Algebraic approach to collective neutrino oscillations

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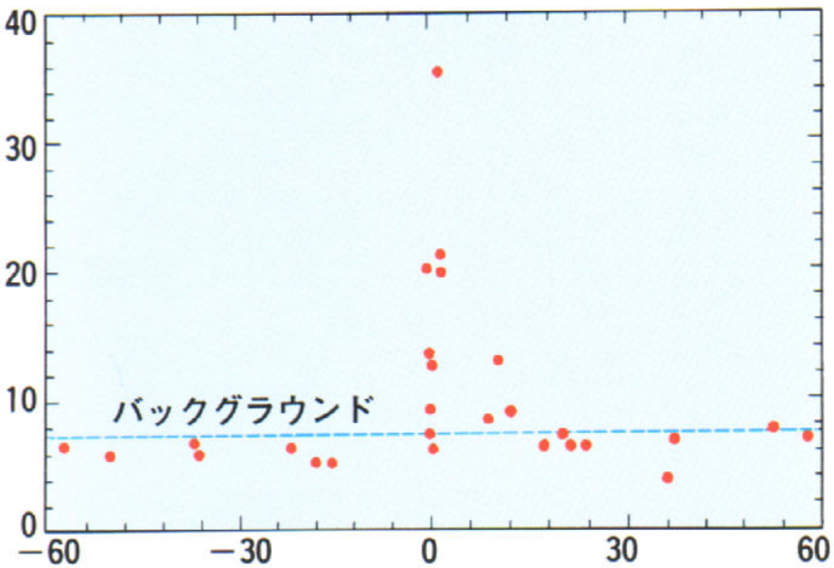
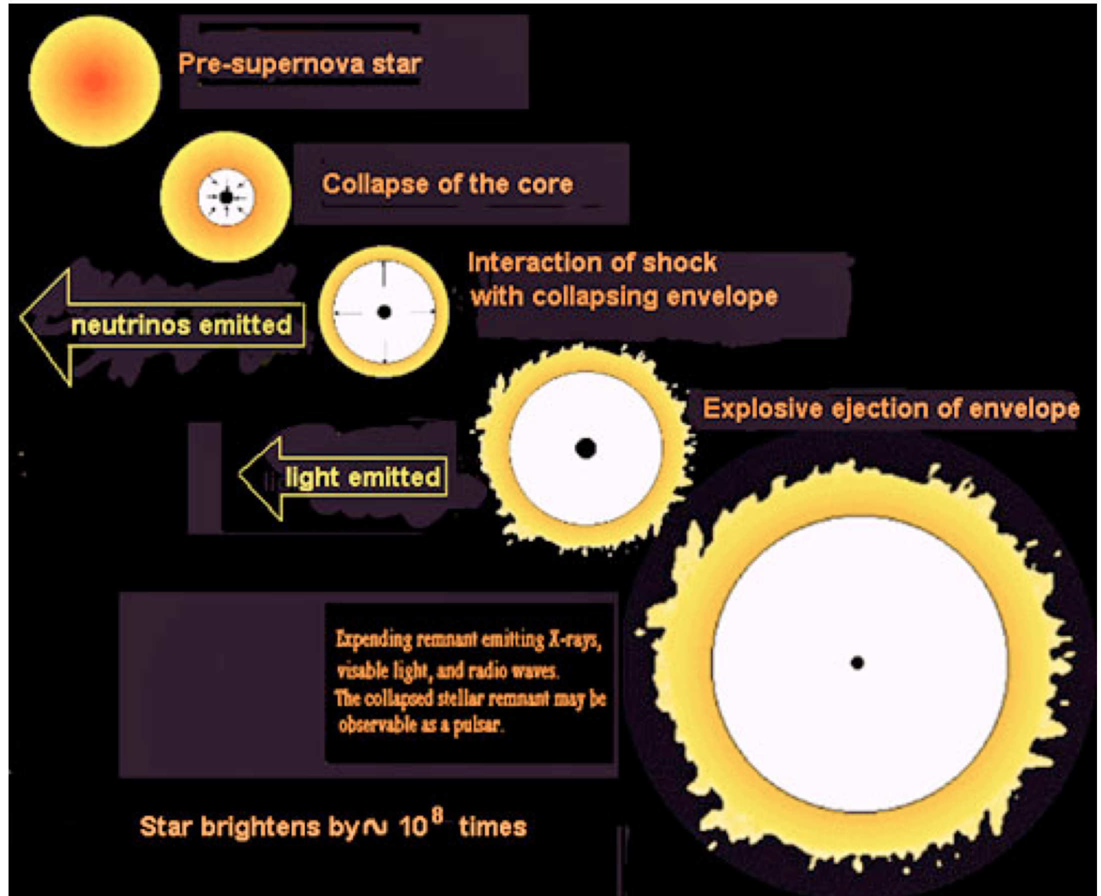
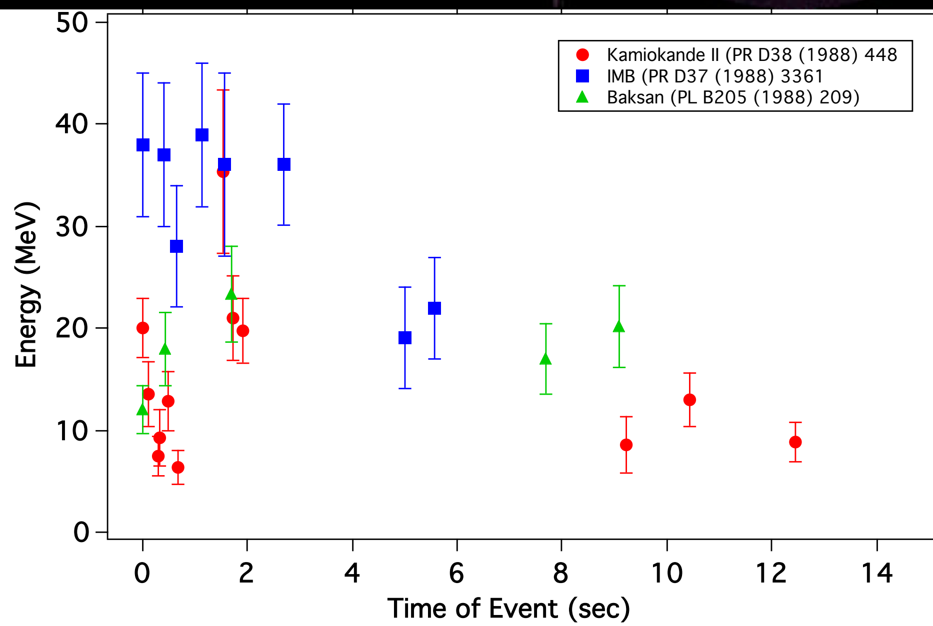
ECT* Workshop on SN neutrinos at the crossroads: astrophysics, oscillations, and detection

May 2019

Neutrinos from core-collapse supernovae

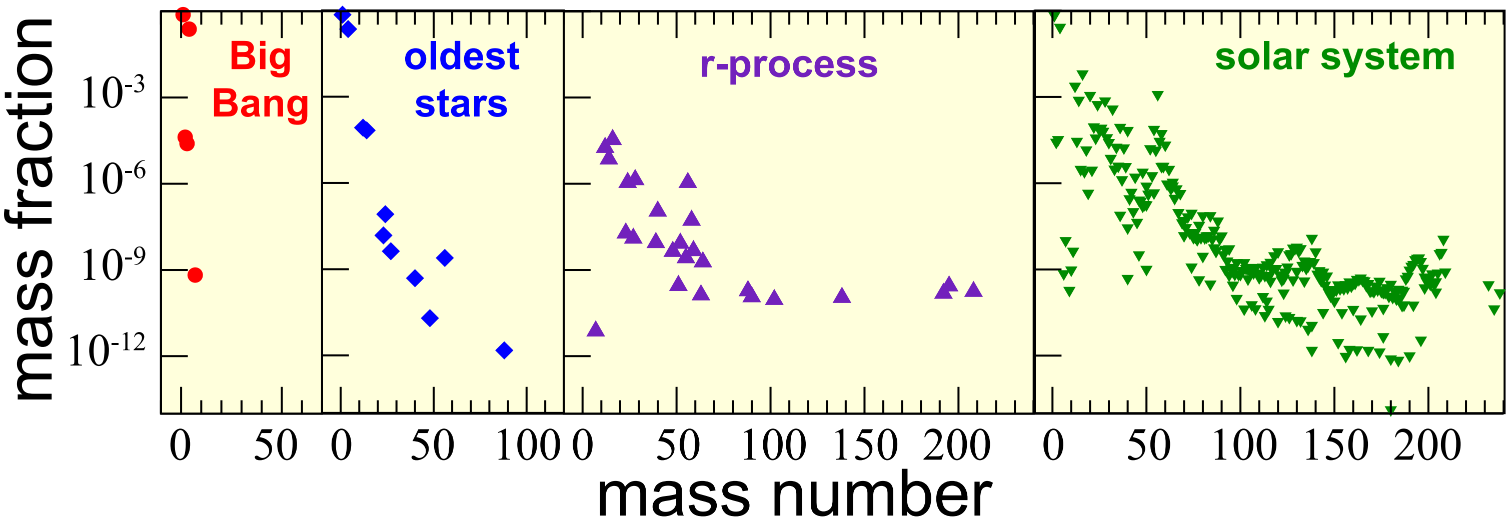


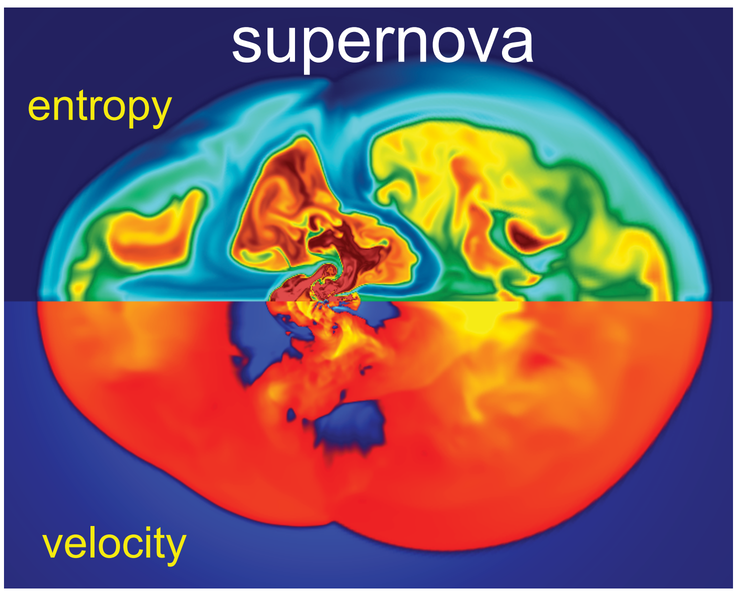


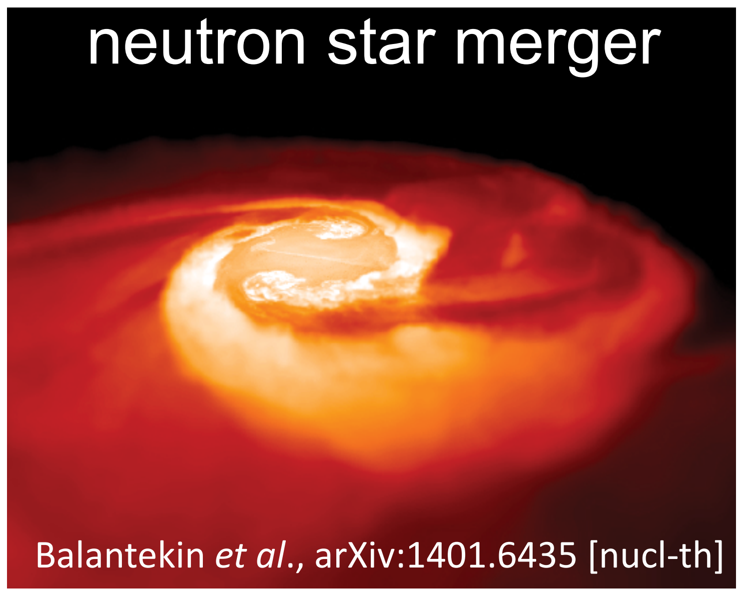


- $M_{prog} \geq$ 8 $M_{sun} \Rightarrow \Delta E \approx 10^{53}$ ergs $\approx 10^{59}$ MeV
- 99% of the energy is carried away by neutrinos and antineutrinos with $10 \leq E_\nu \leq 30$ MeV $\Rightarrow 10^{58}$ neutrinos

The origin of elements





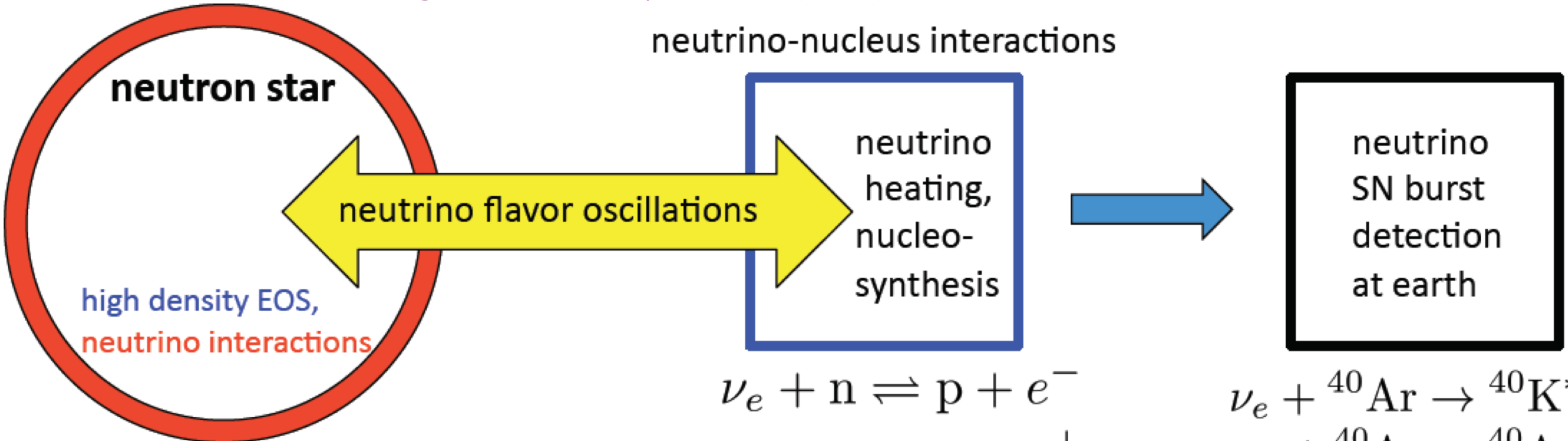


Possible sites for the r-process

Neutrinos not only play a crucial role in the dynamics of these sites, but they also control the value of the electron fraction, the parameter determining the yields of the r-process.

Understanding a core-collapse supernova requires answers to a variety of questions some of which need to be answered, both theoretically and experimentally.

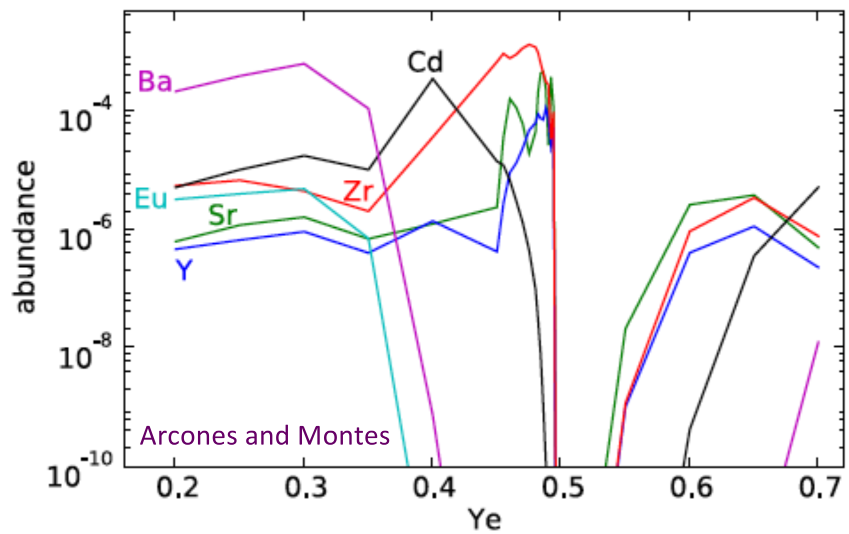
Balantekin and Fuller, Prog. Part. Nucl. Phys. **71** 162 (2013)



$$\nu_e + \mathrm{n} \rightleftharpoons \mathrm{p} + e^-$$
$$\bar{\nu}_e + \mathrm{p} \rightleftharpoons \mathrm{n} + e^+$$

$$\nu_e + {}^{40}\mathrm{Ar} \to {}^{40}\mathrm{K}^* + e^-$$
$$\nu_x + {}^{40}\mathrm{Ar} \to {}^{40}\mathrm{Ar}^* + \nu_x$$

$$Y_e = \frac{N_p}{N_p + N_n} = \frac{1}{1 + \lambda_p / \lambda_n}$$



Collective Neutrino Oscillations

$$H = \sum_{\mathbf{p}} \omega_{\mathbf{p}} \vec{B} \cdot \vec{J}_{\mathbf{p}} + \frac{\sqrt{2} G_F}{V} \sum_{\mathbf{p},\mathbf{q}} (1 - \cos\vartheta_{\mathbf{pq}})\, \vec{J}_{\mathbf{p}} \cdot \vec{J}_{\mathbf{q}}$$

$$\vec{B} = (0, 0, -1)_{\text{mass}} = (\sin 2\theta, 0, -\cos 2\theta)_{\text{flavor}}, \quad \omega_p = \frac{\delta m^2}{2p}$$

$$J_{\mathbf{p}}^{+} = a_1^{\dagger}(\mathbf{p}) a_2(\mathbf{p})$$
$$J_{\mathbf{p}}^{-} = a_2^{\dagger}(\mathbf{p}) a_1(\mathbf{p})$$
$$J_{\mathbf{p}}^{z} = \frac{1}{2} \left(a_1^{\dagger}(\mathbf{p}) a_1(\mathbf{p}) - a_2^{\dagger}(\mathbf{p}) a_2(\mathbf{p}) \right)$$

$$a_e(\mathbf{p}) = \cos\theta\, a_1(\mathbf{p}) + \sin\theta\, a_2(\mathbf{p})$$
$$a_x(\mathbf{p}) = -\sin\theta\, a_1(\mathbf{p}) + \cos\theta\, a_2(\mathbf{p})$$

Two of the adiabatic eigenstates of this equation are easy to find in the single-angle approximation:

$$H = \sum_p \omega_p \vec{B} \cdot \vec{J}_p + \mu(r) \vec{J} \cdot \vec{J}$$

$$|j, +j\rangle = |N/2, N/2\rangle = |\nu_1, \ldots, \nu_1\rangle$$

$$|j, -j\rangle = |N/2, -N/2\rangle = |\nu_2, \ldots, \nu_2\rangle$$

$$E_{\pm N/2} = \mp \sum_p \omega_p \frac{N_p}{2} + \mu \frac{N}{2} \left(\frac{N}{2} + 1 \right)$$

To find the others will take a lot more work

What is the mean-field approximation?

$$\left[\hat{O}_1, \hat{O}_2\right] \cong 0$$

$$\hat{O}_1\hat{O}_2 \approx \hat{O}_1\left\langle\hat{O}_2\right\rangle + \left\langle\hat{O}_1\right\rangle\hat{O}_2 - \left\langle\hat{O}_1\hat{O}_2\right\rangle$$

Expectation values should be calculated with a state $\left|\Psi\right\rangle$ chosen to satisfy:

$$\left\langle\hat{O}_1\hat{O}_2\right\rangle = \left\langle\hat{O}_1\right\rangle\left\langle\hat{O}_2\right\rangle$$

This reduces the two-body problem to a one-body problem:

$$a^\dagger a^\dagger a a \Rightarrow \left\langle a^\dagger a\right\rangle a^\dagger a + \left\langle a^\dagger a^\dagger\right\rangle a a + \text{h.c.}$$

$$\hat{H}_{\nu\nu} = \frac{\sqrt{2}G_F}{V}\int dp\,dq\left(1-\cos\theta_{pq}\right)\vec{\mathbf{J}}_p\cdot\vec{\mathbf{J}}_q \cong \frac{\sqrt{2}G_F}{V}\int dp\,dq\left(1-\cos\theta_{pq}\right)\left\langle\vec{\mathbf{J}}_p\right\rangle\cdot\vec{\mathbf{J}}_q$$

Mean field

Neutrino-neutrino interaction

$$\bar{\psi}_{\nu L}\gamma^{\mu}\psi_{\nu L}\bar{\psi}_{\nu L}\gamma_{\mu}\psi_{\nu L} \Rightarrow \bar{\psi}_{\nu L}\gamma^{\mu}\psi_{\nu L}\left\langle\bar{\psi}_{\nu L}\gamma_{\mu}\psi_{\nu L}\right\rangle + \cdots$$

Antineutrino-antineutrino interaction

$$\bar{\psi}_{\bar{\nu} R}\gamma^{\mu}\psi_{\bar{\nu} R}\bar{\psi}_{\bar{\nu} R}\gamma_{\mu}\psi_{\bar{\nu} R} \Rightarrow \bar{\psi}_{\bar{\nu} R}\gamma^{\mu}\psi_{\bar{\nu} R}\left\langle\bar{\psi}_{\bar{\nu} R}\gamma_{\mu}\psi_{\bar{\nu} R}\right\rangle + \cdots$$

Neutrino-antineutrino interaction

$$\bar{\psi}_{\nu L}\gamma^{\mu}\psi_{\nu L}\bar{\psi}_{\bar{\nu} R}\gamma_{\mu}\psi_{\bar{\nu} R} \Rightarrow \bar{\psi}_{\nu L}\gamma^{\mu}\psi_{\nu L}\left\langle\bar{\psi}_{\bar{\nu} R}\gamma_{\mu}\psi_{\bar{\nu} R}\right\rangle + \cdots$$

Balantekin and Pehlivan, JPG **34**,1783 (2007)

Neutrino-antineutrino can also have an additional mean field

$$\bar{\psi}_{\nu L}\gamma^{\mu}\psi_{\nu L}\bar{\psi}_{\bar{\nu}R}\gamma_{\mu}\psi_{\bar{\nu}R} \Rightarrow \bar{\psi}_{\nu L}\gamma^{\mu}\left\langle \psi_{\nu L}\bar{\psi}_{\bar{\nu}R}\gamma_{\mu}\right\rangle \psi_{\bar{\nu}R} + \cdots$$

However note that

$$\left\langle \psi_{\nu L}\bar{\psi}_{\bar{\nu}R}\gamma_{\mu}\right\rangle \propto m_{\nu} \quad \text{(negligible if the medium is isotropic)}$$

Fuller *et al.*
Volpe

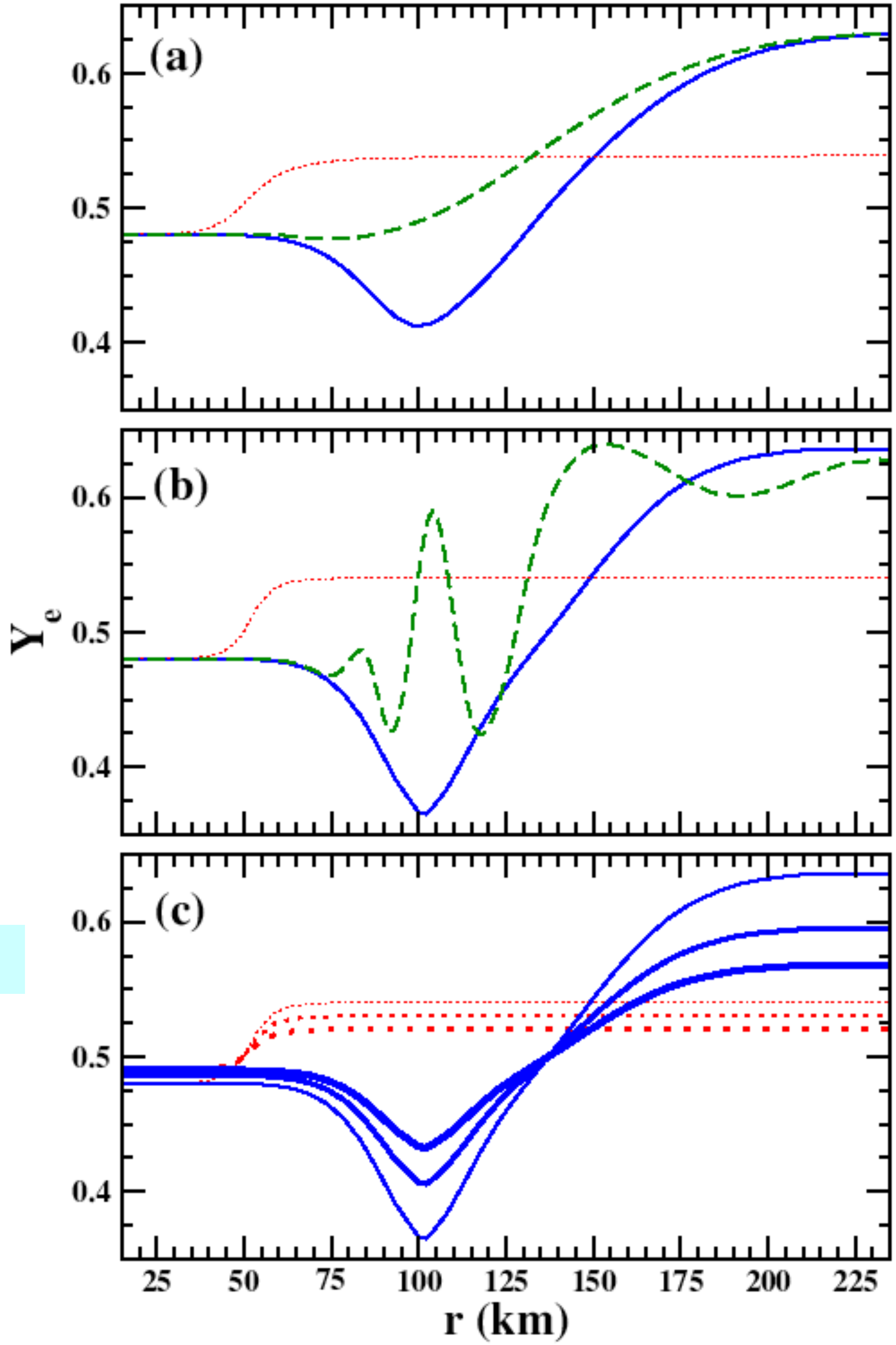
An example of an early mean field calculation

Equilibrium electron fraction with the inclusion of $\nu\nu$ interactions

L^{51} = 0.001, 0.1, 50

Balantekin and Yuksel, 2005

X_α= 0, 0.3, 0.5 (thin, medium, thick lines)



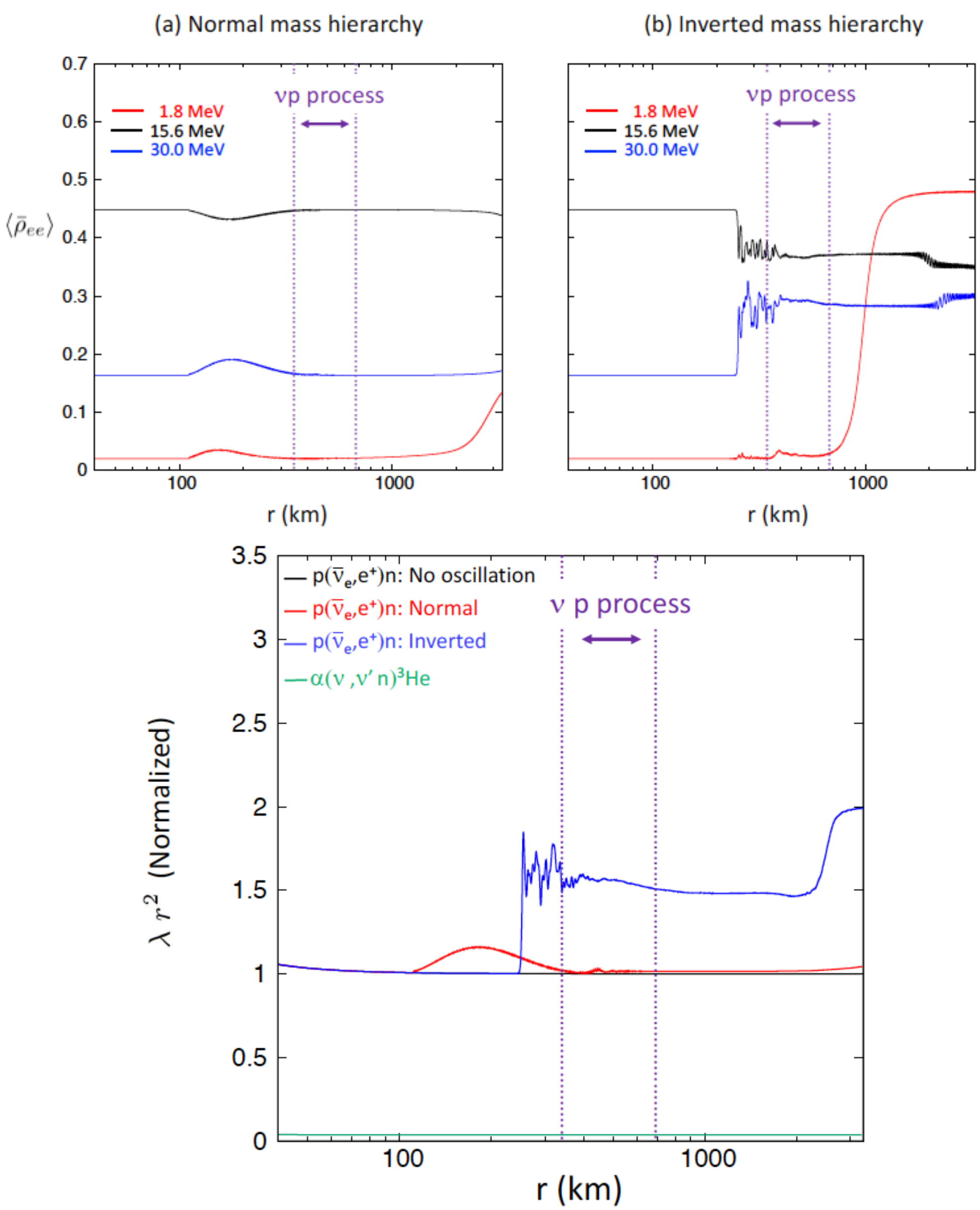
θ_{13}~π/10

θ_{13}~π/20

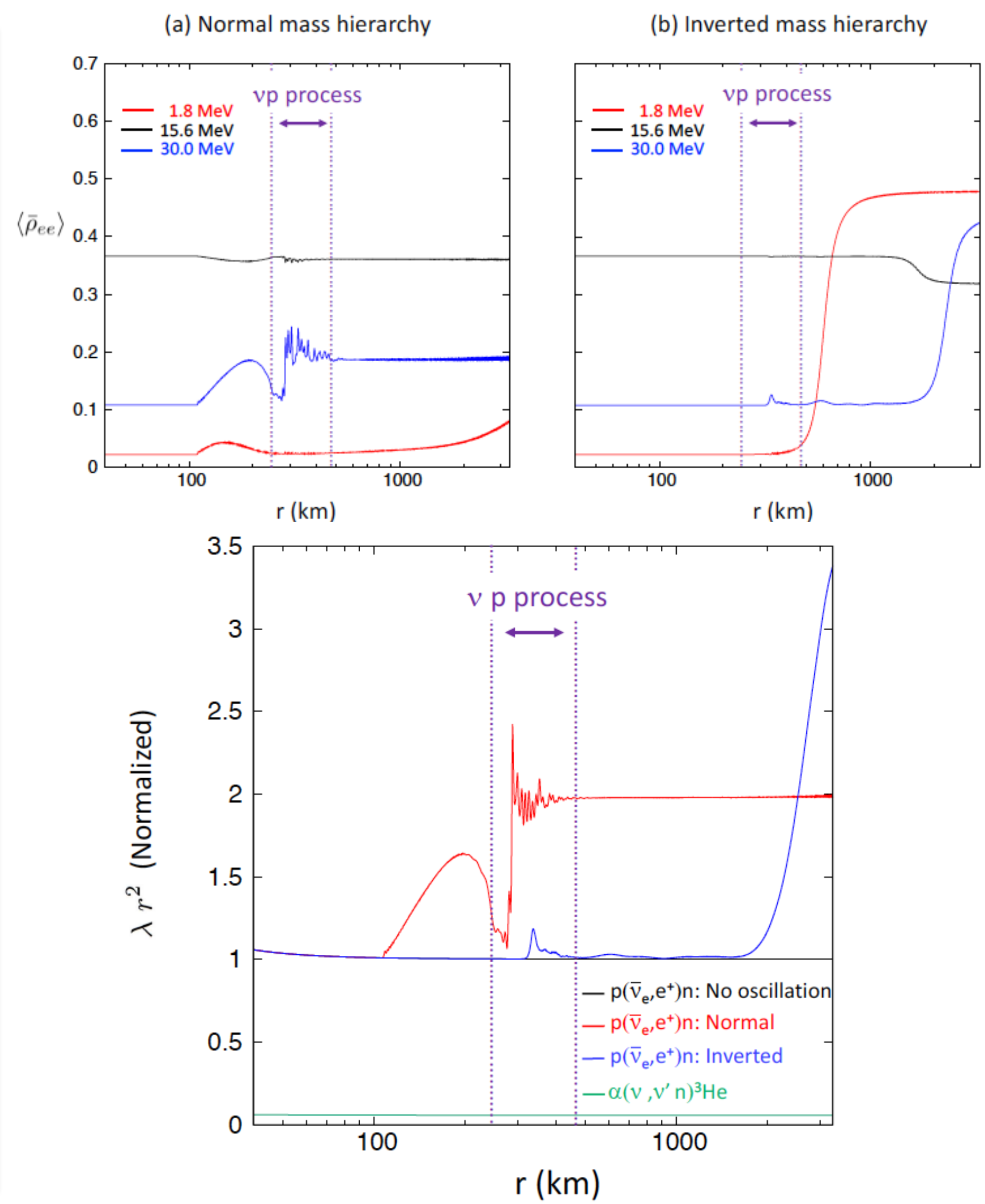
θ_{13}~π/20 with α effect

Collective Oscillations within mean field for the νp process

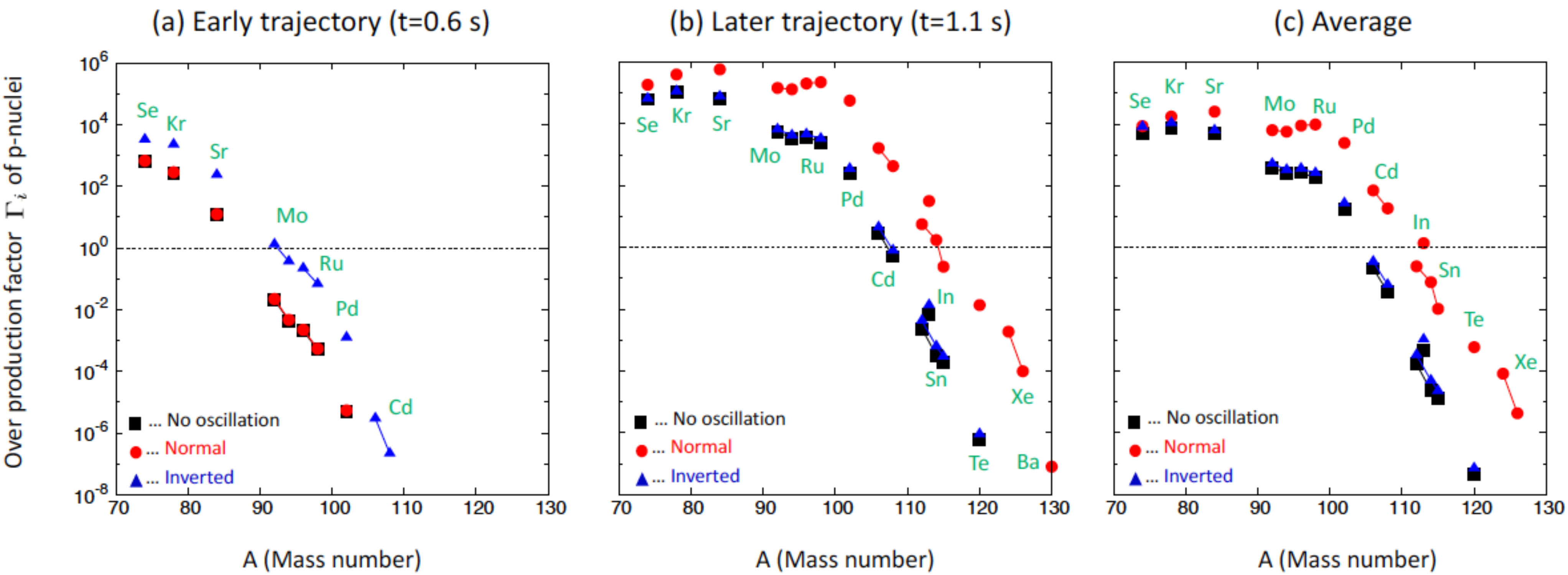
Sasaki *et al.*, Phys.Rev. D96 (2017) 043013



Early outflow (t=0.6 s.)



Later outflow (t=1.1 s.)



Impact of the production of p-nuclei

Sasaki *et al.,* Phys.Rev. D96 (2017) 043013

A system of N particles each of which can occupy k states

Exact Solution → **Mean-field approximation**

Entangled and unentangled states → Only unentangled states

Dimension of Hilbert Space: k^N

Dimension of Hilbert Space: kN

This problem is “exactly solvable” in the single-angle approximation

$$H = \sum_{\mathbf{p}} \omega_{\mathbf{p}} \vec{B} \cdot \vec{J}_{\mathbf{p}} + \frac{\sqrt{2} G_F}{V} \sum_{\mathbf{p},\mathbf{q}} (1 - \cos \vartheta_{\mathbf{pq}}) \, \vec{J}_{\mathbf{p}} \cdot \vec{J}_{\mathbf{q}}$$

$$H = \sum_{p} \omega_p \vec{B} \cdot \vec{J}_p + \mu(r) \vec{J} \cdot \vec{J}$$

The duality between $H_{\nu\nu}$ and BCS Hamiltonians

The ν-ν Hamiltonian

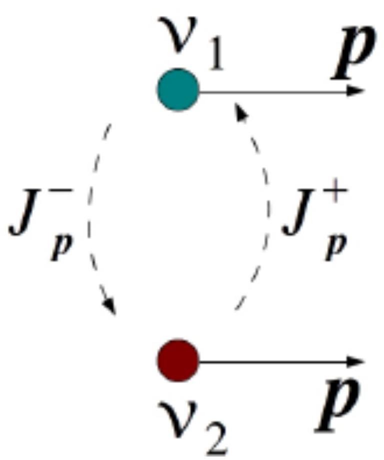
$$\hat{H} = \sum_p \frac{\delta m^2}{2p} \hat{B} \cdot \vec{J}_p + \frac{\sqrt{2} G_F}{V} \vec{J} \cdot \vec{J}$$

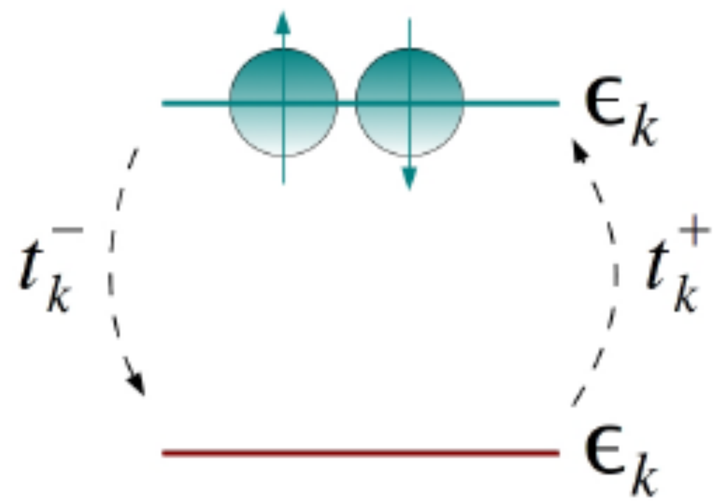
$\Longleftrightarrow$

The BCS Hamiltonian

$$\hat{H}_{\mathrm{BCS}} = \sum_k 2\epsilon_k \hat{t}_k^0 - |G| \hat{T}^+ \hat{T}$$

Same symmetries leading to Analogous (dual) dynamics!
Pehlivan, Balantekin, Kajino, and Yoshida, Phys.Rev. D **84**, 065008 (2011)





This symmetry naturally leads to splits in the neutrino energy spectra and was used to find conserved quantities in the single-angle case.

This problem is "exactly solvable" in the single-angle approximation

Pehlivan, Balantekin, Kajino, Yoshida, PRD**84** (2011) 065008

$$H = \sum_p \omega_p \vec{B} \cdot \vec{J}_p + \mu(r) \vec{J} \cdot \vec{J}$$

$$-\frac{1}{2\mu} - \sum_{p=1}^{M} \frac{j_p}{\omega_p - \zeta_\alpha} = \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^{\kappa} \frac{1}{\zeta_\alpha - \zeta_\beta}$$

$$|\zeta_1, \ldots, \zeta_\kappa\rangle = \mathcal{N}(\zeta_1, \ldots, \zeta_\kappa) \left(\prod_{\alpha=1}^{\kappa} S_\alpha^- \right) |j, +j\rangle$$

$$\vec{S}(\zeta_\alpha) \equiv \sum_p \frac{\vec{J}_p}{\omega_p - \zeta_\alpha}$$

$$E = E_{+N/2} + \sum_{\alpha=1}^{\kappa} \zeta_\alpha - \mu\kappa(N - \kappa + 1)$$

Recall that two of the adiabatic eigenstates of this equation are easy to find:

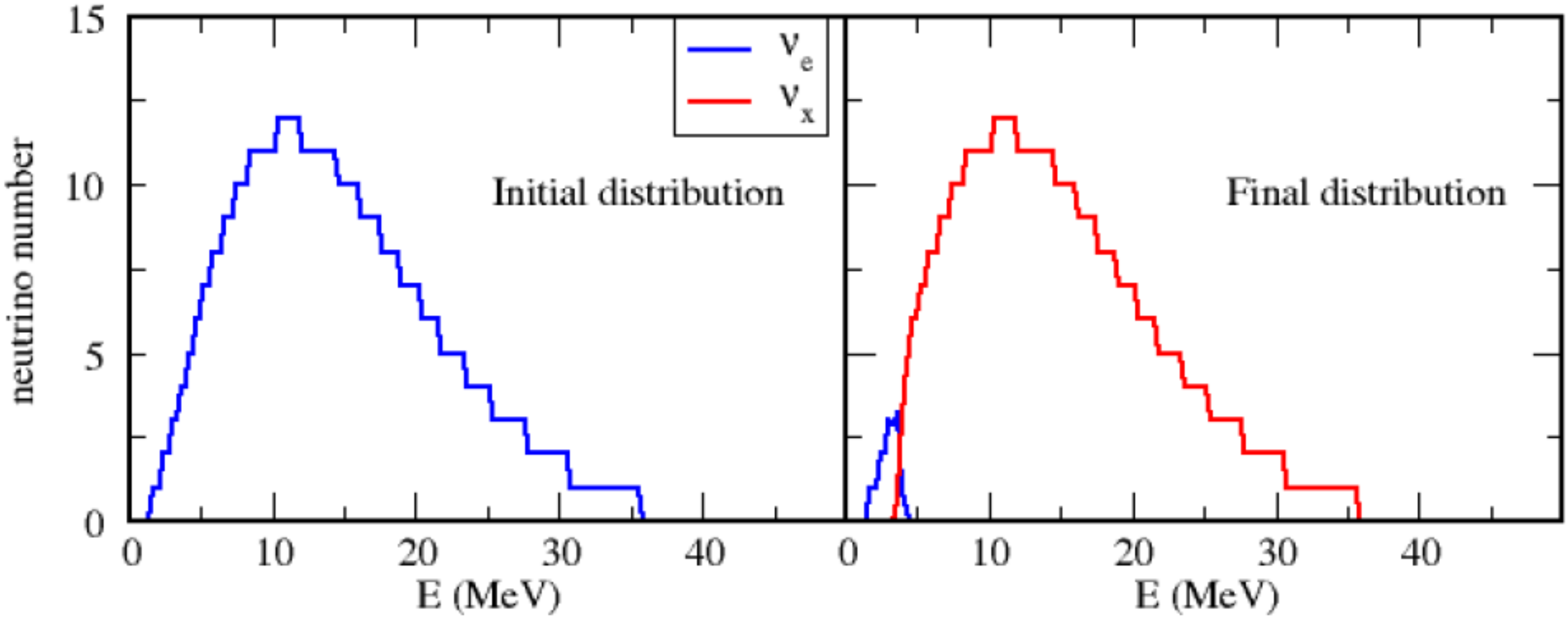
$$H = \sum_p \omega_p \vec{B} \cdot \vec{J}_p + \mu(r) \vec{J} \cdot \vec{J}$$

$$|j, +j\rangle = |N/2, N/2\rangle = |\nu_1, \ldots, \nu_1\rangle$$

$$|j, -j\rangle = |N/2, -N/2\rangle = |\nu_2, \ldots, \nu_2\rangle$$

$$E_{\pm N/2} = \mp \sum_p \omega_p \frac{N_p}{2} + \mu \frac{N}{2} \left(\frac{N}{2} + 1 \right)$$

Away from the mean-field: First adiabatic solution of the *exact* many-body Hamiltonian

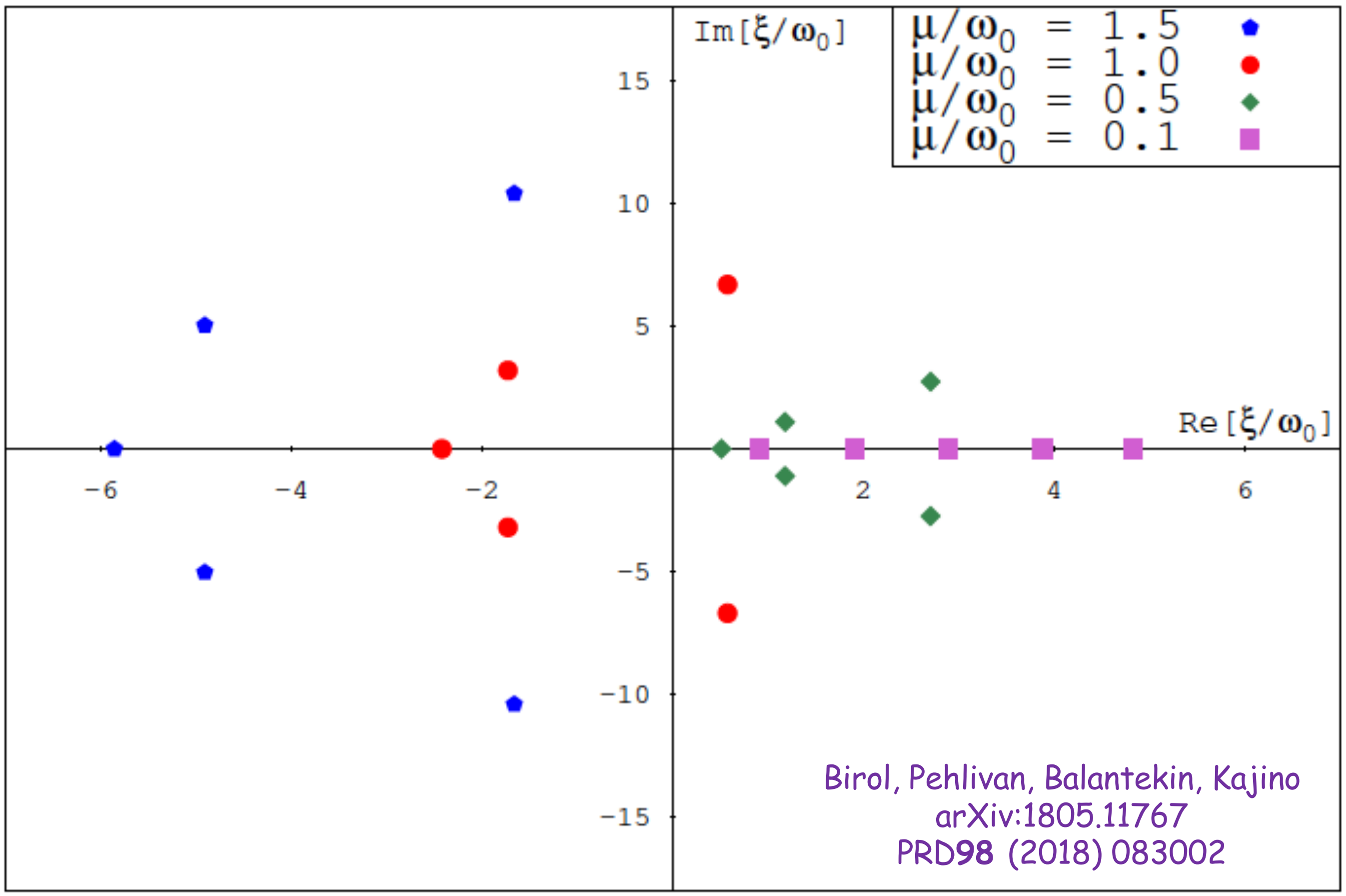


- Solutions of the Bethe ansatz equations for 250 neutrinos. Same behavior as the mean-field.
- Two flavors only
- Inverted hierarchy, no matter effect

2015

$$\mathbf{H} = \sum_{\omega} \omega \mathbf{B} \cdot \mathbf{J}_\omega + \mu(r) \mathbf{J} \cdot \mathbf{J}$$

$$|\psi\rangle_{\mu\to\infty} = \sum_{m=-n/2}^{m=+n/2} c_m \left|\frac{n}{2}, m\right\rangle \longrightarrow |\psi\rangle_{\mu\to 0} = \sum_m c_m \phi_m \left| \underbrace{\nu_1, \nu_1, \cdots, \nu_1}_{n/2+m}; \underbrace{\nu_2, \nu_2, \cdots, \nu_2}_{n/2-m} \right\rangle$$



Birol, Pehlivan, Balantekin, Kajino
arXiv:1805.11767
PRD**98** (2018) 083002

Adiabatic evolution of an initial thermal distribution (T = 10 MeV) of electron neutrinos. 10^8 neutrinos distributed over 1200 energy bins with solar neutrino parameters and normal hierarchy.

Birol, Pehlivan, Balantekin, Kajino
arXiv:1805.11767
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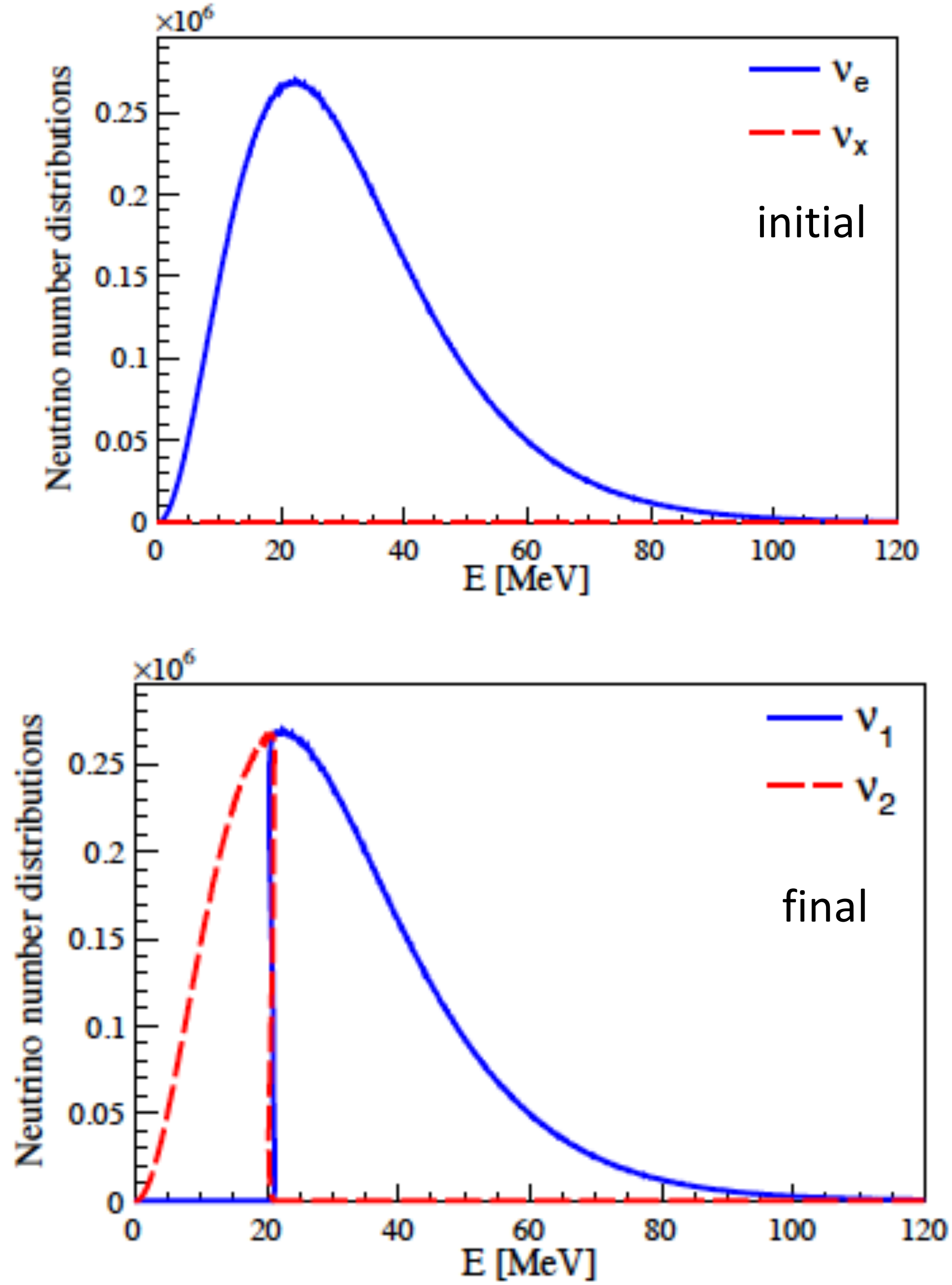
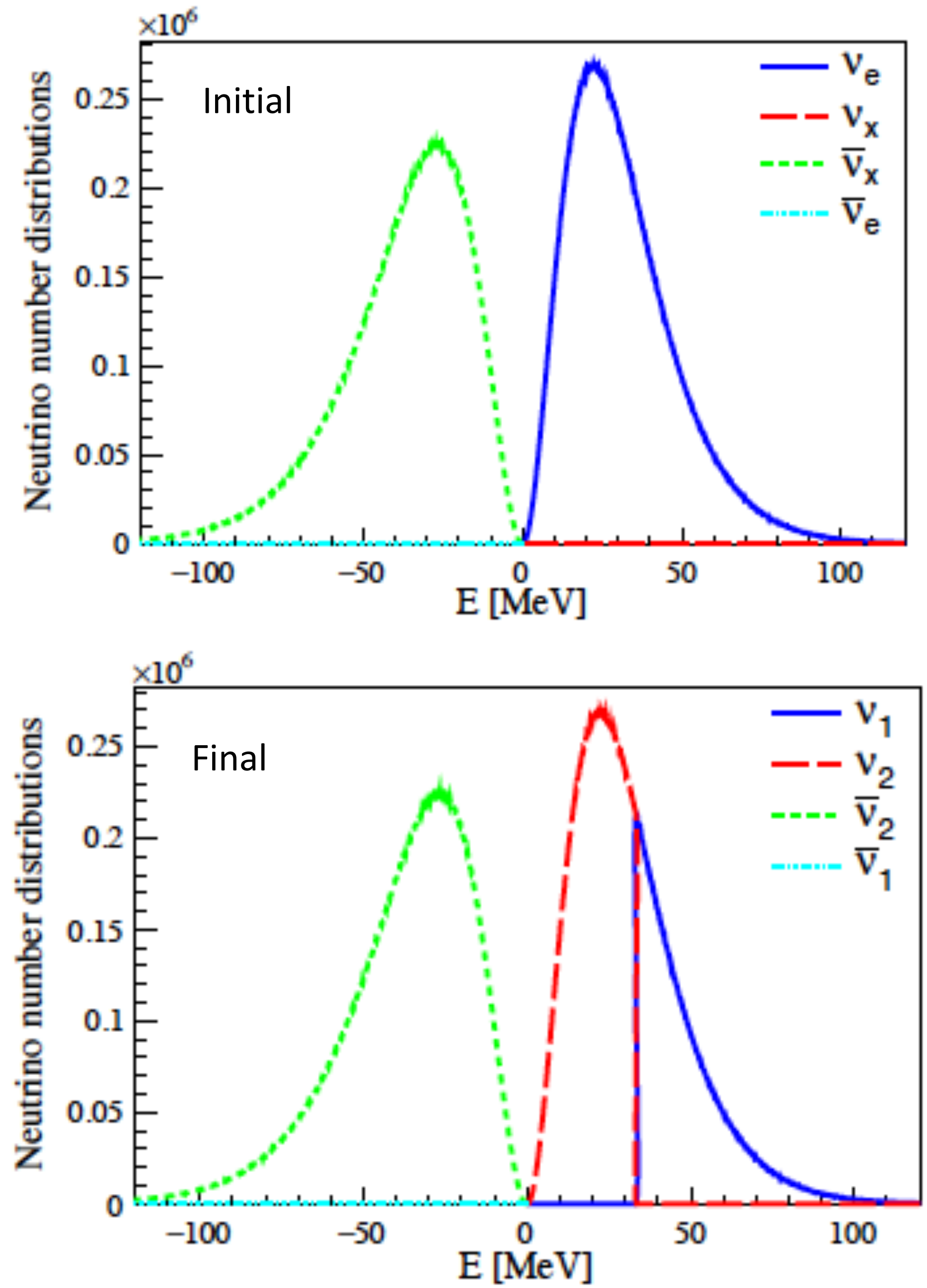


Adiabatic evolution of an initial thermal distribution of electron neutrinos (T=10 MeV) and antineutrinos of another flavor (T=12MeV). 10^8 neutrinos distributed over 1200 energy bins both for neutrinos and antineutrinos with solar neutrino parameters and normal hierarchy.



Birol, Pehlivan, Balantekin, Kajino
arXiv:1805.11767

A more practical approach

$$\Lambda(\lambda) = \sum_{\alpha=1}^{\kappa} \frac{1}{\lambda - \zeta_\alpha}$$

$$\Lambda(\lambda)^2 + \Lambda'(\lambda) + \frac{1}{\mu}\Lambda(\lambda) = \sum_{q=1}^{M} 2j_q \frac{\Lambda(\lambda) - \Lambda(\omega_q)}{\lambda - \omega_q}$$

$$\Lambda_p^2 + (1 - 2j_p)\Lambda'_p + \frac{1}{\mu}\Lambda_p = \sum_{\substack{q=1 \\ q\neq p}}^{M} 2j_q \frac{\Lambda_p - \Lambda_q}{\omega_p - \omega_q}, \quad \Lambda_k = \Lambda(\omega_k)$$

$$E = E_{N/2} - 2\mu \sum_{p=1}^{M} j_p \omega_p \Lambda_p .$$

Patwardhan, Cervia, Balantekin, arXiv:1905.04386

These $\Lambda(\omega_p) = \Lambda_p$ are eigenvalues of the invariants, h_p:

$$h_p = -J_p^z + 2\mu \sum_{\substack{q=1 \\ q \neq p}}^{N} \frac{\mathbf{J}_p \cdot \mathbf{J}_q}{\omega_p - \omega_q}, \qquad [h_p, h_q] = 0, p \neq q$$

$$\sum_p \omega_p h_p = -\sum_{p=1}^{N} \omega_p J_p^z + \mu \sum_{\substack{p,q=1 \\ p \neq q}}^{N} \mathbf{J}_p \cdot \mathbf{J}_q$$

$$h_p|\xi_1, \ldots, \xi_\kappa\rangle = \left(2\mu \sum_{\substack{q=1 \\ q \neq p}}^{N} \frac{j_p j_q}{\omega_p - \omega_q} + j_p - 2\mu j_p \Lambda(\omega_p) \right) |\xi_1, \ldots, \xi_\kappa\rangle$$

Cervia, Patwardhan, BalantekinarXiv:1905.00082

Adiabatic Eigenstates

$$\Lambda(\lambda)^2 + \frac{1}{\mu}\Lambda(\lambda) = \sum_{q=1}^{M} 2j_q \frac{\Lambda(\lambda) - \Lambda(\omega_q)}{\lambda - \omega_q}$$

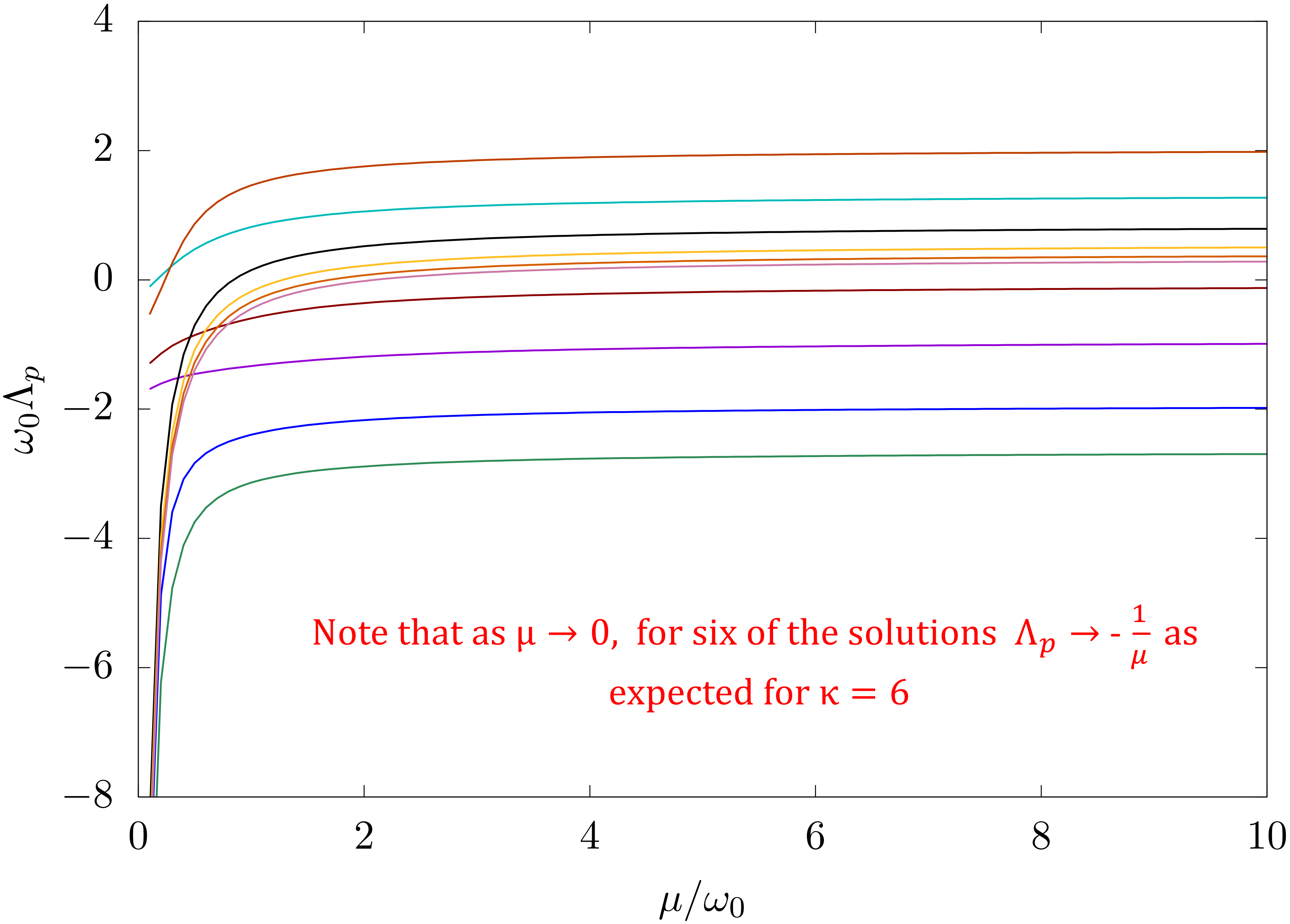
$$S_1^- S_2^- |j, +j\rangle = \frac{1}{2} \sum_{\substack{p,q=1 \\ p \neq q}}^{M} \left(\Lambda_p \Lambda_q + \frac{\Lambda_p - \Lambda_q}{\omega_p - \omega_q} \right) J_p^- J_q^- |j, +j\rangle$$

$$S_1^- S_2^- S_3^- |j, +j\rangle = \frac{1}{3!} \times$$

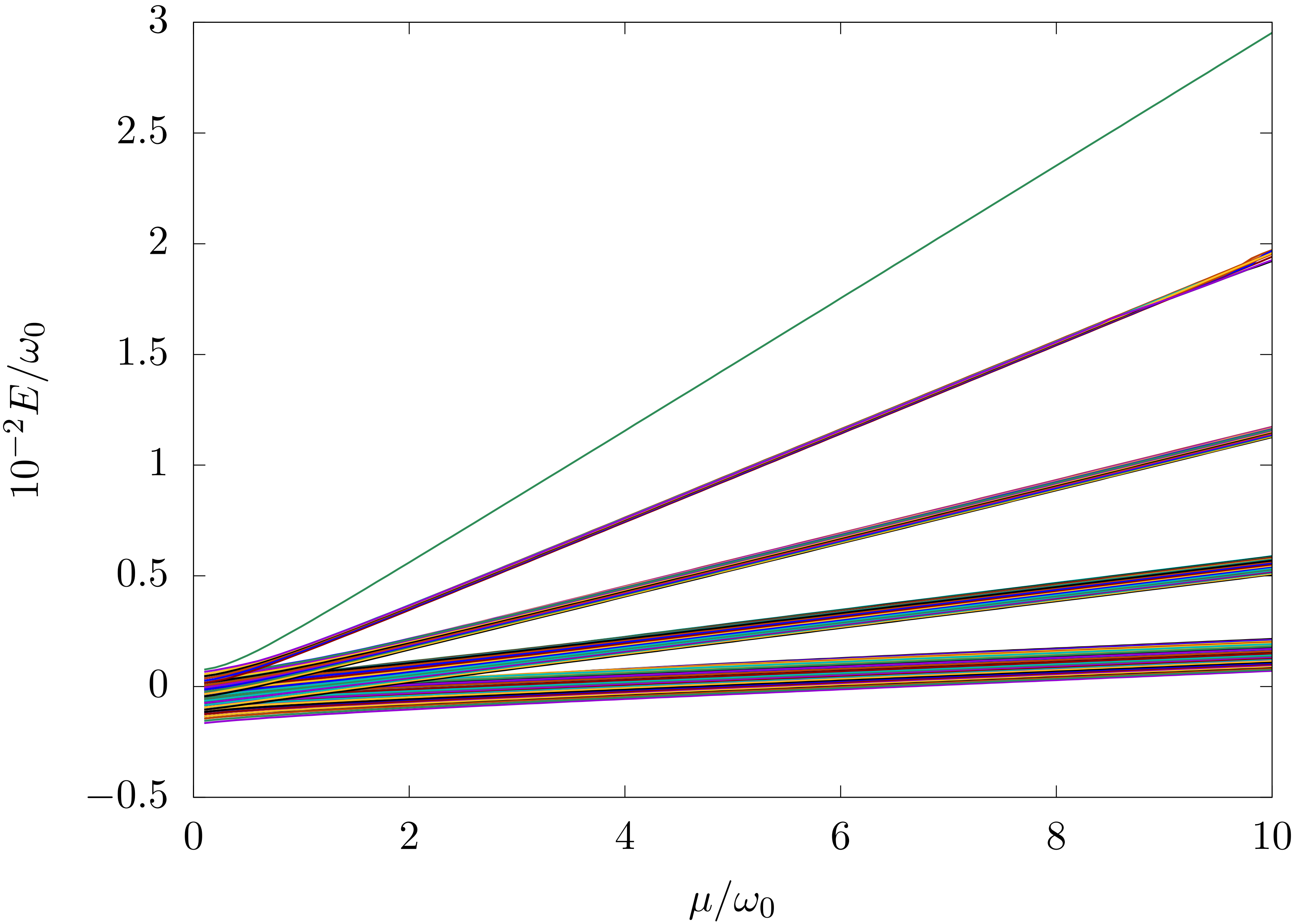
$$\sum_{\substack{p,q,r=1 \\ p,q,r \\ \text{distinct}}}^{M} \left[\Lambda_p \Lambda_q \Lambda_r + 3\Lambda_r \frac{\Lambda_p - \Lambda_q}{\omega_p - \omega_q} + \frac{2}{\omega_p - \omega_q} \left(\frac{\Lambda_p - \Lambda_r}{\omega_p - \omega_r} - \frac{\Lambda_q - \Lambda_r}{\omega_q - \omega_r} \right) \right] J_p^- J_q^- J_r^- |j, +j\rangle$$

Patwardhan, Cervia, Balantekin, arXiv:1905.04386

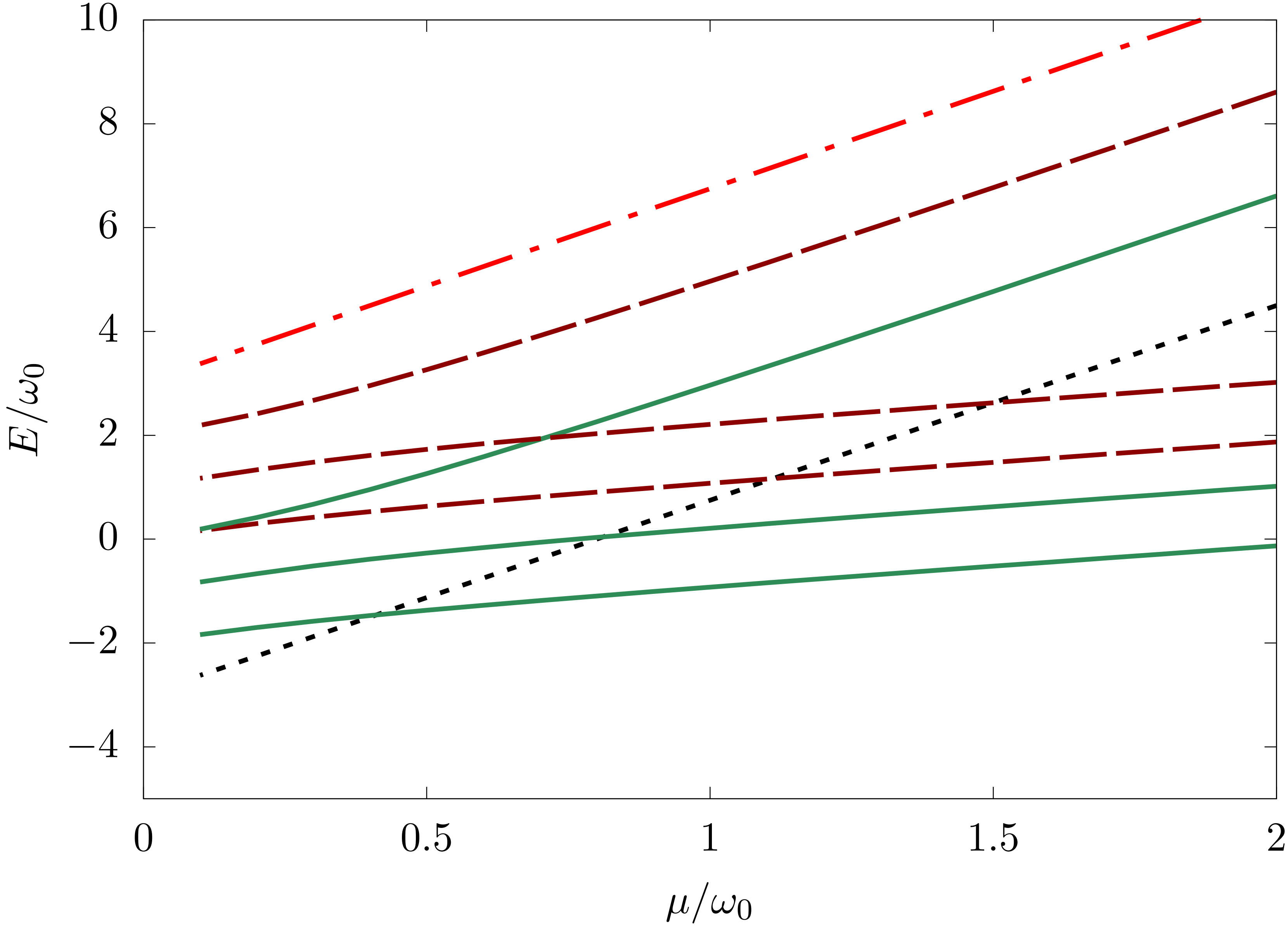
One of the 210 solutions for κ = 6 for 10 neutrinos in 10 bins with $\omega_p = p\omega_0$



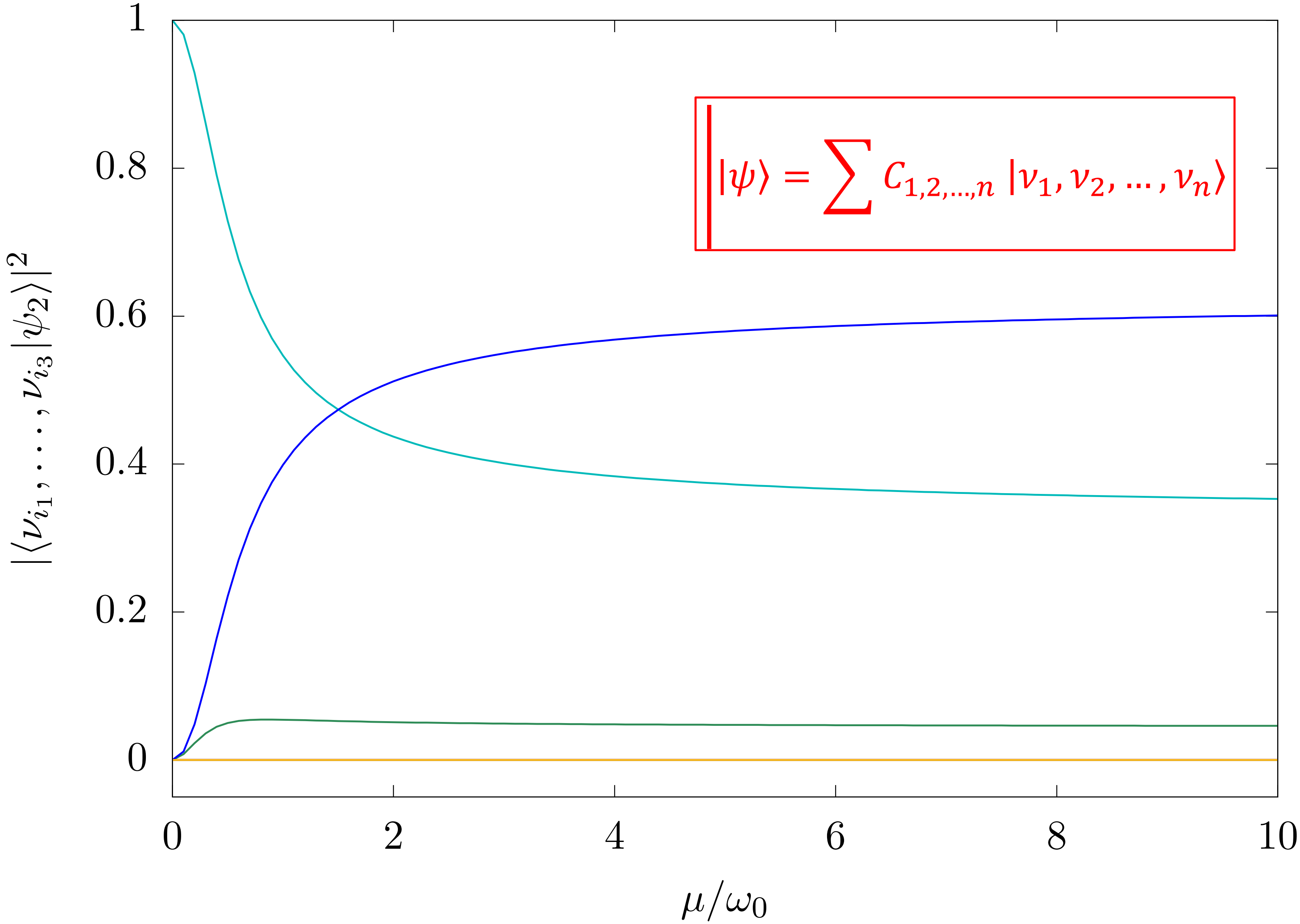
Energy eigenvalues for the same system for κ = 4



All the energy levels for 3 neutrinos in three bins with κ = 0 (black dotted line), κ = 1 (green solid lines), κ = 2 (brown dashed lines), and κ = 3 (red dot-dashed line)



Decomposition of the fourth excited κ = 1 state with 3 neutrinos in 3 bins



CONCLUSIONS

- Full many-body Hamiltonian describing collective neutrino oscillations is exactly solvable in the single-angle approximation in the sense that eigenvalues and eigenvectors can be expressed in terms of solutions of a Bethe ansatz equations.
- However, obtaining solutions of these Bethe ansatz equations is notoriously difficult for neutrinos in more than three energy bins.

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- Many-body literature contains several prescriptions with varying degrees of numerical difficulty to obtain eigenvalues.

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- However, obtaining solutions of these Bethe ansatz equations is notoriously difficult for neutrinos in more than three energy bins.
- Many-body literature contains several prescriptions with varying degrees of numerical difficulty to obtain eigenvalues.
- We provided an approach to calculate eigenstates (crucial for astrophysical calculations) and demonstrated its numerical feasibility. We are now ready to compare with mean-field calculations.
- Next step is to carry out three-flavor, neutrino-antineutrino calculations in realistic supernova scenarios.
- The last two steps will be carried out with the participation of three N3AS postdoctoral fellows: A. Patwardhan, E. Rrapaj, and M. Sen.


Thank you very much!