MODIFIED UNRUH EFFECT FROM GENERALIZED UNCERTAINTY PRINCIPLE

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Generalized Uncertainty Principles (GUPs)

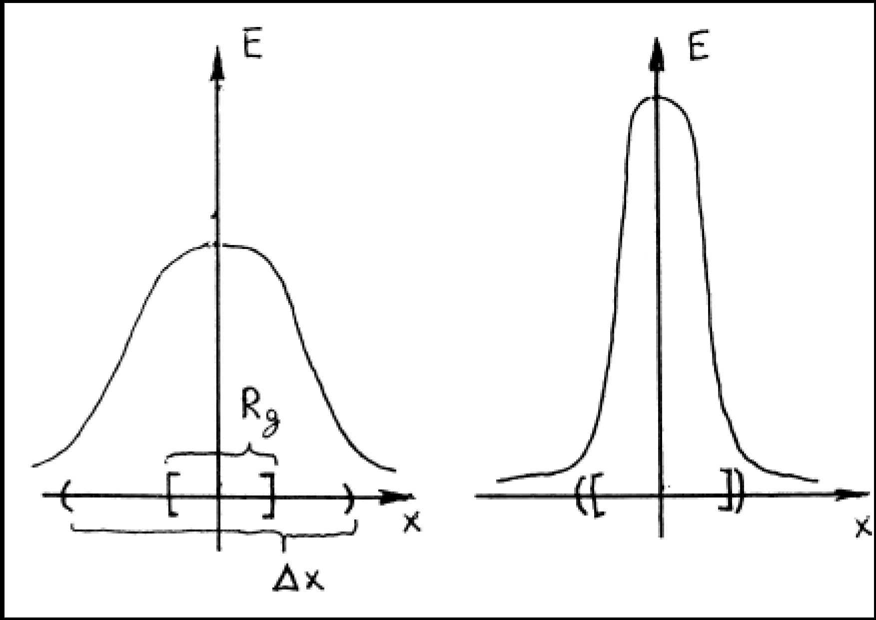
- Research on generalizations of the Heisenberg uncertainty principle has several decades of history

(Bronstein, 1935 - C.N. Yang, 1947 - H.Snyder, 1947 - C.A. Mead, 1964 - F. Karolyhazy, 1966).

- Last 30 years: string theory (Veneziano 1987, Gross 1987) suggested that, in gedanken experiments on high energy string scatterings involving Gravity, the "effective" uncertainty relation should read

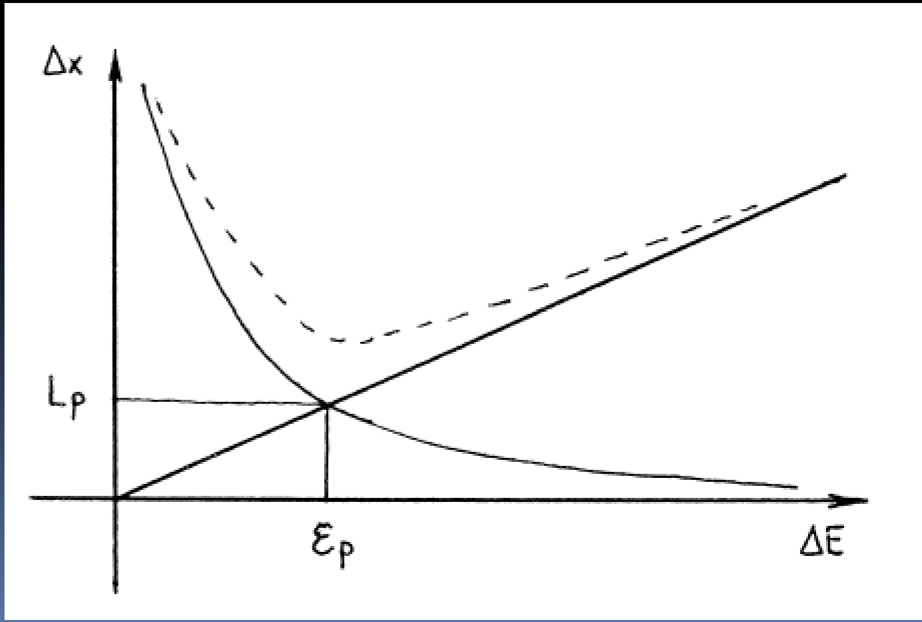
$$\Delta x \geq \frac{\hbar}{2\Delta p} + 2\beta\,\ell_{4n}^2\,\frac{\Delta p}{\hbar},$$

Gedanken Experiments on **formation of MicroBlack Holes** (Scardigli, Chen, Adler 1999) yields a similar relation



$$\Delta x \geq \begin{cases} \dfrac{\hbar c}{2\,\Delta E} & \text{for } \Delta E \leq \epsilon_P \\ \dfrac{2 G \Delta E}{c^4} & \text{for } \Delta E > \epsilon_P \end{cases}$$

$$\Delta x \geq \frac{\hbar c}{2\,\Delta E} + \frac{2 G \Delta E}{c^4}$$



Also: M.Maggiore (1993) arrives to similar relation via a Gedanken experiment on the Hawking radiation emitted by **Large Black Holes**

Kempf, Mann, Brau, Vagenas, etc. (1995, 1999, 2008), translated GUP into a **deformed commutator**

$$[\hat{x}, \hat{p}] = i\hbar \left[1 + \beta \left(\frac{\hat{p}}{m_{\mathrm{p}}} \right)^2 \right]$$

and developed a Deformed Quantum Mechanics.

The dawn of Unruh Effect

"…an accelerated detector even in flat spacetime will detect particles in the vacuum… This result is exactly what one would expect of a detector immersed in a thermal bath of scalar photons of temperature $T_U = \dfrac{\hbar a}{2\pi c k_B}$"

(W.Unruh, PRD 1976).

Unruh Effect: heuristic derivation from HUP

Consider a particle of mass m at rest in an uniformly accelerated frame. The kinetic energy it acquires while the frame moves for a distance δx is

$$E_k = m\,a\,\delta x$$

Suppose this energy is just enough to create N pairs of the same kind of particles from the quantum vacuum (i.e. $E_k \sim 2Nm$, c=1), then the needed distance is $\delta x \sim 2N/a$

Particles and pairs so created are localized inside a space region of width δx, therefore the fluctuation in energy of each single particle is (HUP)

$$\delta E \simeq \frac{\hbar}{2\,\delta x} \simeq \frac{\hbar\,a}{4\,N}$$

If we interpret δE as a classical thermal energy of the particles (i.e. $\frac{3}{2}k_B T \sim \delta E$), then

$$T = \frac{\hbar\,a}{6\,N\,k_{\mathrm{B}}} = T_{\mathrm{U}}$$

for $\mathrm{N} = \frac{\pi}{3} \sim 1$

Modified Unruh effect from GUP: heuristic derivation

We can repeat the same steps, using GUP instead of HUP

$$\delta x \simeq \frac{\hbar}{2\,\delta E} + 2\,\beta\,\ell_{\mathrm{p}}^2\,\frac{\delta E}{\hbar}$$

So we have

$$\delta x \simeq 2\,\frac{N}{a}, \;\; \delta E \simeq \frac{1}{2}\,k_{\mathrm{B}}\,T \quad \Longrightarrow \quad \frac{2\,\pi}{a} \simeq \frac{\hbar}{k_{\mathrm{B}}\,T} + \beta\,\ell_{\mathrm{p}}^2\,\frac{k_{\mathrm{B}}\,T}{\hbar}$$

Inverting the last for T=T(a), and expanding for $\beta\,k_{\mathrm{B}}\,T/m_{\mathrm{p}} \sim \beta\,m/m_{\mathrm{p}} \ll 1$ we get the GUP-modified Unruh temperature:

$$T \simeq T_{\mathrm{U}}\left(1 + \frac{\beta}{4}\,\frac{\ell_{\mathrm{p}}^2\,a^2}{\pi^2}\right) = T_{\mathrm{U}}\left[1 + \frac{\beta}{4}\left(\frac{k_{\mathrm{B}}\,T_{\mathrm{U}}}{m_{\mathrm{p}}}\right)^2\right]$$

Unruh effect from canonical QFT

Quantization of a scalar field in Minkowski space

Plane wave expansion

$$\phi(\mathbf{x}) = \int dk \left[a_k U_k(\mathbf{x}) + a_k^\dagger U_k^*(\mathbf{x}) \right]$$

$$[a_k, a_{k'}^\dagger] = \delta(k - k')$$

$$a_k |0\rangle_{\mathrm{M}} = 0 \quad \forall k$$

$$U_k(\mathbf{x}) = (4\pi\omega_k)^{-\frac{1}{2}} e^{i(k\,x - \omega_k\,t)}$$

$$\omega_k = \sqrt{m^2 + k^2}$$

Lorentz-boost mode expansion (Takagi 1986)

$$\tilde{U}_\Omega^{(\sigma)}(\mathbf{x}) = \int dk\, p_\Omega^{(\sigma)*}(k)\, U_k(\mathbf{x})$$

$$p_\Omega^{(\sigma)}(k) = \frac{1}{\sqrt{2\pi\,\omega_k}} \left(\frac{\omega_k + k}{\omega_k - k} \right)^{i\,\sigma\,\Omega/2}$$

$$\phi(\mathbf{x}) = \int_0^{+\infty} d\Omega \sum_{\sigma=\pm} \left[d_\Omega^{(\sigma)} \tilde{U}_\Omega^{(\sigma)}(\mathbf{x}) + d_\Omega^{(\sigma)\dagger} \tilde{U}_\Omega^{(\sigma)*}(\mathbf{x}) \right]$$

Link boost-mode operators with Minkowski annihilators

$$d_\Omega^{(\sigma)} = \int dk \, p_\Omega^{(\sigma)}(k)\, a_k$$

In these operators the Lorentz boost generator is diagonal

$$M^{(1,0)} = \int d^3\kappa \sum_\sigma \sigma\Omega \left(d_\kappa^{(\sigma)\dagger} d_\kappa^{(\sigma)} + \bar{d}_\kappa^{(\sigma)\dagger} \bar{d}_\kappa^{(\sigma)} \right)$$

Canonical commutators

$$\left[a_k, a_{k'}^\dagger\right] = \delta(k - k') \Longleftrightarrow \left[d_\Omega^{(\sigma)}, d_{\Omega'}^{(\sigma')\dagger}\right] = \delta_{\sigma\sigma'}\delta(\Omega - \Omega')$$

Therefore also $d_\Omega^{(\sigma)}$ are annihilation operators of Minkowski quanta

$$d_\kappa^{(\sigma)} |0\rangle_M = 0$$

The two different field-expansions are equivalent for inertial observers in Minkowski spacetime.

Rindler – Fulling – Unruh quantization in uniformly accelerating frame. Rindler space:

- **Rindler coordinates**

$$t = \xi \sinh \eta, \quad x = \xi \cosh \eta$$

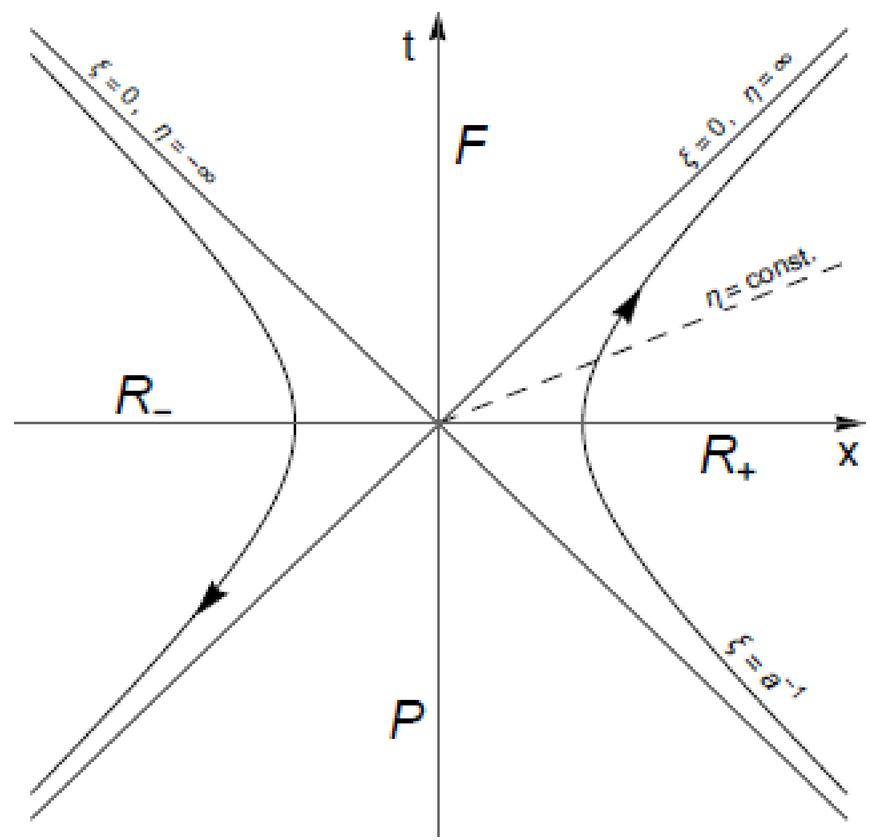
- Rindler vs Minkowski metrics

$$ds^2_{\mathrm{M}} = (dt)^2 - (dx)^2 \rightarrow$$

$$ds^2_{\mathrm{R}} = \xi^2 d\eta^2 - d\xi^2$$

- Worldline of a Rindler observer

$$\eta = a\tau, \quad \xi = \mathrm{const} \equiv a^{-1}$$



For c≠1, $\tau=\eta c/a$, $\eta=a\tau/c$ (dimensionless)

Worldline of uniformly accelerated Rindler observer $\xi(\tau) = c^2/a$

Quantization of the scalar field in Rindler space

$$\phi(\mathbf{x}) = \int_0^{+\infty} d\Omega \sum_{\sigma=\pm} \left[b_\Omega^{(\sigma)} u_\Omega^{(\sigma)}(\mathbf{x}) + b_\Omega^{(\sigma)\dagger} u_\Omega^{(\sigma)*}(\mathbf{x}) \right]$$

$\boldsymbol{\phi}$ expanded on the positive frequency solutions of the KG equation in Rindler coordinates is $u_\Omega^{(\sigma)}(\mathbf{x}) = N_\Omega\, \theta(\sigma\,\xi)\, K_{i\,\Omega}^{(\sigma)}(m\,\xi)\, e^{-i\,\sigma\,\Omega\,\eta}$ (Takagi 1986)

with: proper frequency ω measured by a Rindler observer is linked to Ω by ($c\neq1$)

$$\omega\,\tau = \omega(\eta c/a) = (\omega c/a)\eta \equiv \Omega\,\eta$$

$\omega = a\Omega/c$, $\sigma = \pm$ refers to the right/left wedges $R_\pm$

$K_{i\Omega}$ = modified Bessel function of the second kind, N_Ω = normalization factor.

The ladder operators satisfy the canonical commutation relations

$$\left[b_\Omega^{(\sigma)}, b_{\Omega'}^{(\sigma')\dagger} \right] = \delta_{\sigma\sigma'}\, \delta(\Omega - \Omega')$$

The Rindler vacuum is accordingly defined as $b_\Omega^{(\sigma)}|0\rangle_{\mathrm{R}} = 0$ for all σ and Ω.

Connection between inertial and accelerated observers

Minkowski (boost-mode) quantization:

$$\phi = \int d\Omega \sum_{\sigma=\pm} \left[d_\Omega^{(\sigma)} \, \widetilde{U}_\kappa^{(\sigma)} + d_\Omega^{(\sigma)\dagger} \, \widetilde{U}_\kappa^{(\sigma)*} \right],$$

Rindler quantization:

$$\phi = \int d\Omega \sum_{\sigma=\pm} \left[b_\Omega^{(\sigma)} \, u_\Omega^{(\sigma)} + b_\Omega^{(\sigma)\dagger} \, u_\Omega^{(\sigma)*} \right]$$

↘ ↙

Bogoliubov transformation

$$b_\Omega^{(\sigma)} = [1 + \mathcal{N}(\Omega)]^{1/2} \, d_\Omega^{(\sigma)} + \mathcal{N}(\Omega)^{1/2} \, d_\Omega^{(-\sigma)\dagger}$$

with B.E. distribution

$$\mathcal{N}(\Omega) = \frac{1}{e^{2\pi\Omega} - 1}$$

Spectrum of Rindler quanta in the Minkowski vacuum:

$$\langle 0_\mathrm{M} | \, b_\Omega^{(\sigma)\dagger} \, b_{\Omega'}^{(\sigma')} \, | 0_\mathrm{M} \rangle = \mathcal{N}(\Omega) \, \delta_{\sigma\sigma'} \, \delta(\Omega - \Omega')$$

Uniformly accelerated observer perceives Minkowski vacuum as a **thermal bath** of Rindler quanta with a **temperature proportional to the acceleration**.

UNRUH TEMPERATURE

$$2\pi\,\Omega = \frac{2\pi}{a} \, a\,\Omega = \frac{\hbar\, a\, \Omega}{k_\mathrm{B}\, T_\mathrm{U}} = \frac{\hbar\,\omega}{k_\mathrm{B}\, T_\mathrm{U}}$$

$$T_U = \frac{a}{2\pi} = \frac{\hbar a}{2\pi c k_B}$$

Modified Unruh effect from GUP: QFT derivation

GUP and one-dimensional quantum harmonic oscillator

$$A = \frac{1}{\sqrt{2\,m\,\hbar\,\omega}}\left(m\omega\hat{x} + i\,\hat{p}\right)$$
$$A^{\dagger} = \frac{1}{\sqrt{2\,m\,\hbar\,\omega}}\left(m\omega\hat{x} - i\,\hat{p}\right)$$

$$\hat{x} = \sqrt{\frac{\hbar}{2\,m\,\omega}}(A^{\dagger} + A)$$
$$\hat{p} = i\,\sqrt{\frac{m\,\hbar\,\omega}{2}}(A^{\dagger} - A)$$

Then $[A, A^{\dagger}] = \frac{1}{i\hbar}[\hat{x}, \hat{p}]$ and from $[\hat{x}, \hat{p}] = i\hbar\left[1 + \beta\left(\frac{\hat{p}}{m_{\rm p}}\right)^2\right]$

we get the deformed algebra of the one dimensional harmonic oscillator

$$[A, A^{\dagger}] = \frac{1}{1-\alpha}[1 - \alpha(A^{\dagger}\,A^{\dagger} + A\,A - 2\,A^{\dagger}\,A)]$$

with

$$\alpha = \beta\,\frac{m\,\hbar\,\omega}{2\,m_{\rm p}^2}$$

* Deformed commutator for scalar field in plane-wave representation

Since for a given momentum k, the energy $\hbar\omega_k$ of the scalar field plays the role of the mass ***m*** of the harmonic oscillator, **then**

$$\alpha = \beta \frac{m\,\hbar\,\omega}{2\,m_p^2} \;\longrightarrow\; \tilde{\alpha} = \beta \frac{\hbar^2\omega_k^2}{2\,m_p^2} = 2\,\beta\,\ell_p^2\,\omega_k^2$$

And the deformed commutator becomes:

$$[A_k, A_{k'}^\dagger] = \frac{1}{1-\tilde{\alpha}}\left[1 - \tilde{\alpha}\left(A_k^\dagger A_{k'}^\dagger + A_k A_{k'} - 2A_k^\dagger A_{k'}\right)\right]\delta(k-k')$$

* Deformed commutator for scalar field in boost-mode representation

In the limit of small deformation ($\beta p^2/m_p^2 \ll 1$), is reasonable to assume the same structure for the deformed algebra of the boost modes

$$\left[D_\Omega^{(\sigma)}, D_{\Omega'}^{(\sigma')\dagger}\right] = \frac{1}{1-\gamma}\left[1 - \gamma\left(D_\Omega^{(\sigma)\dagger} D_{\Omega'}^{(-\sigma')\dagger} + D_\Omega^{(\sigma)} D_{\Omega'}^{(-\sigma')} - D_\Omega^{(\sigma)\dagger} D_{\Omega'}^{(\sigma')} - D_\Omega^{(-\sigma)\dagger} D_{\Omega'}^{(-\sigma')}\right)\right]\delta_{\sigma\sigma'}\,\delta(\Omega-\Omega')$$

where now

$$\gamma = \beta\frac{\hbar^2\omega^2}{2\,m_p^2} = \beta\frac{\hbar^2 a^2\,\Omega^2}{2\,m_p^2} = 2\,\beta\,\ell_p^2\,a^2\,\Omega^2$$

being $\omega = a\Omega$ (c=1) the Rindler proper frequency

Note that

- The deforming parameter $\alpha \sim \longrightarrow \gamma$ has been defined to suit the boost mode representation

  (i.e. plane frequency $\omega_k \rightarrow \omega = a\Omega$ boost mode frequency)
- The D-commutator has been defined so that the ladder operators D in the wedges R+, R- are still commuting with each other.

The D-commutator has been symmetrized with respect to $\pm\sigma$ so that

$$\left[D_{\Omega}^{(\sigma)}, D_{\Omega'}^{(\sigma')\dagger}\right] = \left[D_{\Omega}^{(-\sigma)}, D_{\Omega'}^{(-\sigma')\dagger}\right]$$

The deformation of the D-algebra leads to an analogous modification of the commutator of the Rindler B-operators.

The Bogoliubov transformation between B and D is now

$$B_{\Omega}^{(\sigma)} = [1 + \mathcal{N}(\Omega)]^{1/2} D_{\Omega}^{(\sigma)} + \mathcal{N}(\Omega)^{1/2} D_{\Omega}^{(-\sigma)\dagger}$$

GUP effect on the Unruh temperature

Distribution of B-quanta in the Minkowski vacuum

$$\langle 0_{\rm M}|\, B^{(\sigma)\dagger}_{\Omega}\, B^{(\sigma')}_{\Omega'}\, |0_{\rm M}\rangle = \frac{1}{\left(e^{2\pi\Omega}-1\right)\left(1-\gamma\right)}\,\delta_{\sigma\sigma'}\,\delta(\Omega-\Omega') \simeq \frac{1}{e^{2\pi\Omega-\gamma}-1}\,\delta_{\sigma\sigma'}\,\delta(\Omega-\Omega')$$

This can be interpreted as a B-E thermal distribution with a **SHIFTED UNRUH TEMPERATURE** such that

$$2\pi\Omega-\gamma = \frac{\hbar\, a\,\Omega}{k_{\rm B}\, T_{\rm U}}-\gamma \equiv \frac{\hbar\, a\,\Omega}{k_{\rm B}\, T}$$

$$T = \frac{T_{\rm U}}{1-\beta\,\pi\,\Omega\, k_{\rm B}^2\, T_{\rm U}^2/m_{\rm p}^2} \simeq T_{\rm U}\left(1+\beta\,\pi\,\Omega\left(\frac{k_{\rm B}\, T_{\rm U}}{m_{\rm p}}\right)^2\right) = T_{\rm U}\left(1+\beta\,\pi\,\Omega\,\frac{\ell_{\rm p}^2\, a^2}{\pi^2}\right)$$

Remark: T contains an **explicit dependence** on the Rindler frequency $\Omega = \omega/a$

Expected, since the fundamental commutator now depends on p^2 , i.e. on the energy of the considered quantum mode.

Thermodynamic argument (to get rid of Ω):

Small deformations of HUP $\Longrightarrow$ The modified Unruh radiation is still close to thermal black body spectrum $\Longrightarrow$ The majority of Unruh quanta are emitted around the Rindler frequency ω such that $\hbar\omega = k_B T_U \Longrightarrow \Omega \approx 1/2\pi$. For this typical frequency

$$T \simeq T_{\mathrm{U}} \left[1 + \frac{\beta}{2} \left(\frac{k_{\mathrm{B}} T_{\mathrm{U}}}{m_{\mathrm{p}}} \right)^2 \right] = T_{\mathrm{U}} \left(1 + \frac{\beta}{2} \frac{\ell_{\mathrm{p}}^2 a^2}{\pi^2} \right)$$

...which equals, almost numerically, with the heuristic result

$$T \simeq T_{\mathrm{U}} \left(1 + \frac{\beta}{4} \frac{\ell_{\mathrm{p}}^2 a^2}{\pi^2} \right) = T_{\mathrm{U}} \left[1 + \frac{\beta}{4} \left(\frac{k_{\mathrm{B}} T_{\mathrm{U}}}{m_{\mathrm{p}}} \right)^2 \right]$$

Conclusions and outlook

We investigated

- Deviation from thermality of Unruh radiation in the context of GUP
- Small deformations of HUP $\Longrightarrow$ The resulting Unruh distribution still exhibits a thermal spectrum with a modified temperature $$T \simeq T_{\rm U}\left(1 + \beta\, \mathcal{O}(a^2)\right)$$
- Good agreement between the heuristic and the field theoretical approaches
- Possible measures in analogue models: only in models with an effective "m_p" SMALL , and an effective "β" LARGE.
- More formally: What happens beyond the approximation of equal deformed algebras for the A- and D- operators? (Deformation of algebra for field operators)