UNIVERSITÀ DEGLI STUDI DI TRENTO

Superradiant effects in BECs: amplification and instabilities

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ECT* workshop

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BOSE EINSTEIN CONDENSATION

CNR - INO

Analogue gravity in BECs

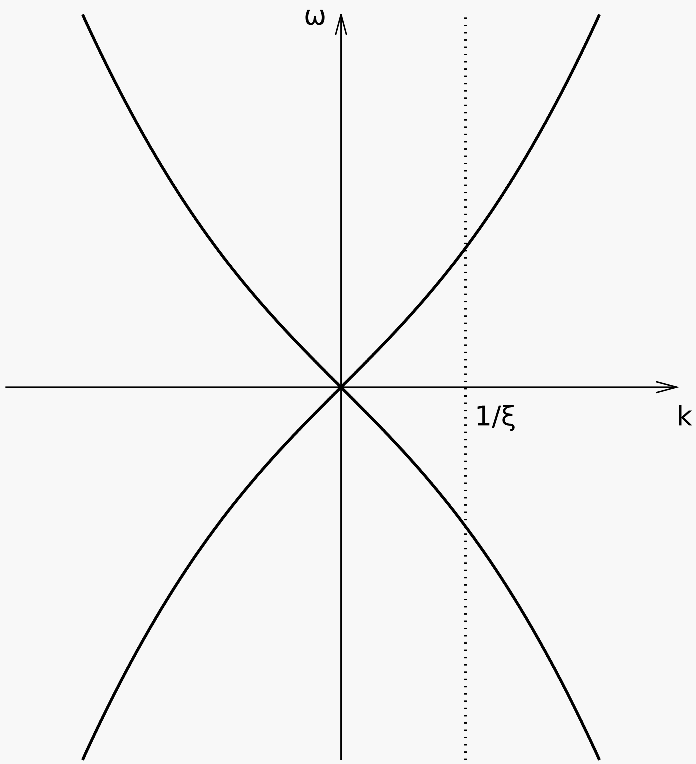
- Gross-Pitaevskii equation: $i\hbar\partial_t\psi = \left(-\frac{\hbar^2\nabla^2}{2M} + V_{\text{ext}} + g|\psi|^2\right)\psi$

$$\psi(t,\mathbf{x}) = \sqrt{n(t,\mathbf{x})}e^{i\Theta(t,\mathbf{x})} \; ; \qquad \mathbf{v} = \frac{\hbar\nabla\Theta}{M} \; ; \qquad c_s = \sqrt{\frac{gn}{M}}$$

- Linear perturbations (Bogoliubov):

$$\hat{\psi} = \psi_0\left(1 + \frac{\delta\hat{n}}{n} + i\delta\hat{\Theta}\right)$$

$$\omega = \pm c_s\sqrt{k^2 + \frac{\xi^2 k^4}{4}}$$



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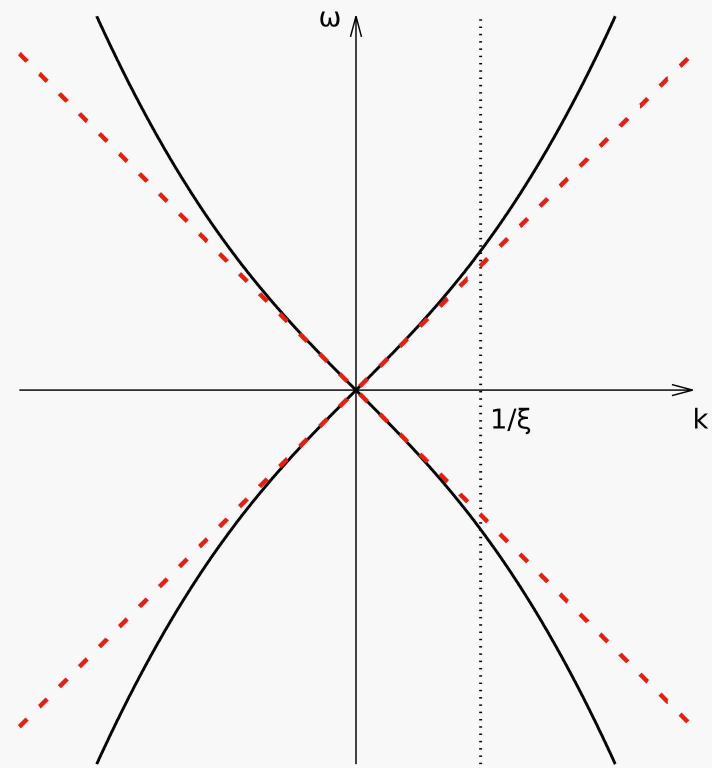
$$\omega = \pm c_s\sqrt{k^2 + \frac{\xi^2 k^4}{4}}$$

Hydrodynamic approximation

$$\lambda \gg \xi = \frac{\hbar}{Mc_s}$$

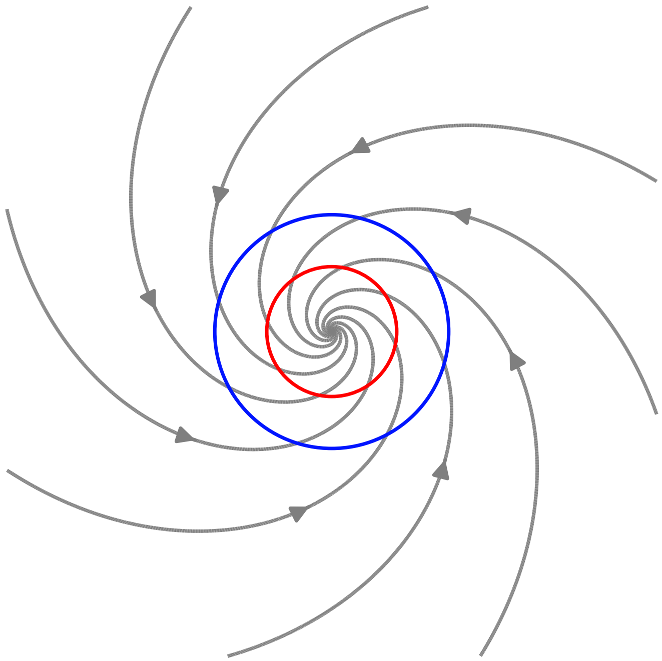
$$\Box(\delta\Theta) = \nabla_\mu\nabla^\mu(\delta\Theta) = 0$$



Rotating acoustic BH: the vortex geometry

$$\mathbf{v} = \frac{A}{r}\hat{r} + \frac{B}{r}\hat{\theta}$$

$$g_{\mu\nu} \propto \begin{bmatrix} -\left(c_s^2 - \frac{A^2+B^2}{r^2}\right) & 0 & -B \\ 0 & \frac{r^2 c_s^2}{r^2 c_s^2 - A^2} & 0 \\ -B & 0 & r^2 \end{bmatrix}$$

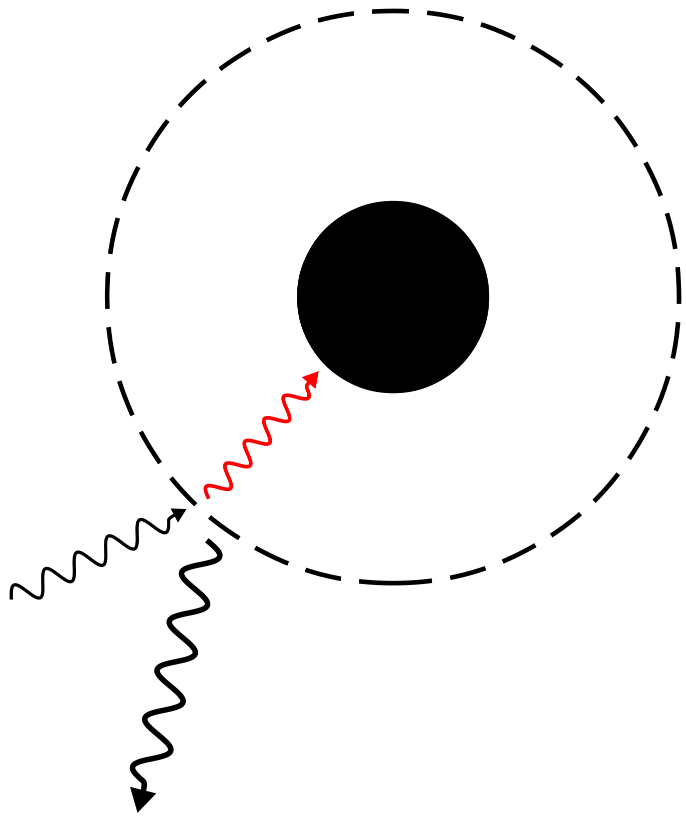


- Acoustic horizon: $r_H^2 = \frac{A^2}{c_s^2}$ $\quad v_r = c_s$
- Ergosurface: $r_E^2 = \frac{A^2 + B^2}{c_s^2}$ $\quad |v| = c_s$

Ref: Visser (1998). Acoustic black holes. Classical and Quantum Gravity, 15(6), 1767

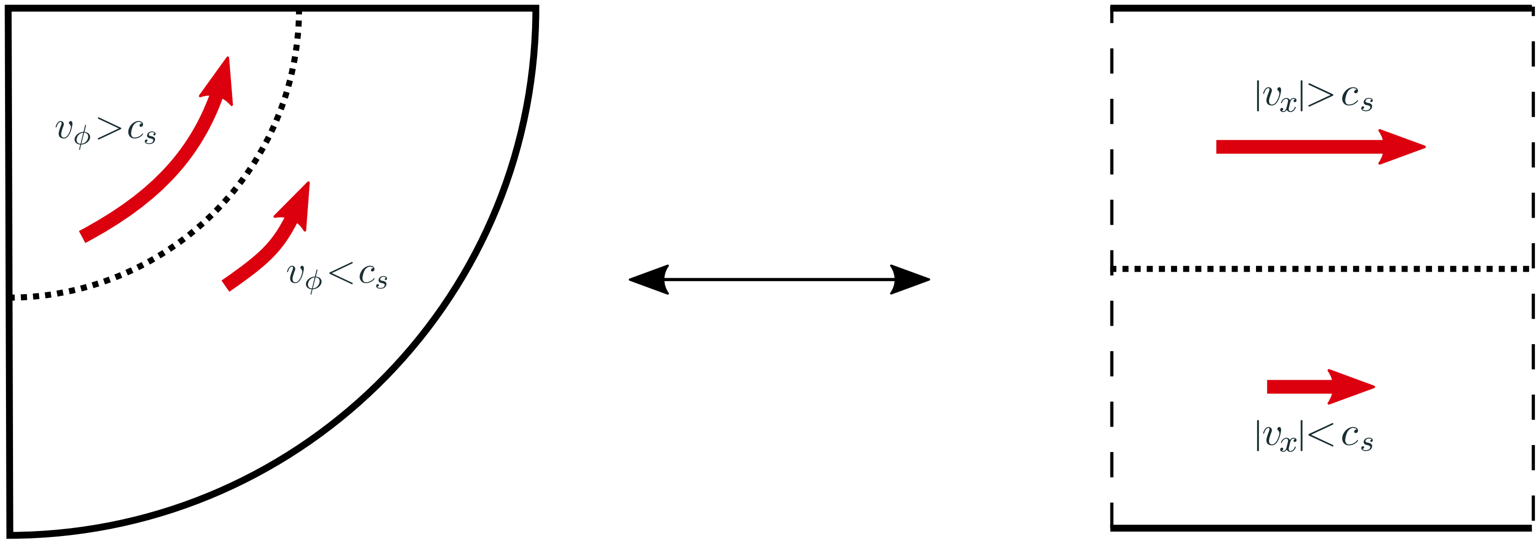
Superradiance

- Amplified reflection by transmission of negative energy



Ref: Brito, Cardoso, Pani (2015). Superradiance. Lect. Notes Phys, 906(1), 1501-06570.

A toy model to understand superradiance



- Can we really do this?

$$\mathbf{v} = \frac{\hbar \nabla \Theta}{M} \implies \nabla \times \mathbf{v} = 0$$

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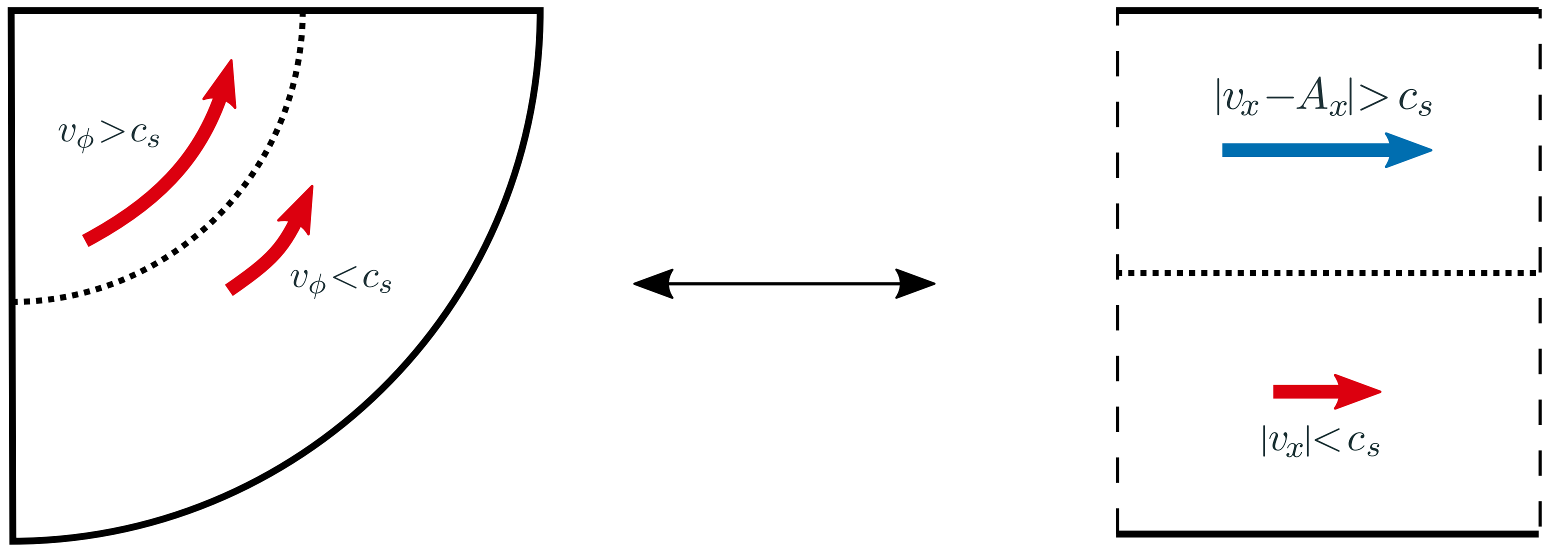


- Can we really do this?

$$\mathbf{v} = \frac{\hbar \nabla \Theta}{M} \implies \nabla \times \mathbf{v} = 0$$

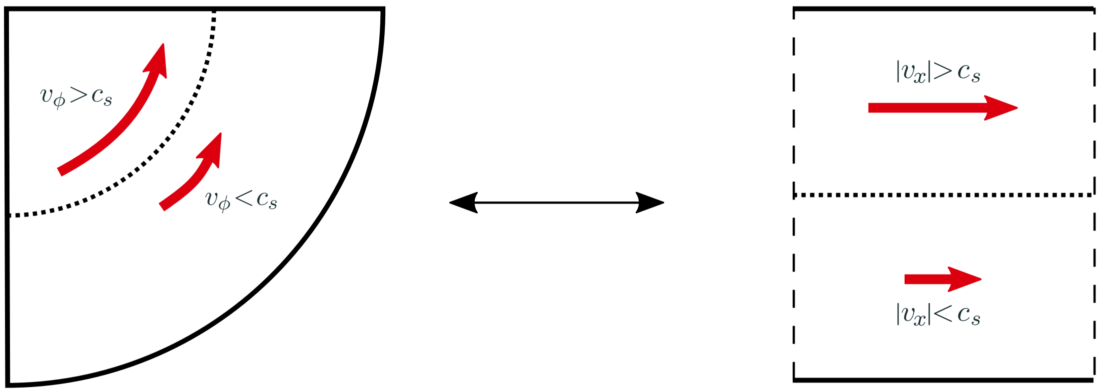
- We can play with synthetic gauge fields:

$$\mathbf{v} = \frac{\hbar \nabla \Theta - \mathbf{A}}{M}$$

For us it is a trick to get rotational flow.

For an introduction Dalibard et al. RMP 2011

A toy model to understand superradiance



$$g_{\mu\nu} = \frac{n^2}{c^2} \begin{bmatrix} -(c^2 - v_x^2) & -v_x & 0 \\ -v_x & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Translational invariance along x: $\phi(t,x,y) = e^{ik_x x}\phi(t,y)$
- Klein-Gordon equation becomes:

$$-\left(\frac{1}{c}\partial_t + i\frac{v_x}{c}k_x\right)^2 \phi + \partial_y^2\phi - k_x^2\phi = 0$$

Ref: Fulling (1989). Aspects of QFT in curved spacetime, Cambridge University Press

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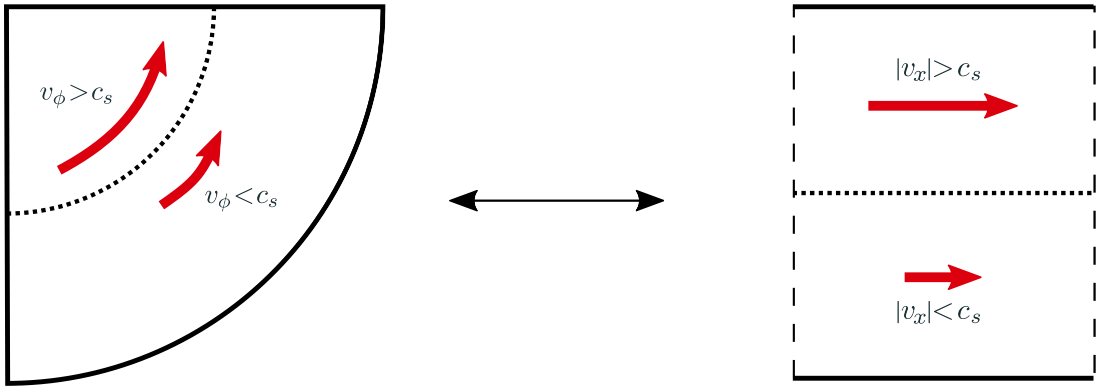

$$g_{\mu\nu} = \frac{n^2}{c^2}\begin{bmatrix} -(c^2 - v_x^2) & -v_x & 0 \\ -v_x & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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- KG for a charged field in an electrostatic potential:

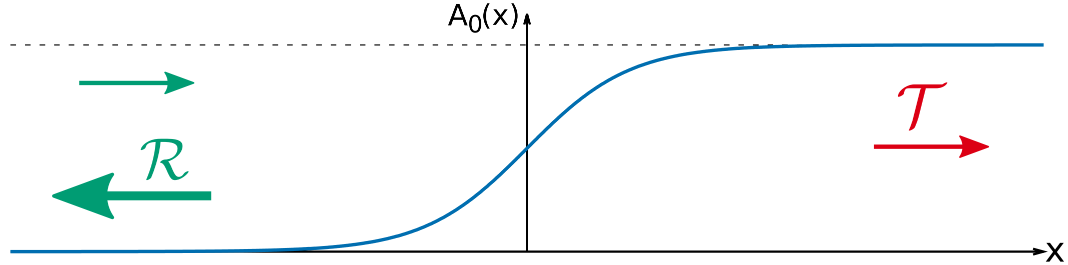
$$-\left(\frac{1}{c}\partial_t + \frac{ie}{\hbar c}A_0\right)^2 \phi + \nabla^2\phi - \frac{m^2c^2}{\hbar^2}\phi = 0$$

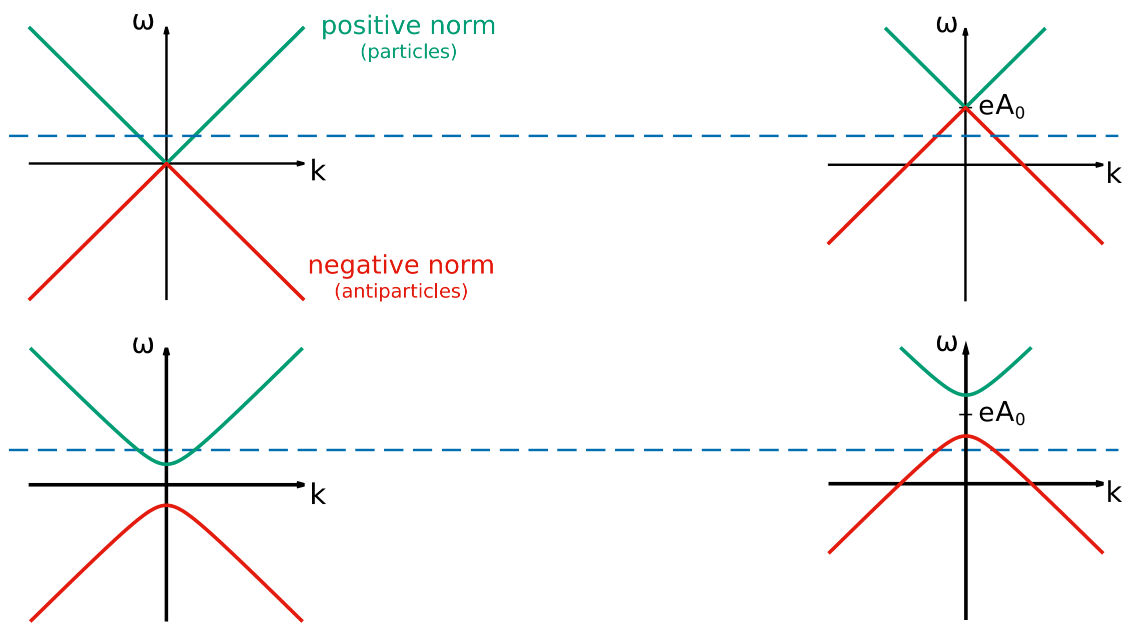
$$m^2 \longleftrightarrow \hbar^2 k_x^2/c^2 \qquad eA_0 \longleftrightarrow \hbar k_x v_x$$

Ref: Fulling (1989). Aspects of QFT in curved spacetime, Cambridge University Press

Bosonic Klein paradox

$$-\left(\frac{1}{c}\partial_t + \frac{ie}{c}A_0\right)^2\phi + \partial_x^2\phi - \frac{m^2c^2}{\hbar^2}\phi = 0$$





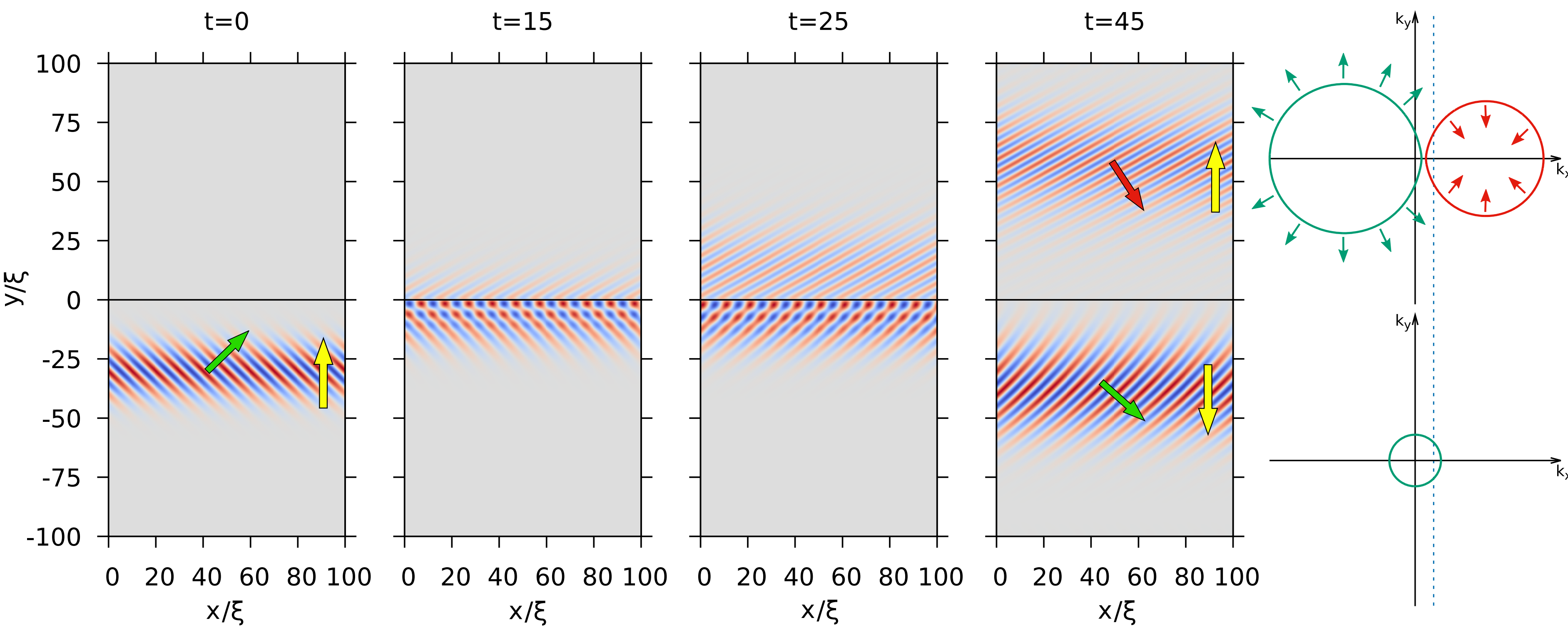
- Massless case: $\mathcal{R} > 1$ for $\omega < eA_0$

- Massive case: $eA_0 > 2mc^2$

- In our 2D configuration with $A_x(y) = \begin{cases} 0 \text{ for } y < 0 \\ A_x \text{ for } y > 0 \end{cases} \implies |A_x| > 2Mc_s$

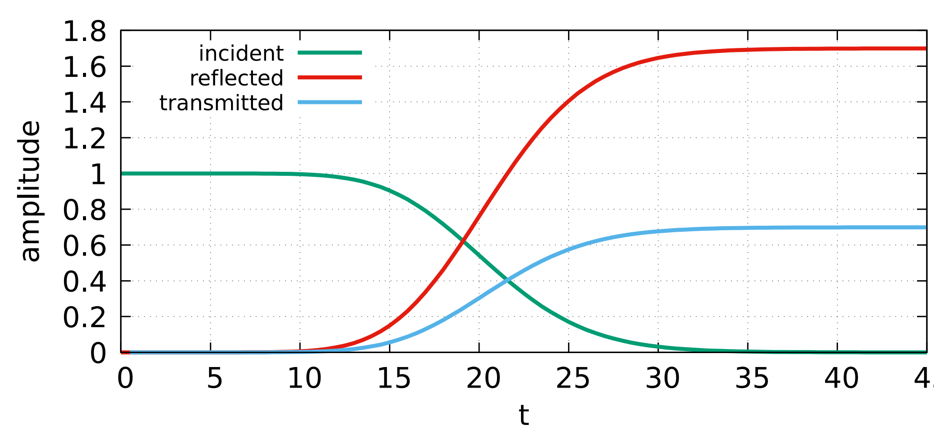
2D GPE simulations

$$i\hbar\partial_t\Psi = \left[\frac{(-i\hbar\nabla - \mathbf{A}(\mathbf{r}))^2}{2M} + V(\mathbf{r}) + g\Psi^2\right]\Psi\ ;\ \ A_x(y) = \begin{cases} 0 \text{ for } y<0 \\ A_x \text{ for } y>0 \end{cases};\ \ V(y) = \begin{cases} 0 \text{ for } y<0 \\ -\frac{A_x^2}{2M} \text{ for } y>0 \end{cases}$$



Energy of an eigenmode

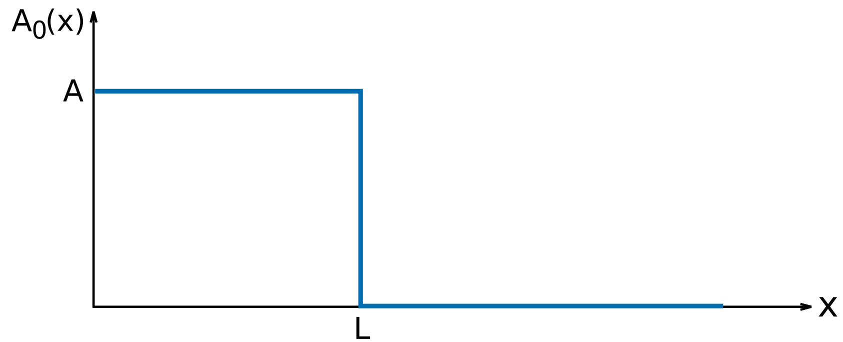
$$E_i = \langle\psi_i|\,\sigma_3\,|\psi_i\rangle\,\epsilon_i$$



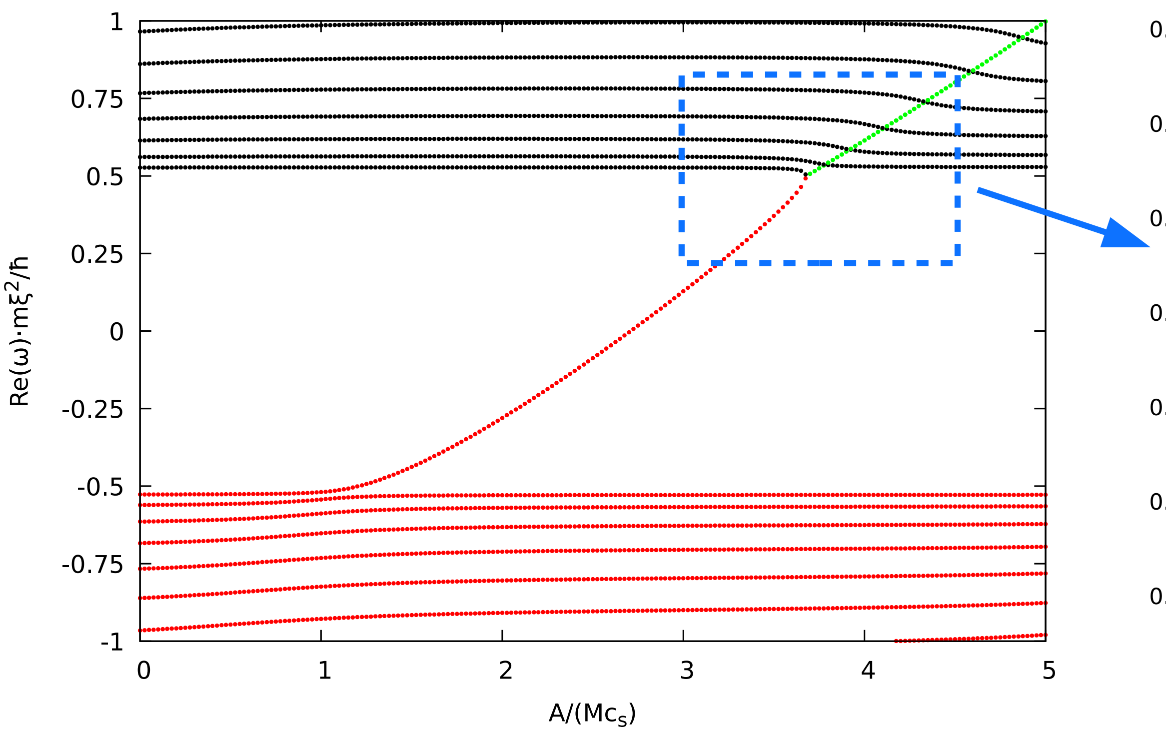
The Schiff-Snyder-Weinberg effect

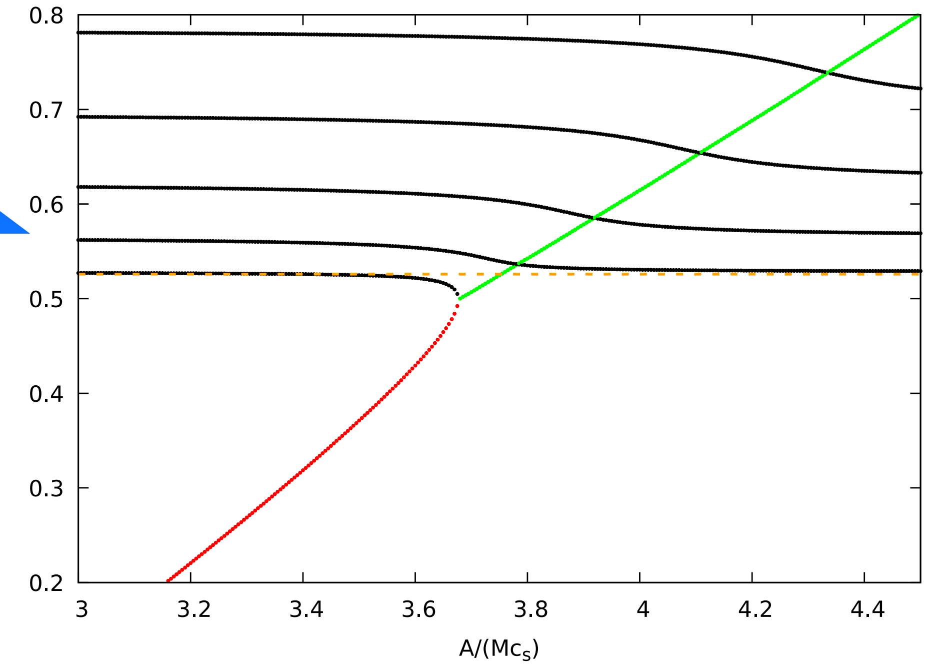
- It is the physics of an electrostatic potential box for a scalar field

[Schiff, Snyder, Weinberg (1940). Physical Review, 57(4), 315.]



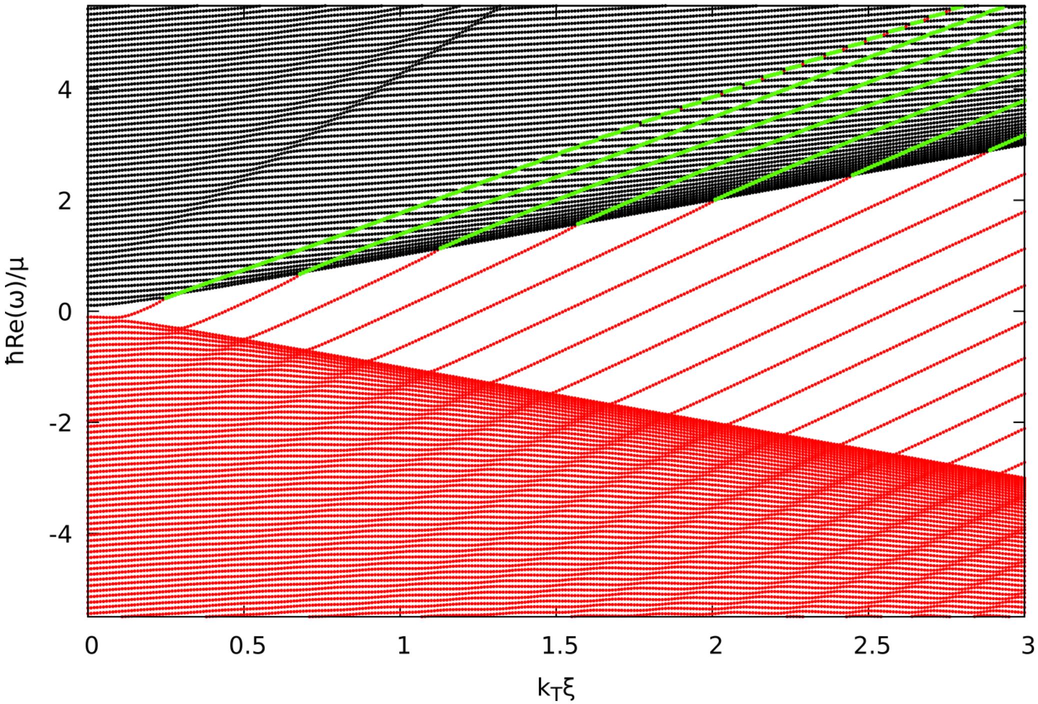
- Diagonalizing the Bogoliubov problem ($L = 2\xi;\ k_T = -0.5/\xi$)

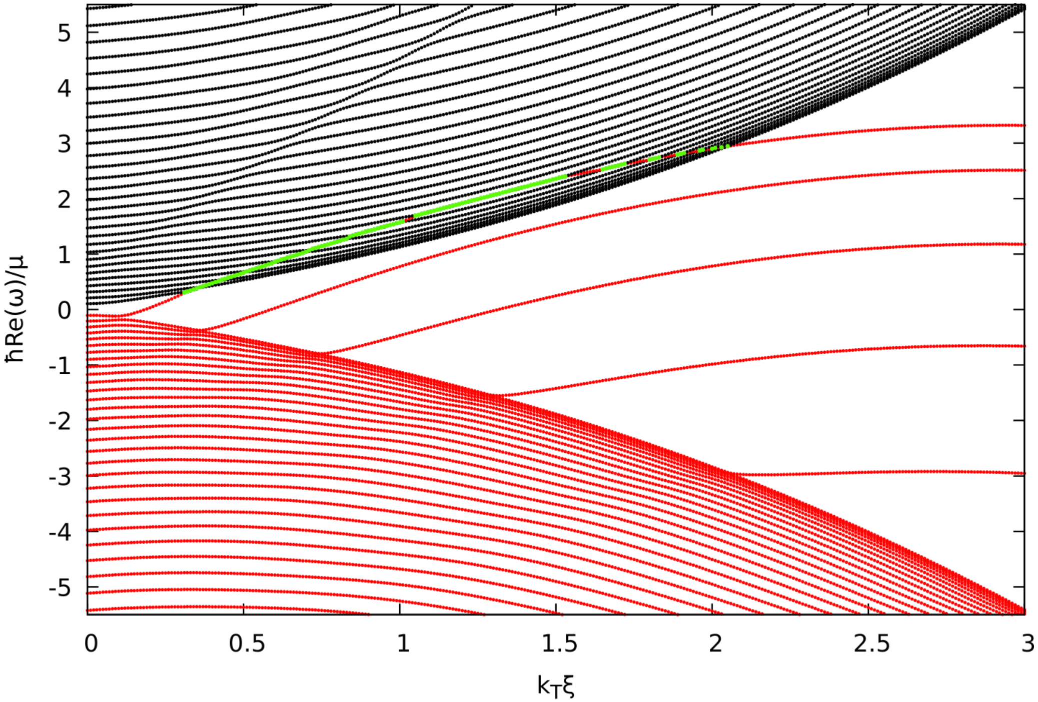




Dispersive SSW effect

- Vary k_T for fixed A $(L = 4\xi;\ A = -3Mc_s)$

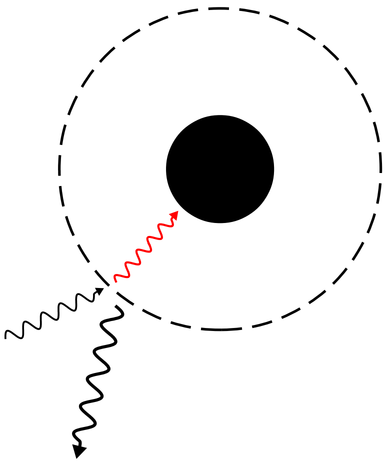




Dispersion bounds the instability range from above!

Ref: Giacomelli, Carusotto. In preparation.

Back to vortices: instabilities



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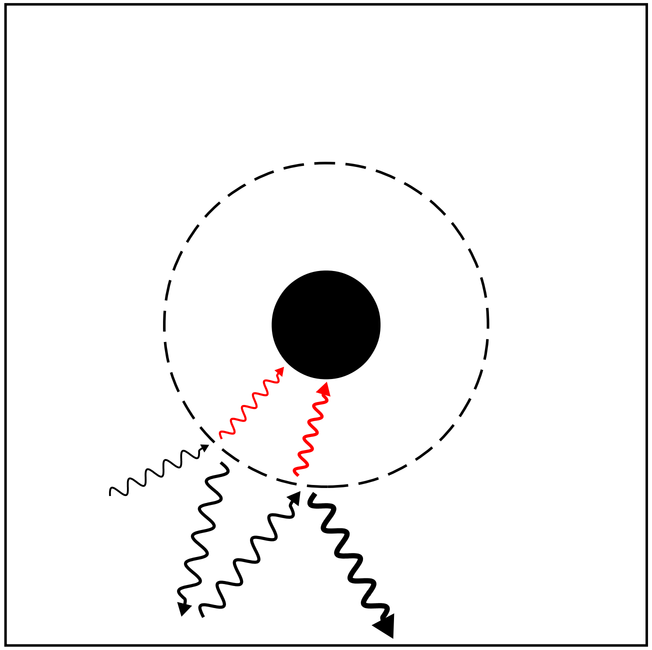
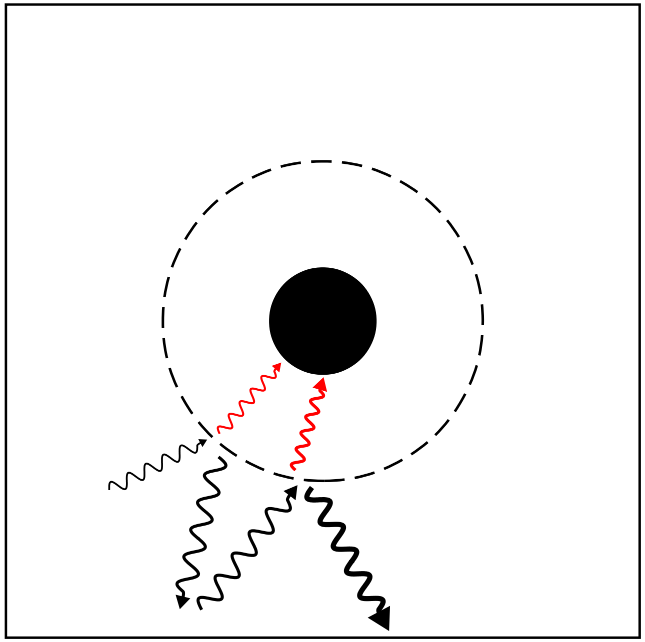


Black hole bomb

Back to vortices: instabilities



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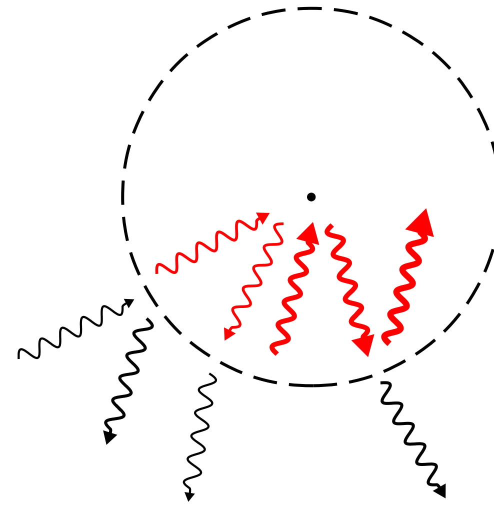
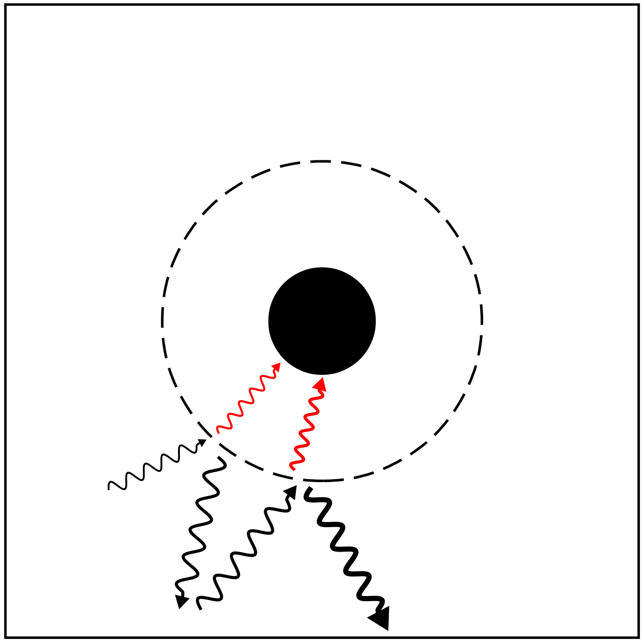


Ergoregion instability

In analogues:

Oliveira et al. PRD 2014

Back to vortices: instabilities



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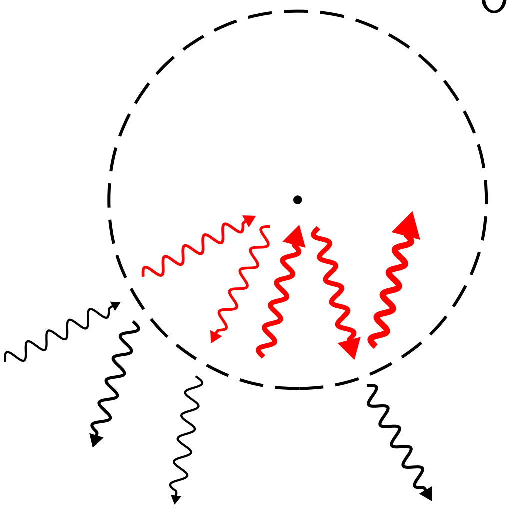


Ergoregion instability

In analogues:
Oliveira et al. PRD 2014

What about quantized vortices?

- In a BEC quantized vortex of charge ℓ ergoregion at $r_E \sim \ell\xi$
- A trap gives a reflecting boundary condition
- Hydrodynamic approximation does not hold!

Vortices in trapped condensates

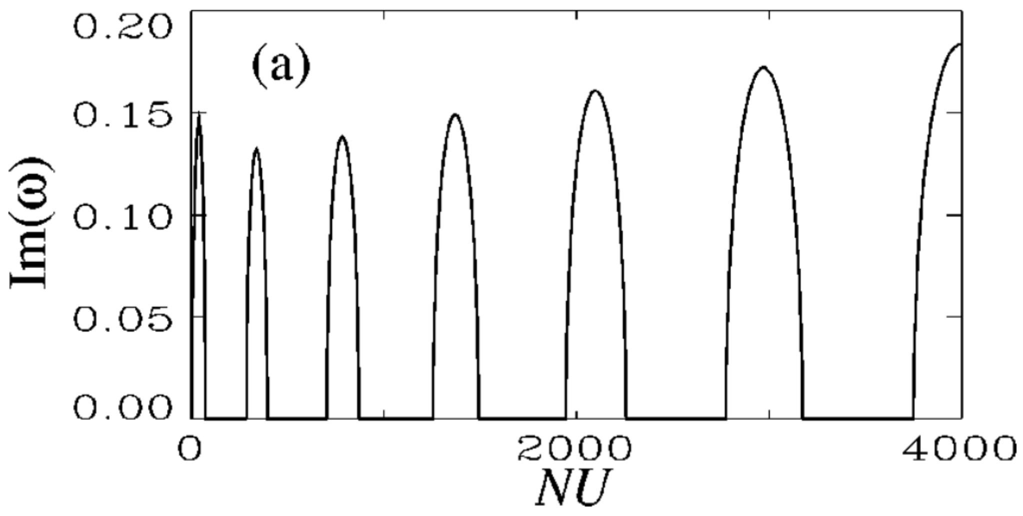
- Multiply charged vortices are energetically unstable
- A trapped singly charged vortex is energetically unstable but dynamically stable

[Rokhsar (1997) PRL, 79(12), 2164]

- Trapped multiply charged vortices can be dynamically unstable

[Pu et al. (1999) PRA, 59(2), 1533]

$$\ell = 2, \ m = 2$$



Study the spectrum as

$$\Psi_0(r,\theta) = e^{i\ell\theta} f(r)$$

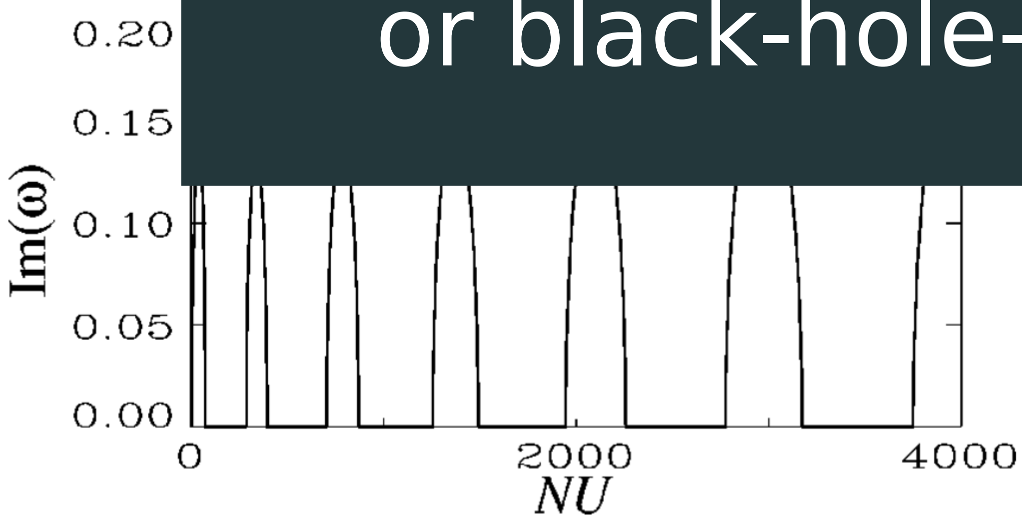
$$\begin{pmatrix} \delta\psi(r,\theta) \\ \delta\psi^*(r,\theta) \end{pmatrix} = e^{im\theta} \begin{pmatrix} e^{i\ell\theta}\phi(r) \\ e^{-i\ell\theta}\phi^*(r) \end{pmatrix}$$

Vortices in trapped condensates

- Multiply charged vortices are energetically unstable
- A trapped singly charged vortex is energetically unstable but dy … 2), 2164]
- Tr … e

  … 2), 1533]

Is the instability of multiply quantized vortices ergoregion-like or black-hole-bomb-like?



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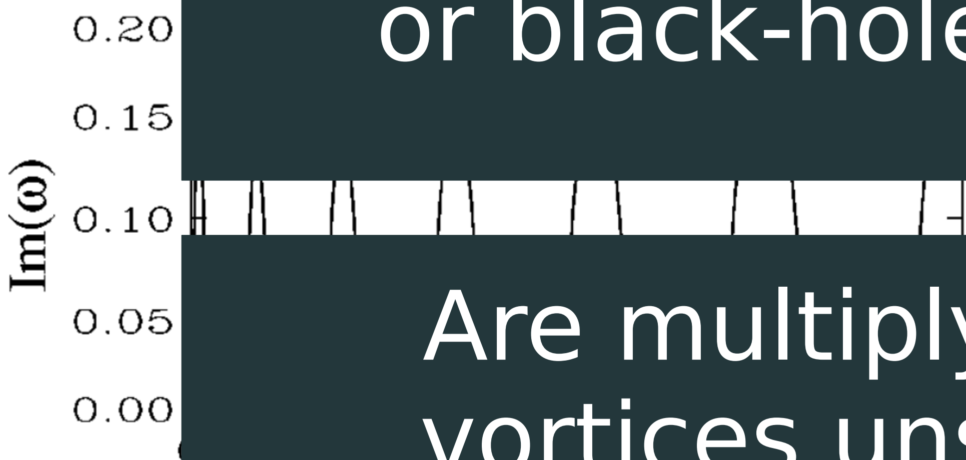
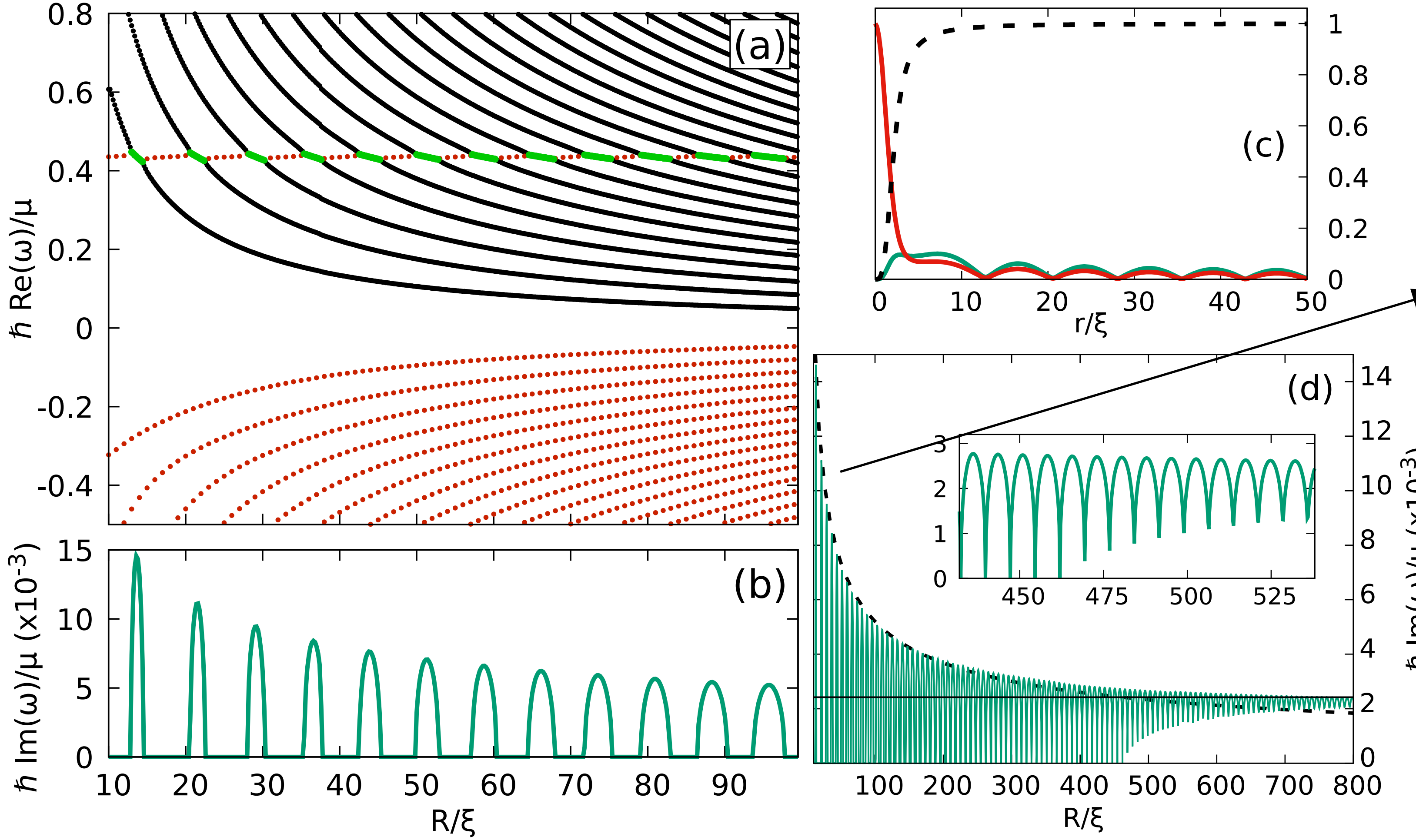


Are multiply quantized vortices unstable in an unbound condensate?

System-size perturbation theory

- Ground state of the radial GPE with Neumann BC at some r=R
- Diagonalize the Bogoliubov problem varying R at fixed m

$$\ell = 2,\ m = 2$$



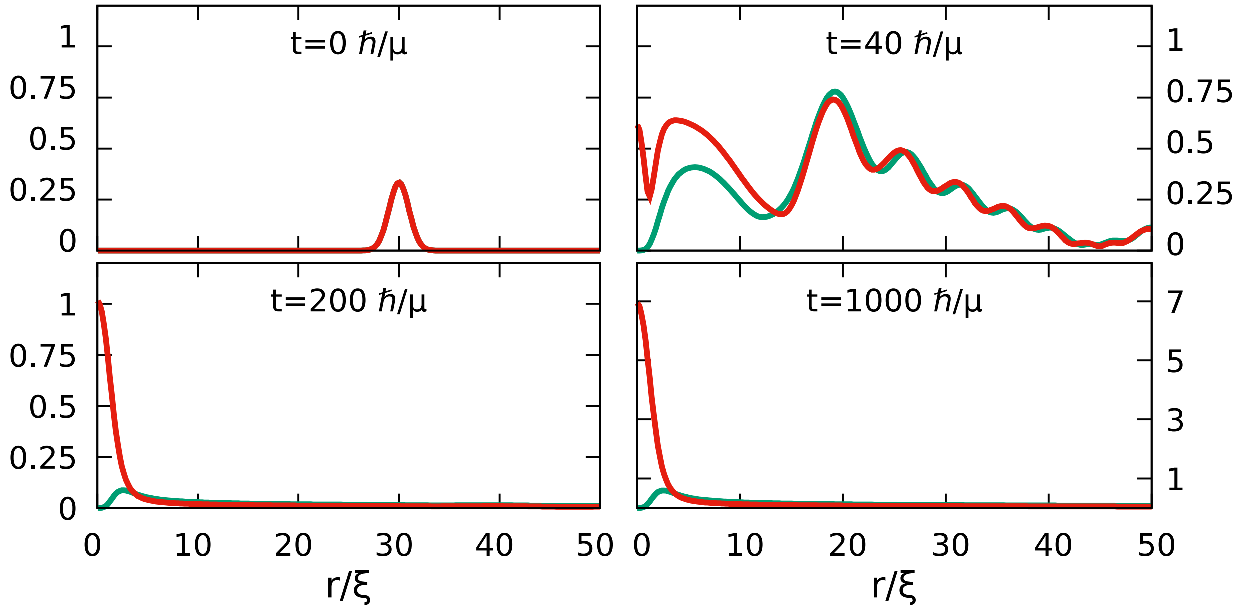
$1/\sqrt{R}$

As an emitter in a cavity

Is the core really inherently stable?

- We always get stationary waves, so also incoming phonons
- Time dependent study of the Bogoliubov problem with absorbing boundary conditions:

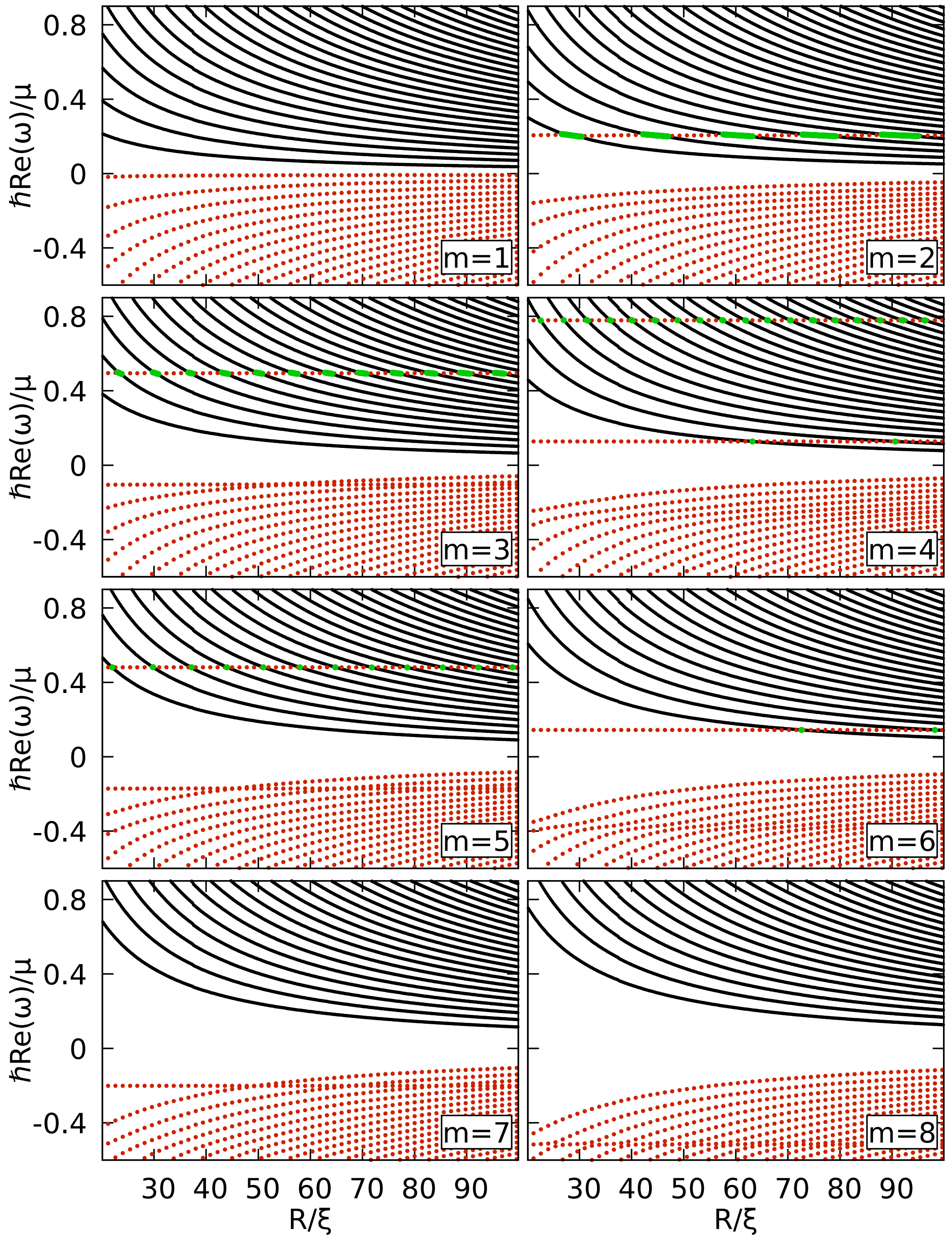
$$\ell = 2,\ m = 2$$



The charge 2 vortex core is inherently dynamically unstable!

Ref: Giacomelli, Carusotto (2019) arXiv:1905.02447

Example: $\ell = 4$



Higher charge vortices

Multiply quantized vortices display dispersive ergoregion instability

- Lower m-s are always the most unstable (as for hydrodynamic)

  [Oliveira et al. PRD 2014]

  Instability not directly related to vortex splitting

- In general for a charge ℓ vortex the unstable channels are

$$2 \leq m \leq 2\ell - 2$$

- For the hydrodynamic vortex no upper bound on m

  [Oliveira et al. PRD 2014]

Dispersion introduces an upper bound on the angular momentum!

Ref: Giacomelli, Carusotto (2019) arXiv:1905.02447

Conclusions

- Superradiance and Klein scattering are essentially the same kind of phenomenon
- Introducing reflecting boundary conditions can make an energetic instability dynamical
- Multiply quantized vortices are intrinsically dynamically unstable via a dispersive ergoregion instability
- The role of dispersion is to suppress the instability at high (angular) momenta

Extra slides

Analogue Gravity in BECs

- Linearization of the GPE (Bogoliubov-de Gennes eq.)

$$i\hbar\partial\Psi_0 = \left[-\frac{\hbar^2}{2m}\nabla^2 + g|\Psi_0|^2 + V_{\text{ext}} - \mu\right]\Psi_0$$

$$\Psi = \sqrt{n}e^{i\Theta}\left(1 + \frac{\delta n}{2n} + i\delta\Theta\right)$$

- Klein-Gordon equation in curved spacetime (stationary metric)

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\; g^{\mu\nu}\partial_\nu\phi\right) = 0$$

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$$\downarrow$$

$$i\hbar\partial_t \begin{pmatrix}\delta\Theta \\ \delta n/n\end{pmatrix} = \mathcal{H}\begin{pmatrix}\delta\Theta \\ \delta n/n\end{pmatrix}$$

$$\mathcal{H} = \begin{bmatrix} -i\frac{\hbar^2}{M}\nabla\Theta\cdot\nabla & -i\hbar^2\frac{1}{n}\nabla(n\nabla) - i\frac{gn}{M} \\ -i\hbar^2\frac{1}{n}\nabla(n\nabla) & -i\frac{\hbar^2}{M}\nabla\Theta\cdot\nabla \end{bmatrix}$$

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$$\downarrow$$

(Feshbach-Villars repr)

$$i\hbar\partial_t \begin{pmatrix}\phi \\ \pi^*\end{pmatrix} = \mathcal{H}\begin{pmatrix}\phi \\ \pi^*\end{pmatrix}$$

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In the hydrodynamic limit the BdG reduces to a massless KG in curved spacetime!

$$g_{\mu\nu} = \frac{n}{c_s}\begin{bmatrix} -(c_s^2 - v^2) & -\mathbf{v}^T \\ -\mathbf{v} & \mathbf{I} \end{bmatrix}\;; \qquad \mathbf{v} = \frac{\hbar\nabla\Theta}{M}\;; \qquad c_s = \sqrt{\frac{gn}{M}}$$

Superradiant scattering in the vortex geometry

- Stationary and axisymmetric solution of the KG equation:

$$\phi(t, r, \theta) = \frac{1}{\sqrt{r}} R(r) e^{im\theta - i\omega t}$$

- Change coordinate: $r \in [r_H, \infty] \longrightarrow r^* \in [-\infty, +\infty]$
  - For $r^* \to \infty$: $R_{+\infty}(r^*) = e^{i\omega r^*} + \mathcal{R} e^{-i\omega r^*}$
  - For $r^* \to -\infty$: $R_H(r^*) = \mathcal{T} \exp\left[i \left(\omega - m \frac{c_s B}{A^2} \right) r^* \right]$
- Matching the two (conservation of the Wronskian):

$$1 - |\mathcal{R}|^2 = \frac{1}{\omega} \left(\omega - m \frac{c_s B}{A^2} \right) |\mathcal{T}|^2,$$

$$\omega < m c_s B / A^2 \implies |\mathcal{R}|^2 > 1$$

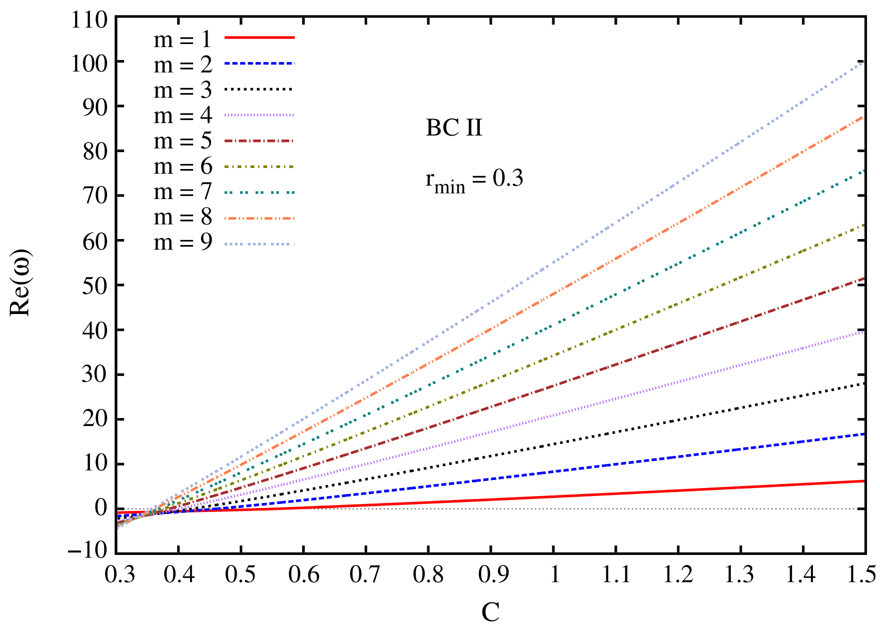
Ref: Basak, Majumdar (2003). Classical and Quantum Gravity, 20(18), 3907

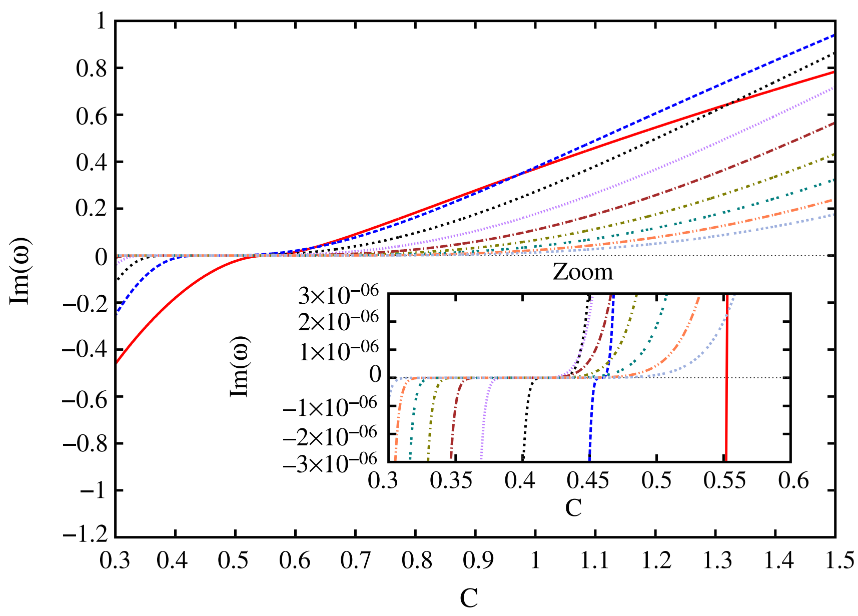
Instability of the hydrodynamic vortex

- What if there is no drain? $\mathbf{v} = \frac{C}{r}\hat{\theta}$
  - Spacetimes with an ergoregion but no horizon are unstable

    [Friedman (1978). Ergosphere instability. Comm. Math. Phys., 63(3), 243-255]
  - Hydrodynamic vortex with reflecting BC at a radius r_{min}

    [Oliveira et al. (2014). Ergoregion instability: The hydrodynamic vortex. PRD, 89(12), 124008]





Eigenmodes of the Bogoliubov problem

- A more familiar shape of the problem: $\Psi = \Psi_0 + \psi$

$$i\hbar\partial_t \begin{pmatrix} \psi \\ \psi^* \end{pmatrix} = \begin{bmatrix} H_{GP} + g|\Psi_0|^2 & g\Psi_0^2 \\ -g\Psi_0^{*2} & -\left(H_{GP} + g|\Psi_0|^2\right)^* \end{bmatrix} \begin{pmatrix} \psi \\ \psi^* \end{pmatrix}$$

- If $\begin{pmatrix} \psi \\ \psi^* \end{pmatrix} = \begin{pmatrix} u_\psi \\ v_\psi \end{pmatrix}$, conserved (nonpositive) inner product

$$\langle\psi|\,\sigma_3\,|\phi\rangle = \int \mathrm{d}\mathbf{x}\left[u_\psi^*(\mathbf{x})u_\phi(\mathbf{x}) - v_\psi^*(\mathbf{x})v_\phi(\mathbf{x})\right]$$

- The energy of an eigenmode is:

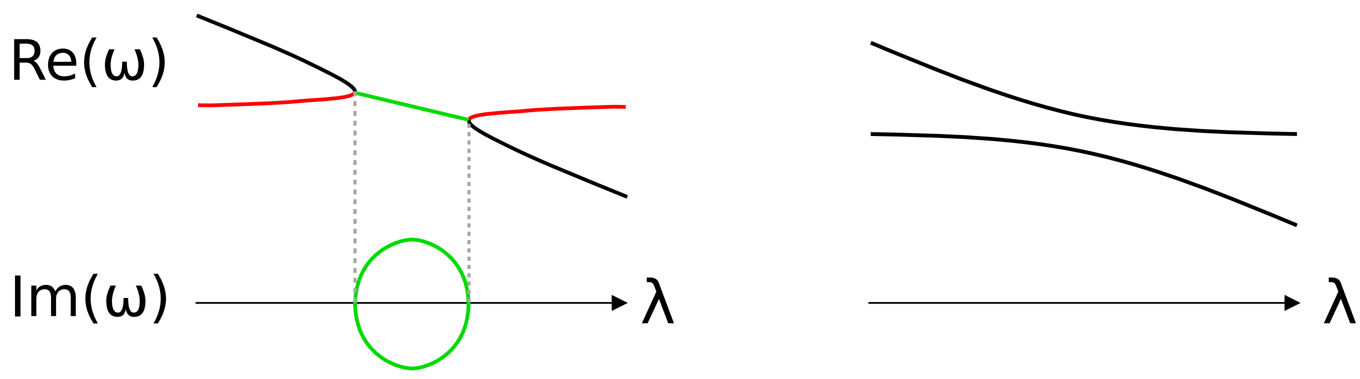
$$E_i = \langle\psi_i|\,\sigma_3\,|\psi_i\rangle\,\epsilon_i$$

  - Negative norm modes with positive frequency have negative energy
  - Positive and negative norm modes with the same eigenvalue can be created at zero energy cost

Pseudo-degenerate modes and instability

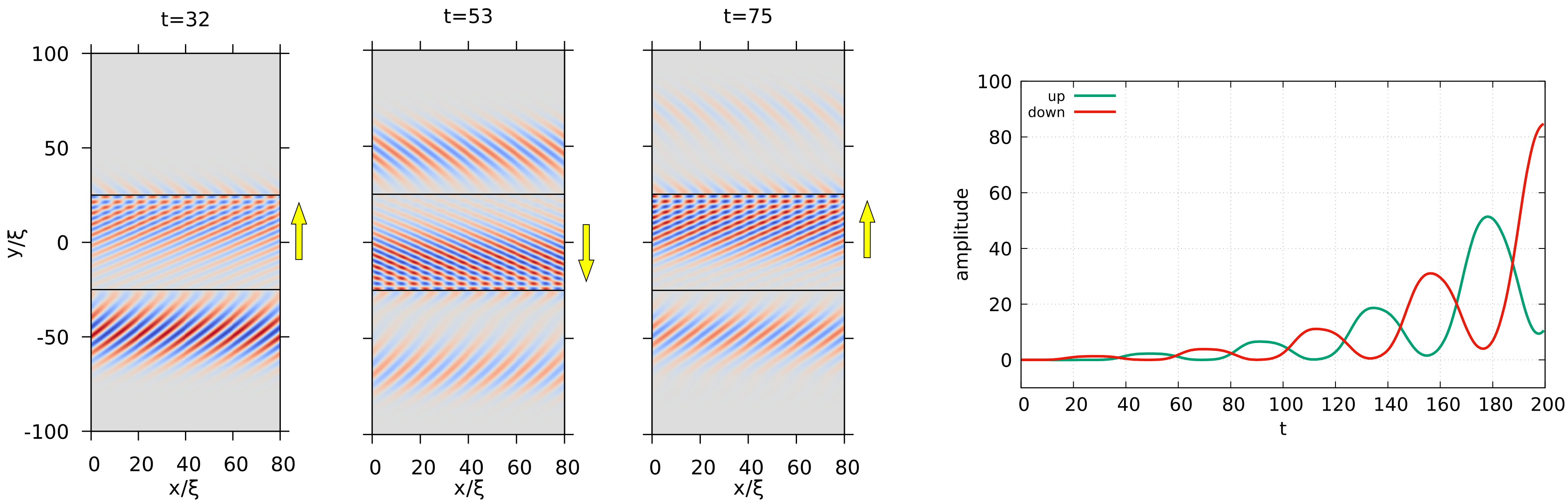
$$i\hbar\partial_t \begin{pmatrix}\psi \\ \psi^*\end{pmatrix} = \begin{bmatrix} H_{GP} + g|\Psi_0|^2 & g\Psi_0^2 \\ -g\Psi_0^{*2} & -\left(H_{GP} + g|\Psi_0|^2\right)^* \end{bmatrix} \begin{pmatrix}\psi \\ \psi^*\end{pmatrix}$$

- The problem is not hermitian $\Longrightarrow$ complex eigenvalues
- For eigenvectors: $(\epsilon_i - \epsilon_j^*) \langle\psi_j| \sigma_3 |\psi_i\rangle = 0$
  - Complex frequency modes have zero norm
- When two modes of opposite norm have the same frequency they become pseudo-degenerate $\epsilon_j = \epsilon_i^*$



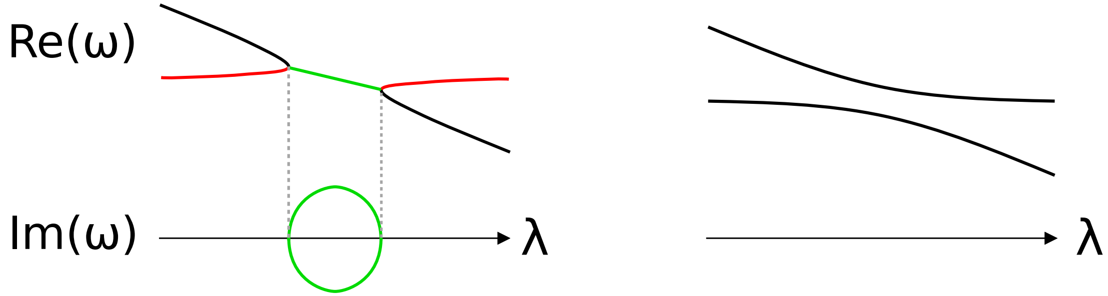
2D GPE simulations

- The amplification does not depend on the direction in which the barrier is crossed



- Trapped negative norm mode causing dynamical instability

- Dynamical instabilities emerge in the spectrum as zero-norm modes



The 2D Bogoliubov problem

$$\Psi_0(r,\theta) = e^{i\ell\theta} f(r)\ ; \qquad \begin{pmatrix} \delta\psi(r,\theta) \\ \delta\psi^*(r,\theta) \end{pmatrix} = e^{im\theta} \begin{pmatrix} e^{i\ell\theta}\phi(r) \\ e^{-i\ell\theta}\phi^*(r) \end{pmatrix}$$

$$i\hbar\partial_t \begin{pmatrix} \phi \\ \phi^* \end{pmatrix} = \begin{bmatrix} D_+ + V_{ext} - \mu + 2g|\Psi_0|^2 & g|\Psi_0|^2 \\ -g|\Psi_0|^2 & -\left(D_- + V_{ext} - \mu + 2g|\Psi_0|^2\right) \end{bmatrix} \begin{pmatrix} \phi \\ \phi^* \end{pmatrix}$$

$$D_\pm = \frac{\hbar^2}{2M}\left(-\partial_r^2 - \frac{\partial_r}{r} + \frac{(\ell \pm m)^2}{r^2}\right)$$

- Particle-antiparticle symmetry: $$\begin{pmatrix} u_{m,i} \\ v_{m,i} \end{pmatrix} \longleftrightarrow \begin{pmatrix} u_{-m,j} \\ v_{-m,j} \end{pmatrix} = \begin{pmatrix} v_{m,i} \\ u_{m,i} \end{pmatrix}$$

$$\epsilon_{m,i} \quad \longleftrightarrow \quad \epsilon_{-m,j} = -\epsilon_{m,i}$$

- The different m-s are decoupled, all the modes at fixed m are independent and the the spectrum at -m is just specular