Neutron Stars & Gravitational Waves

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The Many Faces of Neutron Stars

- Pulsars
- X-ray binaries
- Magnetars
- Binary NS systems
- NS-NS mergers

Typical masses ~1.2-2 M☉
Typical Radius ~9-14 km
Neutron Stars: Mass vs Radius

Static Models

Rotating Models

supramassive

\( M \approx 1.2 M_{TOV} \)

Özel & Freire (2016)

Demorest et al. 2010
Antoniadis et al. 2013

Özel & Freire (2016)
Constraints on Neutron Star Radius

GW observations

Main methods in EM spectrum

- Thermonuclear X-ray bursts (photospheric radius expansion)
- Burst oscillations (rotationally modulated waveform)
- Fits of thermal spectra to cooling neutron stars
- khZ QPOs in accretion disks around neutron stars
- Pericenter precession in relativistic binaries (double pulsar J0737)

Main methods in GW spectrum

- Tidal effects on waveform during inspiral phase of NS-NS mergers
- Tidal disruption in BH-NS mergers
- Oscillations in (early & late) post-merger phase
- Oscillation in the post-collapse phase

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Neutron Stars & “universal relations”

Need for relations between the “observables” and the “fundamentals” of NS physics

<table>
<thead>
<tr>
<th>Observable</th>
<th>Relation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average Density</td>
<td>$\bar{\rho} \sim M / R^3$</td>
</tr>
<tr>
<td>Compactness</td>
<td>$z \sim M / R$  \quad \eta = \sqrt{M^3 / I}$</td>
</tr>
<tr>
<td>Moment of Inertia</td>
<td>$I \sim MR^2$  \quad I \sim J / \Omega$</td>
</tr>
<tr>
<td>Quadrupole Moment</td>
<td>$Q \sim R^5 \Omega^2$</td>
</tr>
<tr>
<td>Tidal Love Numbers</td>
<td>$\lambda \sim I^2 Q$</td>
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</tbody>
</table>
**I-Love-Q relations**

**EOS independent relations** were derived by Yagi & Yunes (2013) for non-magnetized stars in the slow-rotation and small tidal deformation approximations.

... the relations proved to be valid (with appropriate normalizations) even for fast rotating and magnetized stars

The most promising strategy for constraining the physics of neutron stars involves observing their “ringing” (oscillation modes)

- **f-mode**: scales with average density
- **p-modes**: probes the sound speed through out the star
- **g-modes**: sensitive to thermal/composition gradients
- **w-modes**: oscillations of spacetime itself.
- **s-modes**: Shear waves in the crust
- **Alfvèn modes**: due to magnetic field
- **i-modes**: inertial modes associated with rotation (r-mode)

Typically **SMALL AMPLITUDE** oscillations → weak emission of GWs

**UNLESS** they become **unstable due to rotation** (r-mode & f-mode)

\[ l = 2, m = 2 \]
\[ l = 3, m = 3 \]
\[ l = 4, m = 4 \]
p-modes: main restoring force is the pressure (f-mode) (>1.5 kHz)

Inertial modes: (r-modes) main restoring force is the Coriolis force

w-modes: pure space-time modes (only in GR) (>5kHz)

Torsional modes (t-modes) (>20 Hz) shear deformations. Restoring force, the weak Coulomb force of the crystal ions.

... and many more

shear, g-, Alfven, interface, ... modes
Gravitational Wave Asteroseismology: Andersson-Kokkotas 96+

![Graphs showing relationships between different variables related to Gravitational Wave Asteroseismology.](image-url)
The GW signal can be divided into **three distinct phases**

- **inspiral**
- **merger**
- **post-merger ringdown**

(@40Mpc)
I. Direct collapse to BH if 
   $M_{\text{TOT}} > M_{\text{max}}(\Omega)$

II. Formation of an “unstable” NS if 
   $M_{\text{max}}(\Omega) > M_{\text{TOT}} > M_{\text{max}}$

III. Formation of a “stable” NS if 
   $M_{\text{TOT}} < M_{\text{max}}(\Omega)$

- NS-NS mergers will produce: 
  - ~40% prompt BHS
  - ~30% supramassive NS -> BH
  - ~30% Stable NS

- Initial spin near breakup limit $\sim 1\text{ms}$

Differential rotation/turbulence $\rightarrow$ strongly twisted internal field $E_B \geq 10^{50}\text{erg}$


Kiuchi, Sekiguchi, Kyutoku, Shibata 2012
Tidal interactions affect the last part of the inspiral, modifying the orbital motion and the GW emission.

Kokkotas-Schaefer MNRAS 1995
Binary Neutron Star Mergers

Tidal Interaction

Tidal interactions affect the last part of the inspiral, modifying the orbital motion and the GW emission.
The last part of the inspiral signal carries the imprint of the quadrupole tidal deformability

\[ \lambda = -\frac{Q_{ij}}{E_{ij}} = \frac{2}{3} k_2 R^5 \]

Read et al. (2013), Hotozaka et al (2013)...

\[ k_2 : \text{tidal Love number} \]

The leading tidal contribution to the phase evolution is a combination of the two tidal parameters. It is of 5PN order

\[ \tilde{\Lambda} = \frac{16}{13} \left( \frac{m_1 + 12m_2}{m_1 + 2m_2} \right)^4 \Lambda_1 + \left( \frac{m_1 + 12m_2}{m_1 + 2m_2} \right)^4 \Lambda_2 \]

Measurements of \( M_{NS} \) and \( \Lambda \) would be helpful to constrain the NS EOS

With aLIGO

\[ \frac{\Delta R}{R} \sim 10\% \quad \text{at} \quad 100\text{Mpc} \]
Binary Neutron Star Mergers
Tidal Interaction

Probability density for the tidal deformability parameters of the high spin and low spin components inferred from the detected signals using the post-Newtonian model.
Equation of State:
Constraints from GW170817 (Bauswein et al.)

Figure 2. Mass-radius relations of different EoSs with very conservative (red area) and “realistic” (cyan area) constraints of this work for $R_{1.6}$ and $R_{\text{max}}$. Horizontal lines display the limit by Antoniadis & et al. (2013). The dashed line shows the causality limit.
Equation of State: Constraints from X-ray binaries / bursts

Figure 10

The astrophysically inferred (left) EoS and (right) mass-radius relation corresponding to the most likely triplets of pressures that agree with all of the neutron star radius and low energy nucleon-nucleon scattering data and allow for a $M > 1.97 \, M_\odot$ neutron star mass. The light blue bands show the range of pressures and the mass-radius relations that correspond to the region of the $(P_1, P_2, P_3)$ parameter space in which the likelihood is within $e^{-1}$ of its highest value. Around $1.5 \, M_\odot$, this inferred EoS predicts radii between $9.9 - 11.2 \, \text{km}$.

Özel & Freire (2016)
Chandra & Kip in the 60-70s

Together with
• Campollataro, A.
• Ipser, Jim
• Price, Richard
• Hartle, James B.

Study of nearly everything:
➢ Radial pulsations
➢ Non-radial Pulsations
➢ Emission of Gravitational Waves
➢ Slow-Rotation Approximation

Together with:
• Schutz, Bernard F.
• Detweiler S.
• Lindblom L.

Studies of Stability of Relativistic Stars/ellipsoids
➢ Post-Newtonian equilibria and Dynamics (up to 2.5 PN order)
➢ The first hints of the CFS secular instability

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The work till now -I

The “CFS” Instability: Chandrasekhar – Friedman – Schutz 70+
Ergoregion Instability: Friedman, Schutz, Comins 70+
Systematic studies of stability properties of rotating stars and of CFS instability Friedman – Ipser – Parker – Lindblom 80+

Spacetime (w-) modes: Kokkotas-Schutz 1986-90+
Gravitational Wave Asteroseismology: Andersson-KK 90+
R-mode instability: Andersson; Friedman – Morsink 90+

ALL THESE STUDIES WERE DONE FOR:
• Non-Rotating
• Slowly Rotating Stars

Equilibrium configurations of fast rotating GR stars
Hachisu-Komatsu-Eriguchi (1988) more advanced codes later (Cook, Shapiro, Stergioulas, Friedman) 90+

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The work till now -II

Gravitational Wave Asteroseismology: Andersson-KK 96+

Rotation “splits” the spectra

Andersson-KK 1996, -98, -01

Gaertig-KK 2008
Finding order in chaos

Supra-massive

Reliable codes for the background

Perturbation equations not manageable

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Empirical relation connecting the parameters of the rotating neutron stars to the observed frequencies.

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**Frequency**

Although beautiful

20-40% error in frequencies

300% error in damping/growing times

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**Damping/Growth time**

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Doneva, Gaertig, Kokkotas, Krüger (2013)
\[ M \sigma_{i}^{\text{unst}} = [(0.56 - 0.94 \ell) + (0.08 - 0.19 \ell)M \Omega + 1.2(\ell + 1)\eta] \]

The $\ell = 2$ $f$-mode oscillation frequencies and damping as functions of the parameter $\eta$ (moment of inertia)

\[ \eta = \sqrt{M^3 / I} \]
Equilibrium Configuration

\[ ds^2 = -e^{2\nu} dt^2 + e^{2\psi} r^2 \sin^2 \theta (d\phi - \omega dt)^2 + e^{2\mu} (dr^2 + r^2 d\theta^2). \]

\[ (u^t, u^r, u^\theta, u^\phi) = (u^t, 0, 0, \Omega u^t). \]

\[ T^{\mu\nu} = (\epsilon + p) u^\mu u^\nu + pg^{\mu\nu}, \]

Perturbed Einstein Equations & Conservation of Energy-Momentum

\[
\delta G_{\mu\nu} = 8\pi \delta T_{\mu\nu} \\
\delta (\nabla_\nu T^{\mu\nu}) = 0.
\]

Choice of Gauge

\[ \nabla^\nu h_{\mu\nu} = 0. \]

Hilbert/Lorenz

\[-2\delta G_{\mu\nu} = \nabla^\alpha \nabla_\alpha h_{\mu\nu} + 2R^{\alpha}_{\mu\beta\nu} h_{\alpha\beta} - R_{[\mu h_{\nu}\alpha] + Rh_{\mu\nu} - g_{\mu\nu} R^{\alpha\beta} h_{\alpha\beta}.} \]
Due to axi-symmetry we can separate the azimuthal part of the perturbation

$$X(t, r, \theta, \phi) := \tilde{X}(t, r, \theta)e^{im\phi} \quad \Rightarrow \quad \partial_\phi \rightarrow im$$

NOTE: These equations are for NON-ROTATING STARS !!!

+ 8 more for the spacetime
+ 4 for the fluid
Fast Rotating Neutron Star Perturbations
Numerical Results

A characteristic example

\[ h_{\theta\theta} \]

**FLUID PERTURBATIONS**

**SPACETIME PERTURBATIONS**

**W-MODES**

**EOS SLy**
- \( \rho_{\text{cent}} = 1.4 \times 10^{15} \text{ gr/cm}^3 \)
- \( M = 2.175 \, M_\odot \)
- \( R_e = 14.7 \, \text{km} \)
- \( R_e/R_p = 0.6 \)
- \( \Omega = 1.4 \, \text{kHz} \)

\[ \Omega/\Omega_{\text{Kepler}} \sim 0.92 \]

Unstable f-mode

\[ (-)f = 674 \, \text{Hz} \]
\[ (+)f = 3534 \, \text{Hz} \]
Neutron Star Models

Fixed baryon mass & central density sequences

- APR
  3 – 39 fixed $M_b$
  6 – 58 fixed $\rho_c$

- SLy
  7 – 65 fixed $M_b$
  5 – 63 fixed $\rho_c$

- WFF
  3 – 28 fixed $M_b$
  1 – 9 fixed $\rho_c$

- Polytropes
  5 – 52 fixed $\rho_c$

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Fitting Formulae

$\frac{\omega}{\omega_0}$ vs $\frac{T}{W}$

Krüger-KK 2019

Fitting formula:

$\left( \frac{\omega}{\omega_0} \right)_{\text{unst}} = 1 - 27.8 \left( \frac{T}{W} \right) + 301.8 \left( \frac{T}{W} \right)^2 - 1324.4 \left( \frac{T}{W} \right)^3$

$\left( \frac{\omega}{\omega_0} \right)_{\text{stab}} = 1 + 21.9 \left( \frac{T}{W} \right) - 313.7 \left( \frac{T}{W} \right)^2 + 1408.9 \left( \frac{T}{W} \right)^3$

Similar for axis ratios
Fitting Formulae

$\frac{\omega}{\omega_0} \text{ vs } \frac{T}{W}$

Fitting formula:

\[
\begin{align*}
\left( \frac{\omega}{\omega_0} \right)_{\text{corot}} &= 1 + 4.15 \left( \frac{T}{W} \right) - 39.8 \left( \frac{T}{W} \right)^2 \\
\left( \frac{\omega}{\omega_0} \right)_{\text{unst}} &= 1 - 11.2 \left( \frac{T}{W} \right) + 43.9 \left( \frac{T}{W} \right)^2
\end{align*}
\]

Similar for axis ratios
Fitting Formulae

$\omega/\omega_0$ vs $\Omega/\omega_0$

Fitting formula:

$$\omega_0 \approx \alpha + \beta \left(\frac{M}{R^3}\right)^{1/2}$$

Typically $\Omega_{Kepler}/\omega_0 \approx 0.65$

Fixed central density sequence

Krüger-KK 2019
Fitting Formulae

\[ M \omega \] vs \[ \eta \]

**Fixed baryon mass**

**Fixed central density**

\[ \eta = \sqrt{\left( \frac{M}{M_\odot} \right)^3 \left( \frac{I_{cgs}}{10^{45} \text{ g cm}^2} \right)^{-1}} \]

\[ \eta \sim M/R \]

\[ \hat{\Omega} = 5 \]

\[ \hat{\Omega} = 0 \]

\[ \hat{\Omega} = 5 \]

Instability line

For the Cowling case: Doneva+Kokkotas 2015

Non-rotating case: Tsui-Leung 2005

Lattimer-Schutz 2005: I vs M/R

\[ M\omega^{(+)} = \left( -2.38 + 0.43 \hat{\Omega} - 5.07 \cdot 10^{-2} \hat{\Omega}^2 \right) + \left( 3.56 + 2.36 \cdot 10^{-2} \hat{\Omega}^2 \right) \eta \]

\[ M\omega^{(-)} = \left( -2.38 - 0.41 \hat{\Omega} - 2.89 \cdot 10^{-3} \hat{\Omega}^2 \right) + \left( 3.56 + 7.88 \cdot 10^{-3} \hat{\Omega}^2 \right) \eta \]
**Binary Neutron Star Mergers**

**Early: Post-merger Oscillations & GWs**

**GRavitational Waves**

**Neutron Star Oscillations**

- \( l=m=0 \) linear quasi-radial mode
- Quasi-linear combination frequency (\( f_{2-0} \))
- \( l=m=2 \) linear f-mode (\( f_{\text{peak}} \))
- Nonlinear spiral frequency

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Stergioulas et al. (2011)
Bauswein, Stergioulas (2015)
... Krüger, Kokkotas (2019)
Oscillations & Instabilities
In the GW Era

• **Collapse**
  - Torres-Forné, A. et. al (2018, 19) – exitation of f, g-modes.

• **Pre-Merger Phase** (Love number)
  - Wen, De-Hua et.al (2019)
  - Rosofsky, Shawn G. etal (2019)
  - Andersson, Pnigouras (2019)
  - Chakravarti, Andersson (2019)
  - Schmidt, Hinderer (2019)

• **Early post-Merger phase**
  - Bauswein, Stergioulas, Janka 2015-2019

• **Late post-Merger phase**
  - Doneva, KK, Pnigouras (2015)
Formation of a “stable” NS

Slowdown due to three competing mechanisms:

I. Typical dipole B-field spindown

\[ t_{sd} \approx 7 \left( \frac{B_d}{10^{15} G} \right)^{-2} \left( \frac{P}{1 \text{ms}} \right)^2 \text{hr} \]

II. Deformed Magnetar Model

Dall’Osso-Giacomazzo-Perna-Stella 2015

III. Rotational Instabilities

Doneva-Kokkotas-Pnigouras 2015

\[ l = 2, m = 2 \]
Post-Merger NS:
F-mode instability vs Magnetic field

Δt = 10^2 – 10^5 sec

Competition between the B-field and the secular instability

GW frequencies:
WW2a: 920-1000 Hz
APR: 370–810 Hz
WFF2b: 600–780 Hz

Doneva-Kokkotas-Pnigouras 2015
Inverse Problem in the Spectra of Compact Object

Kostas Kokkotas & Sebastian Völkel
Theoretical Astrophysics, University of Tübingen
90s: An interesting observation by Chandrasekhar-Ferrari

1991: A reconsideration of the axial modes

\[ X_{r,r} - \frac{2}{r y} \left( 1 - 2 \epsilon r^2 \frac{y_1}{3y_1 - y} \right) X_{r,r} - \frac{(l-1)(l+2)}{r^2 y} X + \frac{4\sigma^2}{y(3y_1 - y)^2} X = 0, \]

CF 1991

KK 1995
1991: A reconsideration of the axial modes

\[ \frac{d^2}{dr^2} \Psi(r) + (\omega_n^2 - V(r)) \Psi(r) = 0, \]

\[ V(r) = \left(1 - \frac{2M}{r}\right) \left[ \frac{l(l+1)}{r^2} - \frac{6M}{r^3} \right] \]

Black-Holes

\[ V(r) = e^{2\nu} \left[ l(l+1)r + r^3(\rho - p(r)) - 6M(r) \right] \]

Axial oscillations of stars

Völkel-Kokkotas 2017

Kokkotas 1995
An interesting “toy” problem

KK: Les Houches 1995
Perturbations of the spacetime, similar to the QNMs of BHs (Kokkotas+Schutz 1986-1992)

- Frequencies (typical) 5-12kHz
- Damping times (typical) $\geq 0.1\text{ms}$
- For very compact stars they become exciting! (Chandrasekhar+Ferrari 1991)

The creation of an ergosphere signals the onset of an instability (Friedman 1978, Comins+Schutz 1978)

- The dragging is so strong that any timelike backwards moving trajectory gets dragged forward

Growth time of the order of tenths of secs

- It sets in quite early for very compact NS ($R/M \approx 2.26$, $\Omega \approx 0.19\Omega_{\text{Kepler}}$)

No direct way for viscosity to suppress the instability (!)

- Nonlinear saturation (?)

Kokkotas-Ruoff-Andersson 2004
Different types of axial perturbation potentials

\[
\frac{d^2}{dr^*^2} \Psi(r) + \left( \omega_n^2 - V(r) \right) \Psi(r) = 0,
\]

\[
V(r) = \frac{\hbar^2 \nu}{r^3} \left[ l(l + 1)r + r^3(\rho - p(r)) - 6M(r) \right]
\]
Different types of axial spectra

\[
\frac{d^2}{dr^2} \Psi(r) + \left( \omega_n^2 - V(r) \right) \Psi(r) = 0,
\]

\[
V(r) = \frac{e^{2\nu}}{r^3} \left[ l(l+1)r + r^3(\rho - p(r)) - 6M(r) \right]
\]
Axial spectra & Echoes

\[ \frac{d^2}{dr^2} \Psi(r) + \left( \omega_n^2 - V(r) \right) \Psi(r) = 0, \]

\[ V(r) = \frac{e^{2\nu}}{r^3} \left[ l(l+1)r + r^3(\rho - p(r)) - 6M(r) \right] \]

**Figure 2:** Left: Axial perturbations ultra compact stars, Ferrari & Kokkotas (2000). Right: Phenomenological template for parameter estimation: Maselli, Völkel & Kokkotas (2018).
Different types of calculation methods

\[ \frac{d^2}{dr^2} \Psi(r) + (\omega_n^2 - V(r)) \Psi(r) = 0, \quad V(r) = \frac{e^{2\nu}}{r^3} \left[ l(l+1)r + r^3(\rho - p(r)) - 6M(r) \right] \]

- Methods for **direct** QNM calculations:
  Continued fraction, Green’s functions, Time-evolution,…

- But, **reconstructing** potential/properties of the source from the spectrum is different (uniqueness?)

- **WKB** method and **Bohr-Sommerfeld** rules\(^4\) are powerful here
  (approximate, but easier to invert)

\[ \int_{x_0}^{x_1} \sqrt{E_n - V(x)} \, dx = \pi \left( n + \frac{1}{2} \right) - \frac{i}{4} \exp \left( 2i \int_{x_1}^{x_2} \sqrt{E_n - V(x)} \, dx \right) \]

\[ (2) \]

---

\[^4\text{here } E_n \equiv \omega_n^2\]
Inversion Method for Different types of potentials

\[ \frac{d^2}{dr^2} \Psi(r) + \left( \omega_n^2 - V(r) \right) \Psi(r) = 0, \]

\[ V(r) = \frac{e^{2\nu}}{r^3} \left[ (l^2 + 1) + r^2(\rho - p(r)) - 6M(r) \right] \]

\[ \mathcal{L}_1(E) = x_1 - x_0 = 2 \frac{\partial}{\partial E} \int_{E_{\text{min}}}^{E} \frac{n(E') + 1/2}{\sqrt{E - E'}} \, dE' \]

\[ \mathcal{L}_2(E) = x_2 - x_1 = -\frac{1}{\pi} \int_{E}^{E_{\text{max}}} \frac{d(T(E')/dE')}{T(E')\sqrt{E' - E}} \, dE' \]

Figure 3: Völkel & Kokkotas (2017)

Figure 4: Völkel & Kokkotas, (2018)

Wheeler 1976
Inversion Method for stellar potentials

\[ \frac{d^2}{dr^*^2} \Psi(r) + (\omega_n^2 - V(r)) \Psi(r) = 0, \]

\[ V(r) = \frac{e^{2\nu}}{r^3} \left[ l(l+1)r + r^3(\rho - p(r)) - 6M(r) \right] \]

Example for ultra compact constant density star \( C \approx 0.44 \)

Reconstructed axial perturbation potential, constant density star, \( l = 3 \), taken from Völkel and Kokkotas (2017,2).
Inversion Method for DS wormhole

\[ \frac{d^2}{dr^2} \Psi(r) + (\omega_n^2 - V(r)) \Psi(r) = 0, \quad V(r) = \frac{e^{2\nu}}{r^3} \left[ l(l+1)r + r^3(\rho - p(r)) - 6M(r) \right] \]

Example for Damour-Solodukhin wormhole \( C \approx 0.5 \)

Reconstructed scalar perturbation potential, Damour-Solodukhin wormhole, \( l = 3 \), taken from Völkel and Kokkotas (2018,2).
Inversion Method for normal NS

\[
\frac{d^2}{dr^*^2} \Psi(r) + (\omega_n^2 - V(r)) \Psi(r) = 0, \quad V(r) = \frac{e^{2\nu}}{r^3} \left[ l(l+1)r + r^3(\rho - p(r)) - 6M(r) \right]
\]

Example for neutron star polytrope \( C \approx 0.15 \)

Reconstructed axial perturbation potential, neutron star polytrope, \( l = 3 \), Völkel and Kokkotas (2019).
Inversion Problem for Hawking Radiation

Assuming Hawking radiation can be described by

$$\frac{dE}{dt} = \sum_{l} N_{l} |A_{l}|^{2} \frac{\omega}{\exp(\omega/T_{H}) - 1} \frac{d\omega}{2\pi},$$

$T_{H}$ Hawking temperature, $A_{l}$ greybody factors, $N_{l}$ multiplicities

$$\psi = e^{-i\omega r_{*}} + Re^{i\omega r_{*}}, \quad r_{*} \to +\infty,$$
$$\psi = Te^{-i\omega r_{*}}, \quad r_{*} \to -\infty,$$

reflection $R$ and transmission $T$

$$|A_{\ell}|^{2} = 1 - |R_{\ell}|^{2} = |T_{\ell}|^{2}.$$
Analytic approximation given by Gamow formula

$$T(E) = \exp \left( 2i \int_{x_0}^{x_1} \sqrt{E - V(x)} \, dx \right),$$

(5)

$E$ energy, $V(x)$ potential barrier, and $x_0$ and $x_1$ classical turning points\(^5\).

Can be inverted to find width of potential barrier\(^6\)

$$\mathcal{L}(E) \equiv x_1 - x_0 = \frac{1}{\pi} \int_{E}^{E_{\text{max}}} \frac{(d\frac{T(E')}{dE'})}{T(E')\sqrt{E' - E}} \, dE',$$

(6)

But, how do we get the individual greybody factors?

\(^5\)Defined by $E = V(x)$

\(^6\)Cole & Good PRA (1978)
Inversion Problem for Hawking Radiation

Reconstruction of the **Schwarzschild transmissions** $T_l(E)$ from Hawking spectrum fitting

Reconstruction of the **Schwarzschild potential barrier** widths $L_l(E)$ from given transmissions $T_l(E)$.
Inversion Problem for Hawking Radiation

For black hole QNMs, Schutz-Will formula (1985) based on parabolic potential approximation, easily gives fundamental modes, but fails for overtones.

Hawking radiation involves $\textbf{summation}$ over several greybody factors.

Parabolic approximation yields

$$T_1(E) = \left(1 + \exp \left(-\frac{\pi \left(E - V_{\text{max},l}\right)}{\sqrt{a_l}}\right)\right)^{-1},$$

(7)

$T_1(E)$ has non-trivial contribution only around $V_{\text{max},l}$.

For smaller or larger energies, $T_1(E)$ is either 0 or 1, respectively

$$V_{\text{max},l} \equiv V_{\text{BH}}(r_{\text{max},l}^*), \quad a_l \equiv -\frac{V''_{\text{max},l}}{2}.$$
Inversion Problem for Hawking Radiation

Völkel, Konoplya, Kokkotas PRD 2019

\[ I(E) \equiv \sum_l N_l |A_l|^2 \equiv \sum_l I_l(E). \]

Comparison of the exact result (black solid) and parabolic approximation (red dashed) for the normalized energy emission spectrum of the Schwarzschild black hole.
THANK YOU