nabla_{\mathbf{p}}\Omega(\mathbf{r},\mathbf{p}) \right\} - \frac{1}{2}\left\{ \nabla_{\mathbf{p}}\rho(\mathbf{r},\mathbf{p}), \nabla_{\mathbf{r}}\Omega(\mathbf{r},\mathbf{p}) \right\} .$$

Charged current source terms (no scattering implemented yet) have the form

$$\partial_t \rho(\mathbf{r},\mathbf{p})_{\mathrm{coll,CC}} = \left\{ \Gamma(\mathbf{r},\mathbf{p}), \left(1 - \frac{\rho(\mathbf{r},\mathbf{p})}{f_0(\mathbf{r},\mathbf{p})} \right) \right\} ,$$

with $f_0(\mathbf{r},\mathbf{p})$ the equilibrium occupation numbers $\Gamma(\mathbf{r},\mathbf{p})$ the rate which typically projects on one flavor, e.g. the electron flavour.



F. Capozzi et al., Phys.Rev.Lett 122 (2019) 091101 [arXiv:1808.06618

Numerical Setup

Typically we use N_f =2 flavours, momentum modes with equal energy and one spatial dimension x with a one-dimensional array of N_p momentum modes whose velocity projections onto the x (radial) direction are isotropically distributed between -1 and +1,

$$v_x(i_p) = -1 + \frac{1}{N_p} + \frac{i_p - 1}{N_p - 1}\left(2 - \frac{2}{N_p}\right), \quad i_p = 1, \cdots, N_p, \text{with } N_p \text{ even} .$$

The source term then is

$$\partial_t \rho(x, i_p)_{\text{coll,CC}} = f_s(x) f(x, i_p) \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \left(1 - \frac{\rho(x, i_p)}{f_0(x, i_p)}\right) \right\} ,$$

and analogously for anti-neutrinos. The vacuum term is

$$\Omega^0_{v_x} = \frac{\Delta m^2}{4} \begin{pmatrix} \cos 2\theta_0 & -\sin 2\theta_0 \\ -\sin 2\theta_0 & -\cos 2\theta_0 \end{pmatrix} ,$$

where $\Delta m^2 > 0$, $\cos 2\Theta_0 > 0$ corresponds to the inverted hierarchy.
The matter term is

$$\Omega_m(x) = \lambda(x) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} .$$

The self-interaction coefficients are normalised as

$$g_{i,j} = \frac{(1-\delta_{ij})(1-v_i v_j)}{\sum_{kl}(1-\delta_{kl})(1-v_k v_l)/N_p},$$

which assures that the average coupling of one momentum mode summed over all other modes is unity and thus does not depend on N_p, before being multiplied with the characteristic self-coupling μ(x).

As initial conditions we typically put pure flavour eigenstates,

$$\rho(t=0,x,i_p) = f_i(x) f(x,i_p) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \bar{\rho}(t=0,x,i_p) = f_i(x) \bar{f}(x,i_p) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

where $f(x,i_p)$ and $\bar{f}(x,i_p)$ can contain a modulation in i_p and an asymmetry between neutrinos and anti-neutrinos, e.g. a crossing, a switch of sign as a function of direction **n** of the flavour 1 lepton number density

$$G(\mathbf{n}) = \int_0^\infty \frac{dpp^2}{2\pi^2}\left[f_{\nu_1}(p\mathbf{n}) - f_{\bar{\nu}_1}(p\mathbf{n})\right].$$

The modulation function was defined by

$$f(x,i_p)=\frac{1}{2}[1-ah(x,i_p)][1-bh(x,i_p)]\,,\quad \bar{f}(x,i_p)=\frac{1}{2}[1-ah(x,i_p)][1+bh(x,i_p)]\,,$$

where a and b are parameters and

$$h(x,i_p)=\left(2\frac{i_p-1}{N_p-1}-1\right)g(x)\,,$$

with g(x) a function which vanishes at the boundaries, $g(x=0)=g(x=L_x)=0$. Note that

$$\sum_{i_p}\left[f(x,i_p)+\bar{f}(x,i_p)\right]=N_p\,.$$

Usually we set a=0. A crossing can be induced by setting b>0.

At $x=L_x$ the boundary condition for the incoming modes, $v_x<0$, is given in terms of the initial condition (to make them consistent),

$$\rho(t, x = L_x, v_x < 0) = \rho(t = 0, x = L_x, v_x < 0) = f_i(L_x) f(L_x, i_p) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} ,$$

and analogously for anti-neutrinos. Idea is to make them close to zero (no neutrinos coming from outside).
At $x=0$ the boundary conditions for the incoming modes, $v_x<0$, are given in terms of a reflective boundary,

$$\rho(t, x = 0, v_x) = \rho(t, x = 0, -v_x) , \quad \bar{\rho}(t, x = 0, v_x) = \bar{\rho}(t, x = 0, -v_x) .$$

Anti-neutrino initial and equilibrium densities and production rates are typically assumed to be equal to the ones of neutrinos. Flavor perturbations are initially driven by the vacuum frequency.

Partial differential equations are integrated within $0<x<L_x$ and $0<t<t_{max}$.
Mathematica 11.1 was used with the NDSolve routine.

Results

We show the following quantities:

Total number of neutrinos

$$N(t,x) \equiv \sum_{v_x} \mathrm{Tr}\left[\rho(x,v_x) + \bar{\rho}(x,v_x)\right] \ ,$$

off diagonal terms

$$F_{\rm off}(t,x) \equiv \frac{\sum_{v_x} \left|\rho_{12}(x,v_x) + \bar{\rho}_{12}(x,v_x)\right|}{N(t,x)} ,$$

and total flavour asymmetry

$$F_{\rm asym}(t,x) \equiv \frac{\sum_{v_x} \left[\rho_{11}(x,v_x) - \rho_{22}(x,v_x) + \bar{\rho}_{11}(x,v_x) - \bar{\rho}_{22}(x,v_x)\right]}{N(t,x)} .$$

We consider the following cases:

models 1-2 show slow transitions with constant $\mu(x)$ without and with constant $\lambda(x)$, $\Theta_0=0.01$

models 3-6 show slow transitions for $\mu(x)$ profiles, for $\Theta_0=0.01$ and $\Theta_0=34^o$, without and with $\lambda(x)$ matter profile

model 7 shows a fast transition with an angular crossing and large matter term

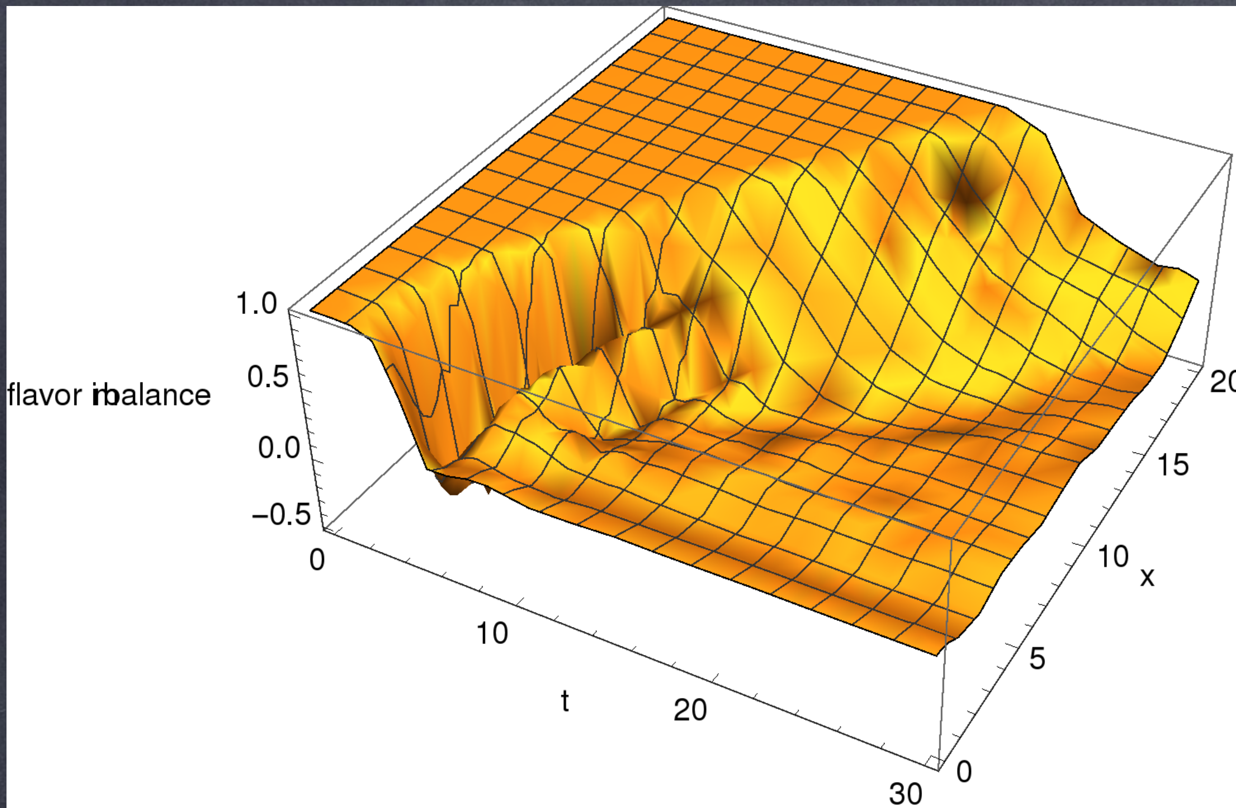
models 8-9 show slow transitions for larger hierarchies of frequencies, $\Theta_0=34^o$, without and with $\lambda(x)$ matter profile

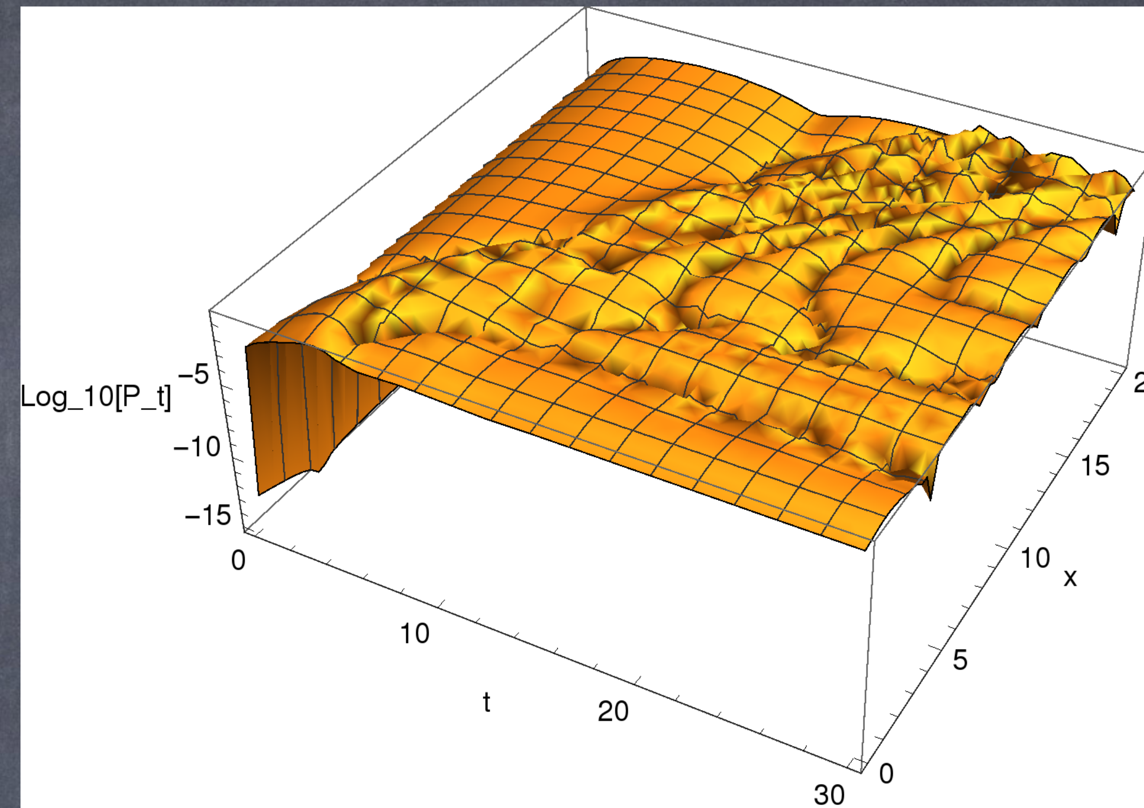
Model 1

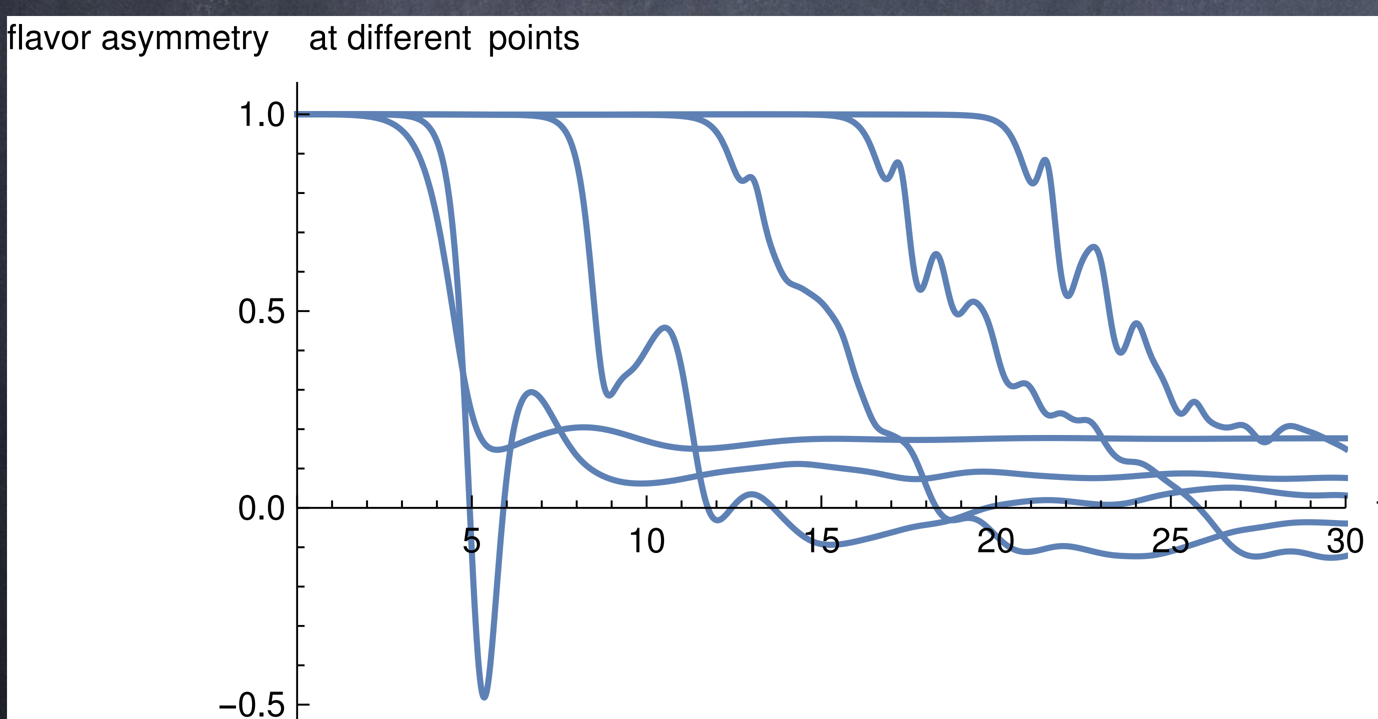
Results for a simulation with N_p=20 isotropically distributed angular modes, pure flavor 1 for neutrinos and anti-neutrinos with initial total number $N(t=0,x)=N_p f_i(x)=20\exp(-x)$ and injected with a rate $f_s(x)=\exp(-x)$ and equilibrium occupation numbers characterised by $f_0(x)=1$ (identical for anti-neutrinos). No anisotropy or crossing of lepton flavor in the momentum modes is assumed here, i.e. $f(x, i_p) = \bar{f}(x, i_p) = 1/2$

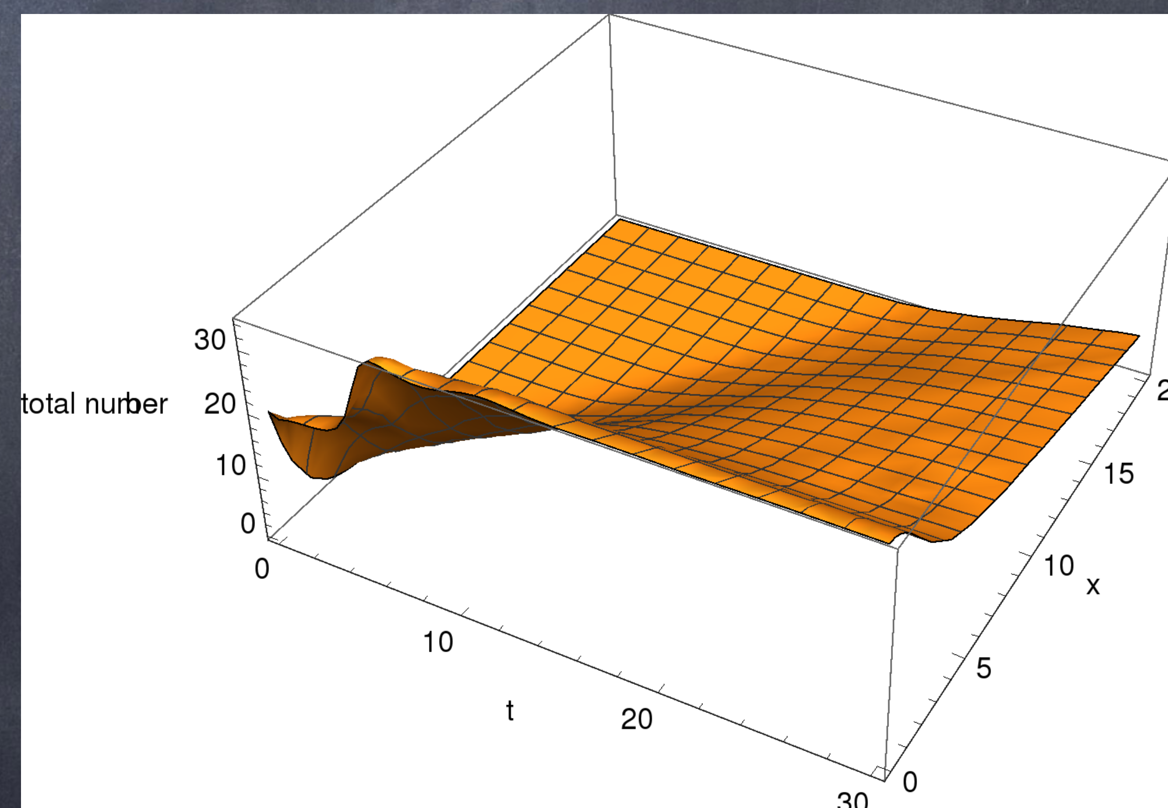
Further, $\Delta m^2=1$, $\Theta_0=0.01$, $\mu(x)=10$, $\lambda(x)=0$, $L_x=20$ with integration up to t=30.

Upper left: Normalized flavor asymmetry.
Upper right: Normalized off-diagonal elements.
Lower left: Cuts through flavor asymmetry from upper left
at x=0, x=4, x=8, x=12, x=16, and x=20.
Lower right: Total neutrino+anti-neutrino number.


flavor imbalance
1.0
0.5
0.0
−0.5
0
10
t
20
30
0
5
10
x
15
20

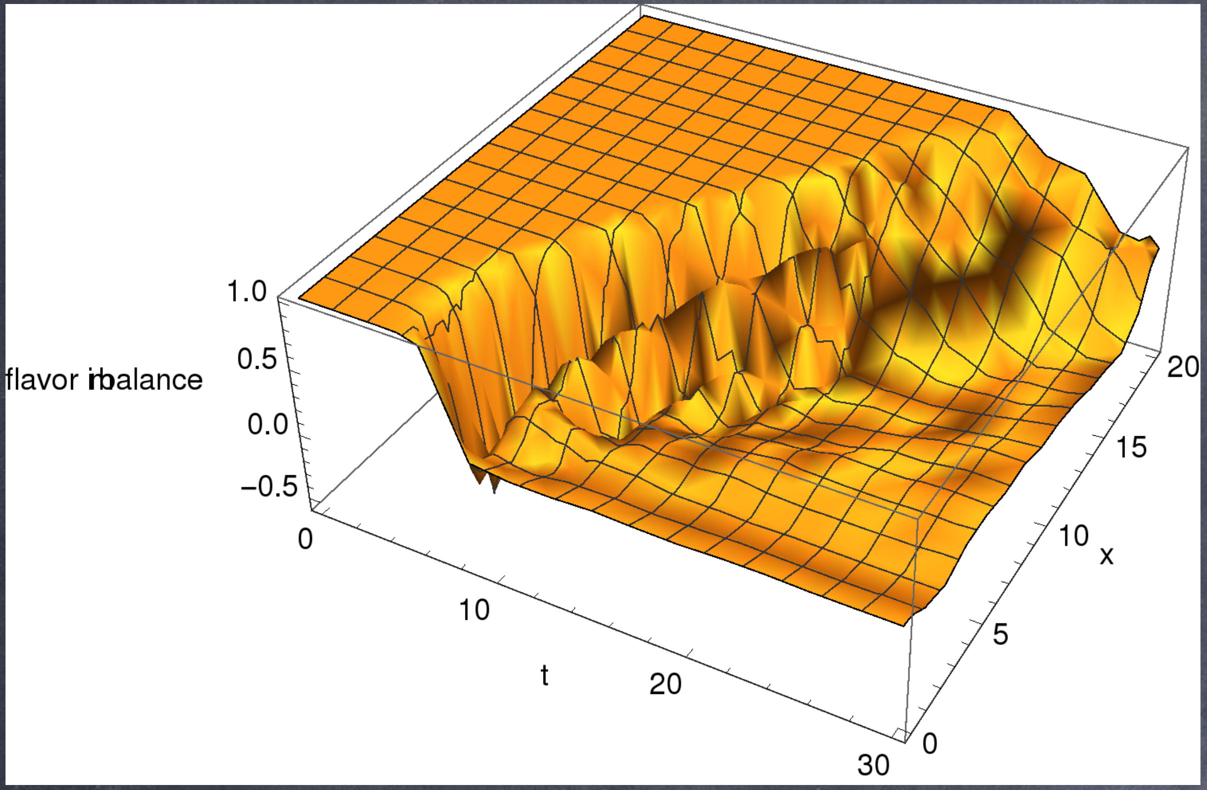

Log_10[P_t]
−5
−10
−15
0
10
t
20
30
0
5
10
x
15
20

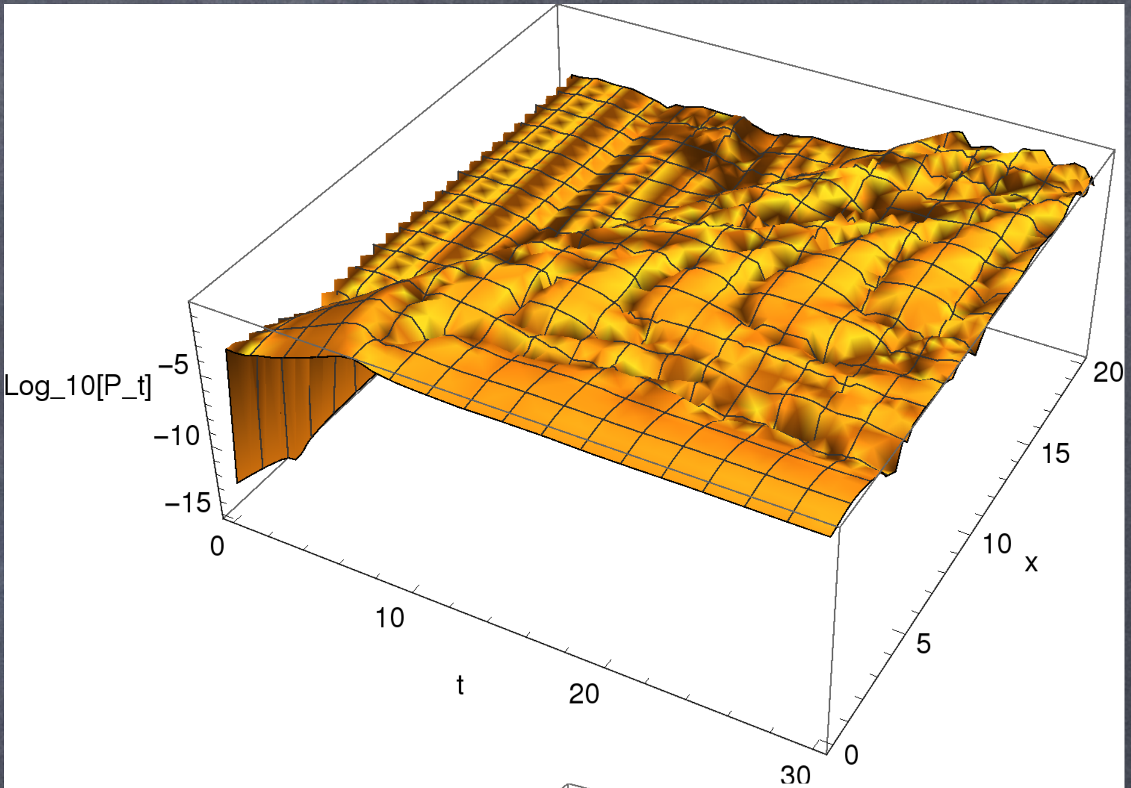

flavor asymmetry at different points
1.0
0.5
0.0
−0.5
5
10
15
20
25
30
t

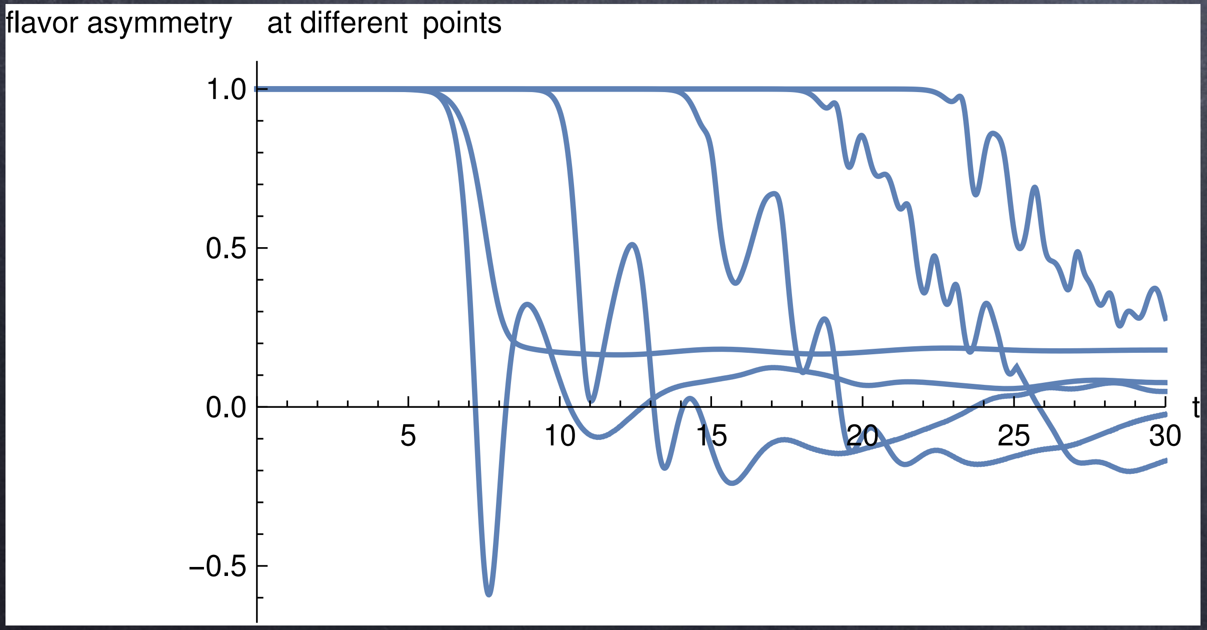

total number
30
20
10
0
0
10
t
20
30
0
5
10
x
15
20

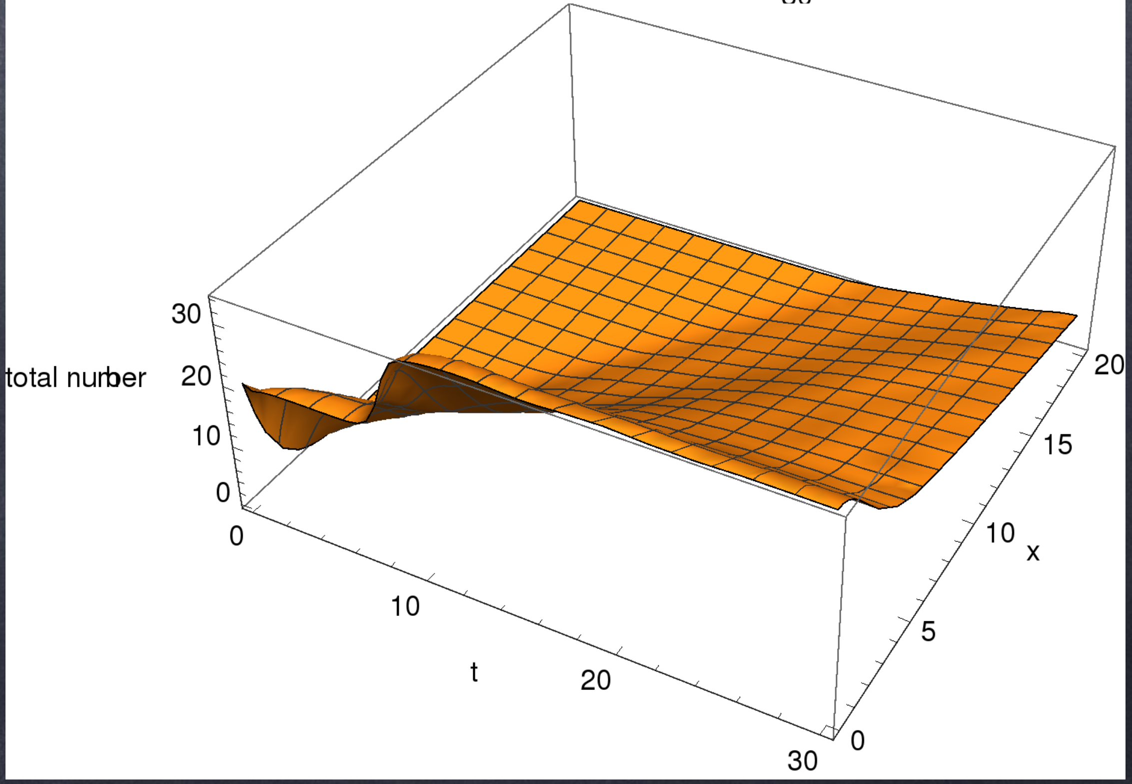
Model 2

Same as Model 1, but a matter potential of the form ƛ(x)=20 was added. No significant effect of matter is observed.







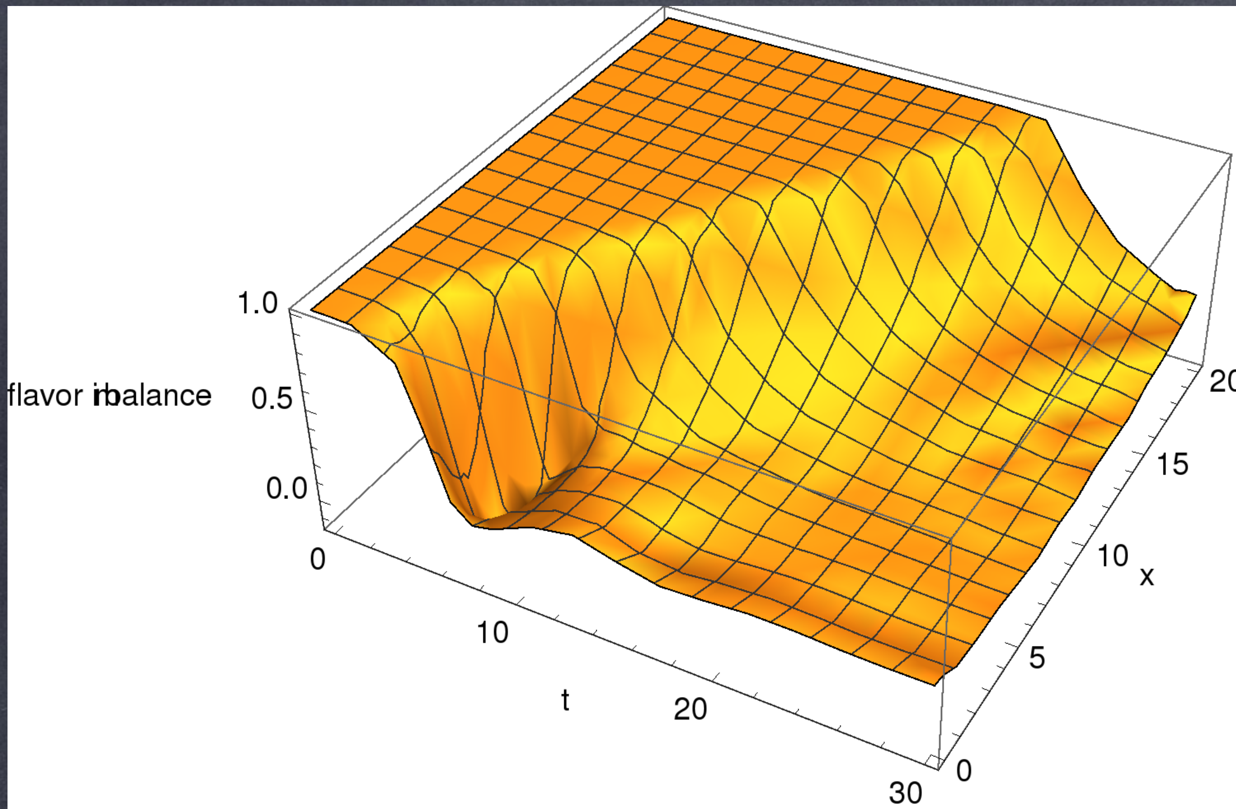


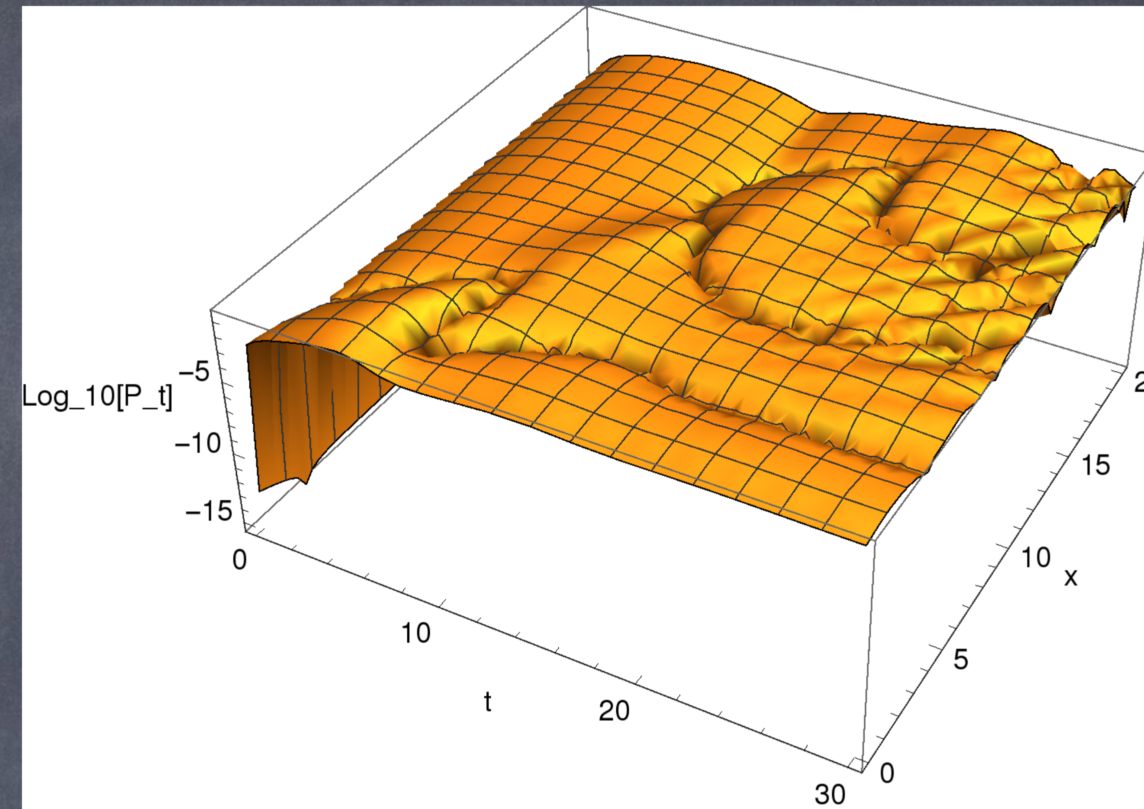
Model 3

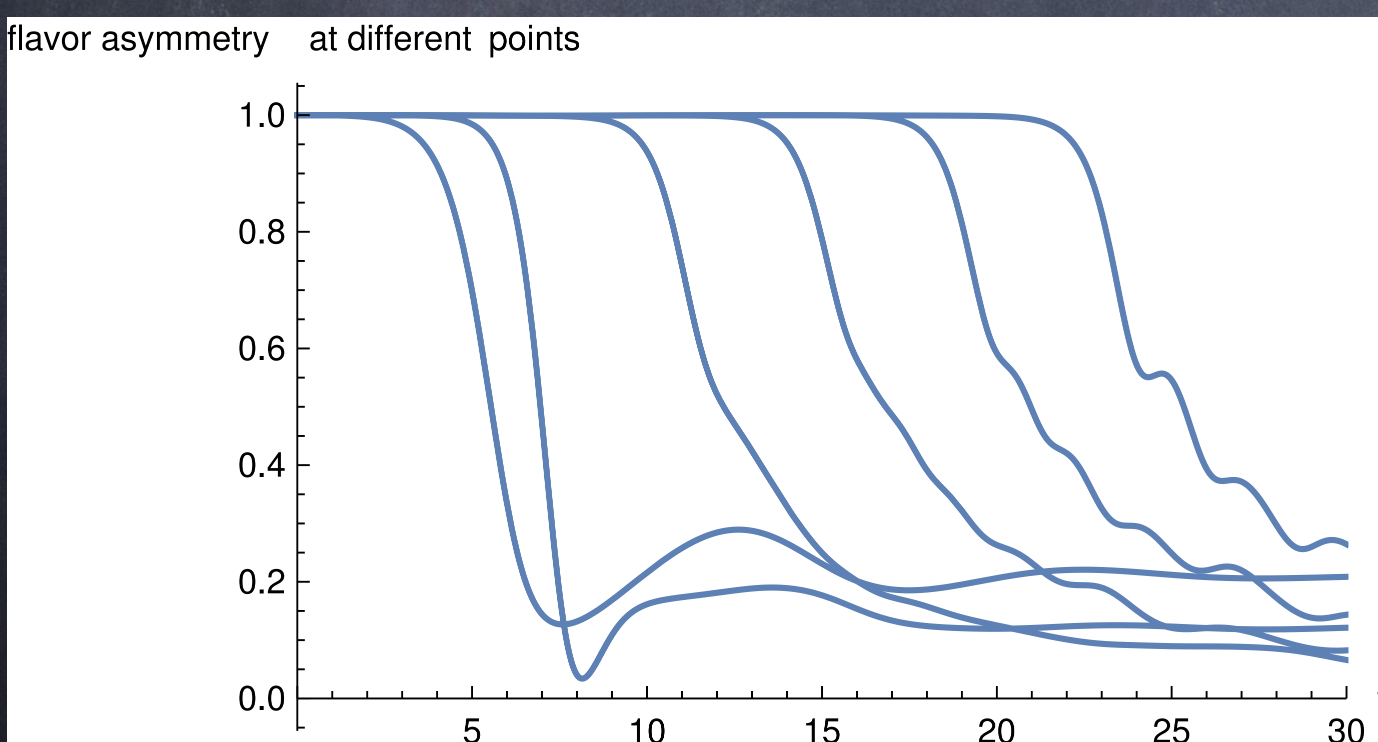
Results for a simulation with N_p=20 isotropically distributed angular modes, pure flavor 1 for neutrinos and anti-neutrinos with initial total number $N(t=0,x)=N_p f_i(x)=20\exp(-x)$ and injected with a rate $f_s(x)=\exp(-x)$ and equilibrium occupation numbers characterised by $f_0(x)=1$ (identical for anti-neutrinos). No anisotropy or crossing of lepton flavor in the momentum modes is assumed here, i.e. $f(x, i_p) = \bar{f}(x, i_p) = 1/2$

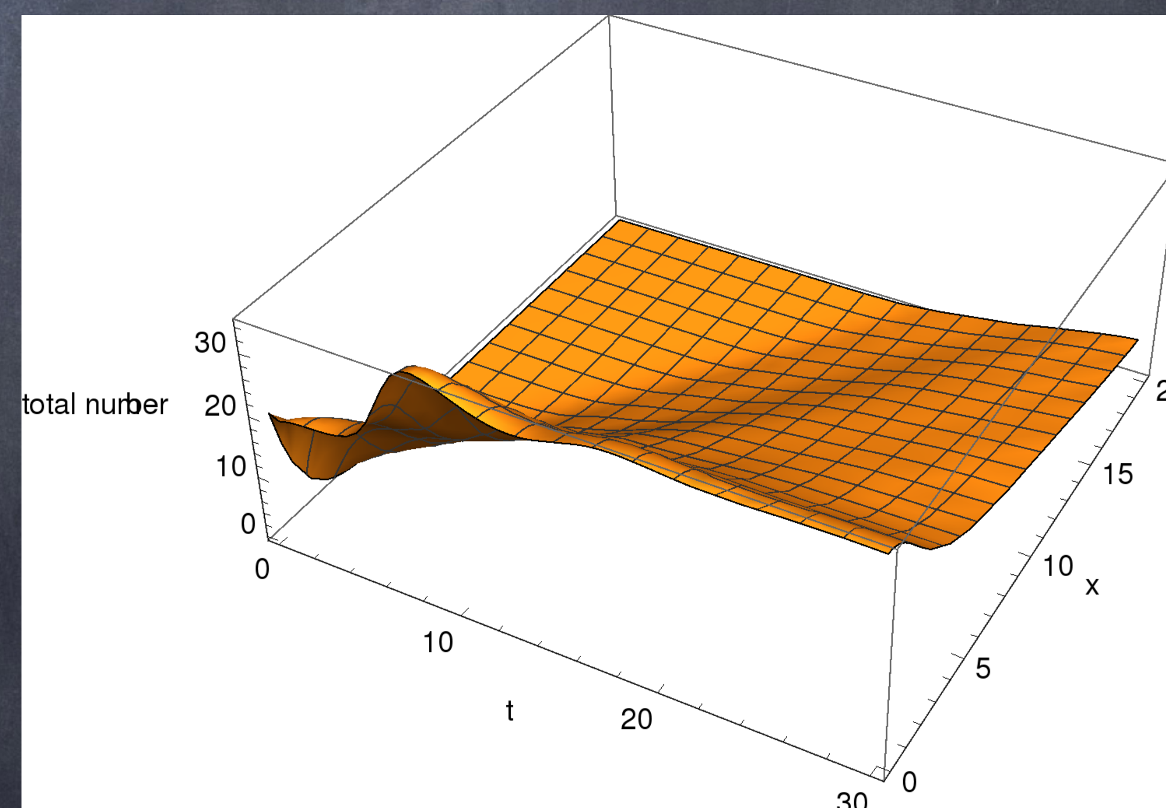
Further, $\Delta m^2=1$, $\Theta_0=0.01$, **$\mu(x)=10/(x/10+1)^4$**, $\lambda(x)=0$, $L_x=20$ with integration up to t=30.

Upper left: Normalized flavor asymmetry.
Upper right: Normalized off-diagonal elements.
Lower left: Cuts through flavor asymmetry from upper left
at x=0, x=4, x=8, x=12, x=16, and x=20.
Lower right: Total neutrino+anti-neutrino number.


flavor imbalance
1.0
0.5
0.0
t
x

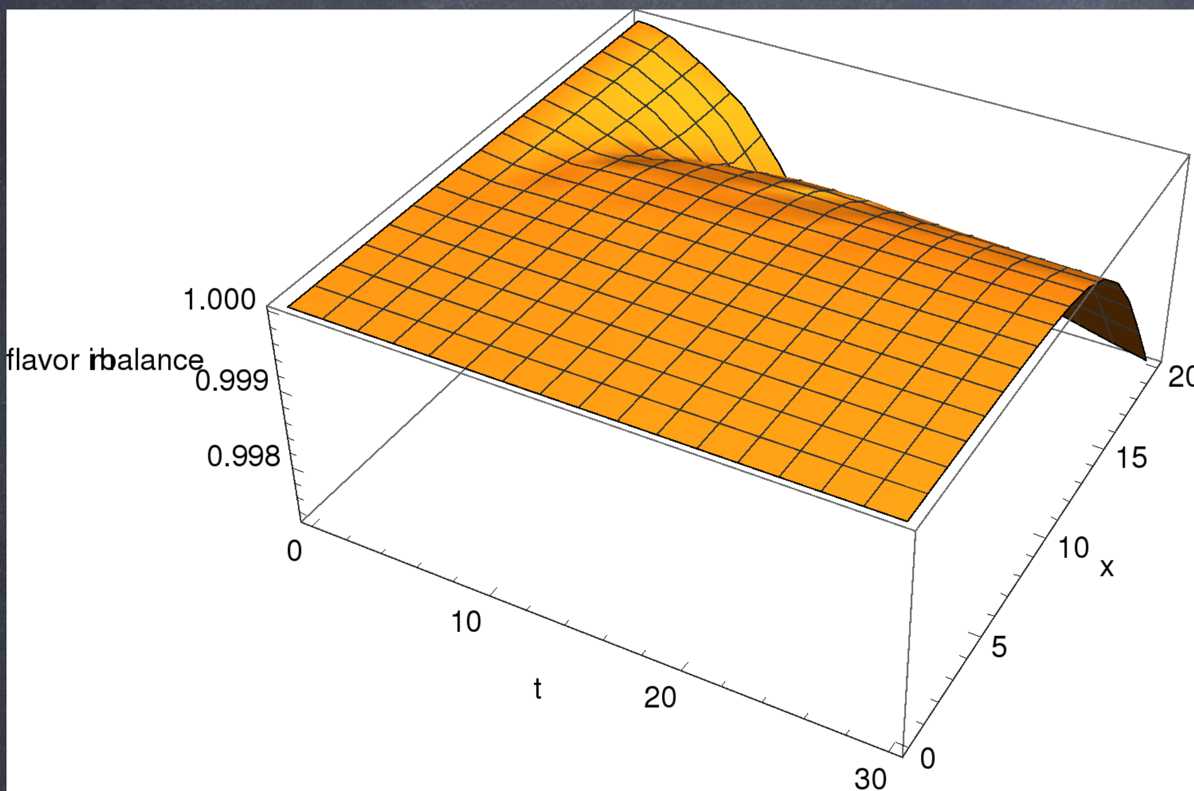

Log_10[P_t]
−5
−10
−15
t
x

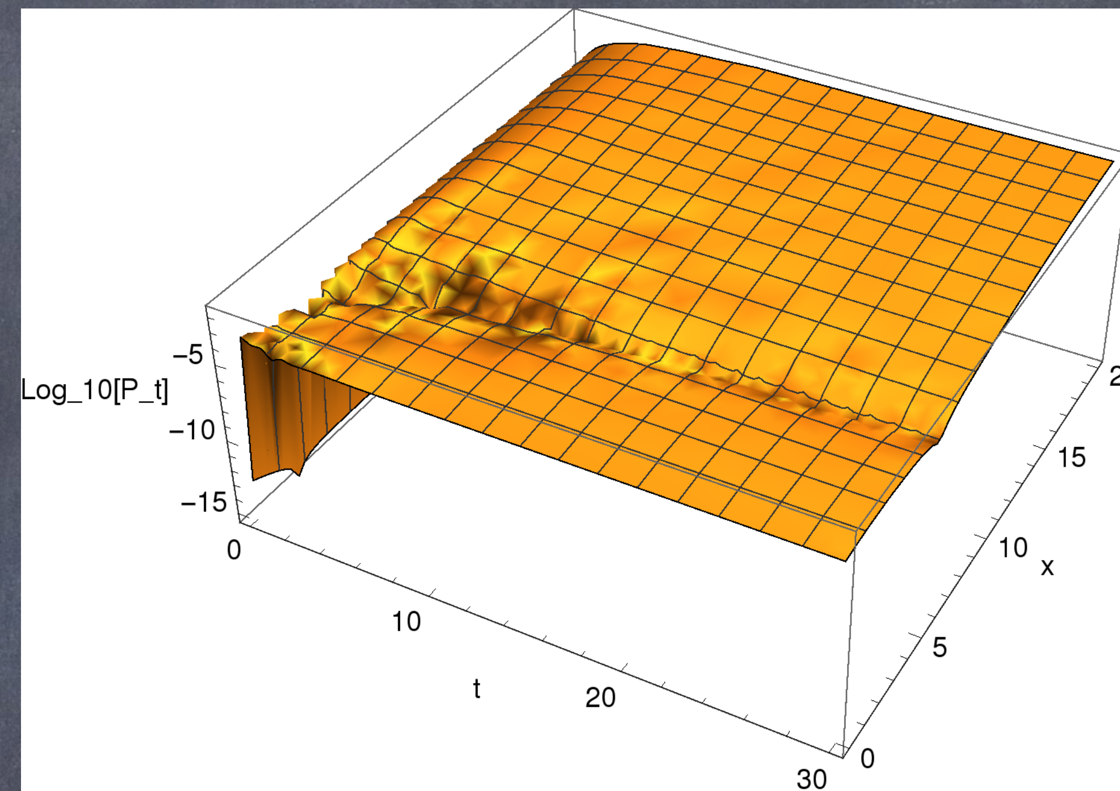

flavor asymmetry at different points
t

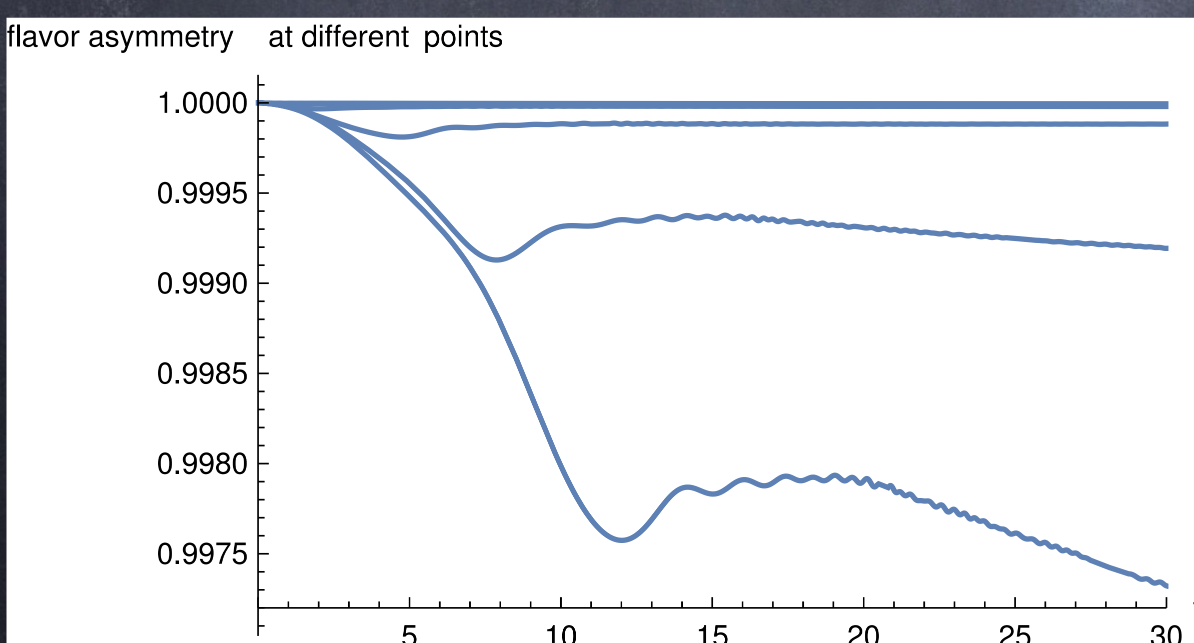

total number
t
x

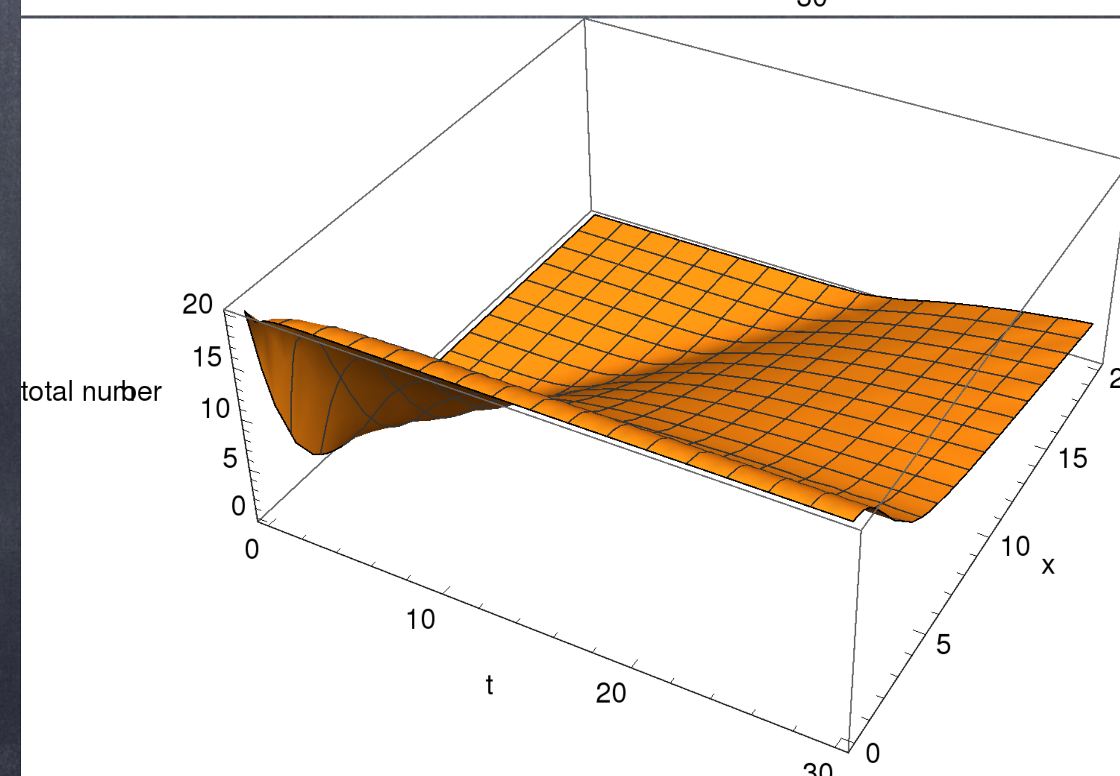
Model 4

Same as Model 1, but a matter potential of the form $\lambda(x)=20/(x/10+1)^4$ was added. A strong matter potential thus suppresses the transition.



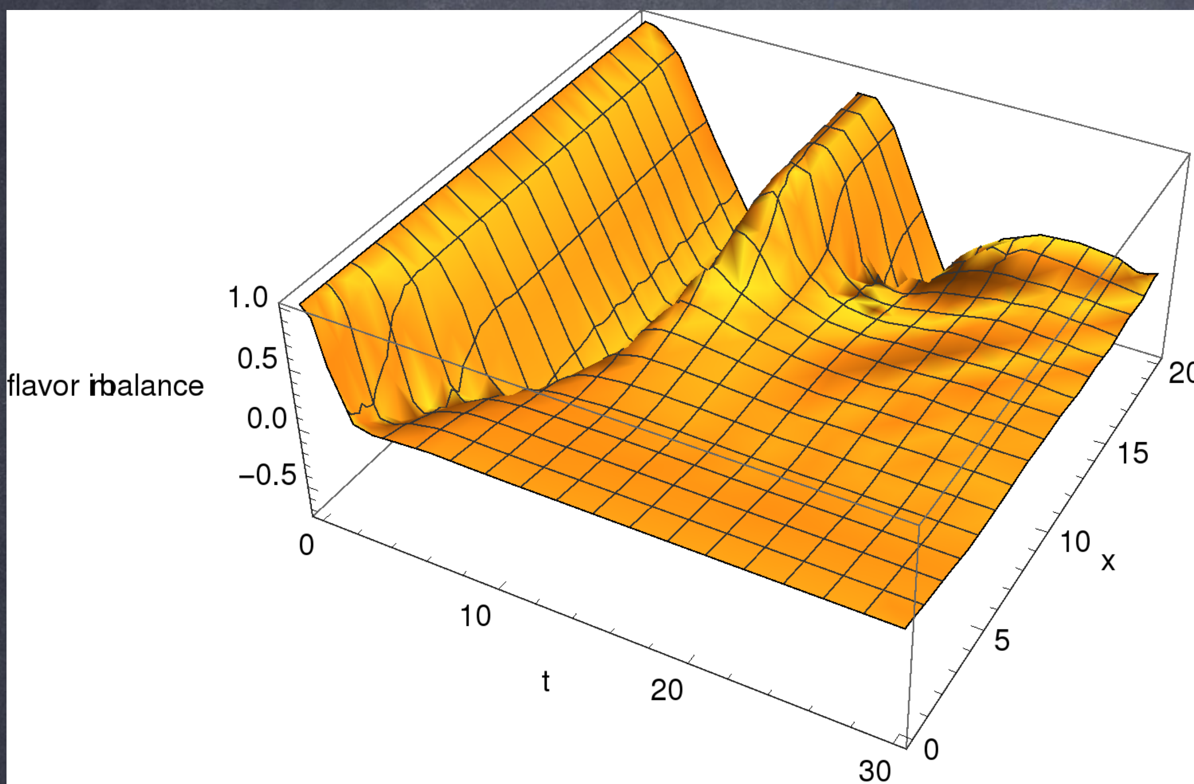


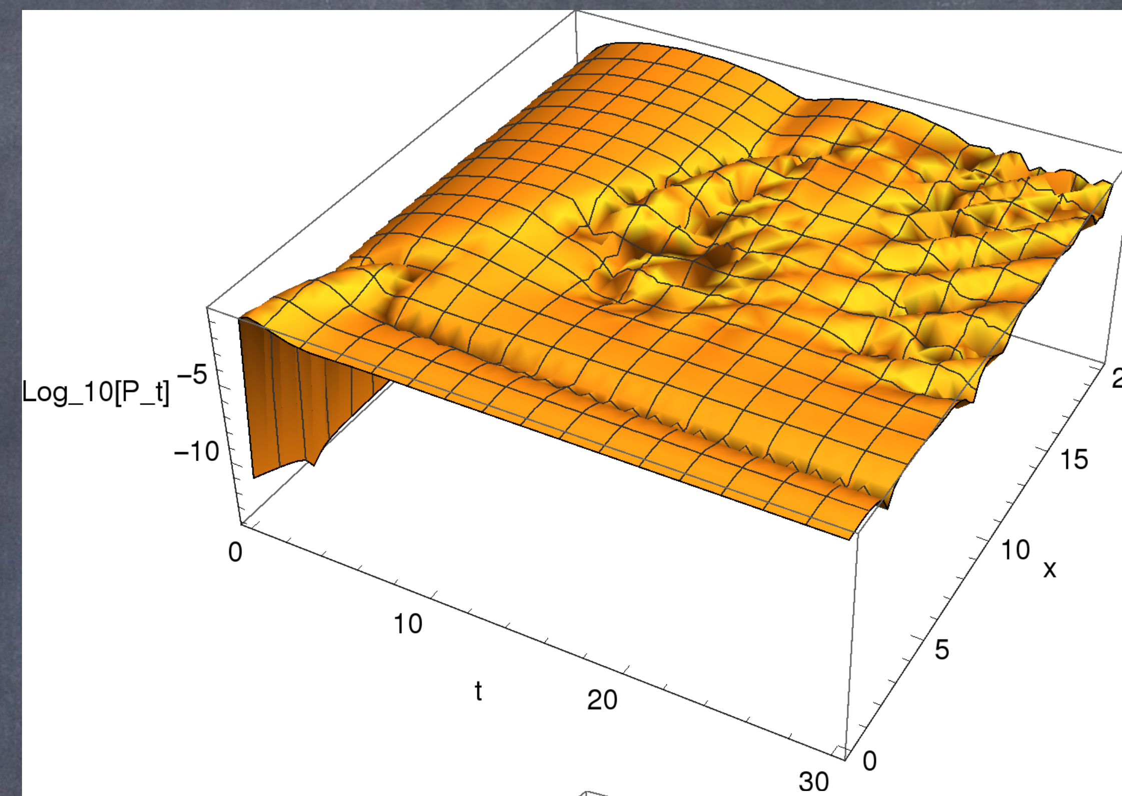


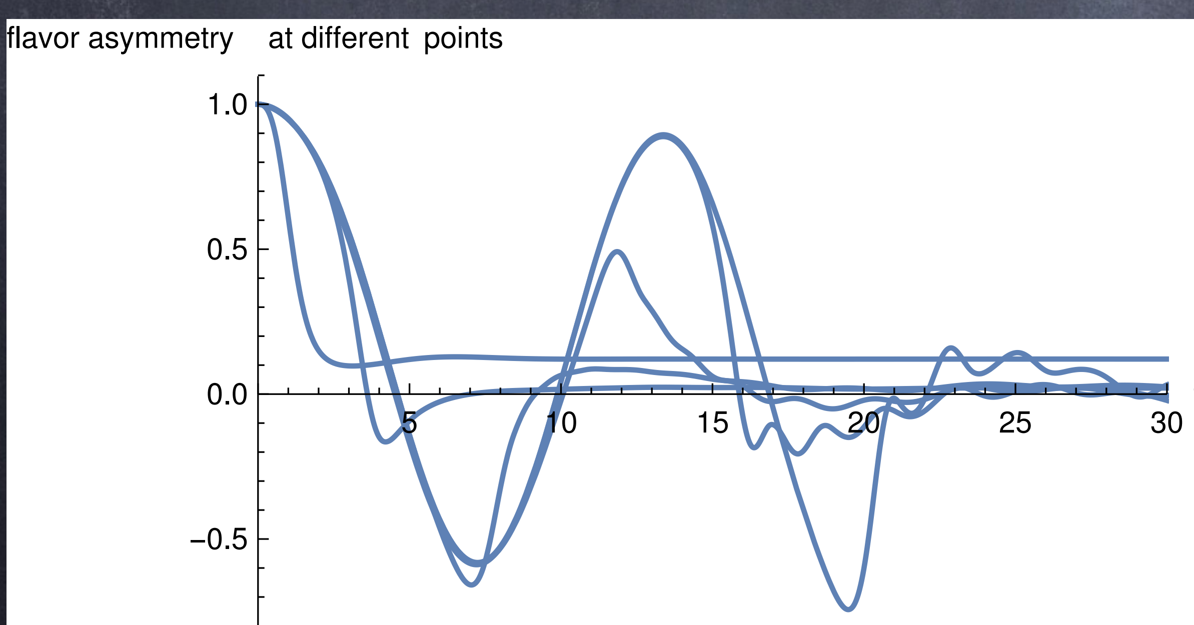


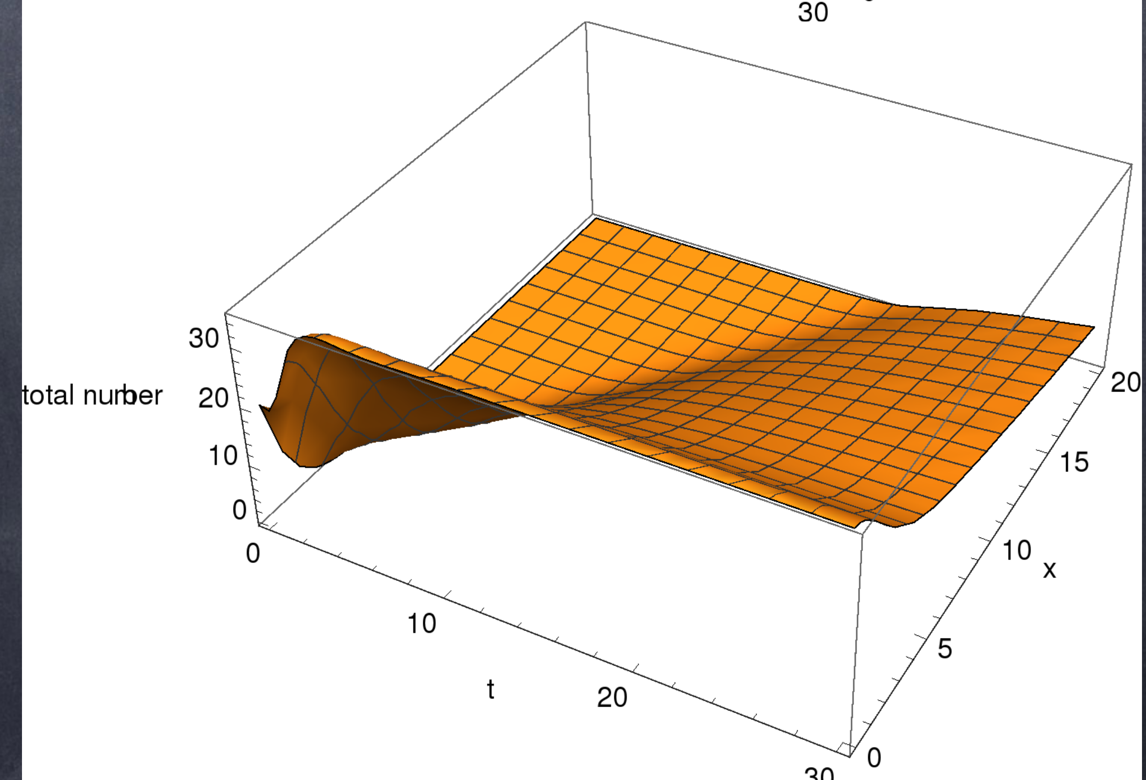
Model 5

Same as Model 1, i.e. no matter potential, but for a realistically large vacuum mixing angle Θ_0=34°



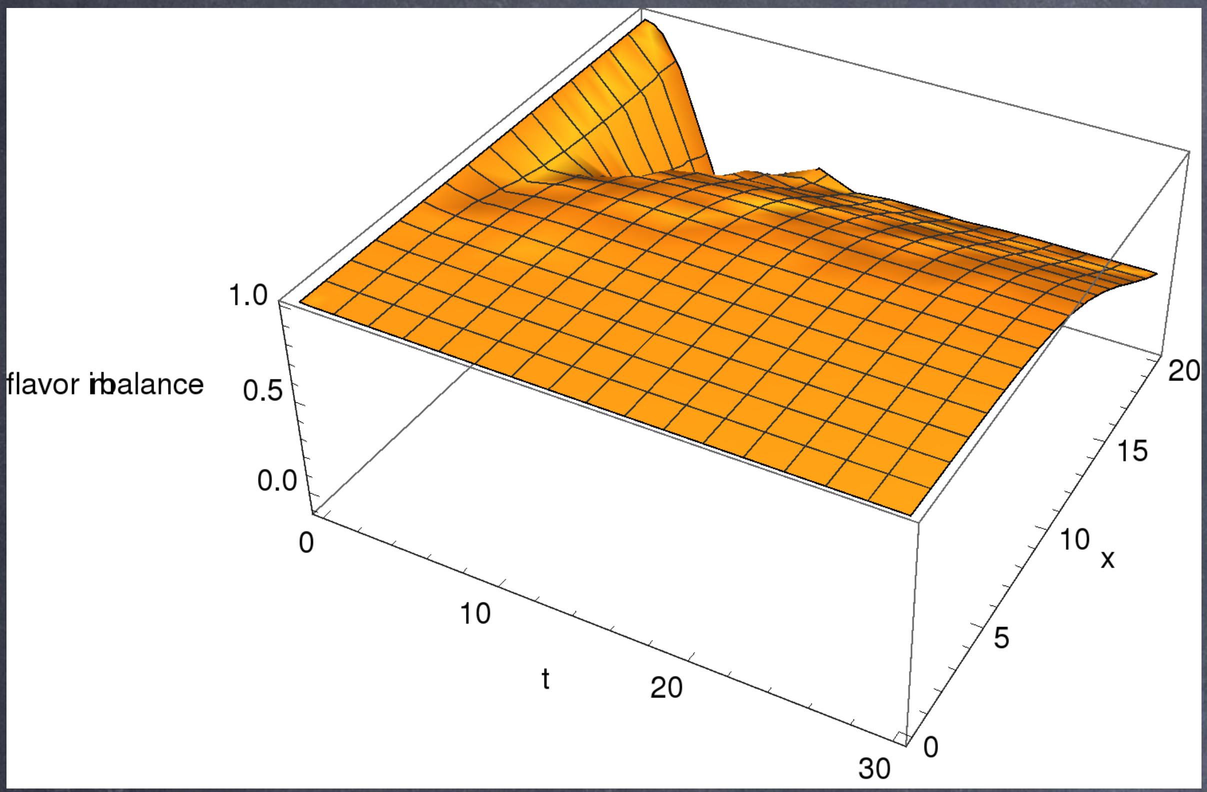


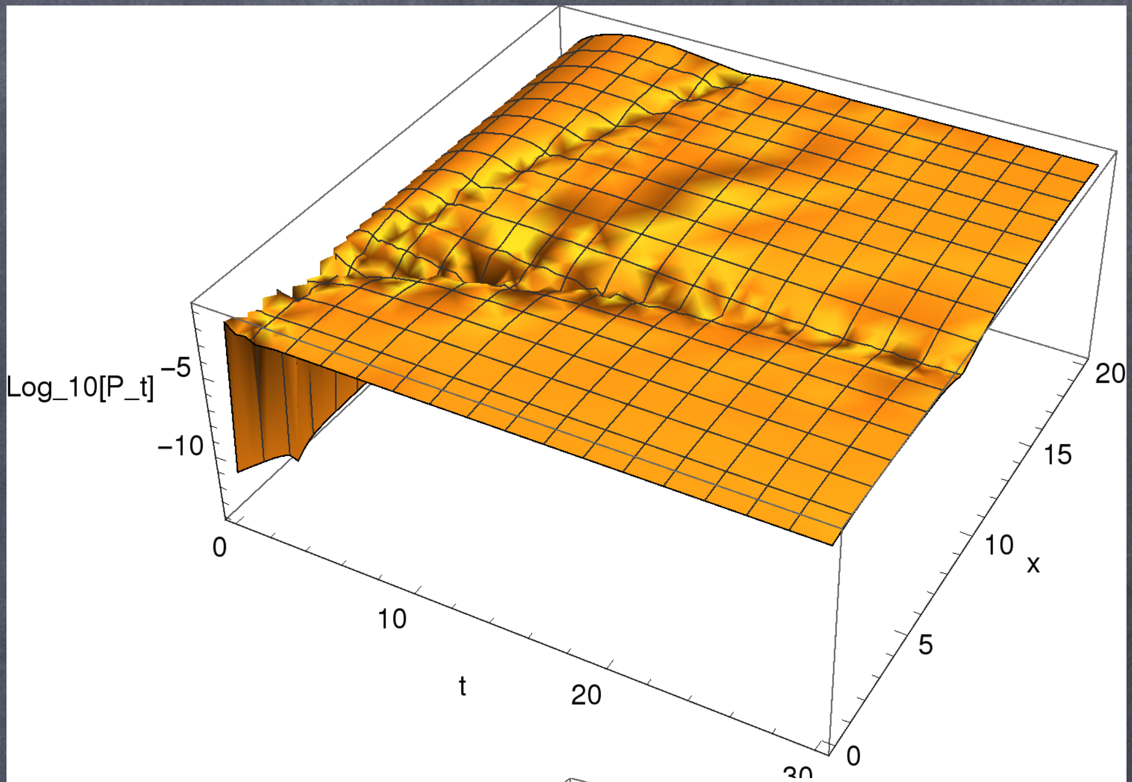


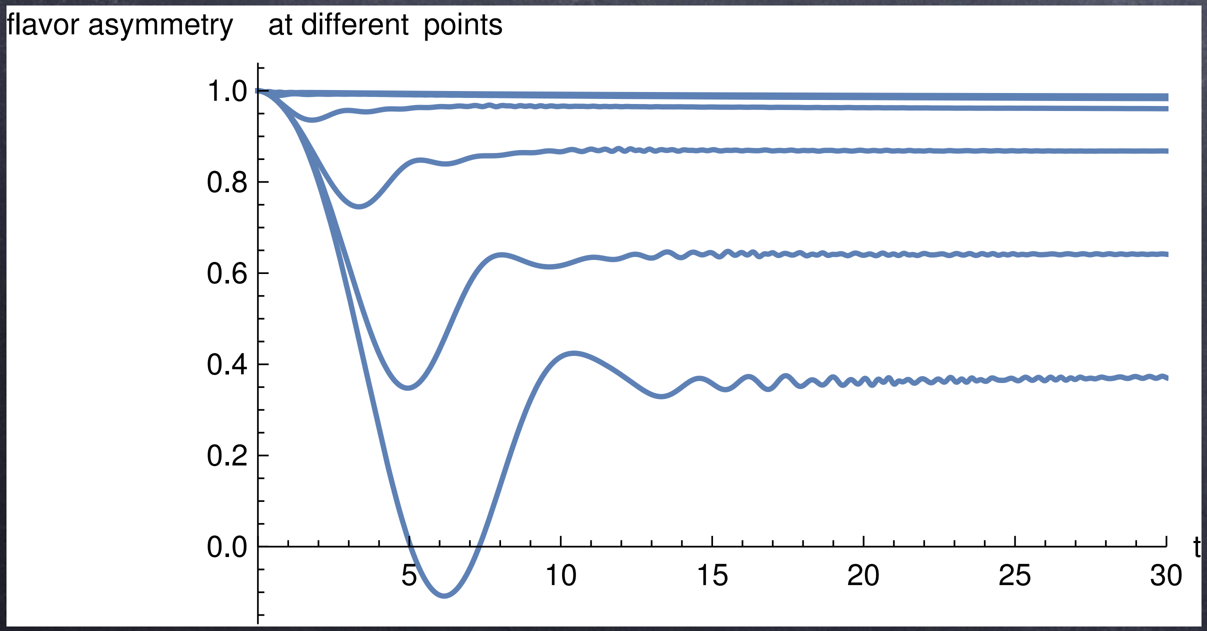


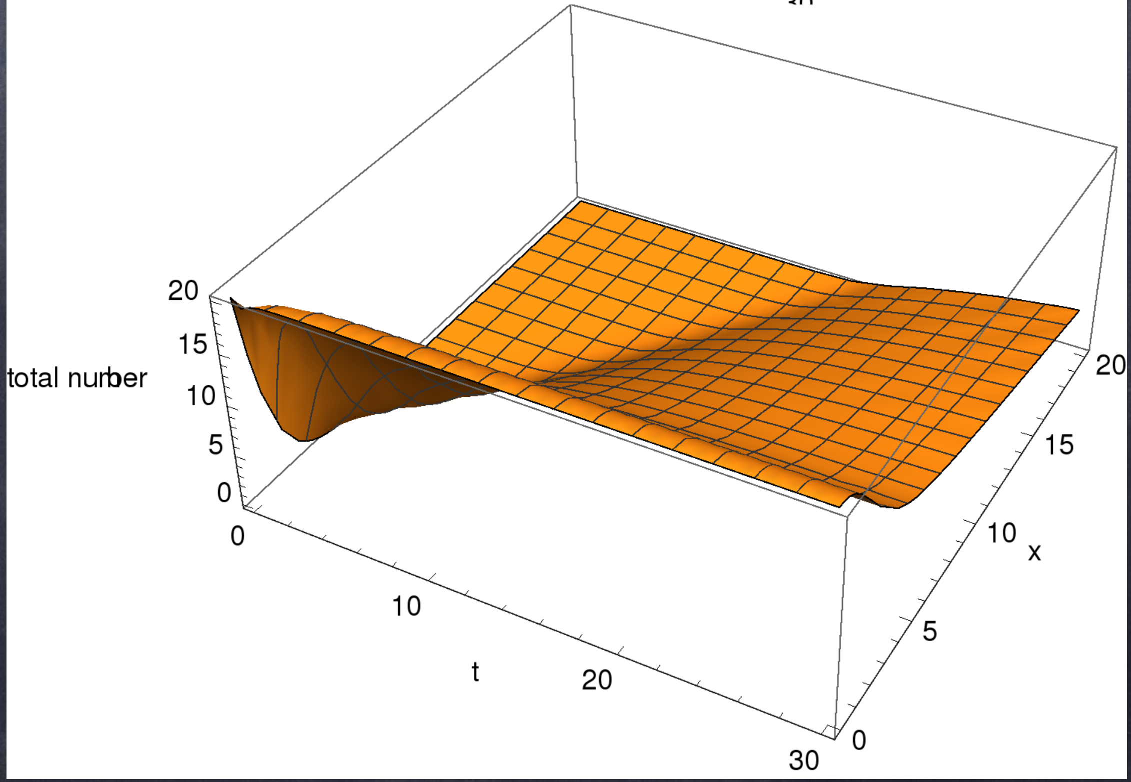
Model 6

Same as Model 2, i.e. for a matter potential of the form $\lambda(x)=20/(x/10+1)^4$, but for a realistically large vacuum mixing angle $\Theta_0=34^o$. Thus for a large vacuum mixing angle a strong matter potential suppresses the transitions less.







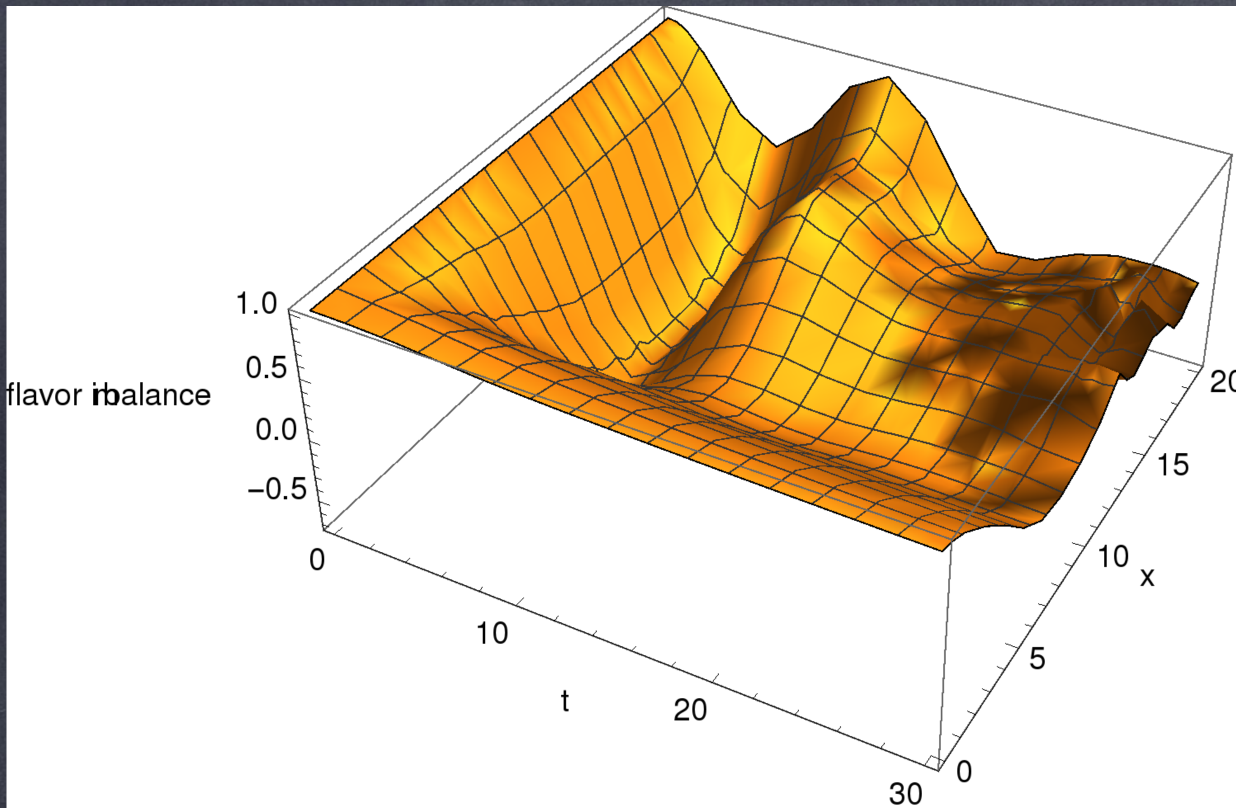


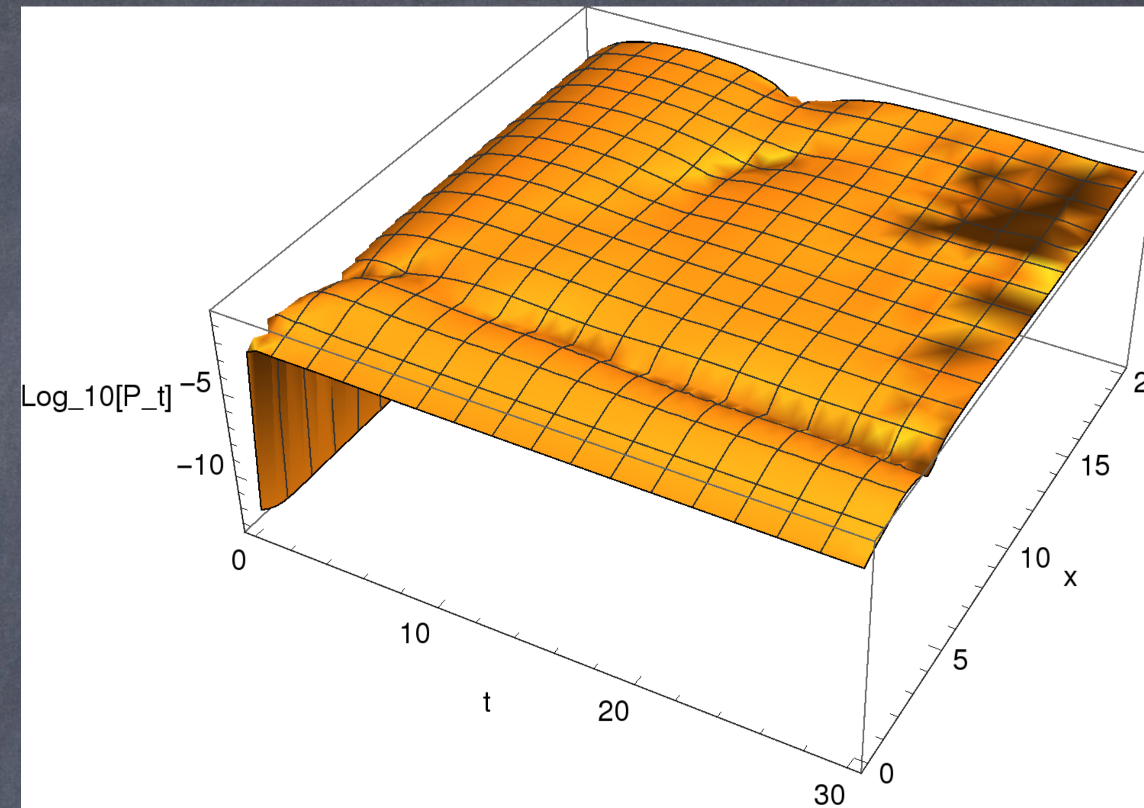
Model 7

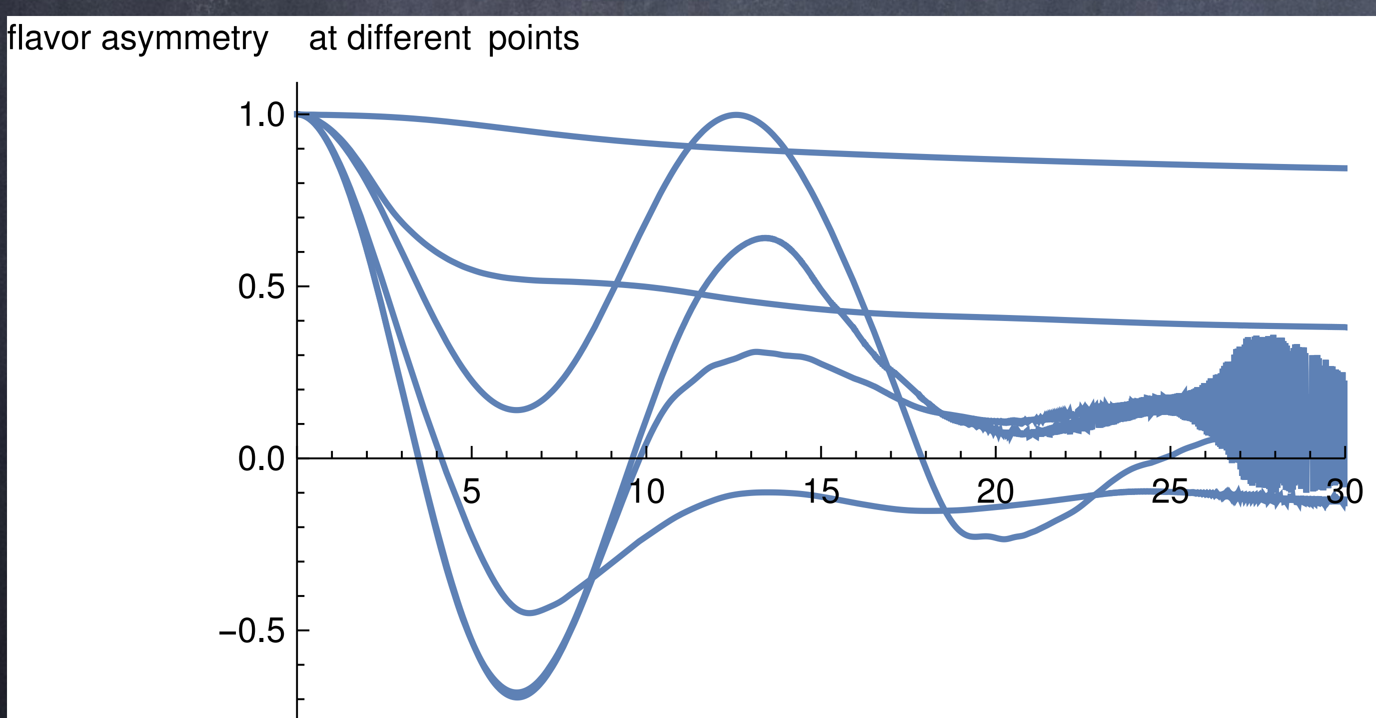
Results for a simulation with N_p=20 isotropically distributed angular modes, pure flavor 1 for neutrinos and anti-neutrinos with initial total number $N(t=0,x)=N_p f_0(L_x)$ and injected with a rate $f_s(x)=\exp(-x)$ and equilibrium occupation numbers characterised by $f_0(x)=1./(x/10+1)^4$ (identical for anti-neutrinos). **A crossing of lepton flavor in the momentum modes is assumed here with b=0.5.**

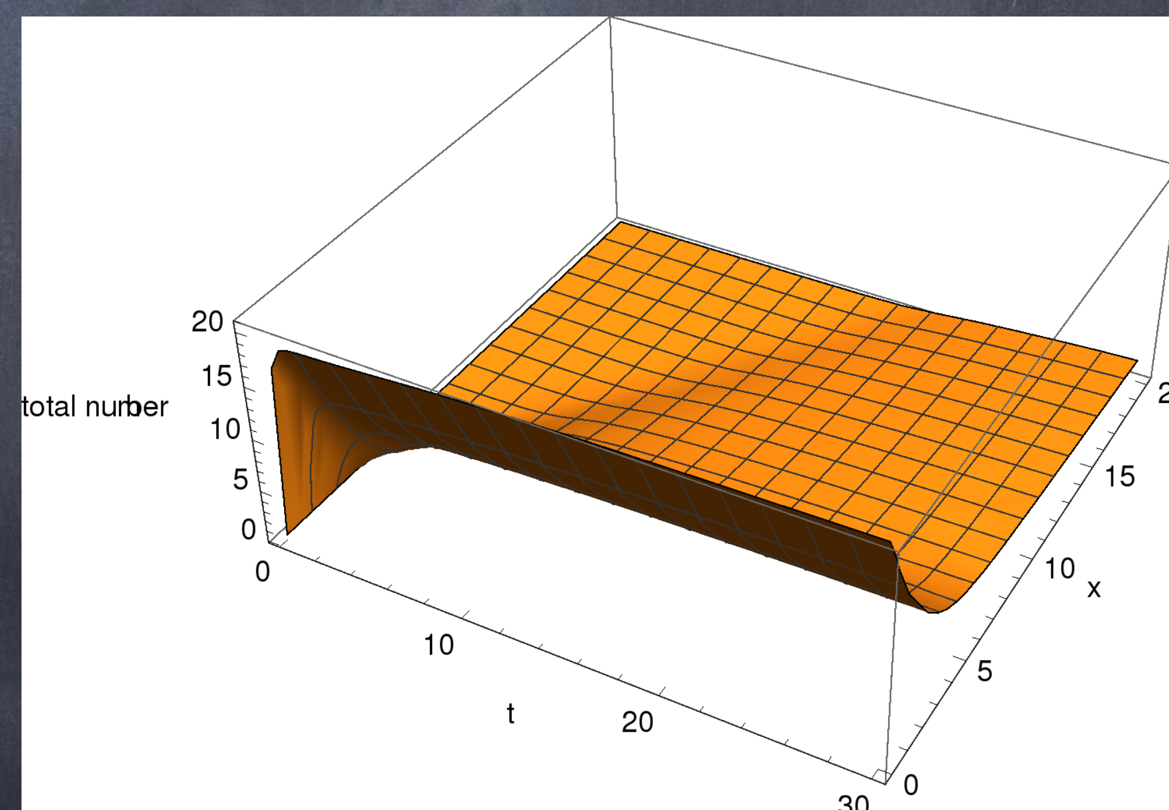
Further, Δm^2=1., $\Theta_0=34^o$, **$\mu(x)=10/(x/10+1)^4$, $\lambda(x)=20\exp(-x)$**, L_x=20 with integration up to t=30.

Upper left: Normalized flavor asymmetry.
Upper right: Normalized off-diagonal elements.
Lower left: equidistant cuts through flavor asymmetry increasing in x from top to bottom
Lower right: Total neutrino+anti-neutrino number.


flavor imbalance
1.0
0.5
0.0
−0.5
0
10
t
20
30
0
5
10
x
15
20


Log_10[P_t]
−5
−10
0
10
t
20
30
0
5
10
x
15
20


flavor asymmetry at different points
1.0
0.5
0.0
−0.5
5
10
15
20
25
30
t

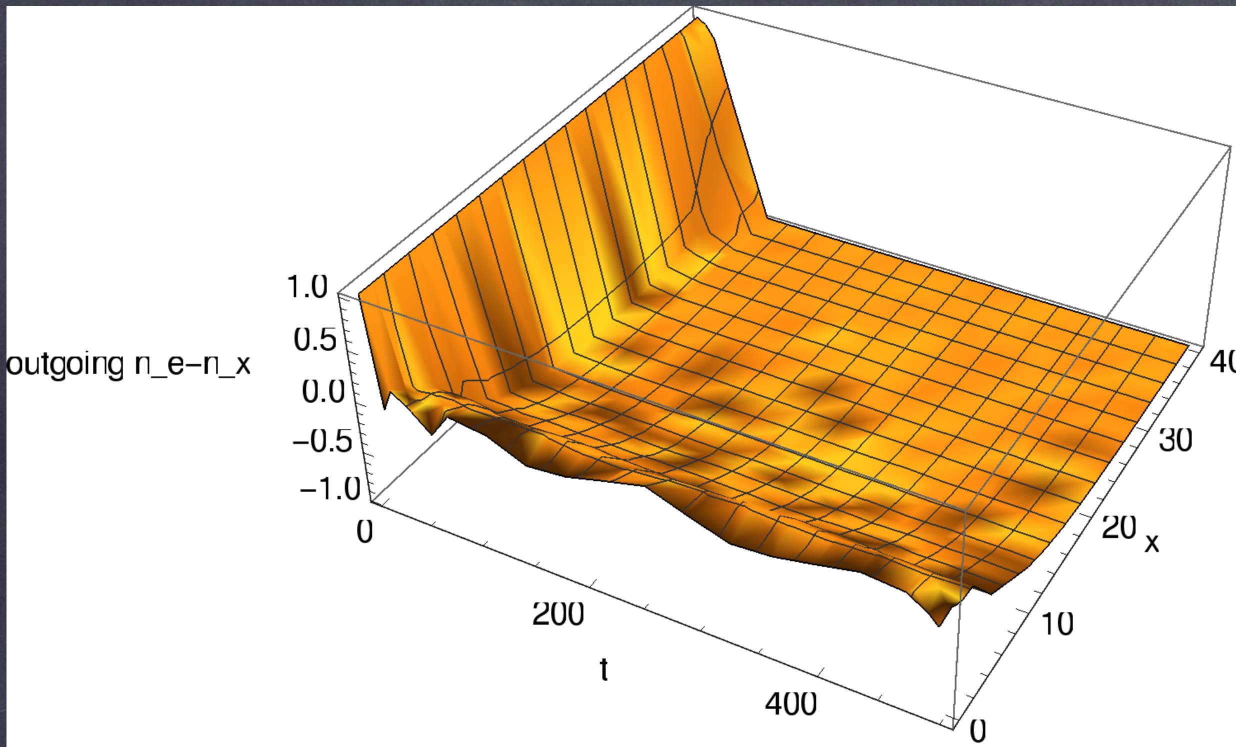

total number
20
15
10
5
0
0
10
t
20
30
0
5
10
x
15
20

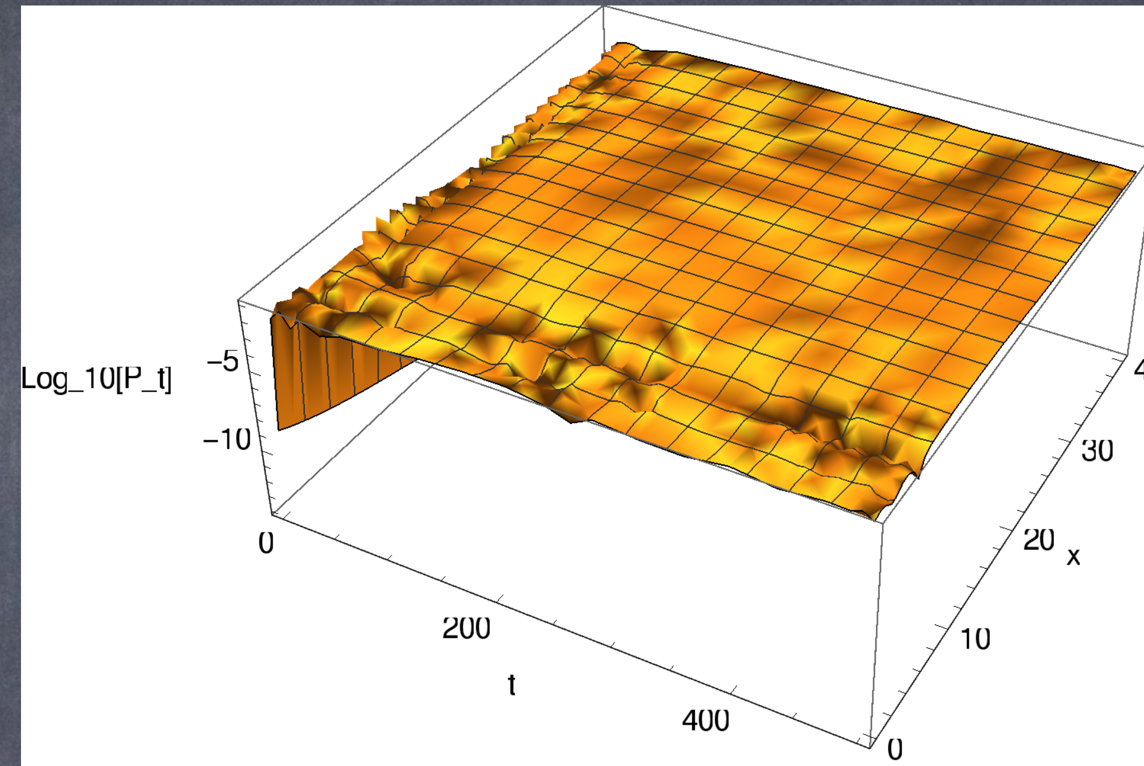
Model 8

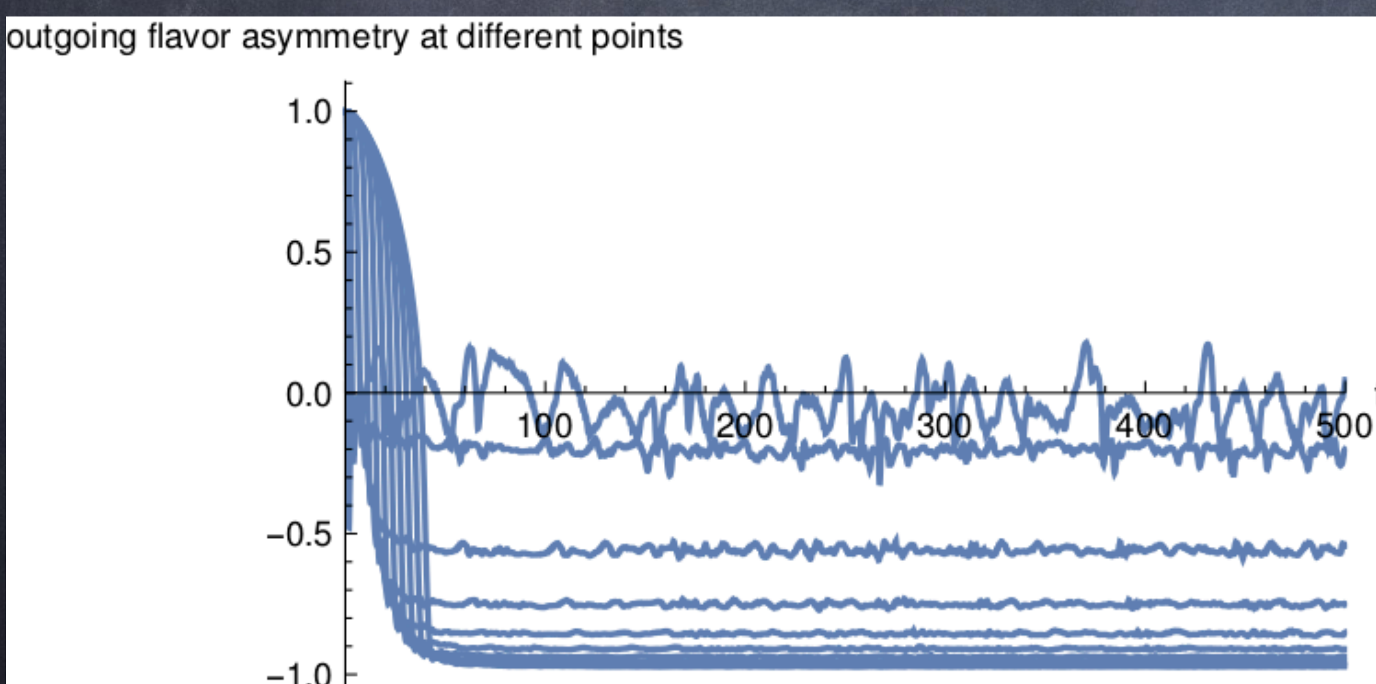
Results for a simulation with N_p=10 isotropically distributed angular modes, pure flavor 1 for neutrinos and anti-neutrinos with initial total number $N(t=0,x)=N_p f_0(L_x)$ and injected with a rate $f_s(x)=0.1/(x/10+1)^4$ and equilibrium occupation numbers characterised by $f_0(x)=0.8/(x/10+1)^4$ (identical for anti-neutrinos). No anisotropy or crossing of lepton flavor in the momentum modes is assumed here, i.e. $f(x, i_p) = \bar{f}(x, i_p) = 1/2$

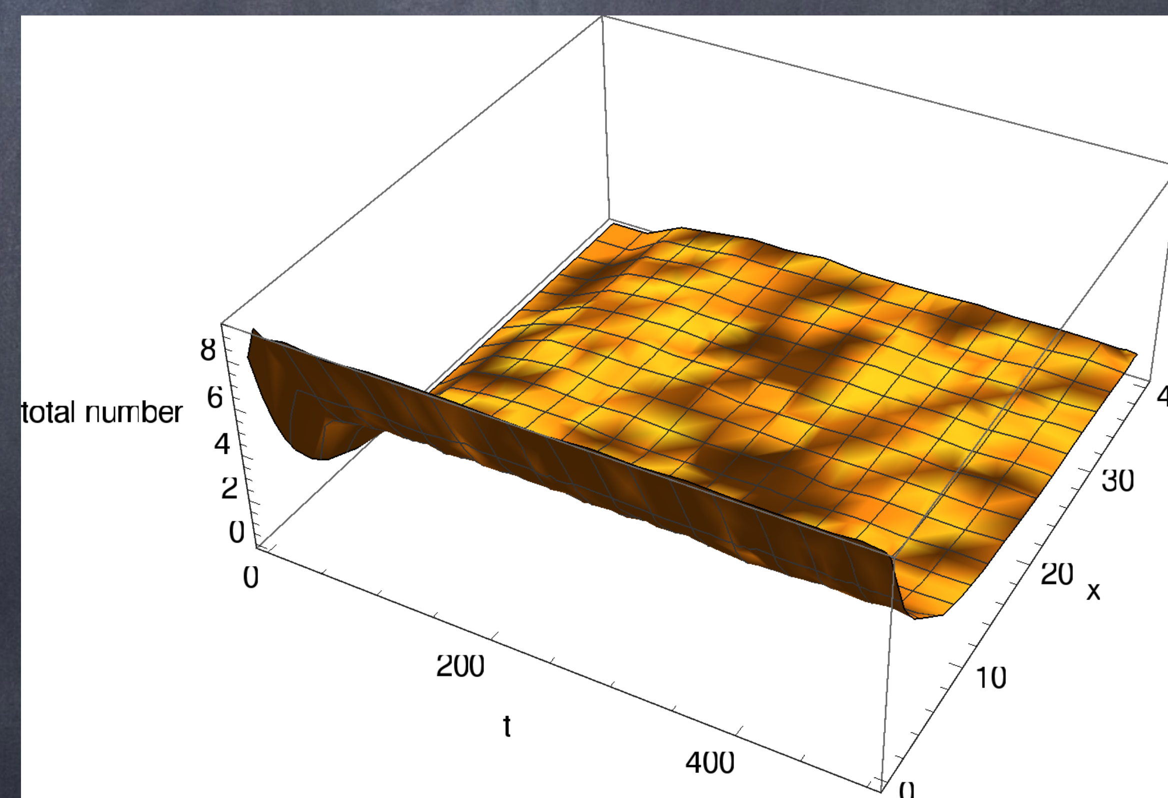
Further, Δm^2=0.1, Θ_0=34^o, $\mu(x)=100/(x/10+1)^4$, $\lambda(x)$=0, L_x=40 with integration up to t=500.

Upper left: Normalized flavor asymmetry.
Upper right: Normalized off-diagonal elements.
Lower left: equidistant cuts through flavor asymmetry increasing in x from top to bottom
Lower right: Total neutrino+anti-neutrino number.


outgoing n_e−n_x
1.0
0.5
0.0
−0.5
−1.0
0
200
t
400
0
10
20
x
30
40

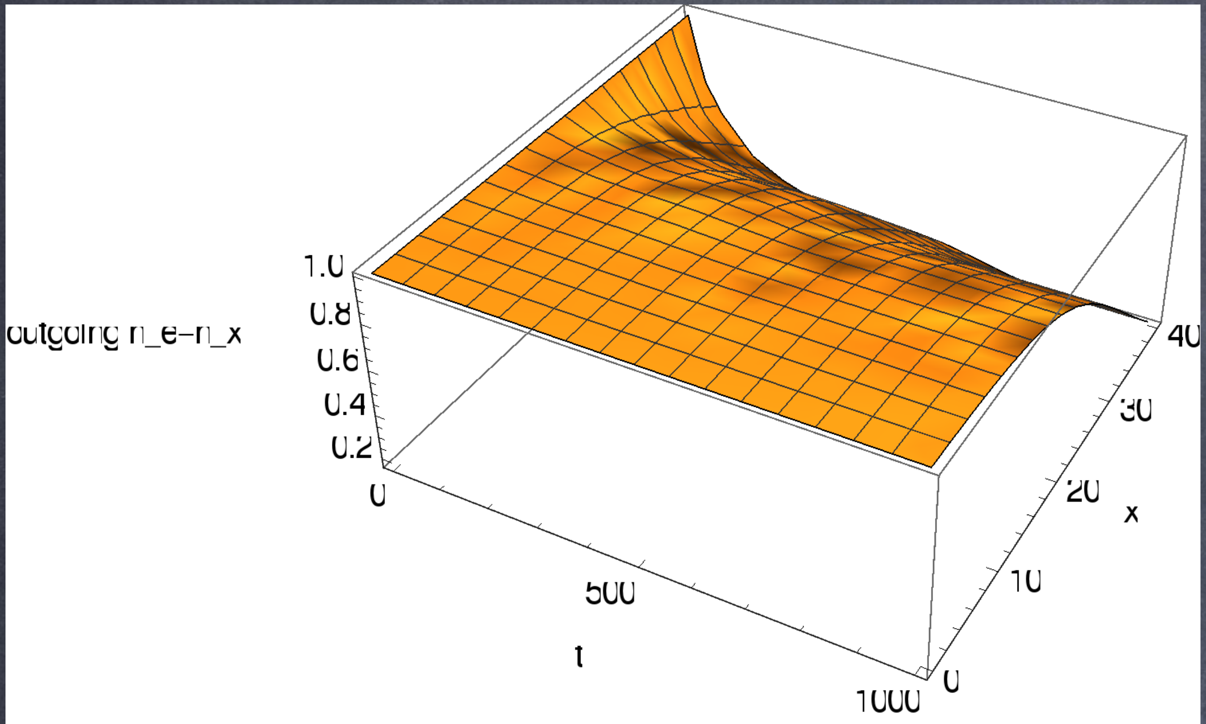

Log_10[P_t]
−5
−10
0
200
t
400
0
10
20
x
30
40

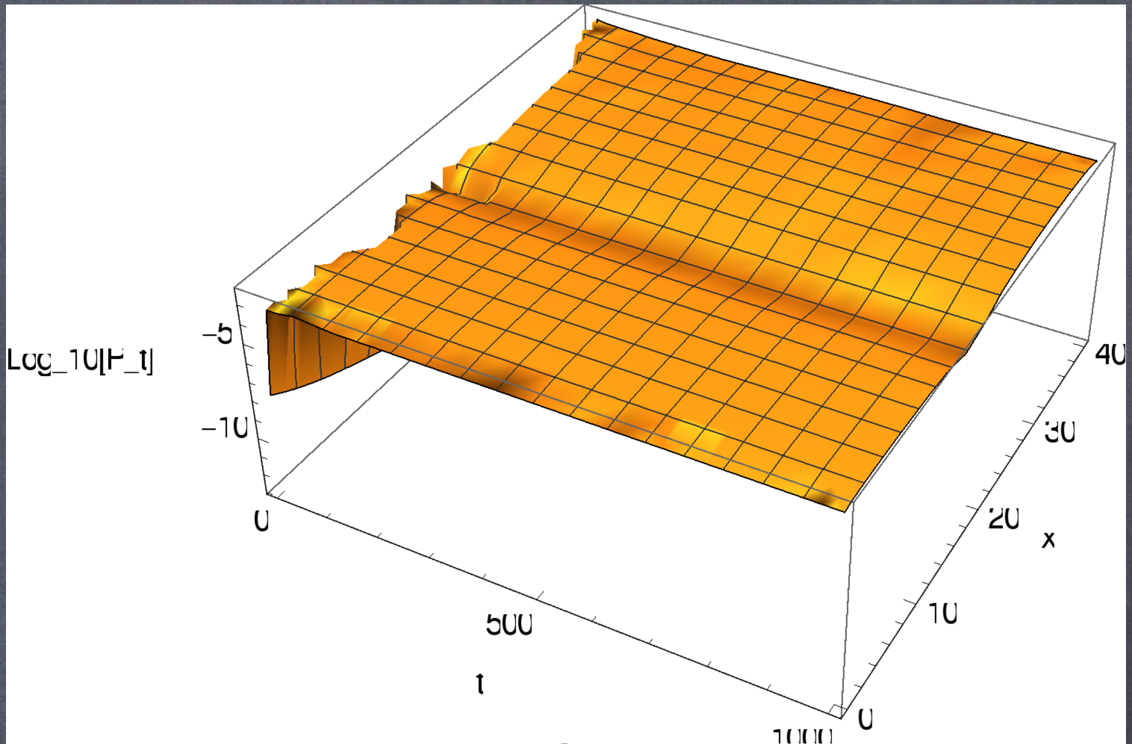

outgoing flavor asymmetry at different points
1.0
0.5
0.0
−0.5
−1.0
100
200
300
400
500
t

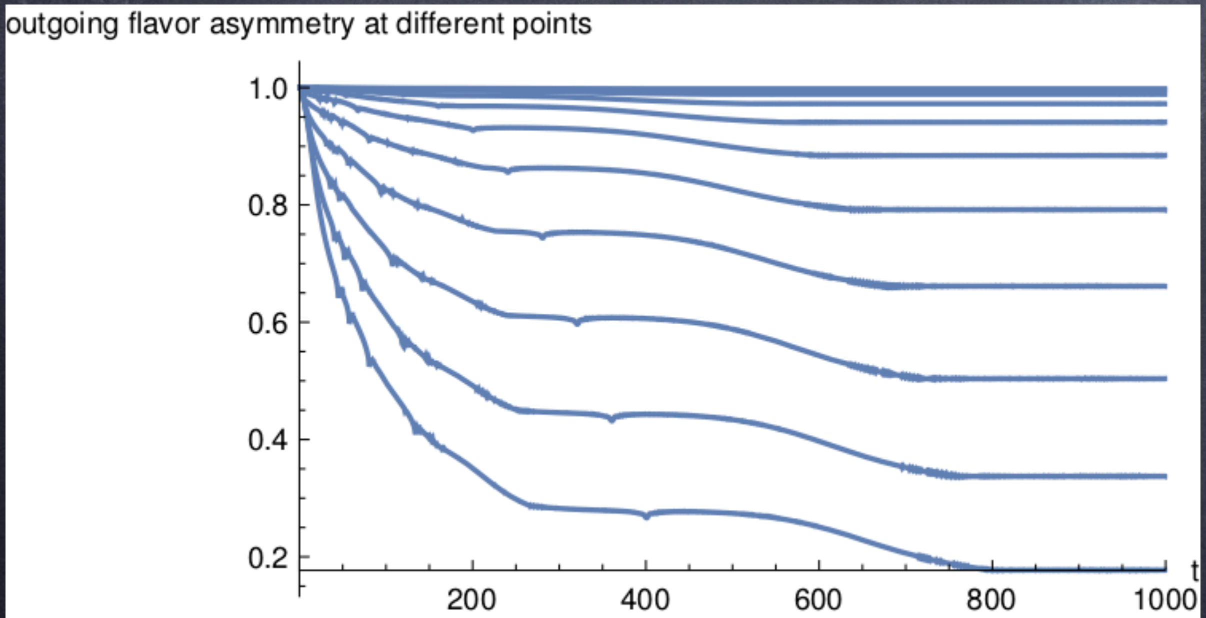

total number
8
6
4
2
0
0
200
t
400
0
10
20
x
30
40

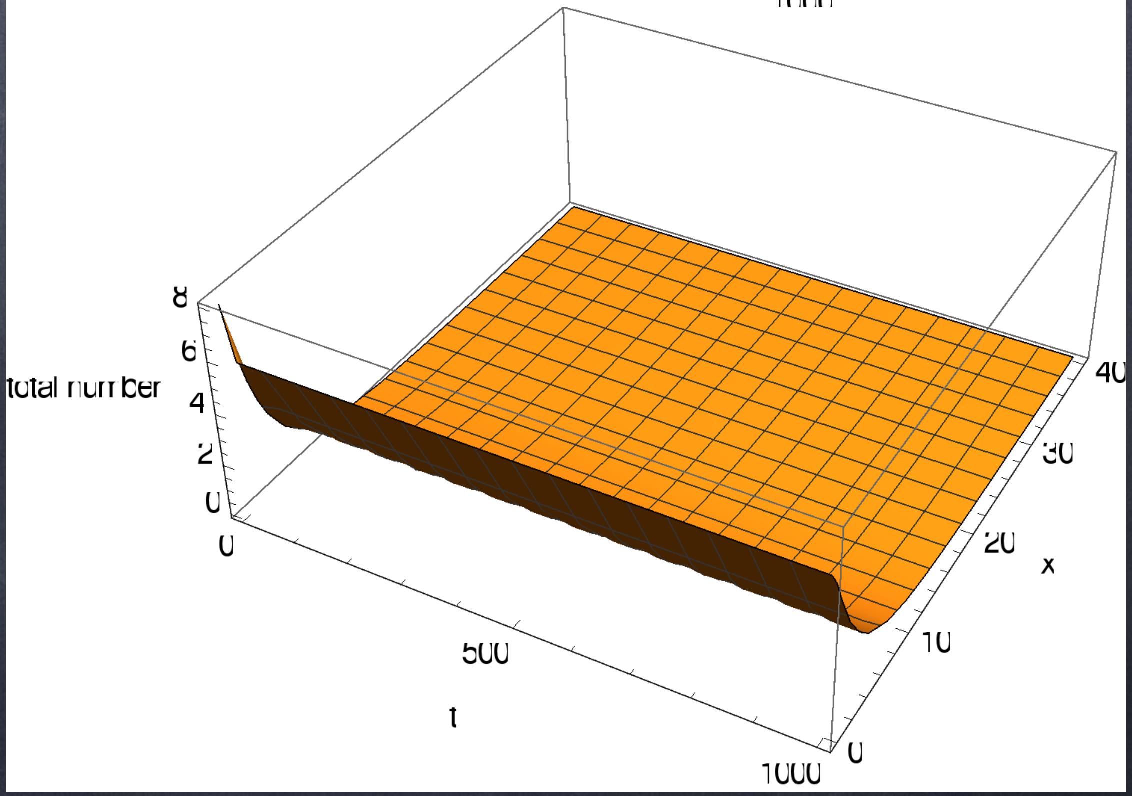
Model 9

Same as Model 5, but a matter potential of the form $\lambda(x)=50/(x/10+1)^4$ was added. A strong matter potential thus suppresses the transition.









Some general Tendencies

A large $\lambda(x)$ seems to strongly suppress transitions for a steep x-profile, but not for a flat/constant profile.

For large θ_0 and steep (exponential) $\mu(x)$ and $\lambda(x)$ profiles, at large x one seems to get essentially vacuum oscillations initially after which the distribution tends to relax to flavor equilibration.

A angular flavor crossing is not necessary/does not make a big difference for flavour conversions for the profiles we considered (slow conversions)

Conclusions

1.) Toy models can be used to understand the interplay between self-interaction, matter, vacuum, source terms and boundary conditions, although in general they are prohibitive for realistic conditions

2.) Consistent initial and boundary conditions are important and sometimes not completely straightforward

3.) Matter terms may not be trivially "rotated away" for profiles with significant slopes; probably depends on relation between profile scale height and oscillation lengths

4.) Flavor crossing may have limited influence