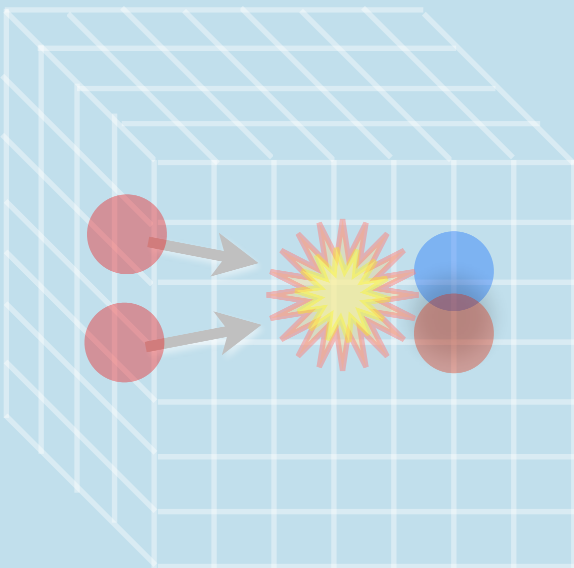
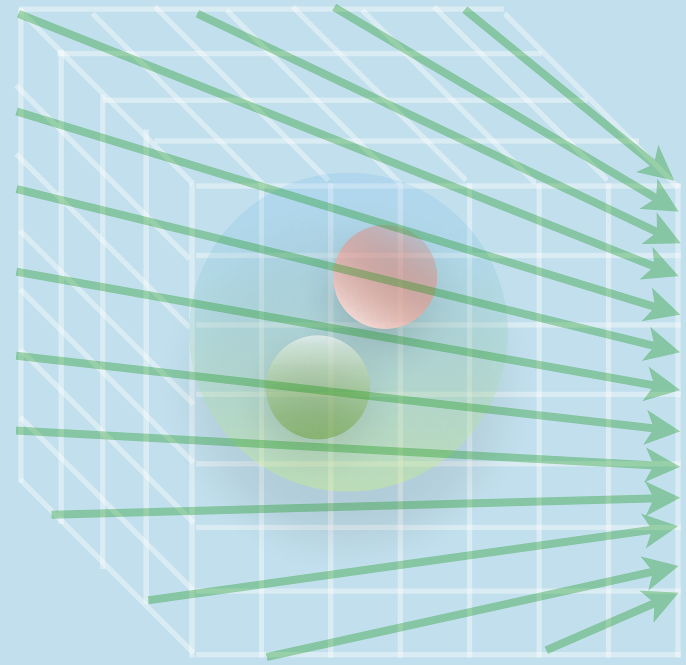
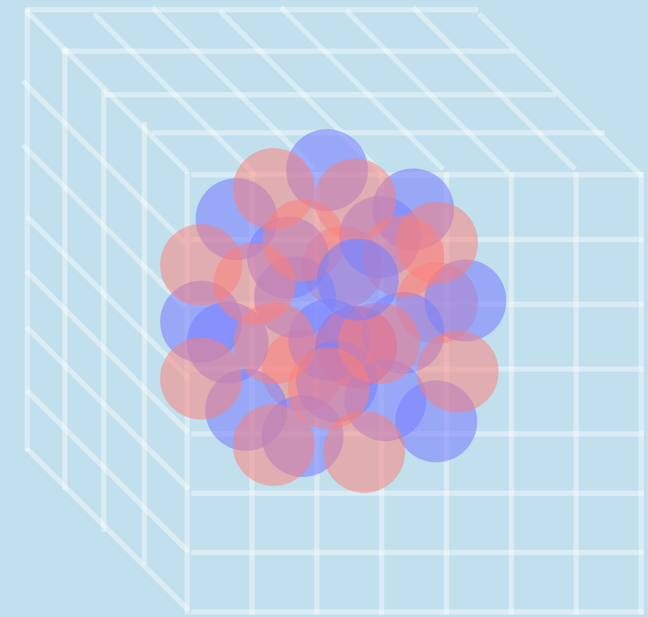
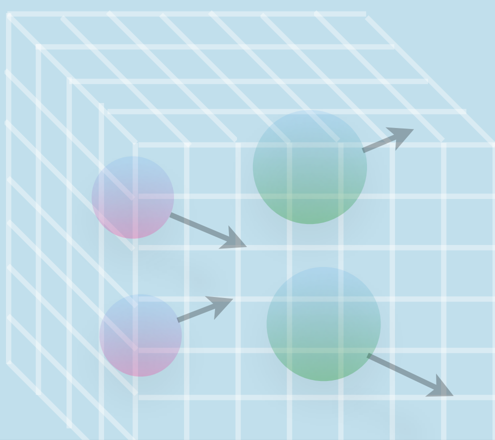
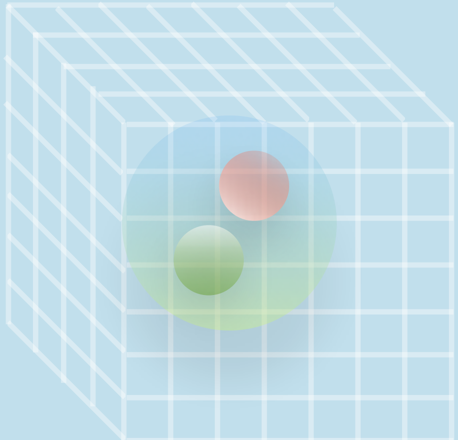






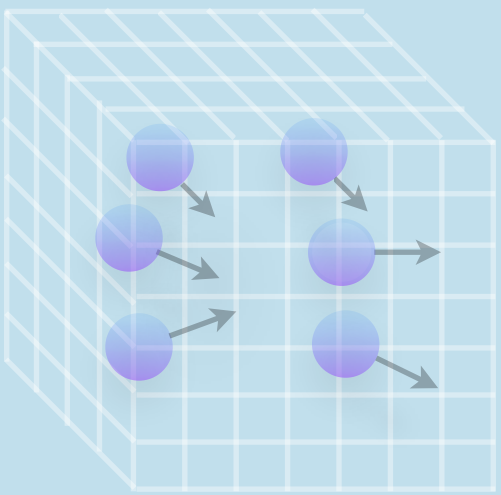


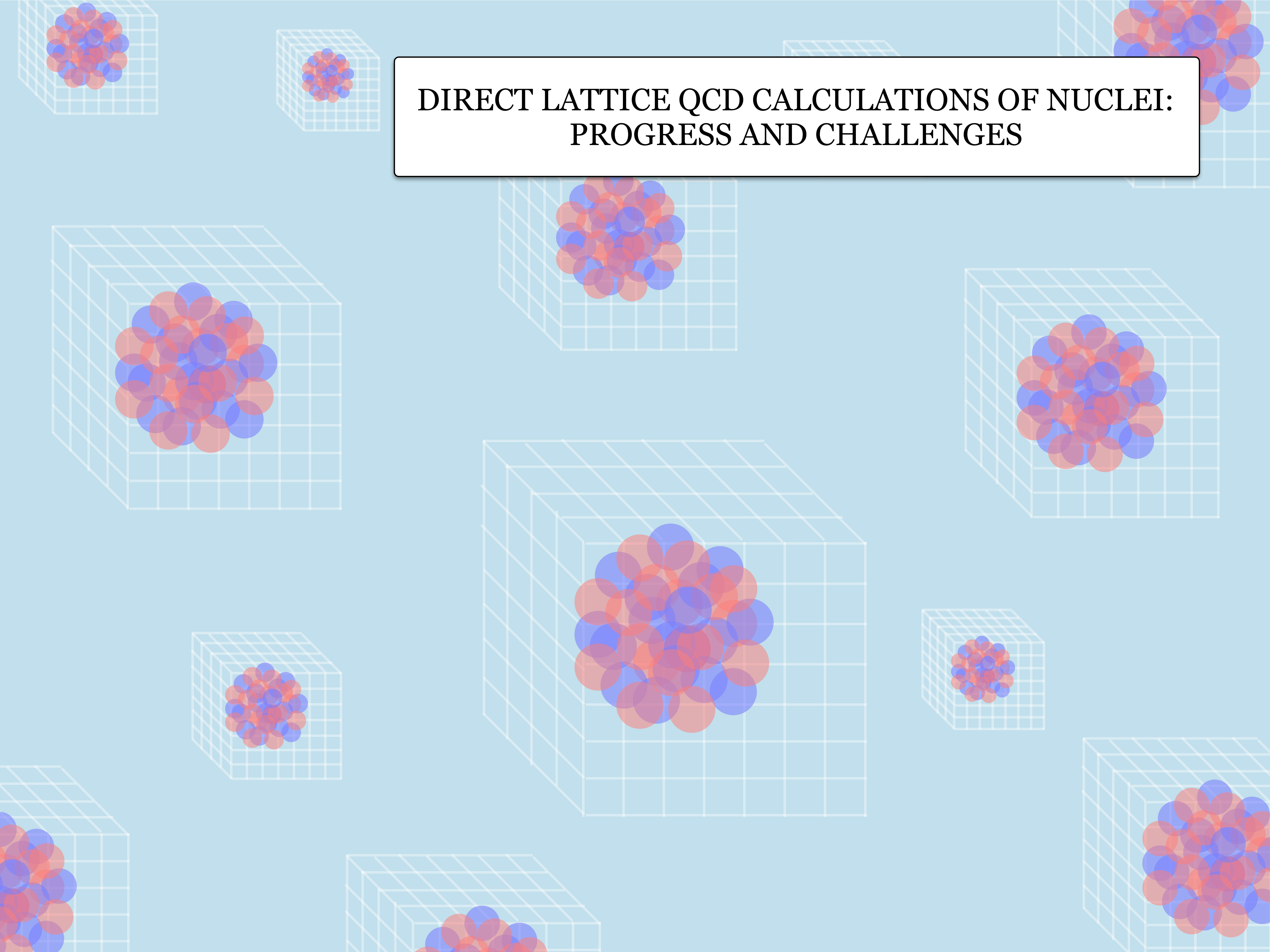


TALENT COURSE ON
FROM QUARKS AND GLUONS TO NUCLEAR FORCES AND STRUCTURE

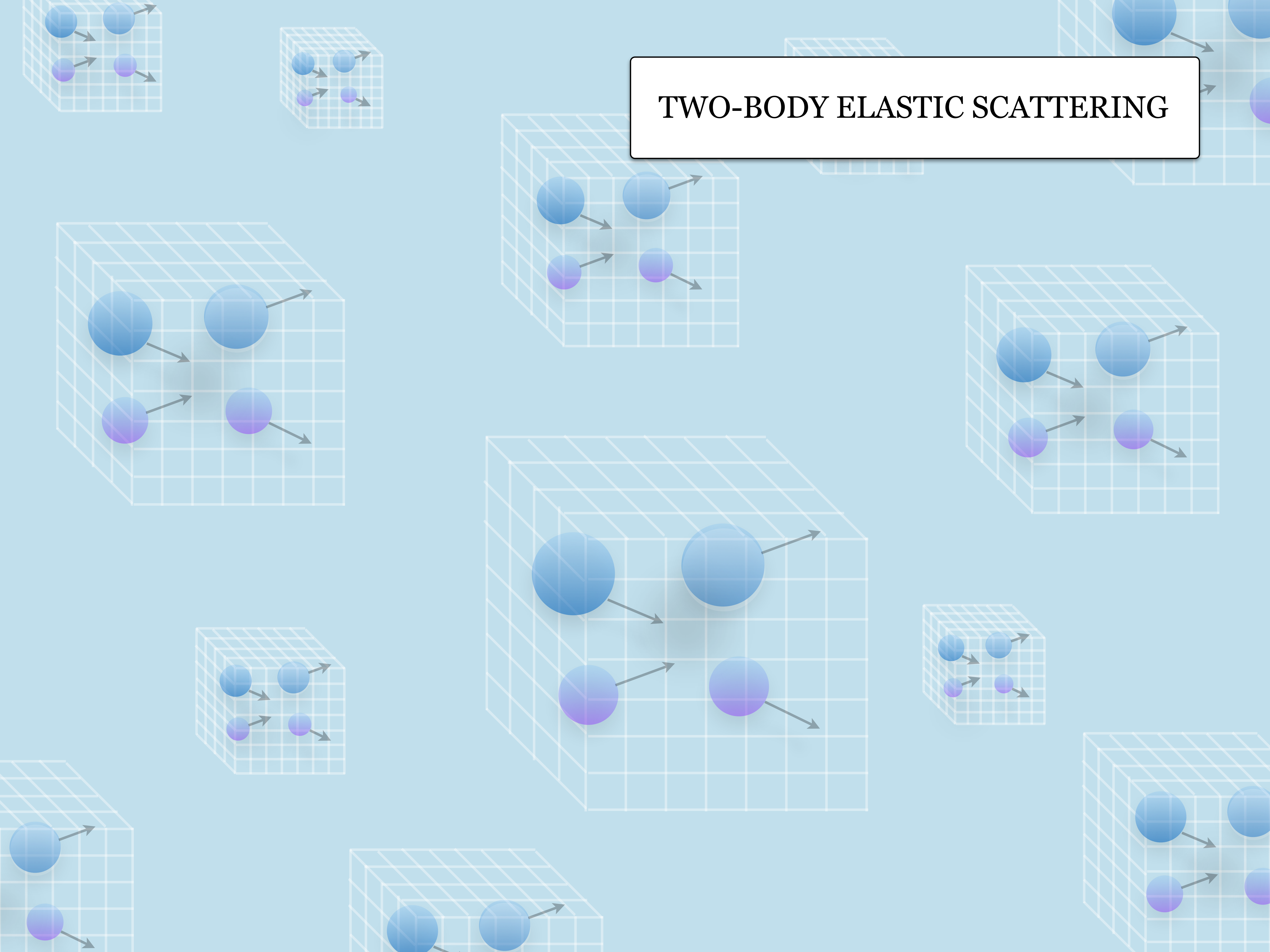
# LATTICE QCD
AND
# MULTI-HADRON PHYSICS

ZOHREH DAVOUDI
UNIVERSITY OF MARYLAND





# DIRECT LATTICE QCD CALCULATIONS OF NUCLEI: PROGRESS AND CHALLENGES

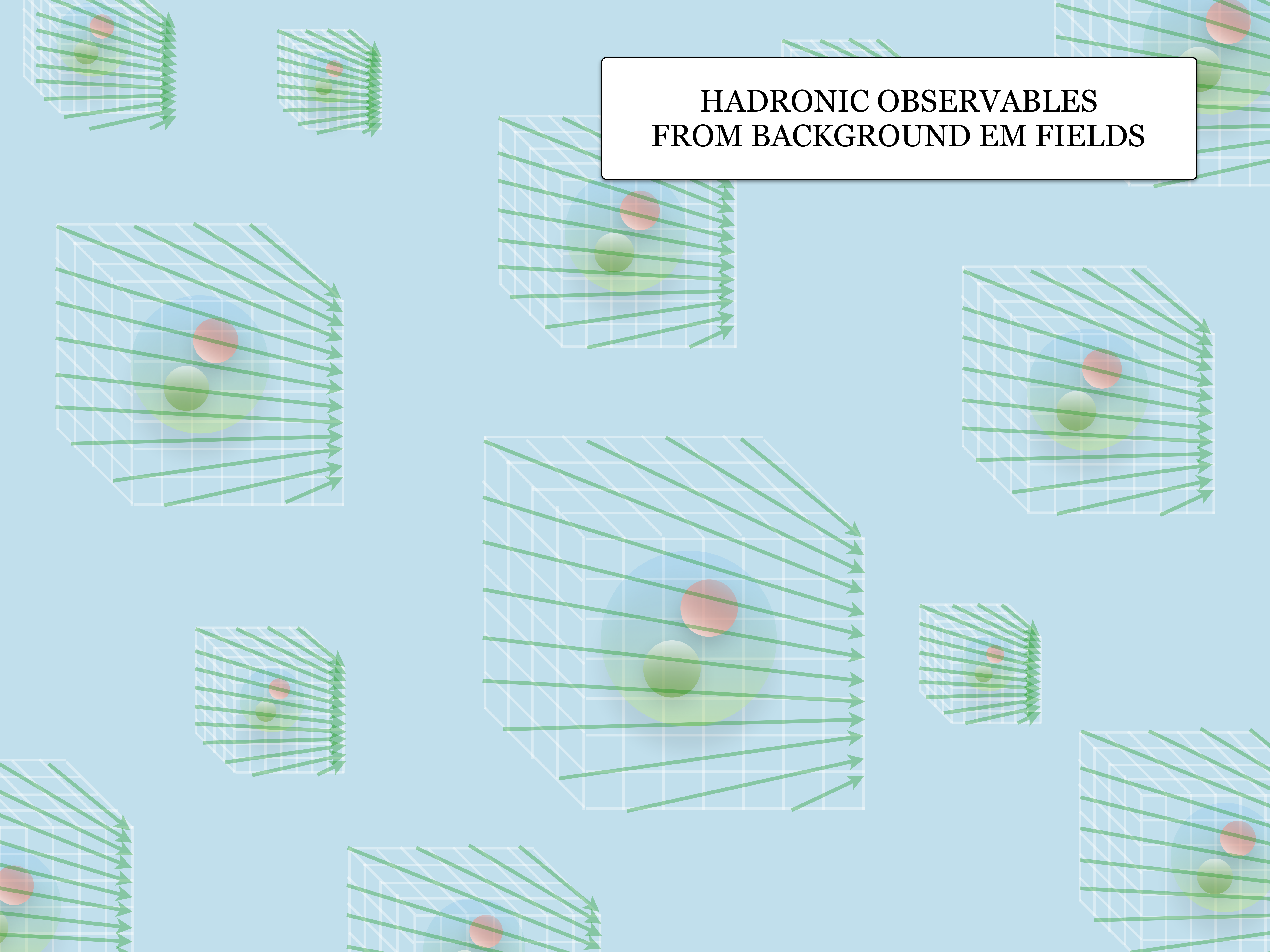


# TWO-BODY ELASTIC SCATTERING

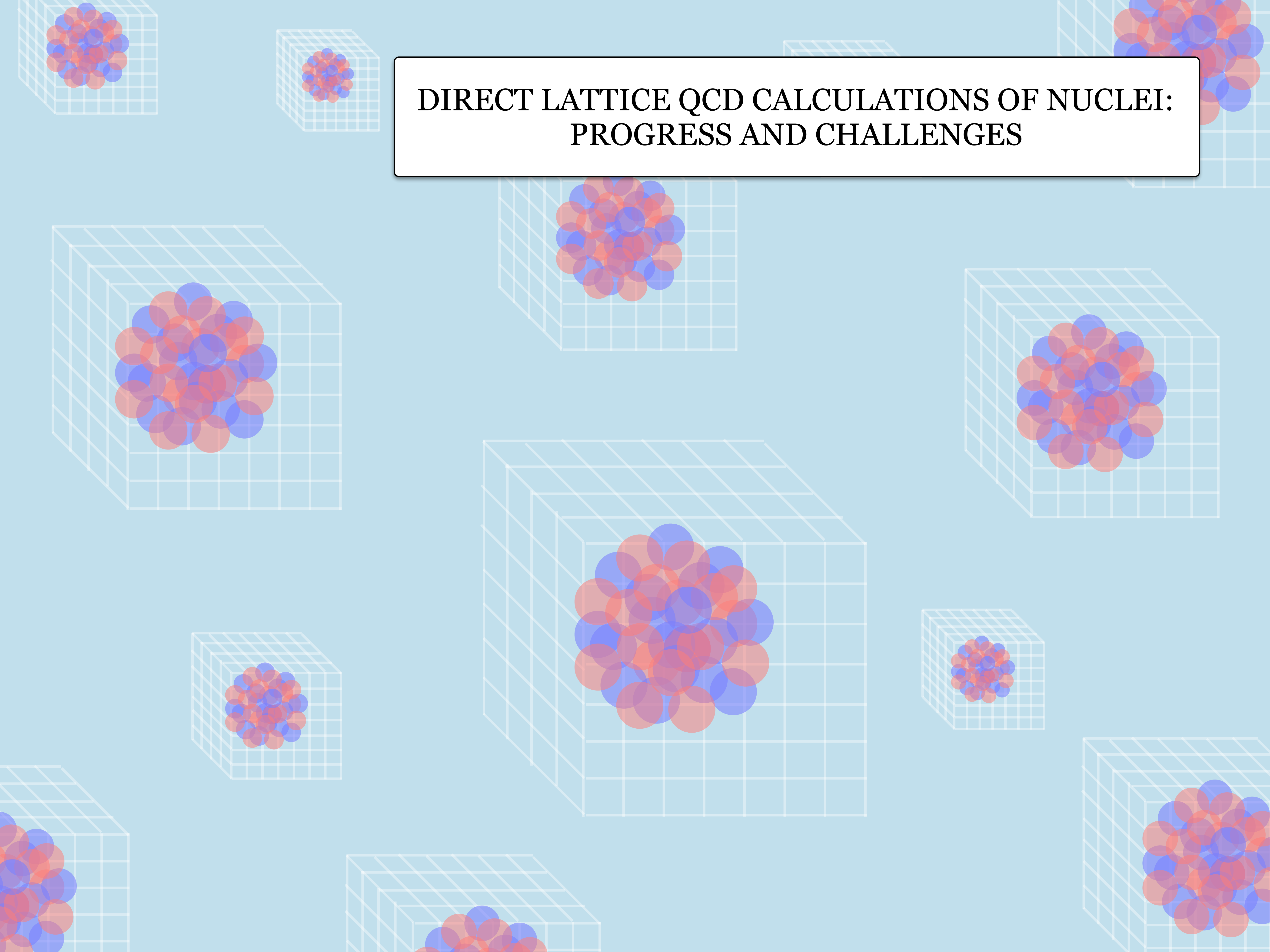
# TWO-BODY INELASTIC SCATTERING

# TWO-NUCLEON OBSERVABLES

# HADRONIC OBSERVABLES FROM BACKGROUND EM FIELDS



# HADRONIC OBSERVABLES FROM THREE-POINT FUNCTIONS



# DIRECT LATTICE QCD CALCULATIONS OF NUCLEI: PROGRESS AND CHALLENGES

# THREE FEATURES MAKE LQCD CALCULATIONS OF NUCLEI HARD:

i) THE COMPLEXITY OF SYSTEMS GROWS RAPIDLY WITH THE NUMBER OF QUARKS.

Detmold and Orginos, Phys. Rev. D 87, 114512 (2013).

See also: Detmold and Savage, Phys.Rev.D82 014511 (2010). Doi and Endres, Comput. Phys. Commun. 184 (2013) 117.

ii) EXCITATION ENERGIES OF NUCLEI ARE MUCH SMALLER THAN THE QCD SCALE.

Beane at al (NPLQCD), Phys.Rev.D79 114502 (2009).
Beane, Detmold, Orginos, Savage, Prog. Part. Nucl. Phys. 66 (2011).
Junnakar and Walker-Loud, Phys.Rev. D87 (2013) 114510.
Briceno, Dudek and Young, Rev. Mod. Phys. 90 025001.

iii) THERE IS A SEVERE SIGNAL-TO-NOISE DEGRADATION.

Paris (1984) and Lepage (1989).

Wagman and Savage, Phys. Rev. D 96, 114508 (2017).
Wagman and Savage, arXiv:1704.07356 [hep-lat].

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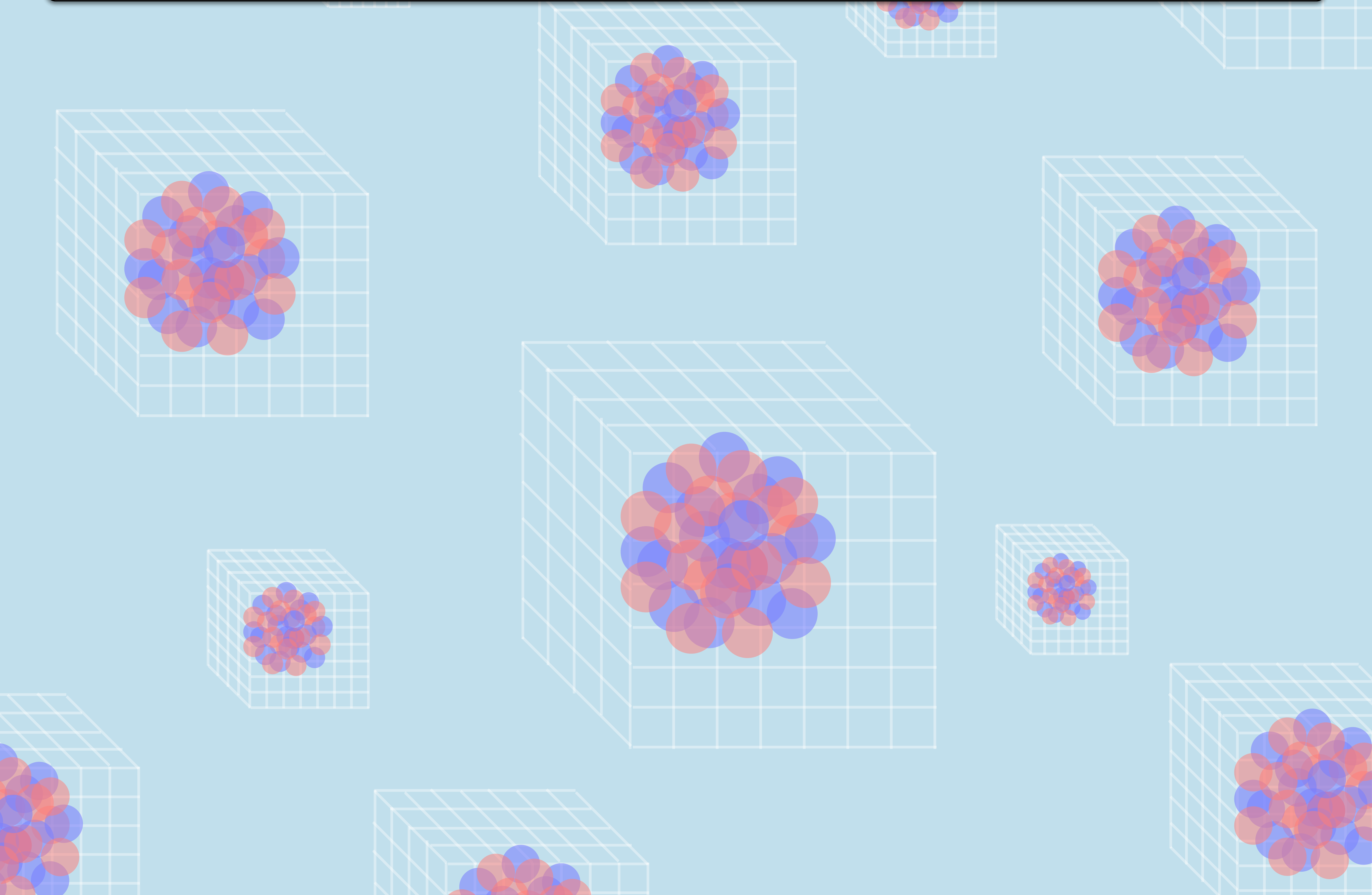
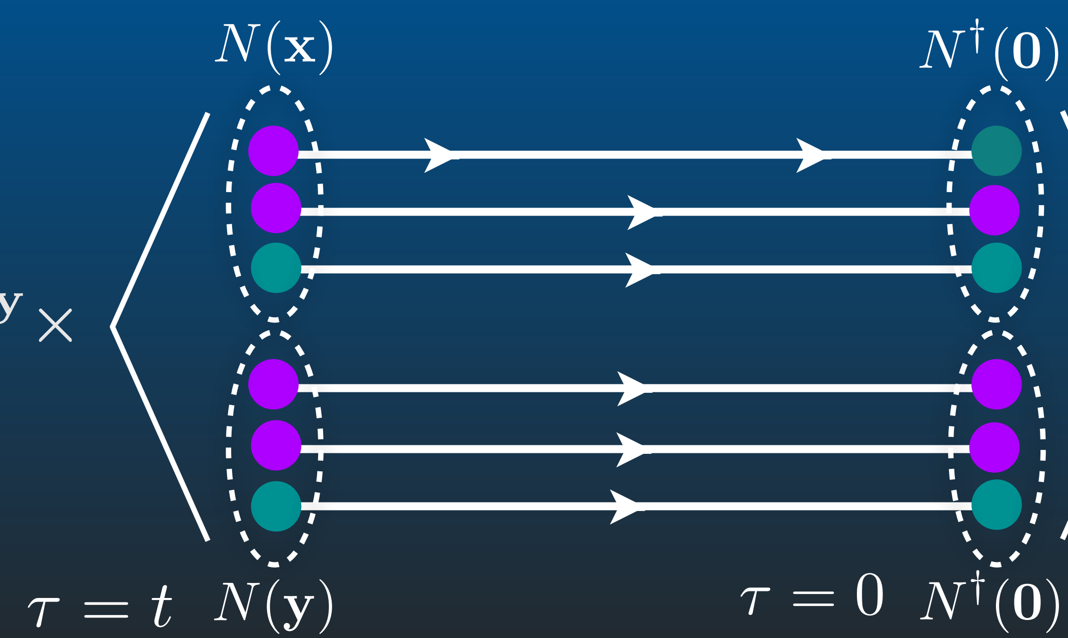


$$C(\mathbf{P};t,t_O) = \sum_{\mathbf{p}_1+\mathbf{p}_2=\mathbf{P}} \sum_{\mathbf{x},\mathbf{y},\mathbf{z}} e^{i\mathbf{p}_1\cdot\mathbf{x}+i\mathbf{p}_2\cdot\mathbf{y}} \times$$



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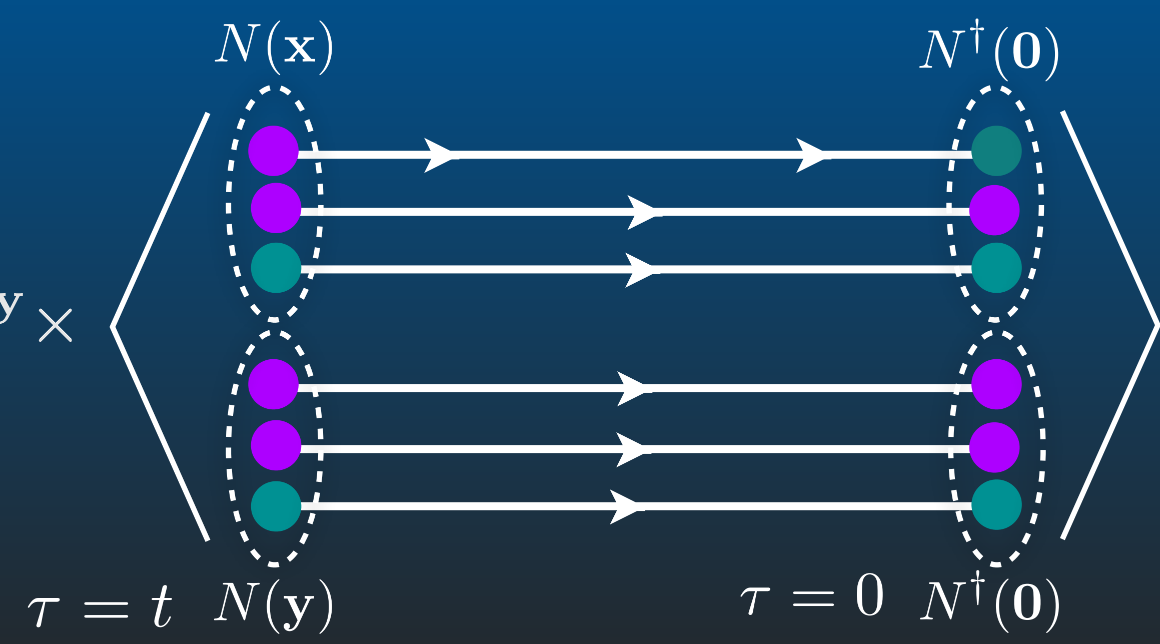
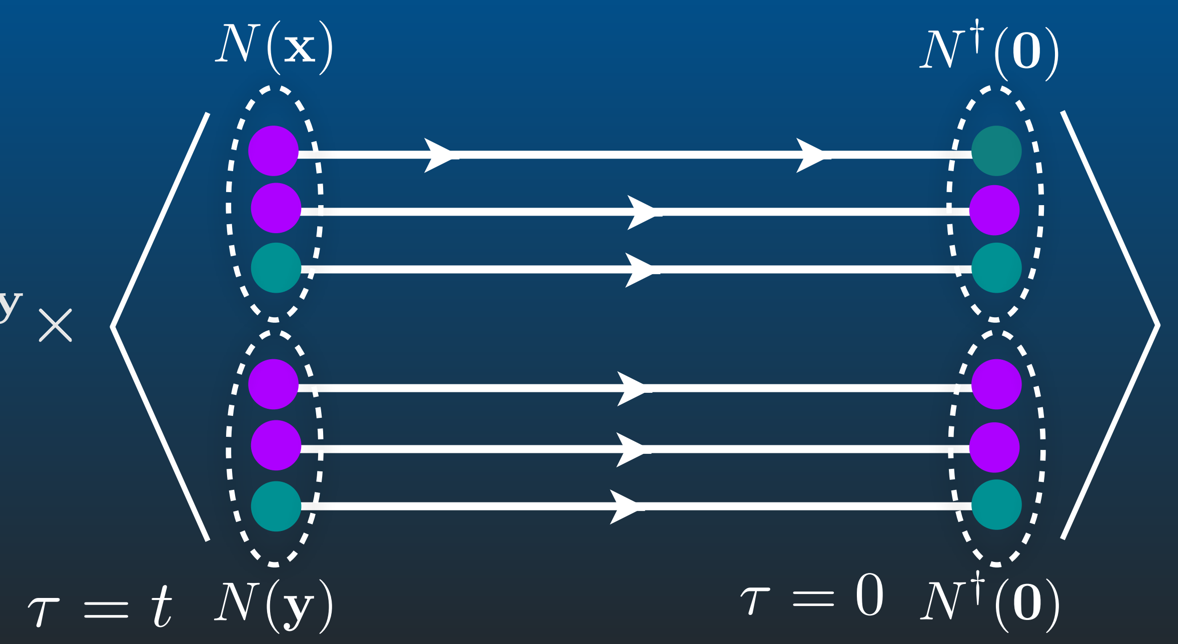



COMPLEXITIES OF QUARK-LEVEL INTERPOLATING FIELDS

COMPLEXITIES OF QUARK CONTRACTIONS

Detmold and Orginos (2013).

$$C(\mathbf{P}; t, t_O) = \sum_{\mathbf{p}_1+\mathbf{p}_2=\mathbf{P}} \sum_{\mathbf{x},\mathbf{y},\mathbf{z}} e^{i\mathbf{p}_1\cdot\mathbf{x}+i\mathbf{p}_2\cdot\mathbf{y}} \times$$




# COMPLEXITIES OF QUARK-LEVEL INTERPOLATING FIELDS

Detmold and Orginos (2013).

Number of terms in the interpolating operators of a nucleus?

$$\bar{\mathcal{N}}^h = \sum_{\mathbf{a}} w_h^{a_1, a_2 \cdots a_{n_q}} \bar{q}(a_1)\bar{q}(a_2)\cdots\bar{q}(a_{n_q})$$

Collective indices: color, spinor, flavor and lattice site

As many quark interpolators as needed to represent a given system, e.g., 6 quarks for *NN(3S1)*.

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The wave-function must be totally anti-symmetric:

$$\frac{N!}{(N-n_q)!}$$

Removing permutations:

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More simplification is possible too:

$$\bar{\mathcal{N}}^h = \sum_{k=1}^{N_w} \tilde{w}_h^{(a_1, a_2 \cdots a_{n_q}), k} \sum_{\mathbf{i}} \epsilon^{i_1, i_2, \cdots, i_{n_q}} \bar{q}(a_{i_1})\bar{q}(a_{i_2})\cdots\bar{q}(a_{i_{n_q}})$$

New weight factors factoring in other constraints such as color singletness, parity, angular momentum, strangeness.

Easier to work with baryon blocks and tabulate the corresponding weights:

$$\bar{\mathcal{N}}^h = \sum_{k=1}^{M_w} \tilde{W}_h^{(b_1,b_2\cdots b_A),k} \sum_{\mathbf{i}} \epsilon^{i_1,i_2,\cdots,i_A} \bar{B}(b_{i_1})\bar{B}(b_{i_2})\cdots\bar{B}(b_{i_A})$$

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Baryon interpolators

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Baryon interpolators

Example: A=2, *P=+, J=0, I=1, L=0, s=0*

$$\frac{1}{\sqrt{2}}(n_\uparrow n_\downarrow - n_\downarrow n_\uparrow) \qquad \frac{1}{2}(n_\uparrow p_\downarrow - n_\downarrow p_\uparrow + p_\uparrow n_\downarrow - p_\downarrow n_\uparrow) \qquad \frac{1}{\sqrt{2}}(p_\uparrow p_\downarrow - p_\downarrow p_\uparrow)$$

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---

Quark-level weight can then be obtained by equality:

$$\bar{B}(b) = \sum_{k=1}^{N_{B(b)}} \tilde{w}_b^{(a_1,a_2,a_3),k} \sum_{\mathbf{i}} \epsilon^{i_1,i_2,i_3} \bar{q}(a_{i_1})\bar{q}(a_{i_2})\bar{q}(a_{i_3})$$

$$\begin{aligned}\bar{\mathcal{N}}^h &= \sum_{k=1}^{M_w} \tilde{W}_h^{(b_1,b_2\cdots b_A),k} \sum_{\mathbf{i}} \epsilon^{i_1,i_2,\cdots,i_A} \bar{B}(b_{i_1})\bar{B}(b_{i_2})\cdots\bar{B}(b_{i_A}) \\ &= \sum_{k=1}^{N_w} \tilde{w}_h^{(a_1,a_2\cdots a_{n_q}),k} \sum_{\mathbf{i}} \epsilon^{i_1,i_2,\cdots,i_{n_q}} \bar{q}(a_{i_1})\bar{q}(a_{i_2})\cdots\bar{q}(a_{i_{n_q}})\end{aligned}$$

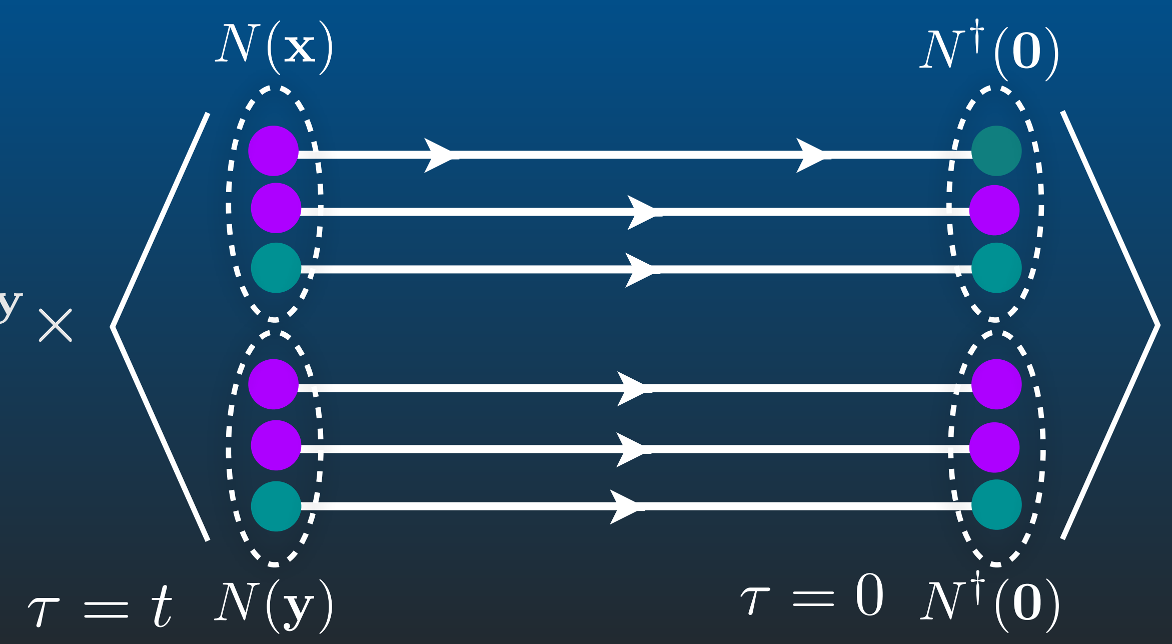
## EXERCISE 1

Show that the number of terms in the simplest quark-level interpolating operator for the proton (constructed at a single point) is 9.

## BONUS EXERCISE 1

Show that the number of terms in the simplest quark-level interpolating operator for the deuteron (constructed at a single point) is 21.

$$C(\mathbf{P}; t, t_O) = \sum_{\mathbf{p}_1+\mathbf{p}_2=\mathbf{P}} \sum_{\mathbf{x},\mathbf{y},\mathbf{z}} e^{i\mathbf{p}_1\cdot\mathbf{x}+i\mathbf{p}_2\cdot\mathbf{y}} \times$$




# COMPLEXITIES OF QUARK CONTRACTIONS

Detmold and Orginos (2013).

Naively the number of quark contractions for a nucleus goes as:

$$(2N_p + N_n)!\ (N_p + 2N_n)!$$

How bad is this?
Example: Consider radium-226 isotope.
the number of contractions required is ~ $10^{1425}$

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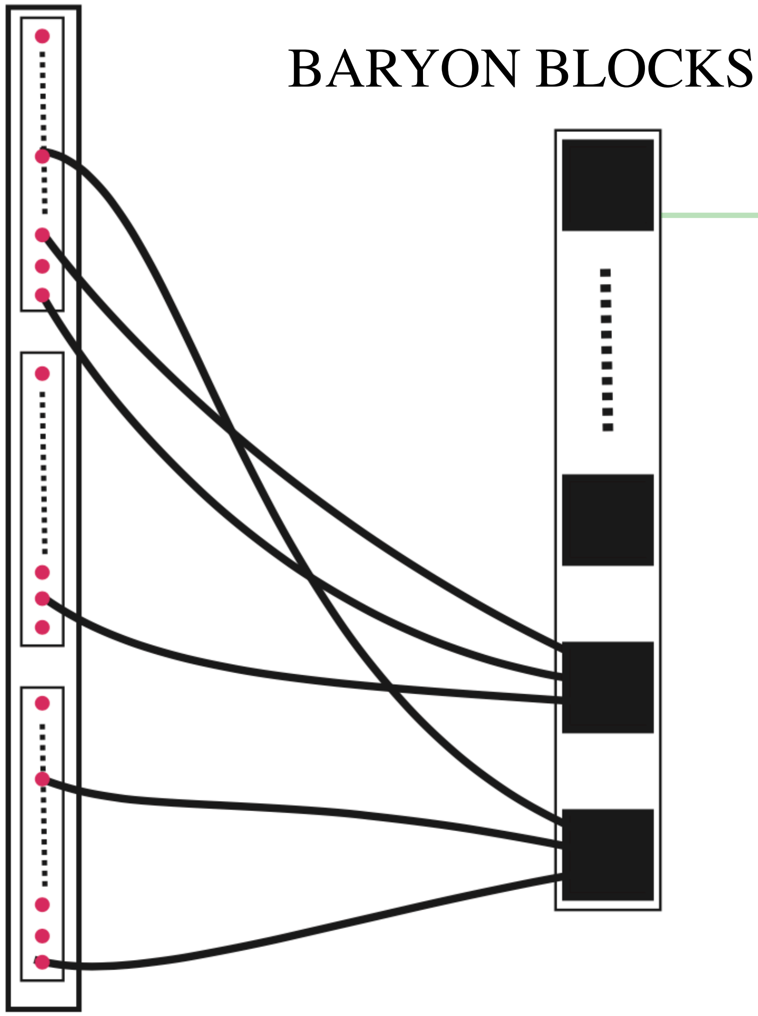
An example of a more efficient algorithm:




$$\mathcal{B}_b^{a_1,a_2,a_3}(\mathbf{p}, t; x_0) = \sum_{\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{x}} \sum_{k=1}^{N_{B(b)}} \tilde{w}_b^{(c_1,c_2,c_3),k} \sum_{\mathbf{i}} \epsilon^{i_1,i_2,i_3} S(c_{i_1}, x; a_1, x_0) S(c_{i_2}, x; a_2, x_0) S(c_{i_3}, x; a_3, x_0)$$

Can also start propagators at different locations.

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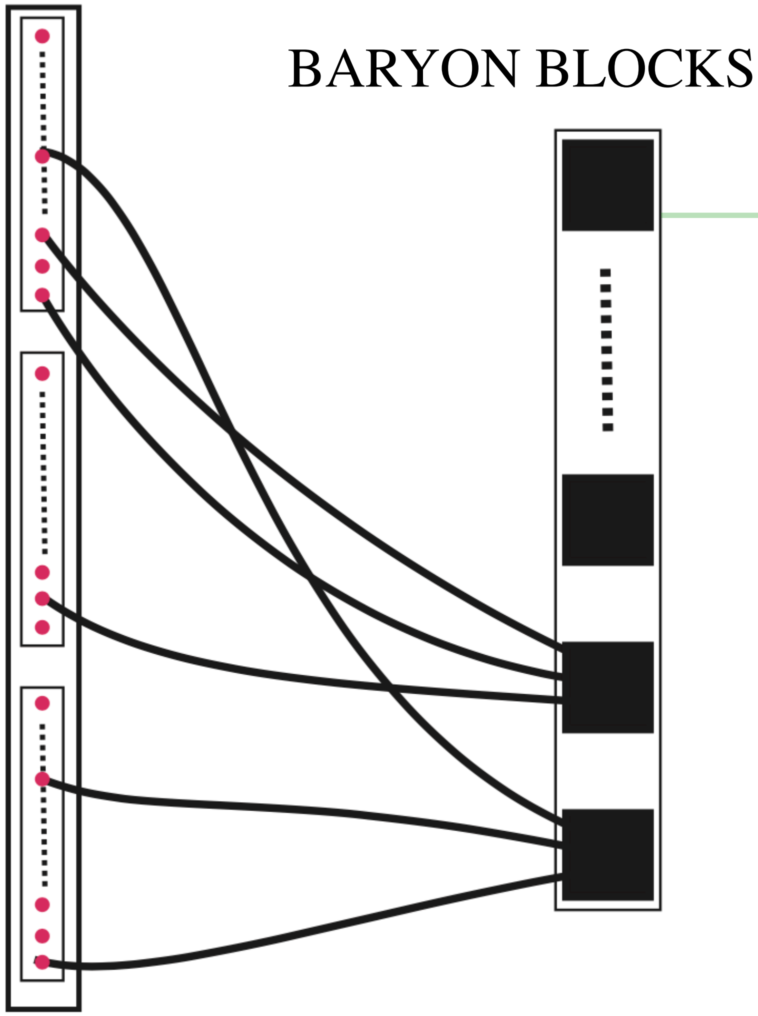


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Can also start propagators at different locations.

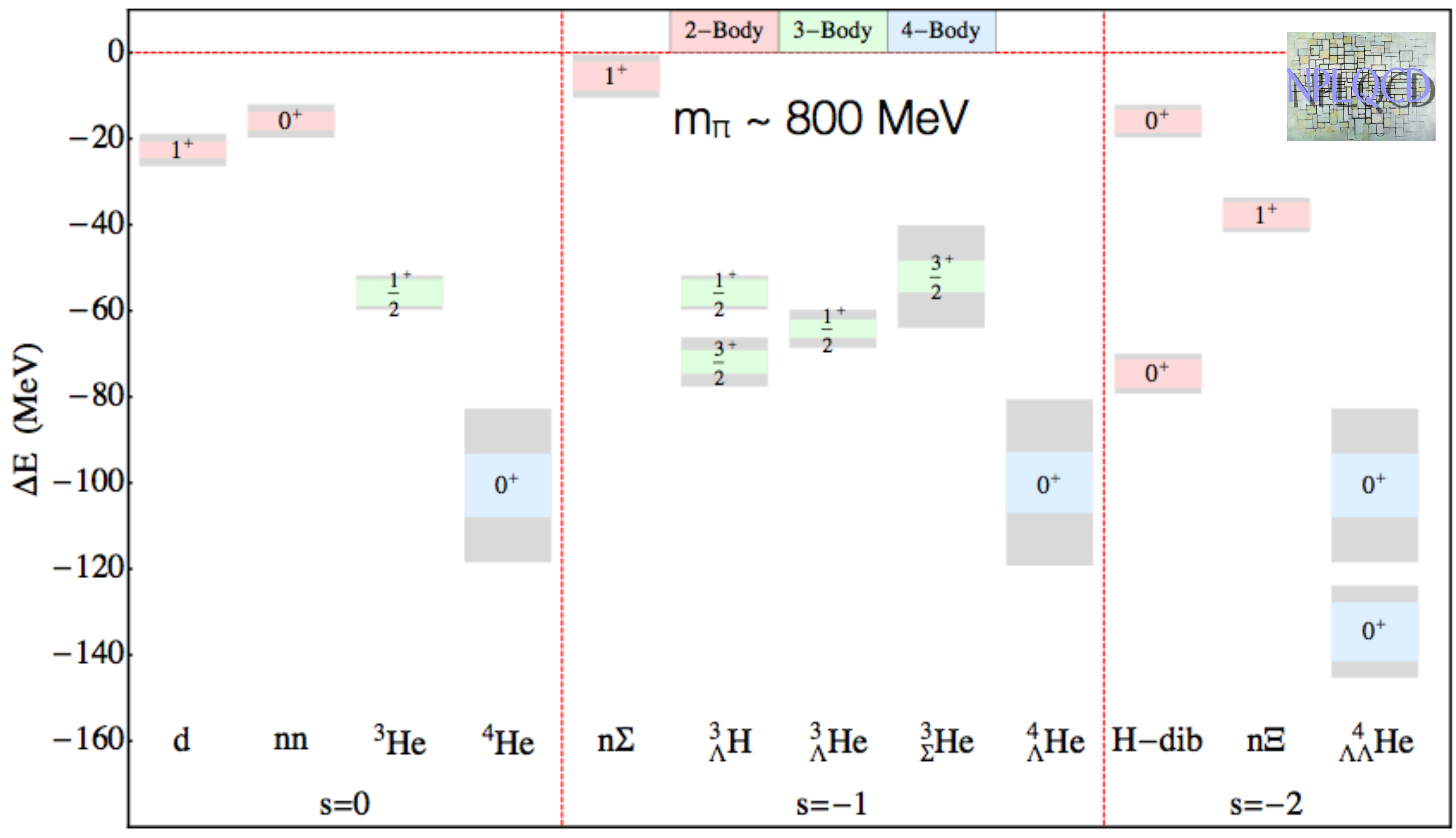
The new scaling is: $M_w \cdot N_w \cdot \frac{(3A)!}{(3!)^A}$

$M_w$: Number of terms in the sink

$N_w$: Number of terms in the source

# NUCLEI OBTAINED FROM SUCH AN APPROACH (AT A HEAVIER QUARK MASSES)

$N_f = 3,\ m_\pi = 0.806\ \text{GeV},\ a = 0.145(2)\ \text{fm}$

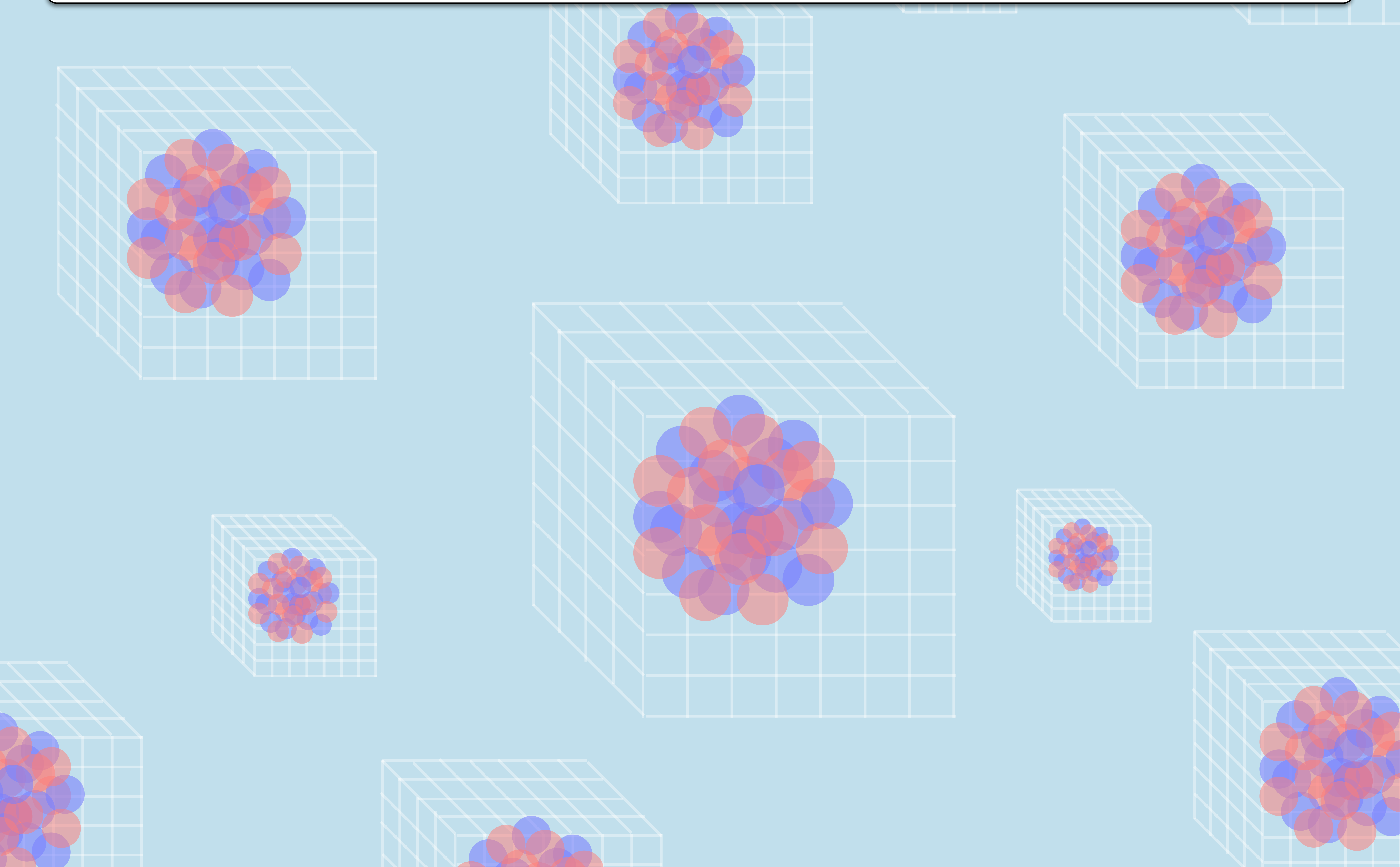



Beane, et al. (NPLQCD), Phys.Rev. D87 (2013) , Phys.Rev. C88 (2013)

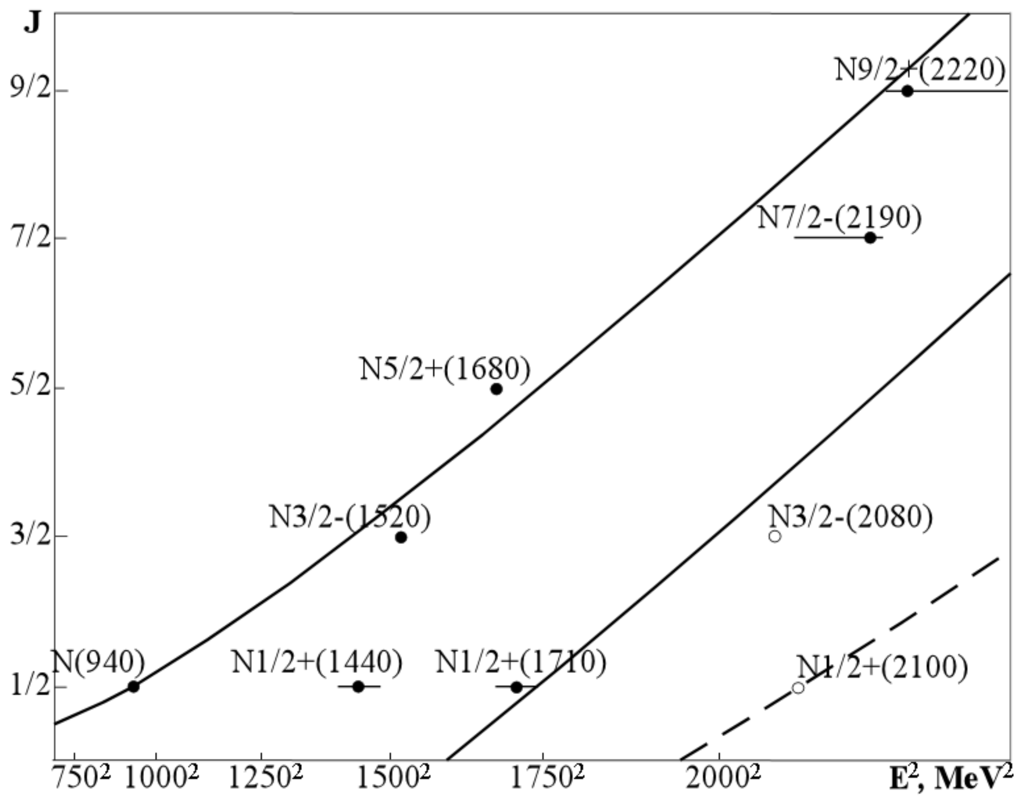
## EXERCISE 2

According to the naive counting, how many contractions are required for a nucleus at the source and sink with atomic numbers A = 4, 8, 12, 16? How many contractions are there with the use of the efficient algorithm described? There are even more optimal algorithms that lead to a polynomial scaling with the number of the quarks.

ii) EXCITATION ENERGIES OF NUCLEI ARE MUCH SMALLER THAN THE QCD SCALE.

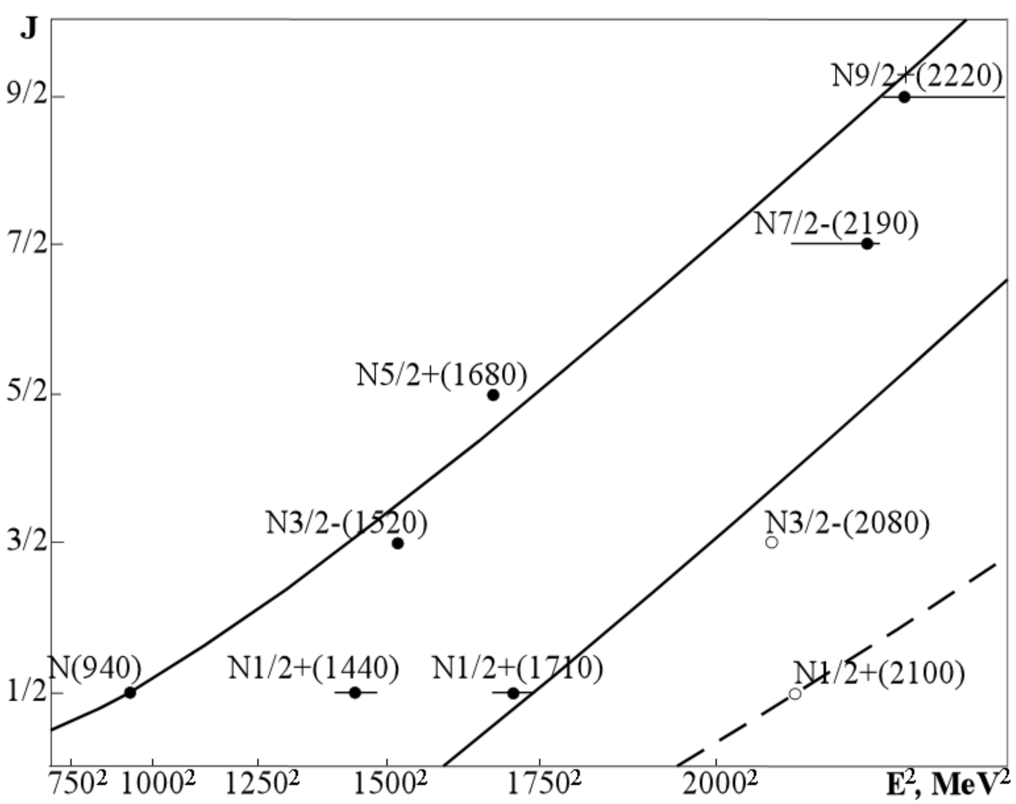


# Nucleon excitations




Kulikov, Dmitry A. et al., Central Eur.J.Phys. 11 (2013) .

## Nucleon excitations

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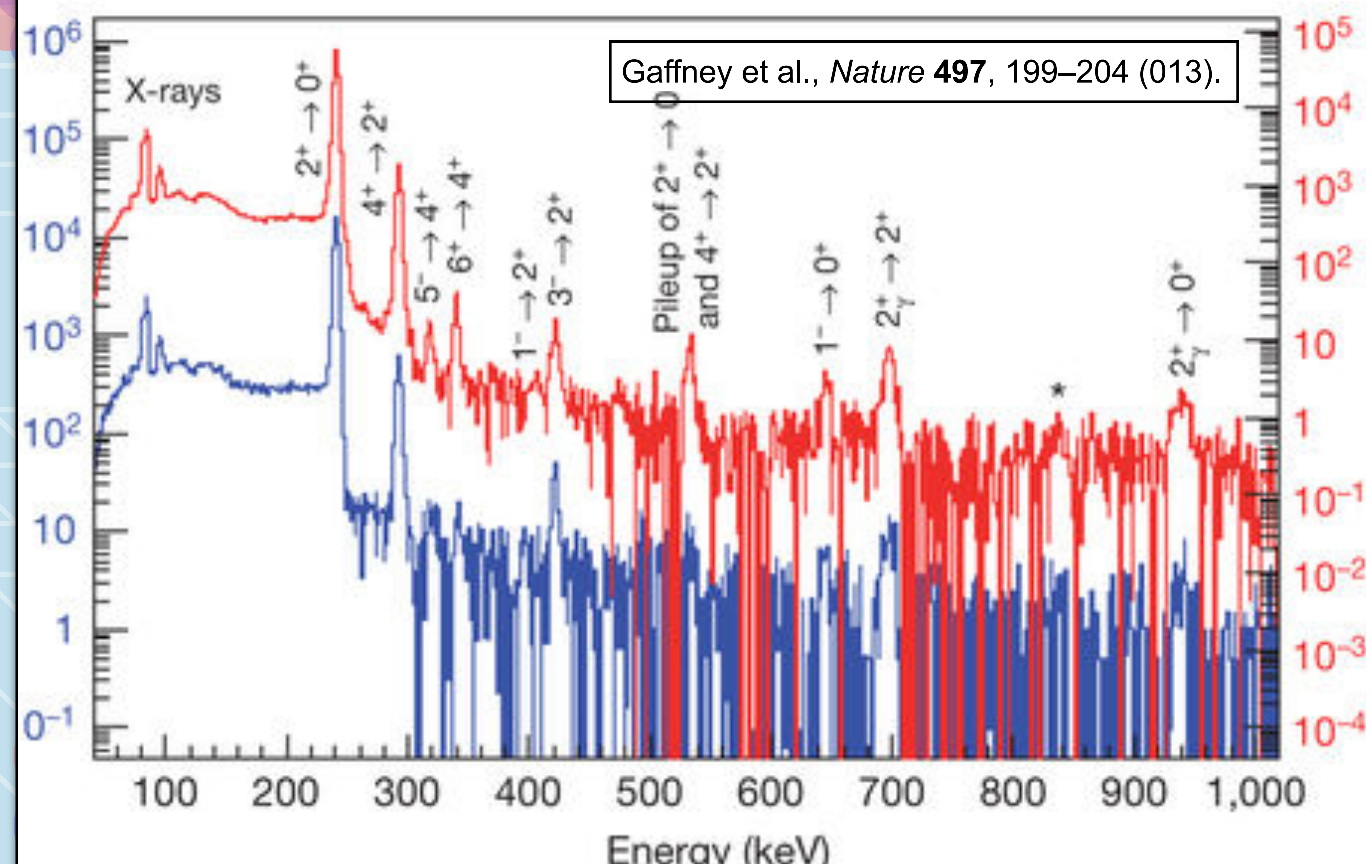
## Nuclear excitations of two pear-shaped nuclei (radium and radon)



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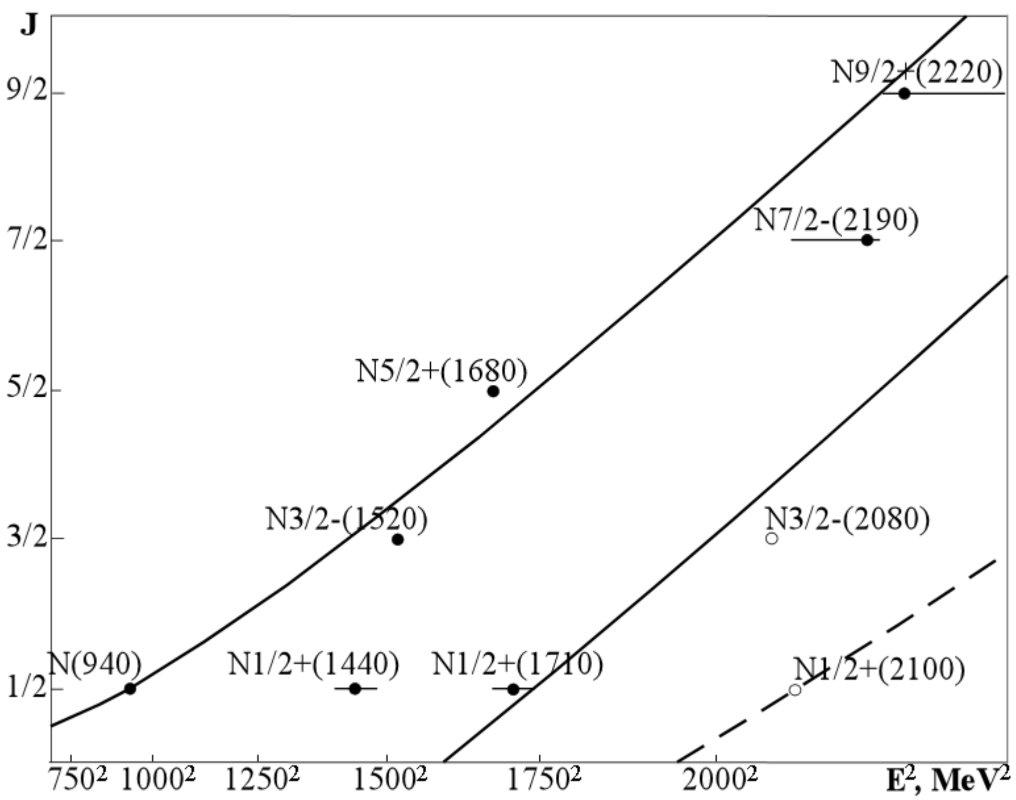


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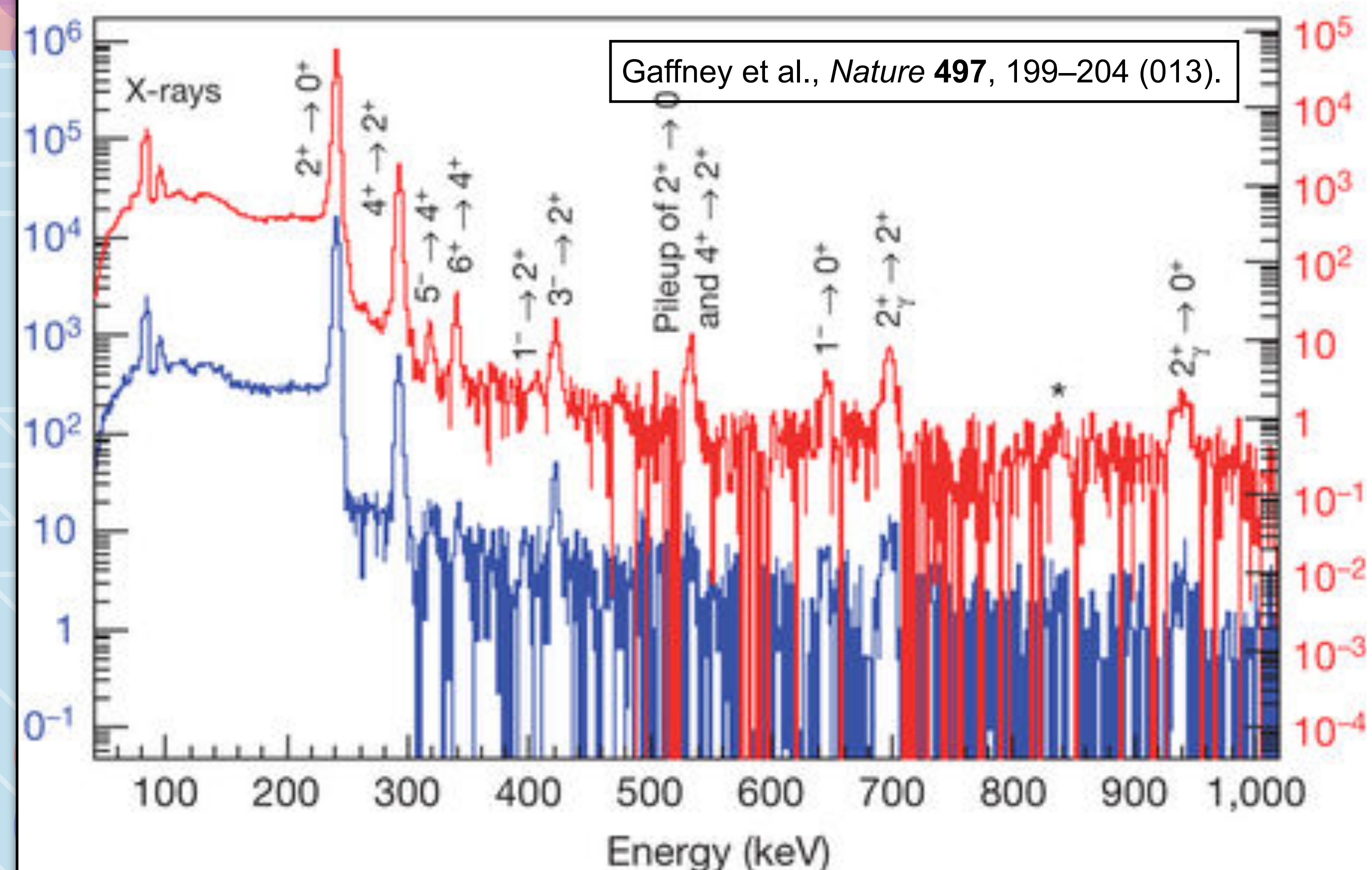



Gaffney et al., *Nature* **497**, 199–204 (013).

Getting radium directly from QCD will remain challenging for a long time! One should first compute A = 2, 3, 4 systems well. This is till not that easy: $B\_d = 2$ MeV!

## EXERCISE 3

With a given amount of computational resources, you have achieved a 1% statistical uncertainty on the extracted mass of the nucleon from your lattice QCD calculation. By what factor should you increase your computing resources (your statistics) to also achieve a 1% statistical uncertainty on the binding energy of the deuteron?

# SO WHAT TO DO?

- With the most naive operators with similar overlaps to all states, unreasonably large times are needed to resolve nuclear energy gaps. See exercise 4!

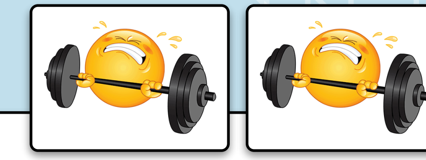
- The key to success of this program is in the use of good interpolating operators for nuclei. Since nucleons retain their identity in nuclei, forming baryon blocks at the sink turns out to be very advantageous. See the previous section.

- Ideally need to use a large set of operators for a variational analysis, but this has remained too costly in nuclear calculations. Applications in mesonic sector: Briceno, Dudek and Young, Rev. Mod. Phys. 90 025001.

- Methods such as matrix Prony that eliminate the excited states in linear combinations of interpolators or correlations functions have shown to be useful.

A good review: Beane, Detmold, Orginos, Savage, Prog. Part. Nucl. Phys. 66 (2011).

# EXERCISE 4



Consider a simple two-state model in the spectral decomposition of an Euclidean two-point function. Demonstrate that the time scale to reach the ground state of the model with a finite statistical precision can depend highly on the corresponding overlap factor for the state. It is sufficient to show this numerically and for a set of chosen energies and overlap factors.

# VARIATIONAL METHOD

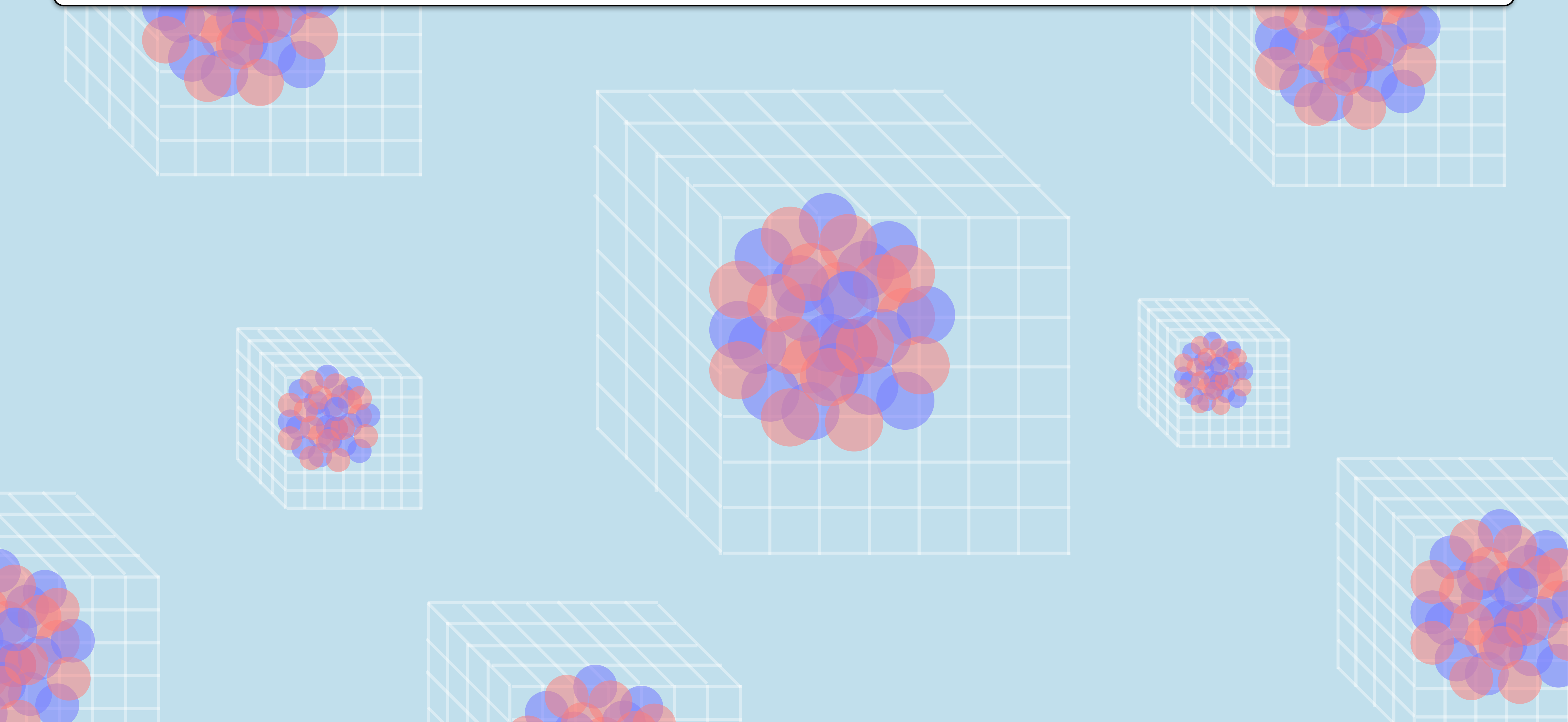
Form a matrix of correlation functions with a number of interpolators:

$$C_{i,j}(t) = \langle 0|\mathcal{O}_i(t)\mathcal{O}_j(0)|0\rangle$$

Michael (1985)
Luescher and Wolf (1990)

Solve the eigenvalue equation for a reasonably chosen initial time:

$$C(t)v_k = \lambda_k C(t_0)v_k \qquad \lim_{t-t_0\to\infty} \lambda_k = e^{-E_k t}$$



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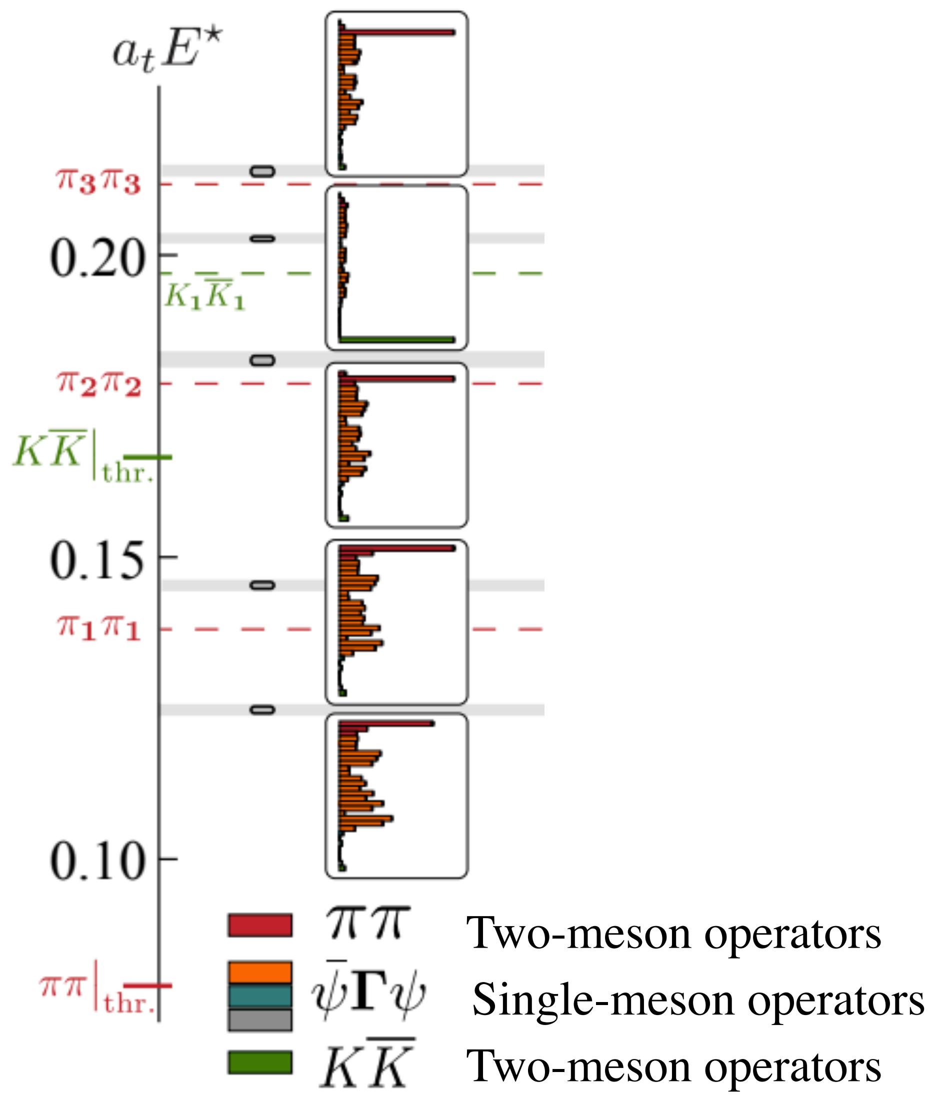
## An example

Meson spectroscopy in the P-wave $\pi\pi - K\overline{K}$ channel:




Wilson et al (HadSpec), Phys. Rev. D 92, 094502 (2015). Briceno, Dudek and Young, Rev. Mod. Phys. 90 025001.

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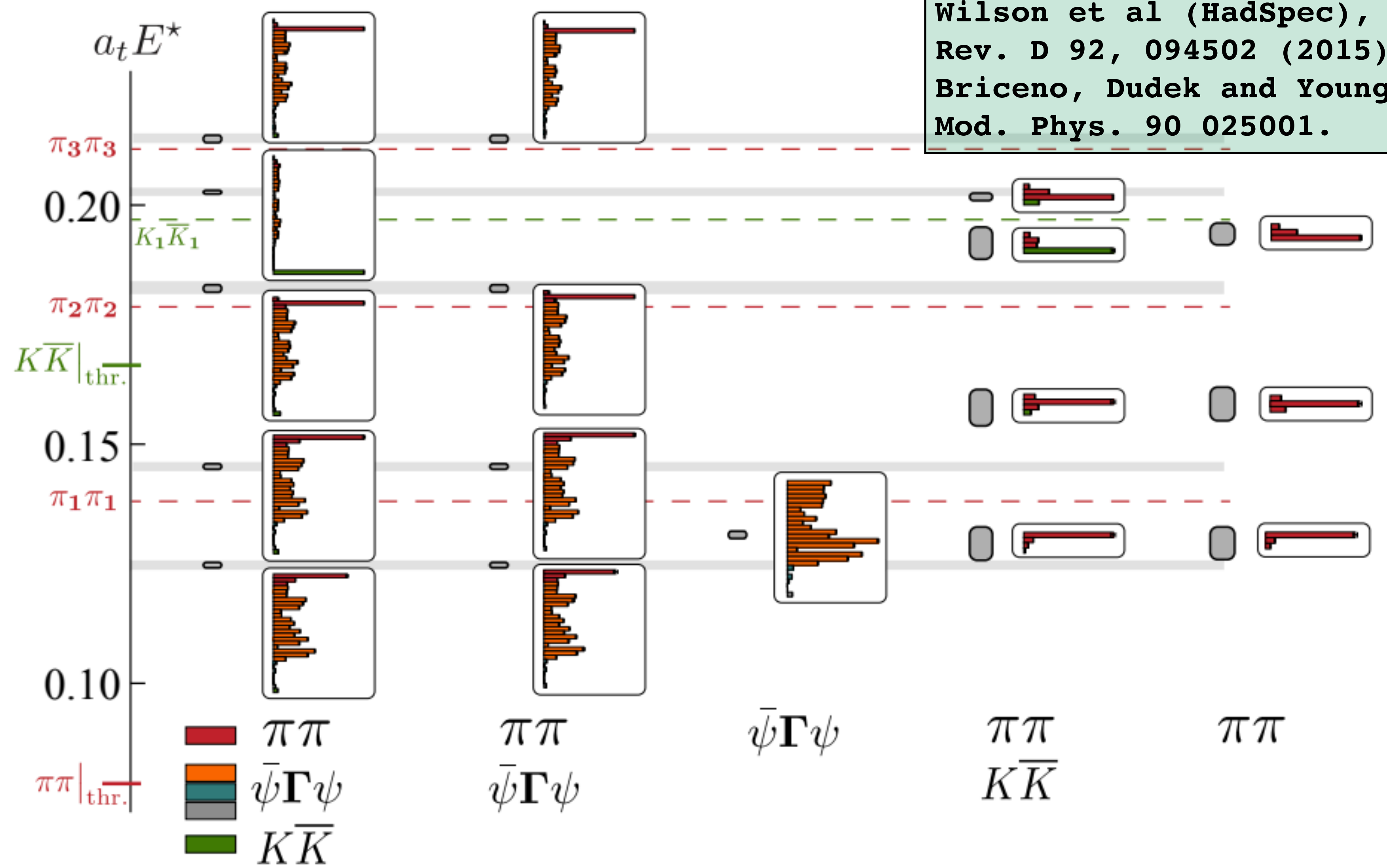

# MATRIX PRONY

de Prony (1795)

The method is useful when the correlation function matrix is not square or positive-definite matrix necessarily. It finds suitable linear combination of the correlates that are dominated by single exponentials.

Consider: $$y(t) = \begin{pmatrix} C_{PS}(t) \\ C_{SS}(t) \end{pmatrix}$$

With the ansatz: $$y(t+\tau) = \hat{T}(\tau) y(t)$$

A "transfer matrix" defined as: $$\hat{T}(\tau) = M^{-1}(\tau) V$$

This implies that: $$M(\tau) y(t+\tau) y^T(t) = V y(t) y^T(t)$$

Which can be satisfied by: $$M(\tau) = \left( \sum_{t=t_0}^{t_0+\Delta t} y(t+\tau) y^T(t) \right)^{-1}, \quad V = \left( \sum_{t=t_0}^{t_0+\Delta t} y(t) y^T(t) \right)^{-1}$$

Finally: $$\hat{T}(\tau) q_n = (\lambda_n)^\tau q_n\,, \qquad \text{with } \lambda_n = e^{-E_n}$$

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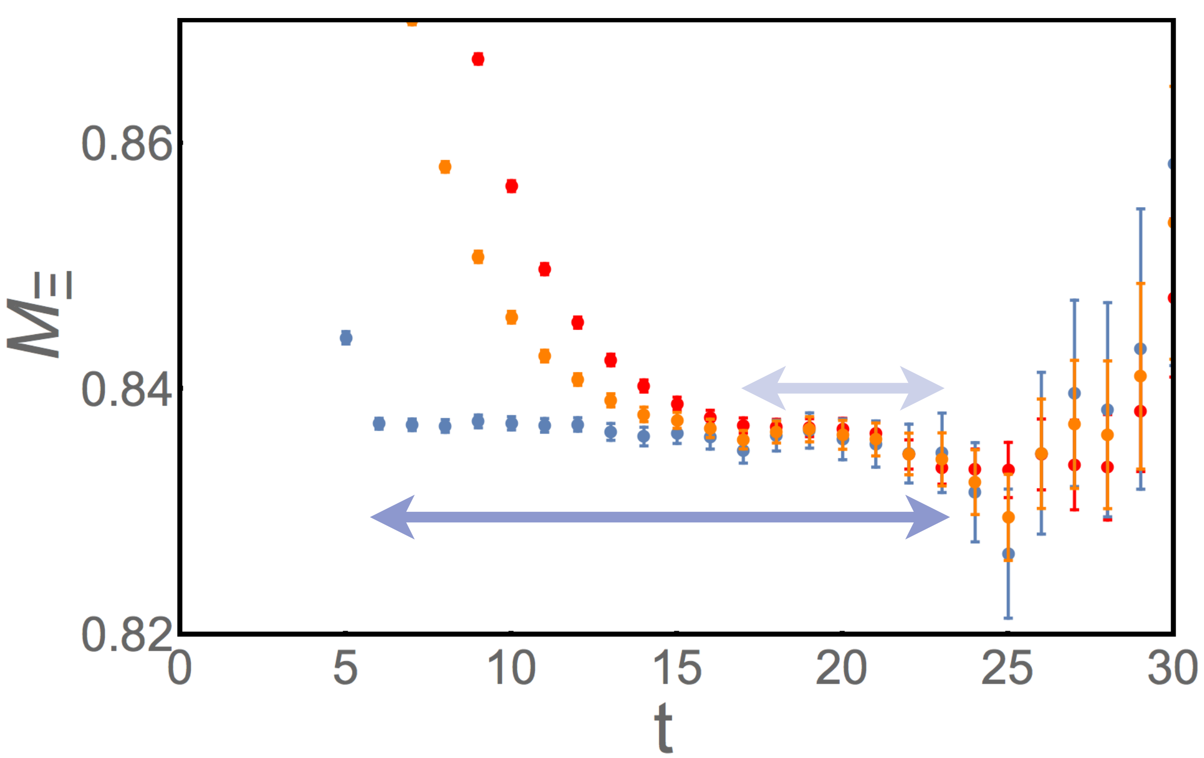
Or: $$M(\tau) \sum_{t=t_0}^{t_0+\Delta t} y(t+\tau) y^T(t) = V \sum_{t=t_0}^{t_0+\Delta t} y(t) y^T(t)$$

Which can be satisfied by: $$M(\tau) = \left( \sum_{t=t_0}^{t_0+\Delta t} y(t+\tau) y^T(t) \right)^{-1}, \quad V = \left( \sum_{t=t_0}^{t_0+\Delta t} y(t) y^T(t) \right)^{-1}$$

Finally: $$\hat{T}(\tau) q_n = (\lambda_n)^\tau q_n \,, \qquad \text{with } \lambda_n = e^{-E_n}$$

# An example

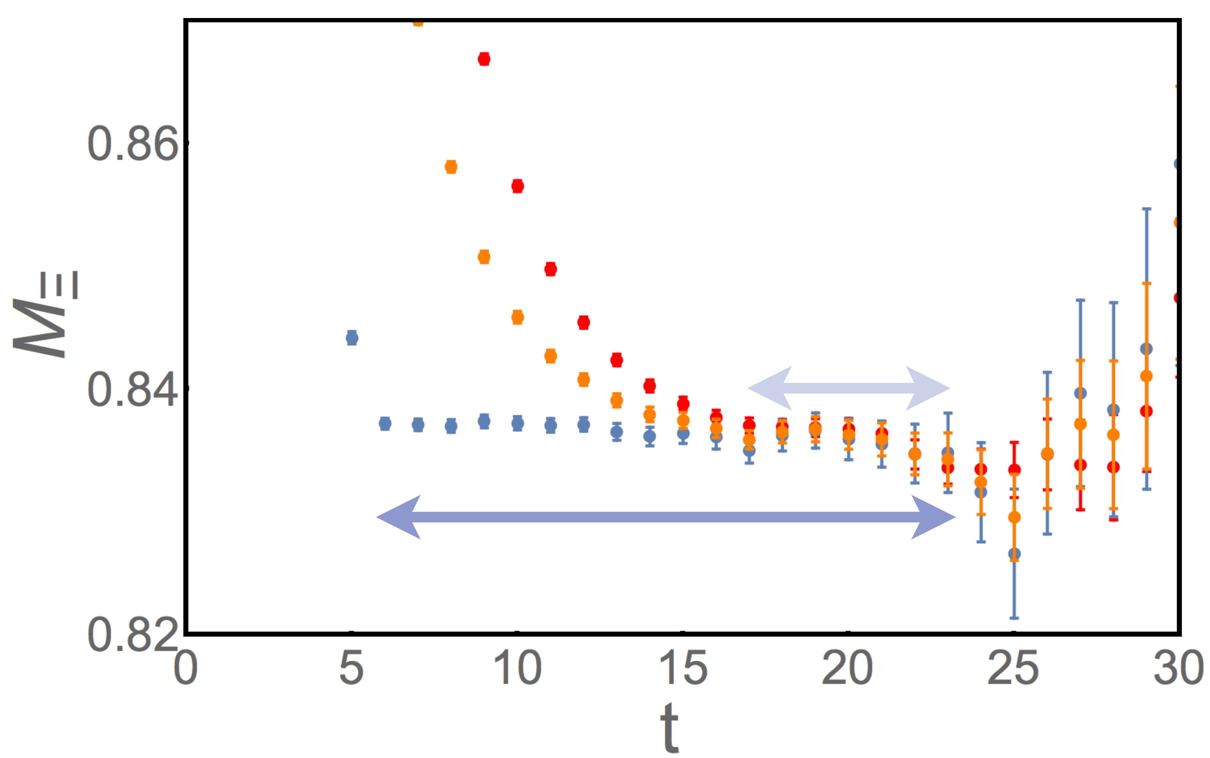
Linear combos. at the level of correlation functions




Beane et al (NPLQCD), Phys.Rev.D79:114502 (2009).

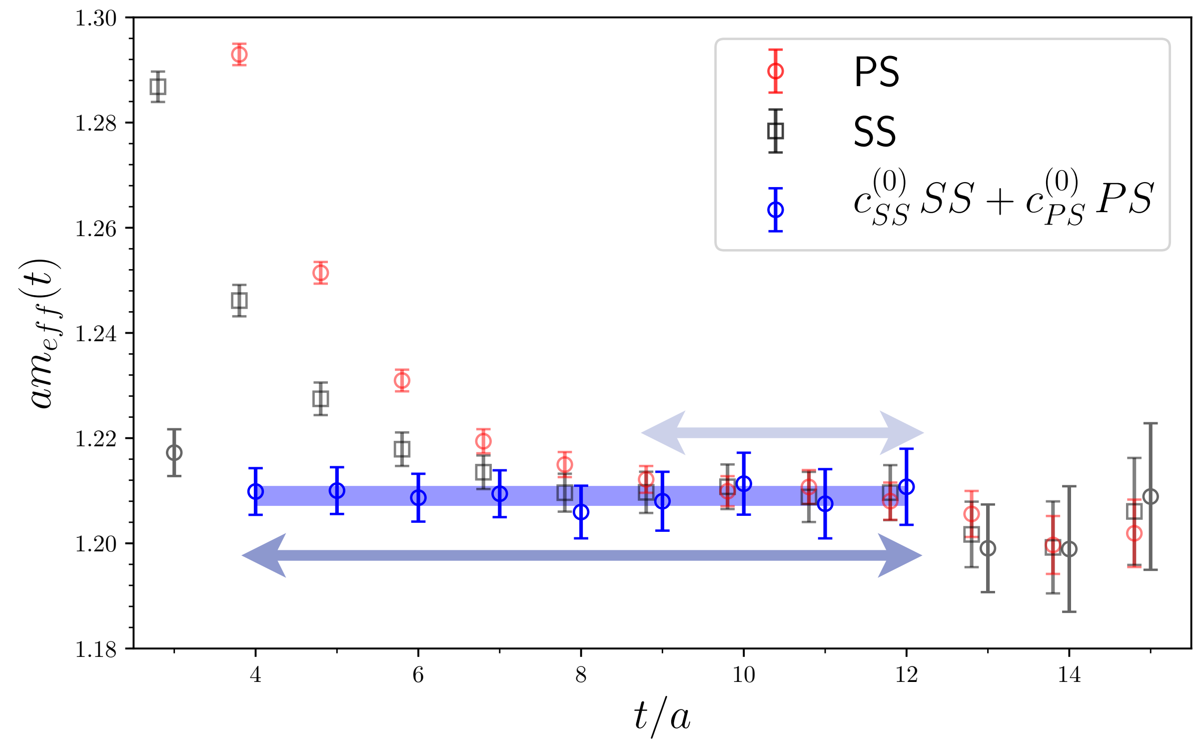
# An example

Linear combos. at the level of correlation functions



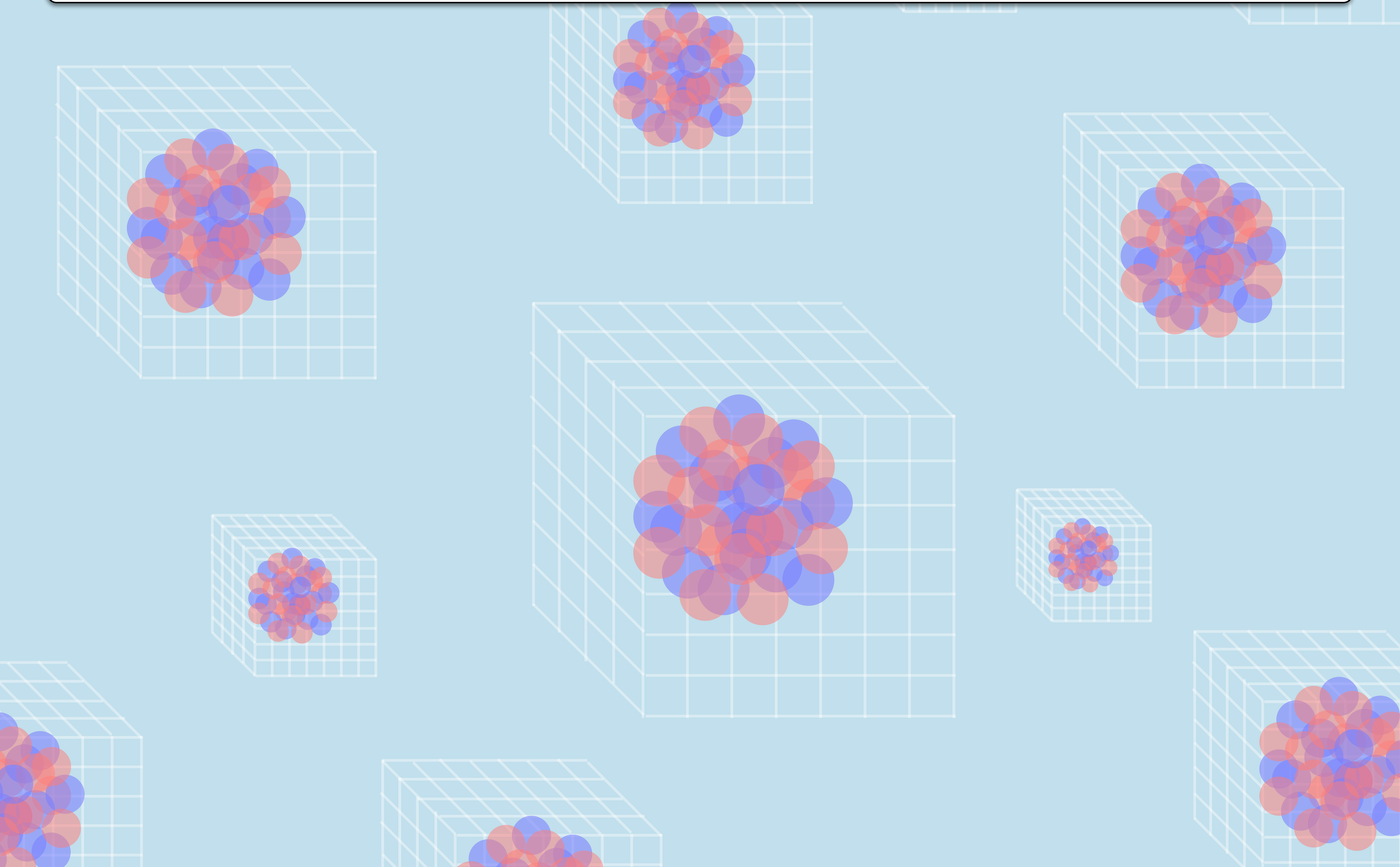

Beane et al (NPLQCD), Phys.Rev.D79:114502 (2009).

Berkowitz et al (CalLatt), arXiv:1710.05642(2017).



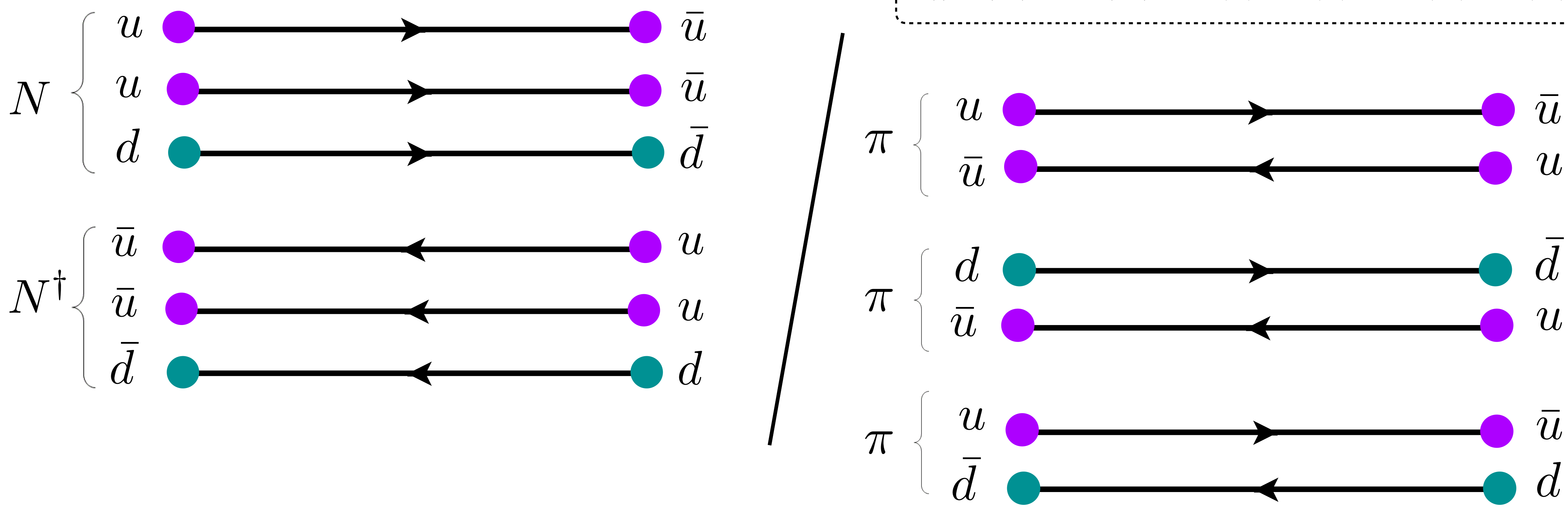

Linear combos. at the level of sink construction

iii) THERE IS A SEVERE SIGNAL-TO-NOISE DEGRADATION.



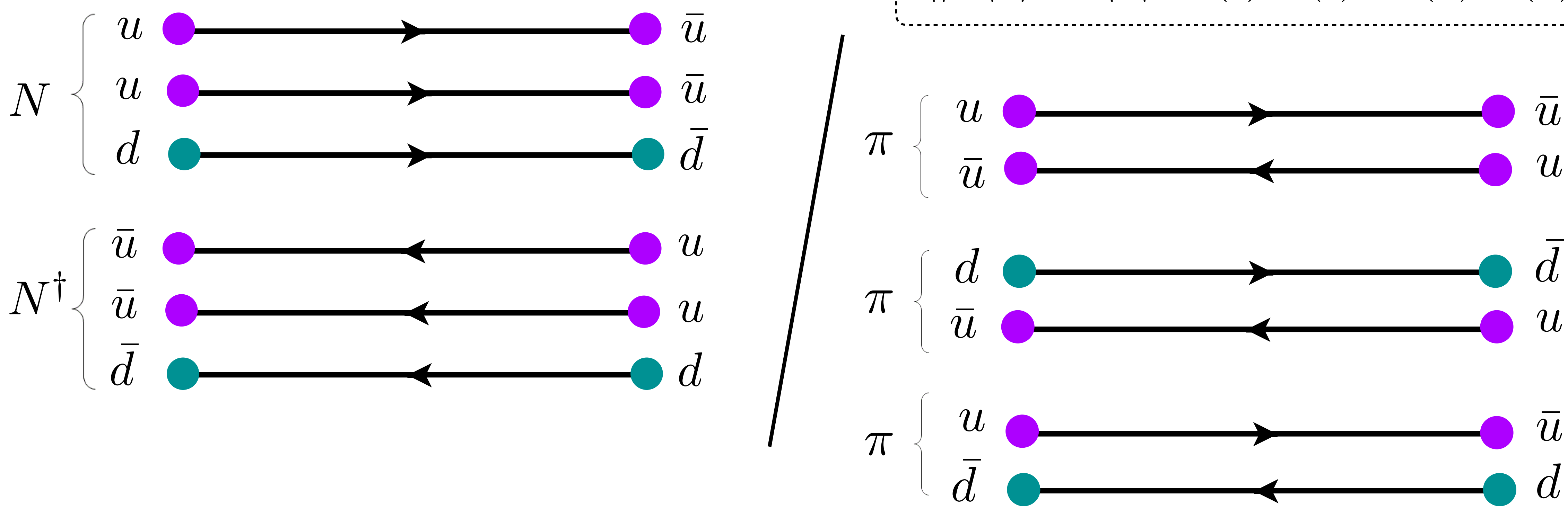
# The origin of noise

$$\langle |C|^2 \rangle = \langle 0 | N^\dagger(t) N(t) N^\dagger(0) N(0) | 0 \rangle$$



# The origin of noise

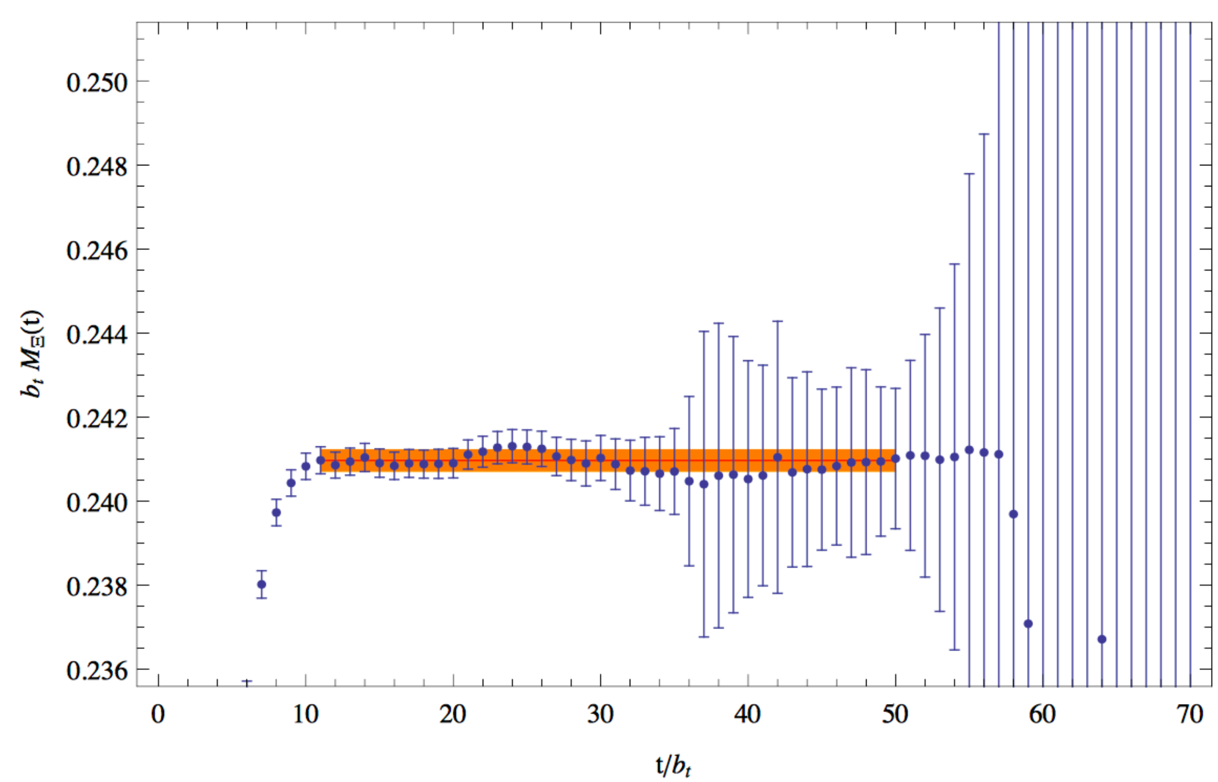
$$\langle |C|^2 \rangle = \langle 0 | N^\dagger(t) N(t) N^\dagger(0) N(0) | 0 \rangle$$




The ground-state of the variance correlator is three pions and not two nucleons:

$$\mathrm{StN}(C_i) \sim \frac{\langle C_i \rangle}{\sqrt{\langle |C_i|^2 \rangle}} \sim e^{-(M_N - \frac{3}{2} m_\pi)t}$$

`Parisi (1984) and Lepage (1989).`



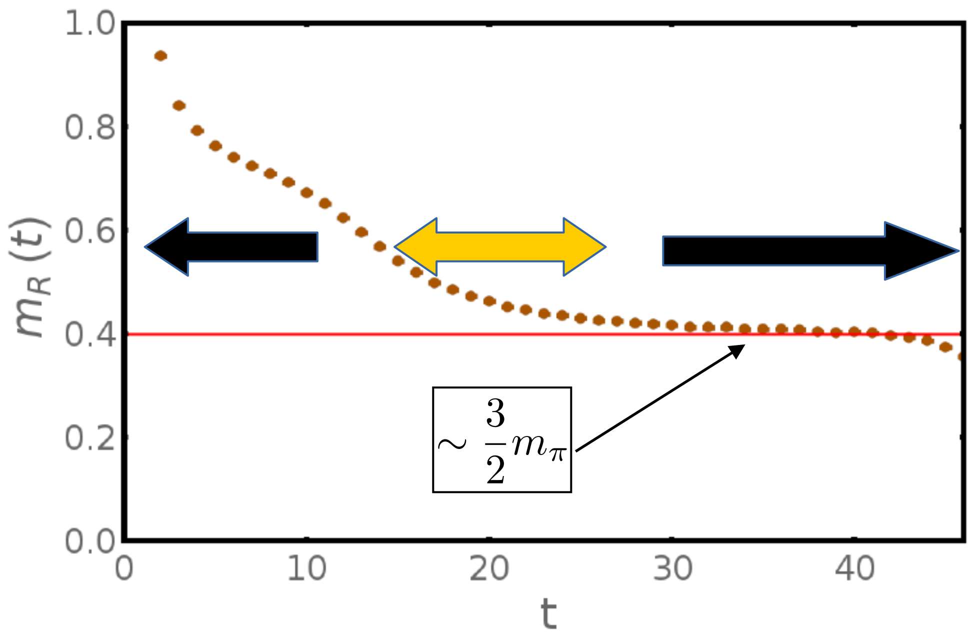
Can we understand better the noise in nuclear correlation function and control it?

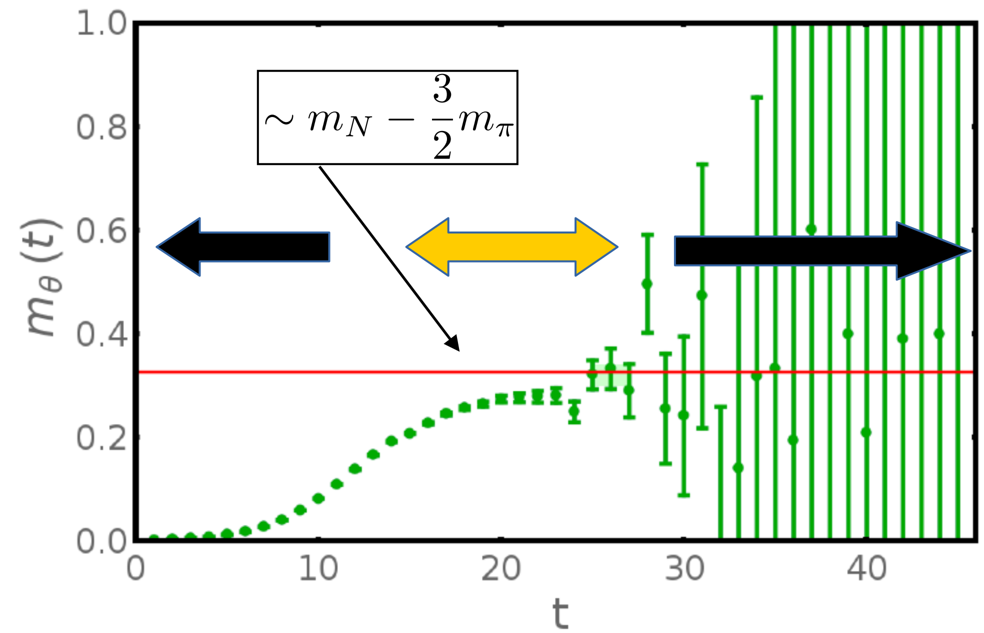
$$\text{StN} \sim e^{-(m_N - \frac{3}{2}m_\pi)\Delta t}$$

Wagman and Savage (2016,2017).

Let's consider the magnitude and the phase of the correlation functions:

$$C_i(t) = e^{R_i(t) + i\theta_i(t)}$$



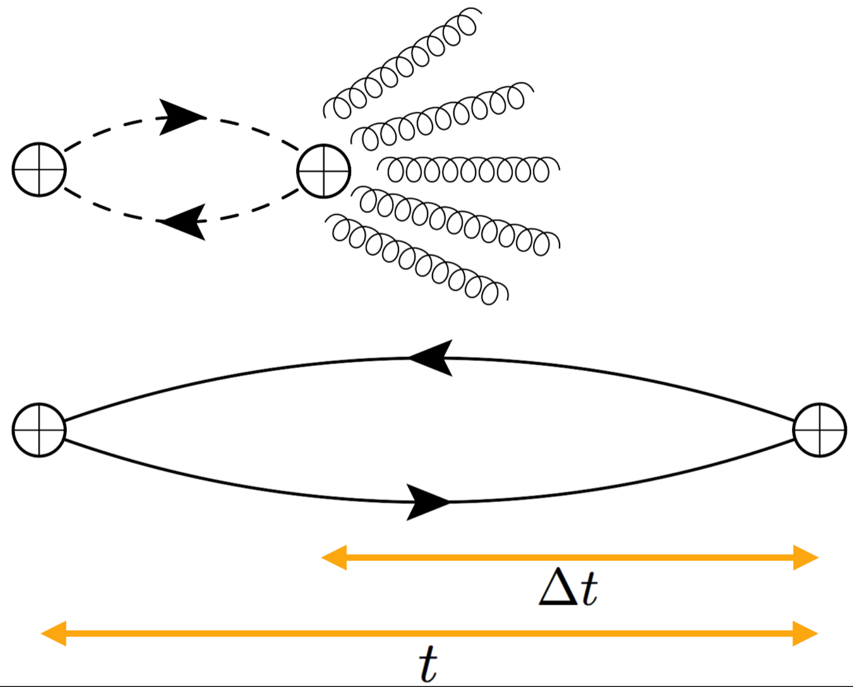




$$m_R(t) = \ln\left(\frac{\langle e^{R_i(t)}\rangle}{\langle e^{R_i(t+1)}\rangle}\right)$$

$$m_\theta(t) = \ln\left(\frac{\langle e^{i\theta_i(t)}\rangle}{\langle e^{i\theta_i(t+1)}\rangle}\right)$$

A phase reweighting method seems to work:

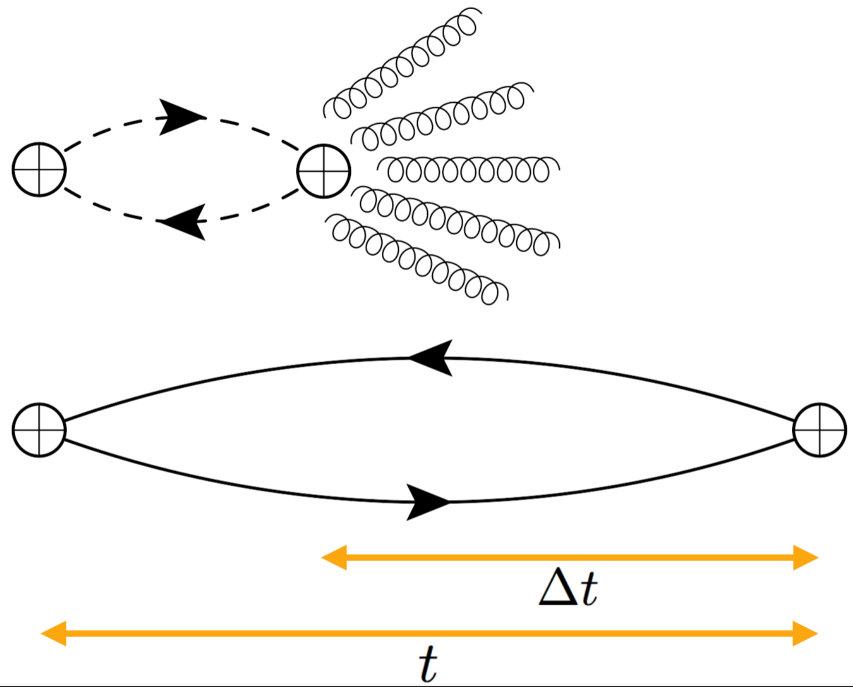
$$G^{\theta}(t, \Delta t) = \langle e^{-i\theta(t-\Delta t)} C(t) \rangle$$



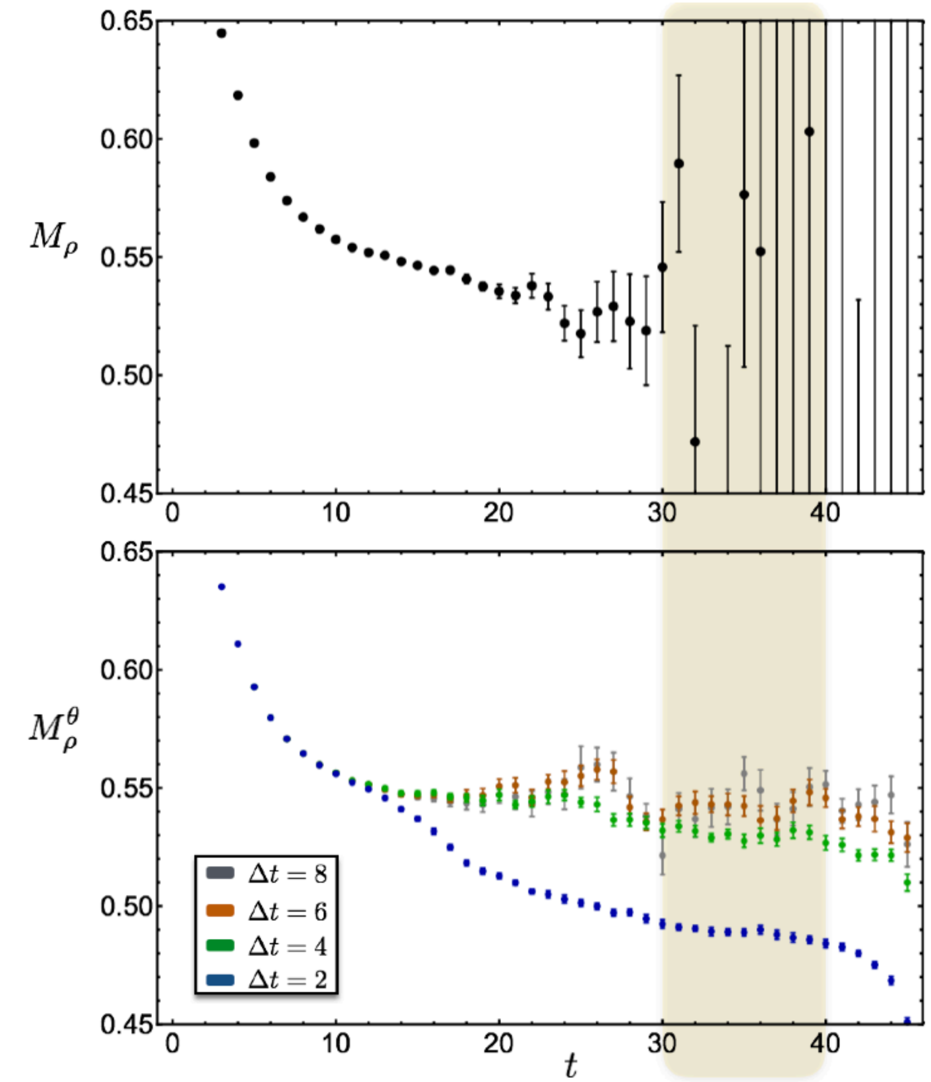

$$M_\rho^\theta(t, \Delta t) \;=\; M_\rho \;+\; c\, \delta M_\rho e^{-\delta M_\rho \Delta t} \;+\; \ldots$$

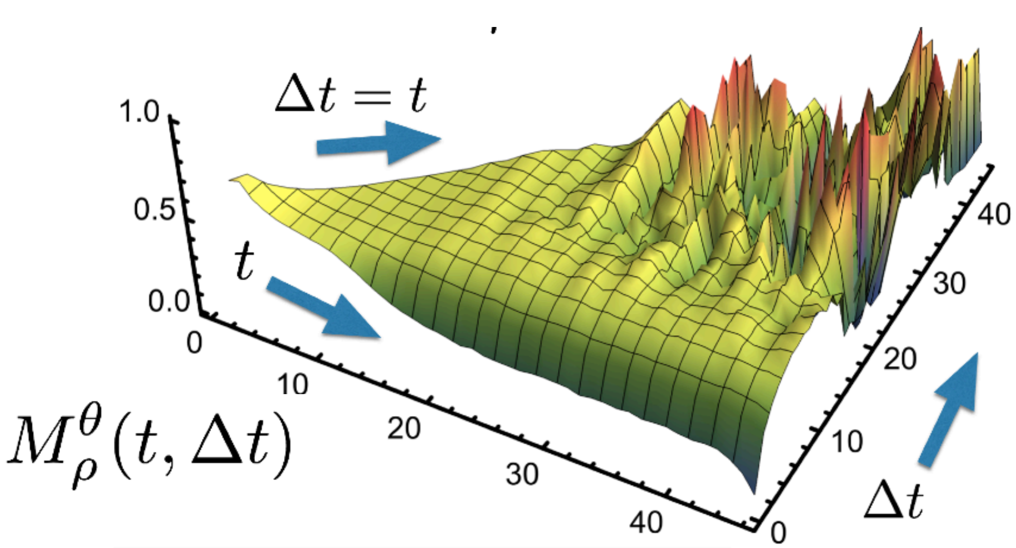
# A phase reweighting method seems to work:

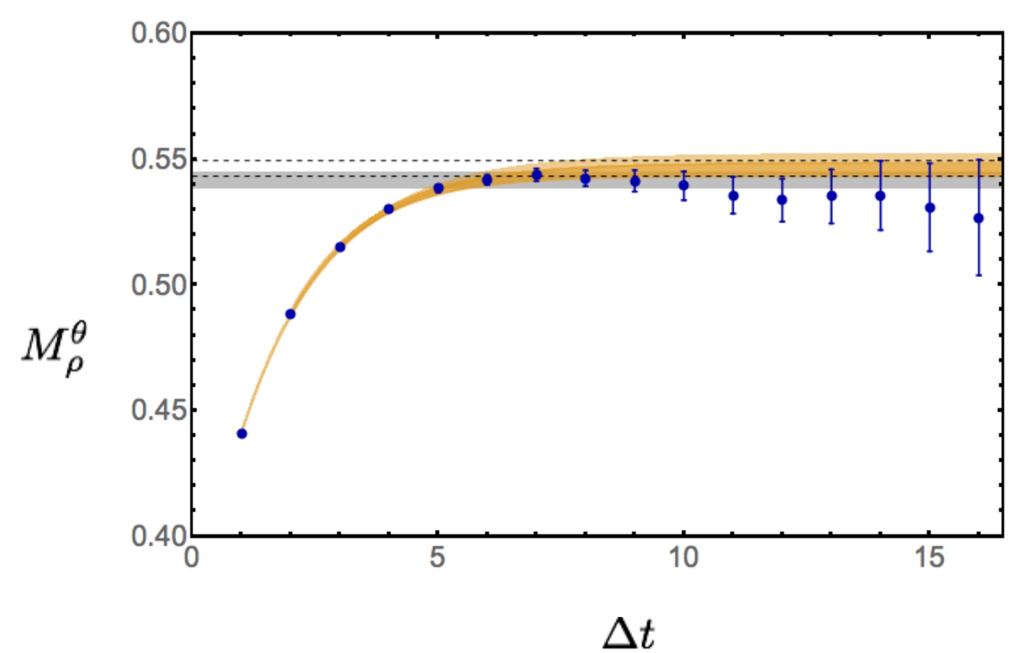
$$G^{\theta}(t, \Delta t) = \langle e^{-i\theta(t-\Delta t)} C(t) \rangle$$




$$M_{\rho}^{\theta}(t, \Delta t) \;=\; M_{\rho} \;+\; c\,\delta M_{\rho} e^{-\delta M_{\rho} \Delta t} \;+\; \ldots$$



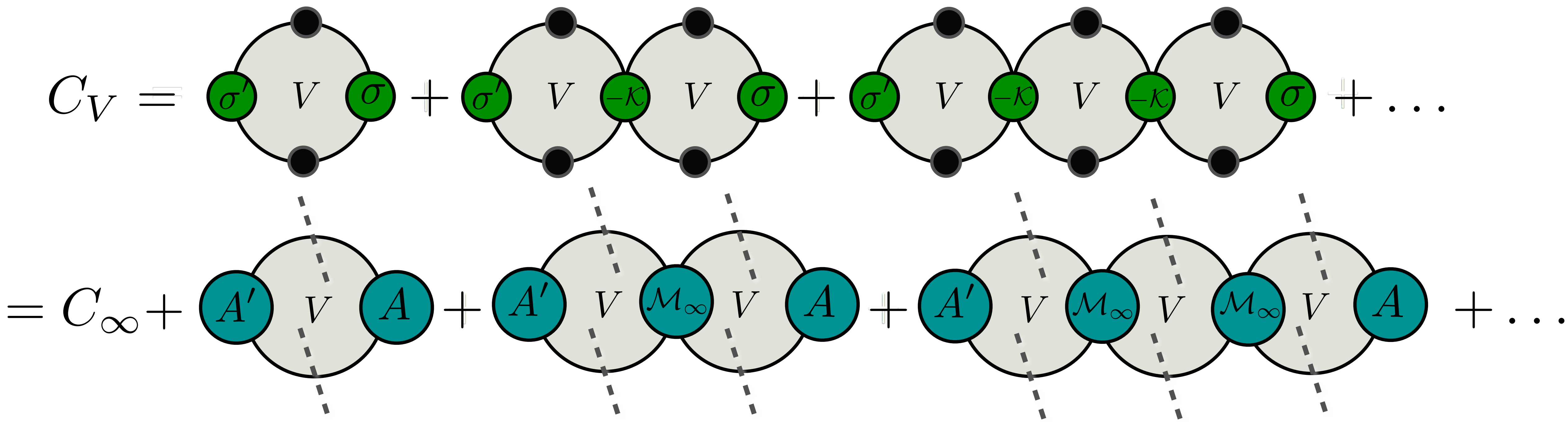
DESPITE CHALLENGES, PROGRESS HAS BEEN MADE. LQCD COMBINED WITH EFTS IS ON RIGHT TRACK TO DELIVER RESULTS ON IMPORTANT NUCLEAR PHYSICS QUANTITIES.

IN THE NEXT TWO LECTURES, WE WILL GO THROUGH A FEW EXAMPLES THAT DEMONSTRATE SUCH A PROGRESS.

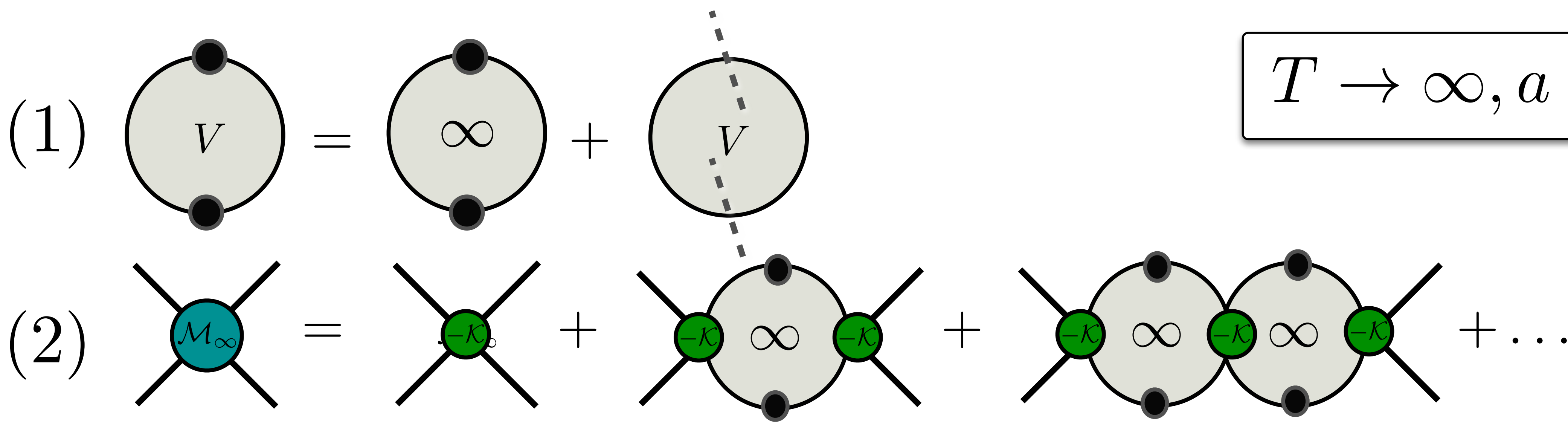
QUESTIONS?

# TWO-BODY ELASTIC SCATTERING

Let's review the Luescher's method first (see module I for more details).
A QFT derivation goes as follows:

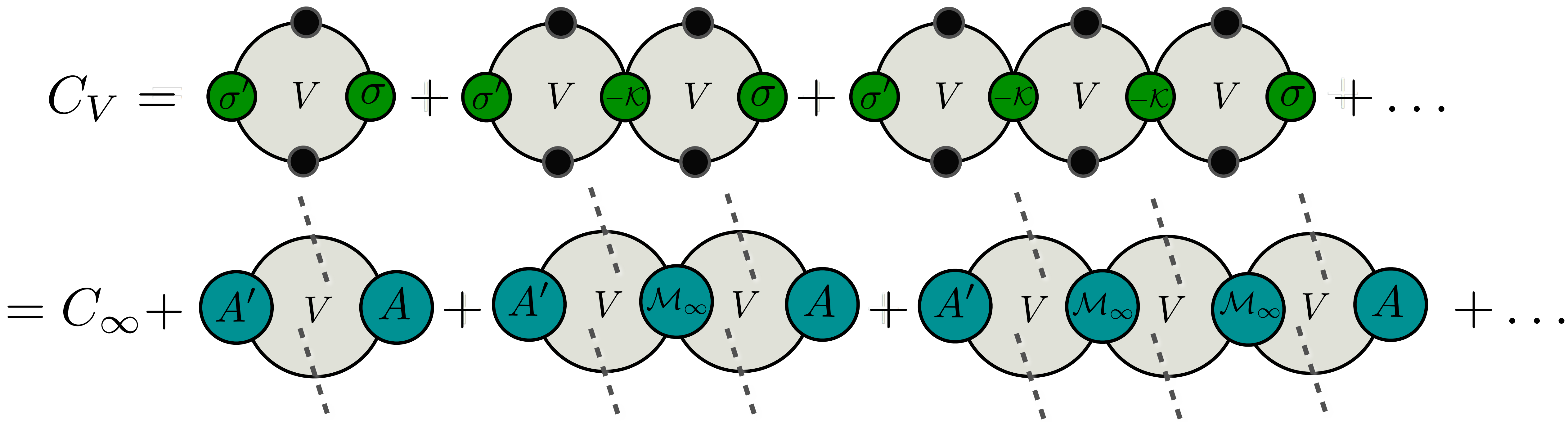



Kim, Sachrajda and Sharpe, Nucl.Phys.B727(2005)218-243.




$$T \to \infty, a \to 0$$

Let's review the Luescher's method first (see module I for more details).
A QFT derivation goes as follows:

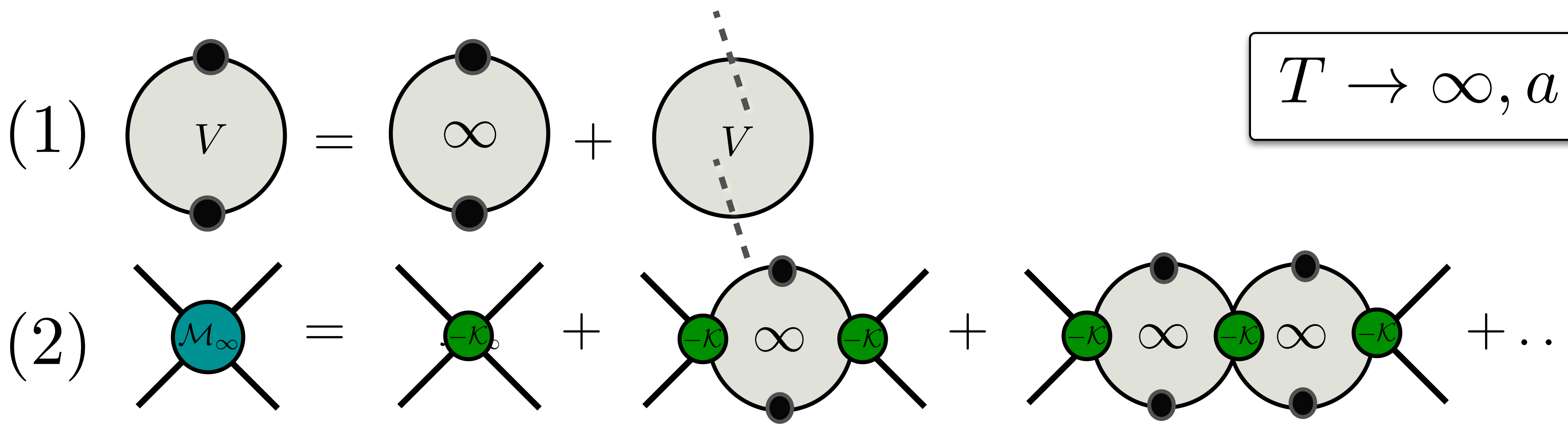



$$\det\left[\delta\mathcal{G}^V(E^*) + \mathcal{M}^{-1}(E^*)\right] = 0$$

Kim, Sachrajda and Sharpe, Nucl.Phys.B727(2005)218-243.

Finite-volume function

Scattering amplitude




$T \to \infty, a \to 0$

## Elastic amplitude more closely…

CM energy

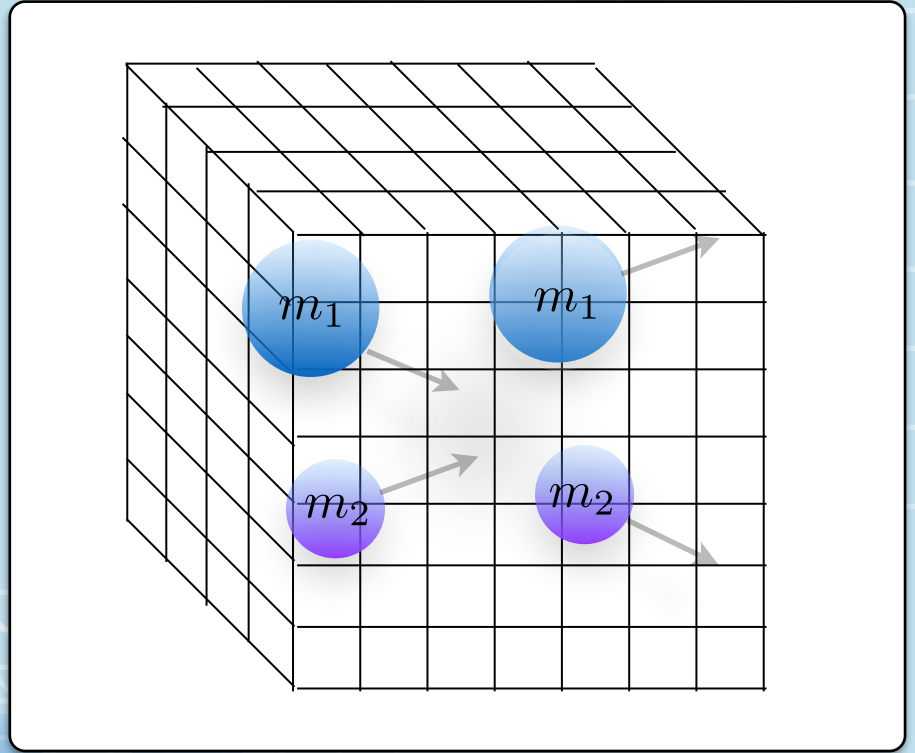
Phase shift

$$(\mathcal{M})_{l_1,m_1;l_2,m_2} = \delta_{l_1,l_2}\delta_{m_1,m_2}\frac{8\pi E^*}{nq^*}\frac{e^{2i\delta^{(l)}(q^*)}-1}{2i}$$

Symmetry factor

$$q^{*2} = \frac{1}{4}\left(E^{*2} - 2(m_1^2+m_2^2) + \frac{(m_1^2-m_2^2)^2}{E^{*2}}\right)$$

$$\det\left[\delta\mathcal{G}^V(E^*) + \mathcal{M}^{-1}(E^*)\right] = 0$$



## Finite-volume function more closely…

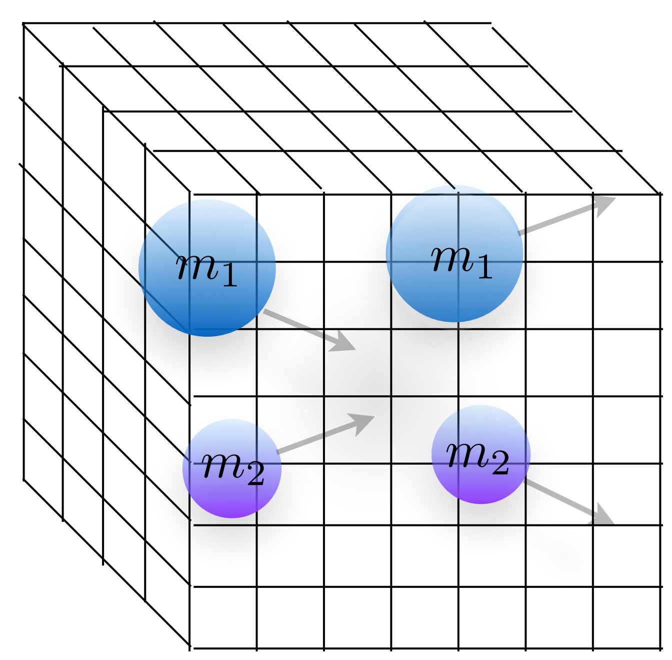
$$(\delta\mathcal{G}^V)_{l_1,m_1;l_2,m_2} = i\frac{q^* n}{8\pi E^*}\left(\delta_{l_1,l_2}\delta_{m_1,m_2} + i\frac{4\pi}{q^*}\sum_{l,m}\frac{\sqrt{4\pi}}{q^{*l}}c^{\mathbf{P}}_{lm}(q^{*2})\int d\Omega^* Y^*_{l_1 m_1} Y^*_{lm} Y_{l_2 m_2}\right)$$

$$c^{\mathbf{P}}_{lm}(x) = \frac{1}{\gamma}\left[\frac{1}{L^3}\sum_{\mathbf{k}} - \mathcal{P}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\right]\frac{\sqrt{4\pi}Y_{lm}(\hat{k}^*)\ k^{*l}}{k^{*2} - x}$$

$$\mathbf{k}^* = \gamma^{-1}\left[\mathbf{k}_{\parallel} - \frac{1}{2}(1 + \frac{m_1^2 - m_2^2}{{E^*}^2})\mathbf{P}\right] + \mathbf{k}_{\perp}$$

ZD and Savage, Phys. Rev.D84,114502(2011).

$$\det\left[\delta\mathcal{G}^V(E^*) + \mathcal{M}^{-1}(E^*)\right] = 0$$



# Finite-volume function more closely…

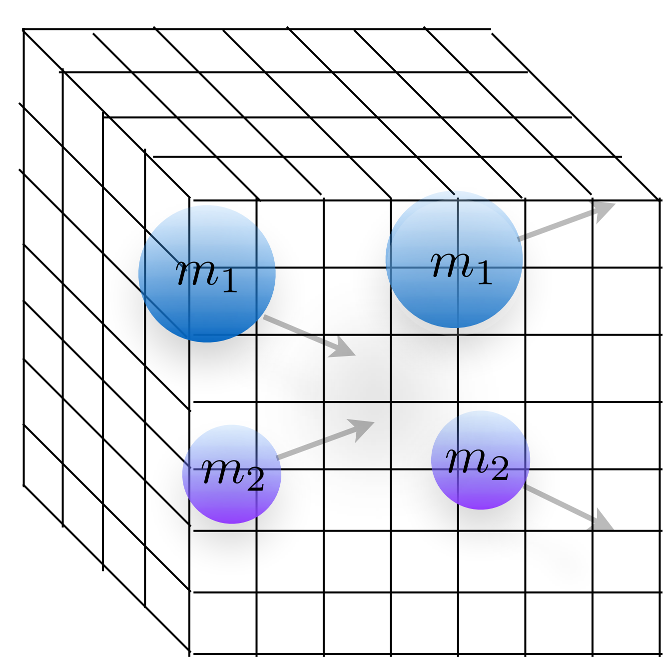
$$(\delta\mathcal{G}^V)_{l_1,m_1;l_2,m_2} = i\frac{q^* n}{8\pi E^*}\left(\delta_{l_1,l_2}\delta_{m_1,m_2} + i\frac{4\pi}{q^*}\sum_{l,m}\frac{\sqrt{4\pi}}{q^{*l}}c^{\mathbf{P}}_{lm}(q^{*2})\int d\Omega^* Y^*_{l_1m_1}Y^*_{lm}Y_{l_2m_2}\right)$$

$$c^{\mathbf{P}}_{lm}(x) = \frac{1}{\gamma}\left[\frac{1}{L^3}\sum_{\mathbf{k}} - \mathcal{P}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\right]\frac{\sqrt{4\pi}Y_{lm}(\hat{k}^*)\ k^{*l}}{k^{*2} - x}$$

$$\mathbf{k}^* = \gamma^{-1}\left[\mathbf{k}_{\|} - \frac{1}{2}\left(1 + \frac{m_1^2 - m_2^2}{E^{*2}}\right)\mathbf{P}\right] + \mathbf{k}_{\perp}$$

ZD and Savage, Phys. Rev.D84,114502(2011).

$$\det\left[\delta\mathcal{G}^V(E^*) + \mathcal{M}^{-1}(E^*)\right] = 0$$




S-wave approximation, valid at low energies:

$$q^* \cot \delta^{(0)} = 4\pi c_{00}(q^{*2})$$

S-wave phase shift

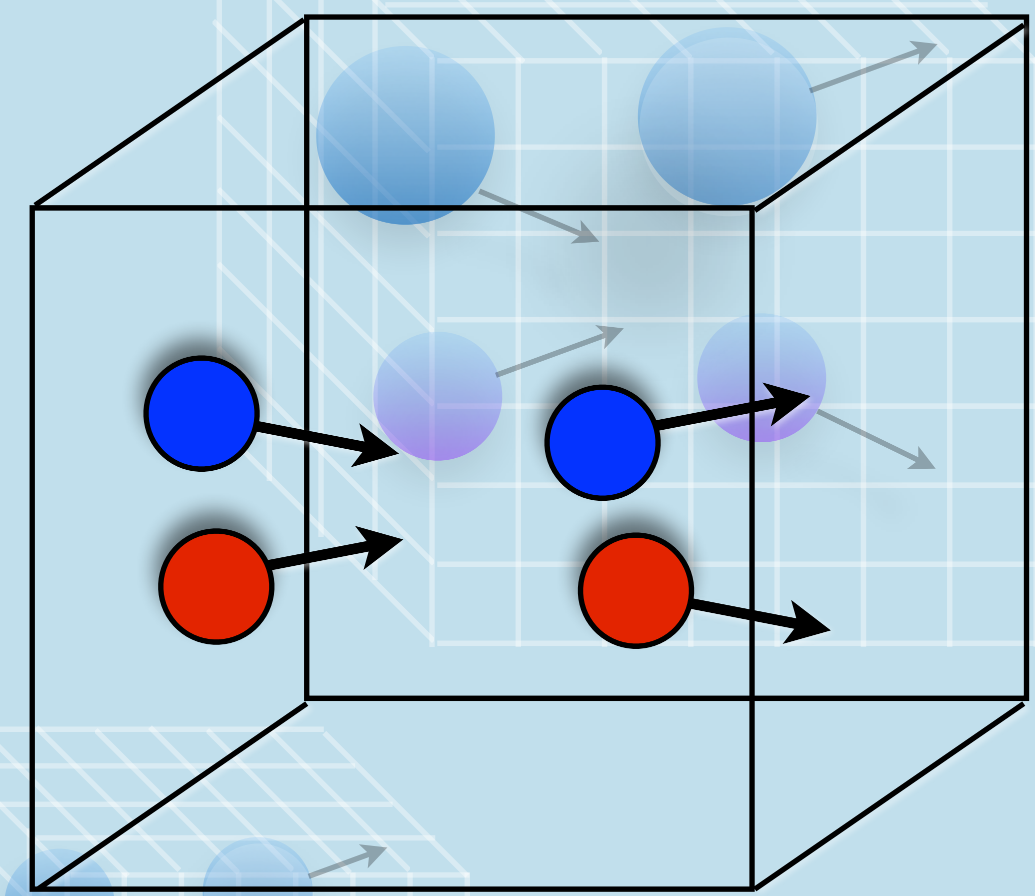
Now let's see an application of Luescher's method to obtain elastic scattering amplitudes of two hadrons from lattice QCD: `Wagman et al.(NPLQCD), Phys.Rev.D 96,114510(2017).`

## Two-baryon states with SU(3) symmetry

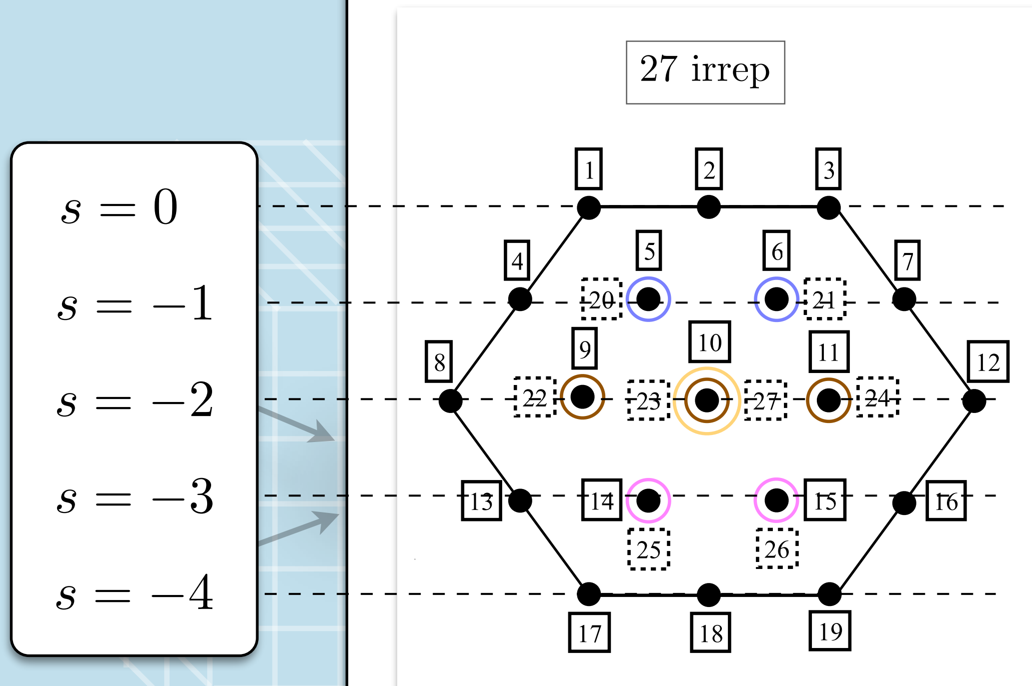
$$\{n, p, \Sigma^+, \Sigma^0, \Sigma^-, \Xi^0, \Xi^+, \Lambda\}$$

SU(3) decomposition of states: $8 \otimes 8 = 27 \oplus 10 \oplus \overline{10} \oplus 8_S \oplus 8_A \oplus 1$

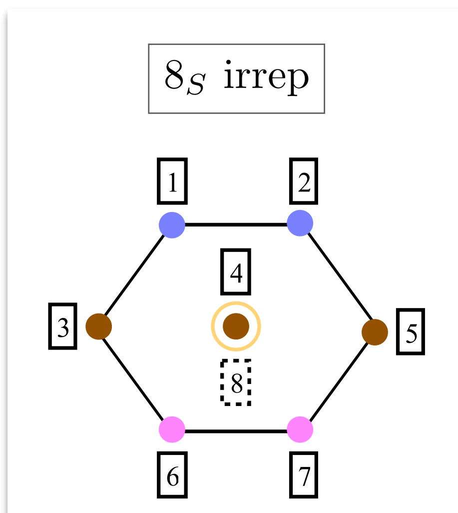
Let's see what these states are…



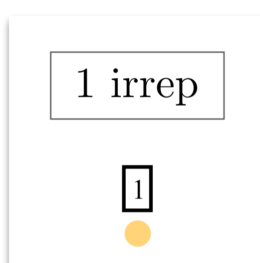
# SPIN-SINGLET STATES




| | Flavor channel | | Flavor channel |
|---|---|---|---|
| 1 | $nn$ | 14 | $-\sqrt{\frac{2}{3}}\Sigma^0\Xi^- + \sqrt{\frac{1}{3}}\Sigma^-\Xi^0$ |
| 2 | $\frac{1}{\sqrt{2}}(np+pn)$ | 15 | $\sqrt{\frac{1}{3}}\Sigma^+\Xi^- + \sqrt{\frac{2}{3}}\Sigma^0\Xi^0$ |
| 3 | $pp$ | 16 | $\Sigma^+\Xi^0$ |
| 4 | $\Sigma^- n$ | 17 | $\Xi^-\Xi^-$ |
| 5 | $\sqrt{\frac{2}{3}}\Sigma^0 n + \sqrt{\frac{1}{3}}\Sigma^- p$ | 18 | $\frac{1}{\sqrt{2}}(\Xi^-\Xi^0 + \Xi^0\Xi^-)$ |
| 6 | $-\sqrt{\frac{1}{3}}\Sigma^+ n + \sqrt{\frac{2}{3}}\Sigma^0 p$ | 19 | $\Xi^0\Xi^0$ |
| 7 | $\Sigma^+ p$ | 20 | $\Lambda n / -\sqrt{\frac{1}{3}}\Sigma^0 n + \sqrt{\frac{2}{3}}\Sigma^- p$ |
| 8 | $\Sigma^-\Sigma^-$ | 21 | $\Lambda p / \sqrt{\frac{2}{3}}\Sigma^+ n + \sqrt{\frac{1}{3}}\Sigma^0 p$ |
| 9 | $\frac{1}{\sqrt{2}}(\Sigma^-\Sigma^0 + \Sigma^0\Sigma^-)$ | 22 | $\Lambda\Sigma^- / \Xi^- n$ |
| 10 | $\frac{1}{\sqrt{6}}(\Sigma^-\Sigma^+ - 2\Sigma^0\Sigma^0 + \Sigma^+\Sigma^-)$ | 23 | $\Lambda\Sigma^0 / \frac{1}{\sqrt{2}}(\Xi^- p - \Xi^0 n)$ |
| 11 | $\frac{1}{\sqrt{2}}(\Sigma^0\Sigma^+ + \Sigma^+\Sigma^0)$ | 24 | $\Lambda\Sigma^+ / \Xi^0 p$ |
| 12 | $\Sigma^+\Sigma^+$ | 25 | $\Lambda\Xi^- / \sqrt{\frac{1}{3}}\Sigma^0\Xi^- + \sqrt{\frac{2}{3}}\Sigma^-\Xi^0$ |
| 13 | $\Sigma^-\Xi^-$ | 26 | $\Lambda\Xi^0 / -\sqrt{\frac{2}{3}}\Sigma^+\Xi^- + \sqrt{\frac{1}{3}}\Sigma^0\Xi^0$ |
| 27 | $\frac{1}{\sqrt{3}}(\Sigma^+\Sigma^- + \Sigma^0\Sigma^0 + \Sigma^-\Sigma^+) / \frac{1}{\sqrt{2}}(\Xi^0 n + \Xi^- p) / \Lambda\Lambda$ | | |

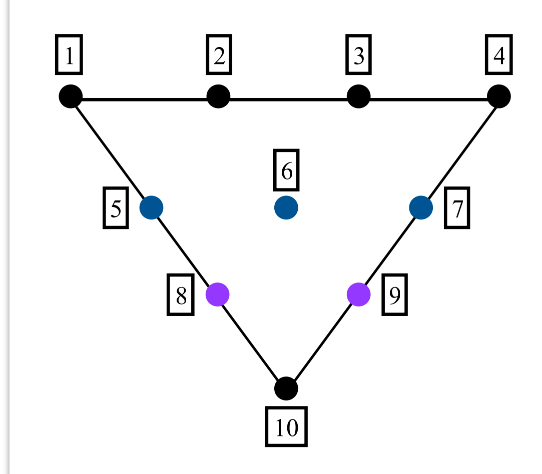



| | Flavor channel |
|---|---|
| 1 | $\Lambda n / -\sqrt{\frac{1}{3}}\Sigma^0 n + \sqrt{\frac{2}{3}}\Sigma^- p$ |
| 2 | $\Lambda p / \sqrt{\frac{2}{3}}\Sigma^+ n + \sqrt{\frac{1}{3}}\Sigma^0 p$ |
| 3 | $\Lambda\Sigma^- / \Xi^- n$ |
| 4 | $\Lambda\Sigma^0 / \frac{1}{\sqrt{2}}(\Xi^- p - \Xi^0 n)$ |
| 5 | $\Lambda\Sigma^+ / \Xi^0 p$ |
| 6 | $\Lambda\Xi^- / \sqrt{\frac{1}{3}}\Sigma^0\Xi^- + \sqrt{\frac{2}{3}}\Sigma^-\Xi^0$ |
| 7 | $\Lambda\Xi^0 / -\sqrt{\frac{2}{3}}\Sigma^+\Xi^- + \sqrt{\frac{1}{3}}\Sigma^0\Xi^0$ |
| 8 | $\frac{1}{\sqrt{3}}(\Sigma^+\Sigma^- + \Sigma^0\Sigma^0 + \Sigma^-\Sigma^+) / \frac{1}{\sqrt{2}}(\Xi^0 n + \Xi^- p) / \Lambda\Lambda$ |

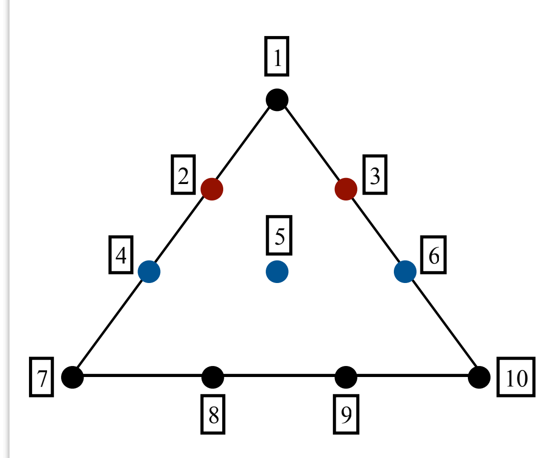



| | Flavor channel |
|---|---|
| 1 | $\frac{1}{\sqrt{3}}(\Sigma^+\Sigma^- + \Sigma^0\Sigma^0 + \Sigma^-\Sigma^+) / \frac{1}{\sqrt{2}}(\Xi^0 n + \Xi^- p) / \Lambda\Lambda$ |

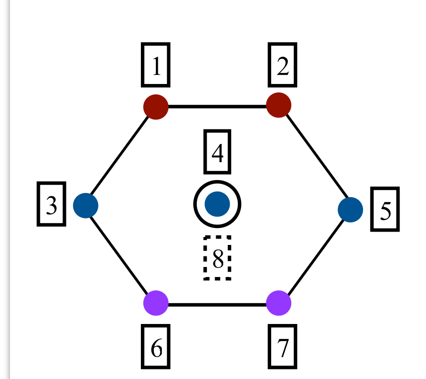
# SPIN-TRIPLET STATES




| | Flavor channel |
|---|---|
| 1 | $\Sigma^- n$ |
| 2 | $\sqrt{\frac{2}{3}}\Sigma^0 n + \sqrt{\frac{1}{3}}\Sigma^- p$ |
| 3 | $-\sqrt{\frac{1}{3}}\Sigma^+ n + \sqrt{\frac{2}{3}}\Sigma^0 p$ |
| 4 | $\Sigma^+ p$ |
| 5 | $\frac{1}{\sqrt{2}}(\Sigma^-\Sigma^0 - \Sigma^0\Sigma^-)/\Xi^- n/\Lambda\Sigma^-$ |
| 6 | $\frac{1}{\sqrt{2}}(\Sigma^-\Sigma^+ - \Sigma^+\Sigma^-)/\frac{1}{\sqrt{2}}(\Xi^- p - \Xi^0 n)/\Lambda\Sigma^0$ |
| 7 | $\frac{1}{\sqrt{2}}(\Sigma^0\Sigma^+ - \Sigma^+\Sigma^0)/\Xi^0 p/\Lambda\Sigma^+$ |
| 8 | $\sqrt{\frac{1}{3}}\Sigma^0\Xi^- + \sqrt{\frac{2}{3}}\Sigma^-\Xi^0/\Lambda\Xi^-$ |
| 9 | $-\sqrt{\frac{2}{3}}\Sigma^+\Xi^- + \sqrt{\frac{1}{3}}\Sigma^0\Xi^0/\Lambda\Xi^0$ |
| 10 | $\frac{1}{\sqrt{2}}(\Xi^0\Xi^- \text{-} \Xi^-\Xi^0)$ |




| | Flavor channel |
|---|---|
| 1 | $\frac{1}{\sqrt{2}}(pn - np)$ |
| 2 | $-\sqrt{\frac{1}{3}}\Sigma^0 n + \sqrt{\frac{2}{3}}\Sigma^- p/\Lambda n$ |
| 3 | $\sqrt{\frac{2}{3}}\Sigma^+ n + \sqrt{\frac{1}{3}}\Sigma^0 p/\Lambda p$ |
| 4 | $\frac{1}{\sqrt{2}}(\Sigma^-\Sigma^0 - \Sigma^0\Sigma^-)/\Xi^- n/\Lambda\Sigma^-$ |
| 5 | $\frac{1}{\sqrt{2}}(\Sigma^-\Sigma^+ - \Sigma^+\Sigma^-)/\frac{1}{\sqrt{2}}(\Xi^- p - \Xi^0 n)/\Lambda\Sigma^0$ |
| 6 | $\frac{1}{\sqrt{2}}(\Sigma^0\Sigma^+ - \Sigma^+\Sigma^0)/\Xi^0 p/\Lambda\Sigma^+$ |
| 7 | $\Sigma^-\Xi^-$ |
| 8 | $-\sqrt{\frac{2}{3}}\Sigma^0\Xi^- + \sqrt{\frac{1}{3}}\Sigma^-\Xi^0$ |
| 9 | $\sqrt{\frac{1}{3}}\Sigma^+\Xi^- + \sqrt{\frac{2}{3}}\Sigma^0\Xi^0$ |
| 10 | $\Sigma^+\Xi^0$ |



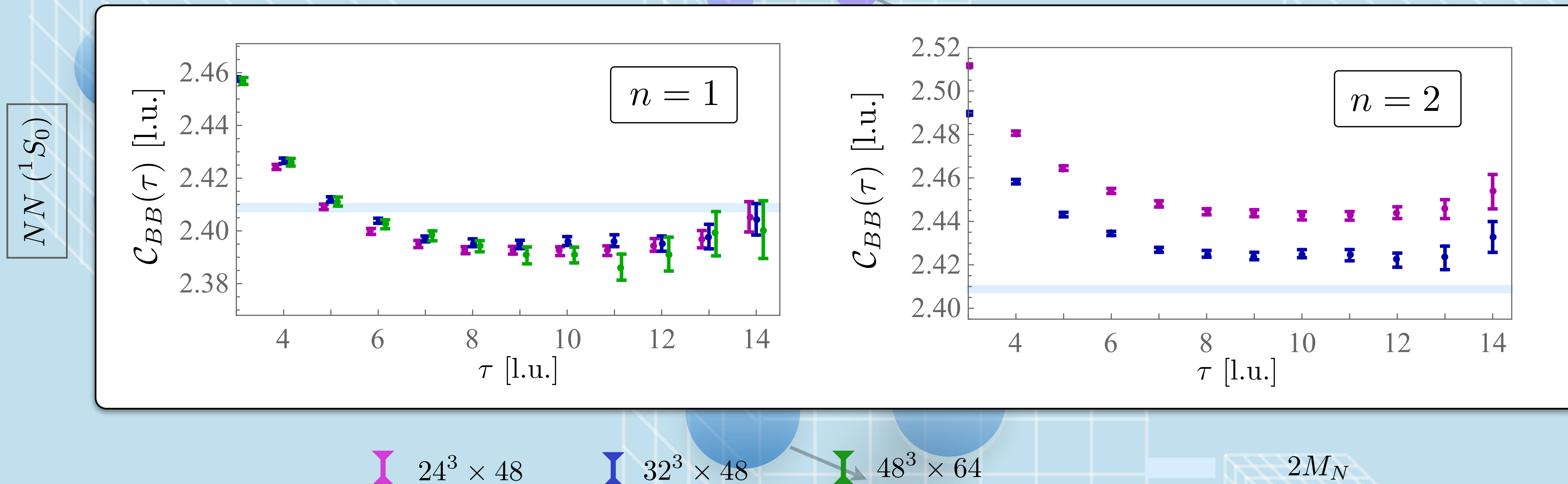

| | Flavor channel |
|---|---|
| 1 | $-\sqrt{\frac{1}{3}}\Sigma^0 n + \sqrt{\frac{2}{3}}\Sigma^- p/\Lambda n$ |
| 2 | $\sqrt{\frac{2}{3}}\Sigma^+ n + \sqrt{\frac{1}{3}}\Sigma^0 p/\Lambda p$ |
| 3 | $\frac{1}{\sqrt{2}}(\Sigma^-\Sigma^0 - \Sigma^0\Sigma^-)/\Xi^- n/\Lambda\Sigma^-$ |
| 4 | $\frac{1}{\sqrt{2}}(\Sigma^-\Sigma^+ - \Sigma^+\Sigma^-)/\frac{1}{\sqrt{2}}(\Xi^- p - \Xi^0 n)/\Lambda\Sigma^0$ |
| 5 | $\frac{1}{\sqrt{2}}(\Sigma^0\Sigma^+ - \Sigma^+\Sigma^0)/\Xi^0 p/\Lambda\Sigma^+$ |
| 6 | $\sqrt{\frac{1}{3}}\Sigma^0\Xi^- + \sqrt{\frac{2}{3}}\Sigma^-\Xi^0/\Lambda\Xi^-$ |
| 7 | $-\sqrt{\frac{2}{3}}\Sigma^+\Xi^- + \sqrt{\frac{1}{3}}\Sigma^0\Xi^0/\Lambda\Xi^0$ |
| 8 | $\frac{1}{\sqrt{2}}(\Xi^0 n + \Xi^- p)$ |

## BONUS EXERCISE 2

Construct the tables shown for the flavor decomposition of two octet-baryon states in each irreducible representation of the SU(3) group.

# Step I: Obtain the lowest-lying spectra

$N_f = 3,\ m_\pi = 0.806\ \text{GeV},\ a = 0.145(2)\ \text{fm}$

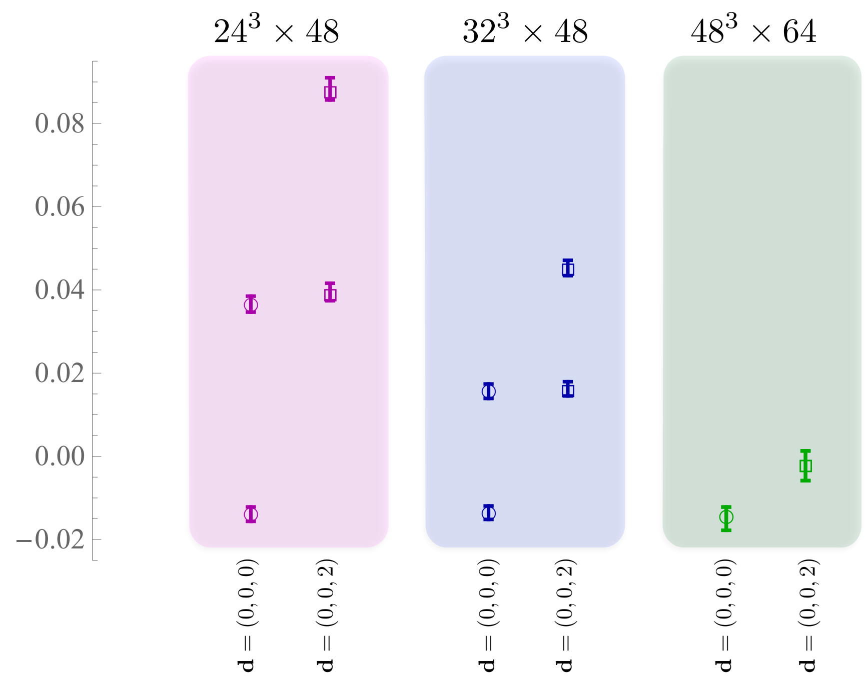



$$C_{\hat{\mathcal{O}},\hat{\mathcal{O}}'}(\tau;\mathbf{d}) = \sum_{\mathbf{x}} e^{2\pi i \mathbf{d}\cdot\mathbf{x}/L} \langle 0|\hat{\mathcal{O}}'(\mathbf{x},\tau)\hat{\mathcal{O}}^\dagger(\mathbf{0},0)|0\rangle = \mathcal{Z}_0'\mathcal{Z}_0^\dagger e^{-E^{(0)}\tau} + \mathcal{Z}_1'\mathcal{Z}_1^\dagger e^{-E^{(1)}\tau} + \ldots$$

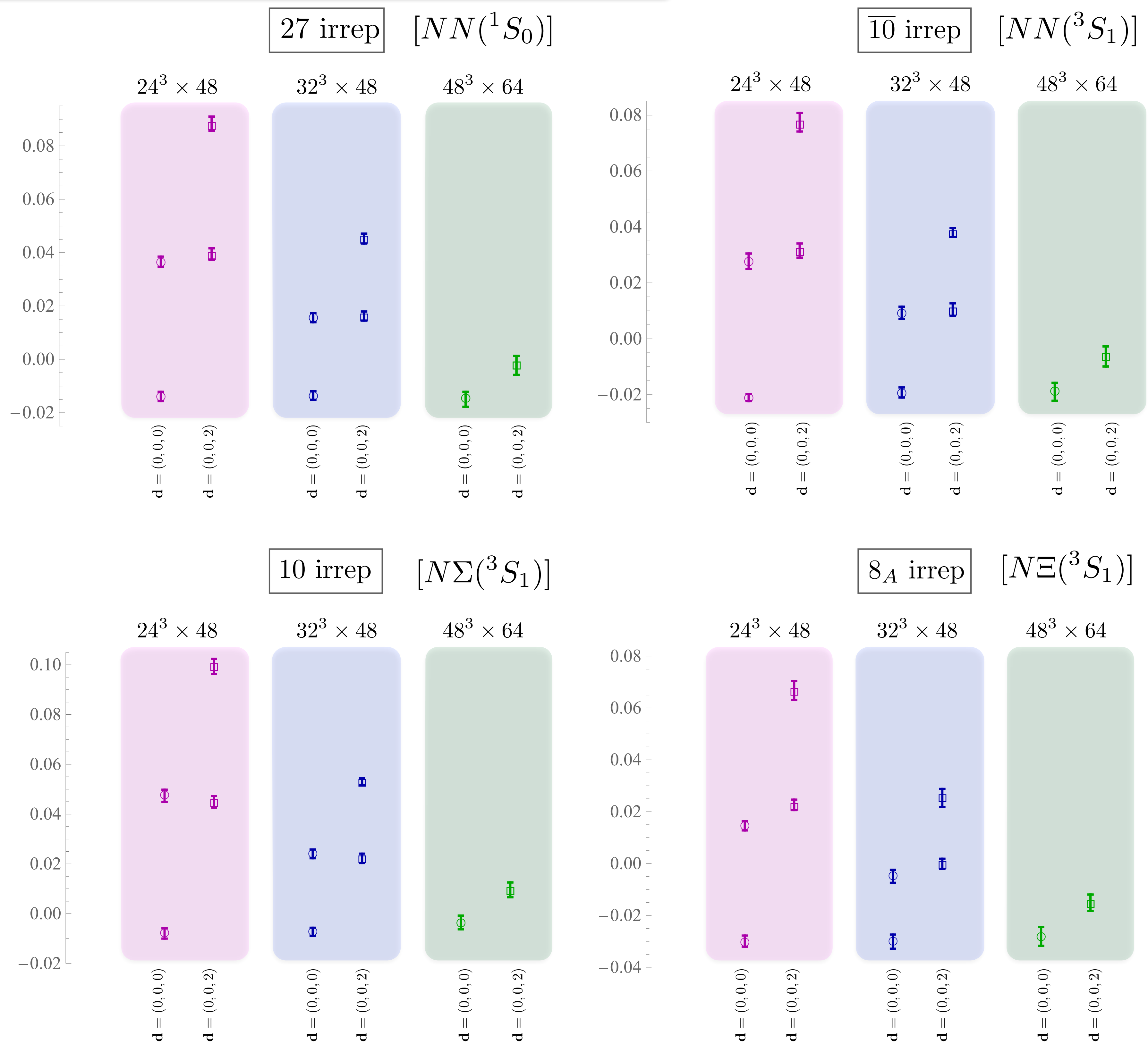
Beane et al (NPLQCD), arXiv:1705.09239, Wagman et al (NPLQCD), arXiv:1706.06550.

# Step I: Obtain the lowest-lying spectra

27 irrep $[NN(^1S_0)]$

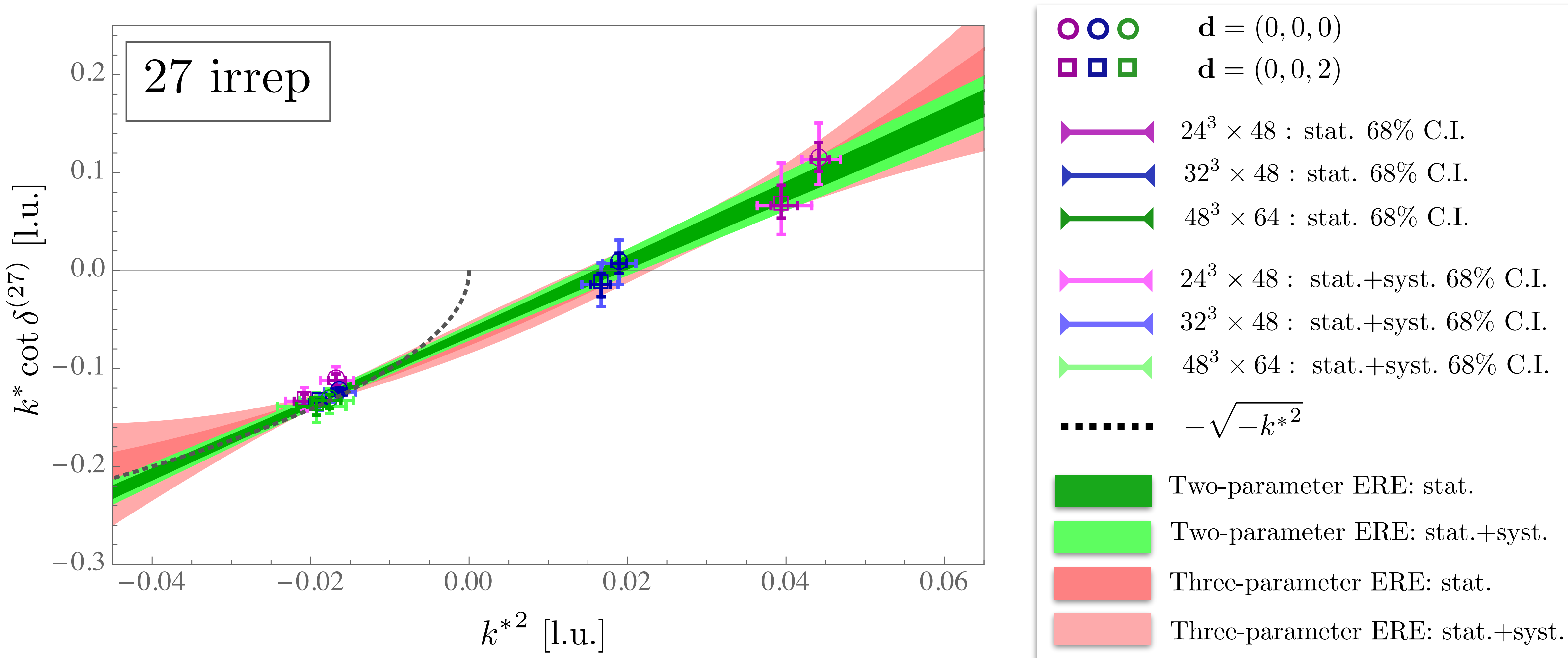


# Step I: Obtain the lowest-lying spectra



Step II: Feed the energies to the Luescher's equation and obtain the S-wave scattering phase shifts.

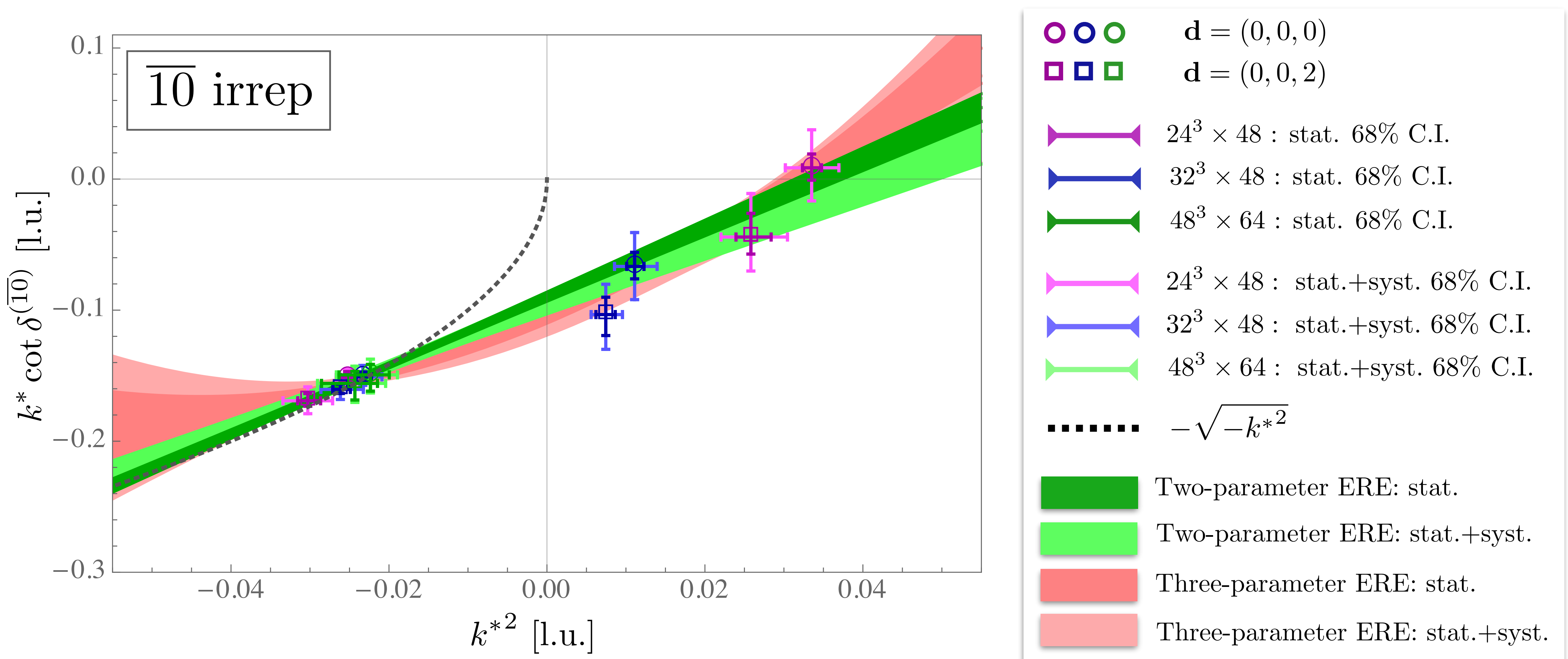
$N_f = 3,\ m_\pi = 0.806\ \text{GeV},\ a = 0.145(2)\ \text{fm}$




$$B = 20.6^{(+1.8)(+2.8)}_{(-2.4)(-1.6)}\ \text{MeV}$$

Step II: Feed the energies to the Luescher's equation and obtain the S-wave scattering phase shifts.

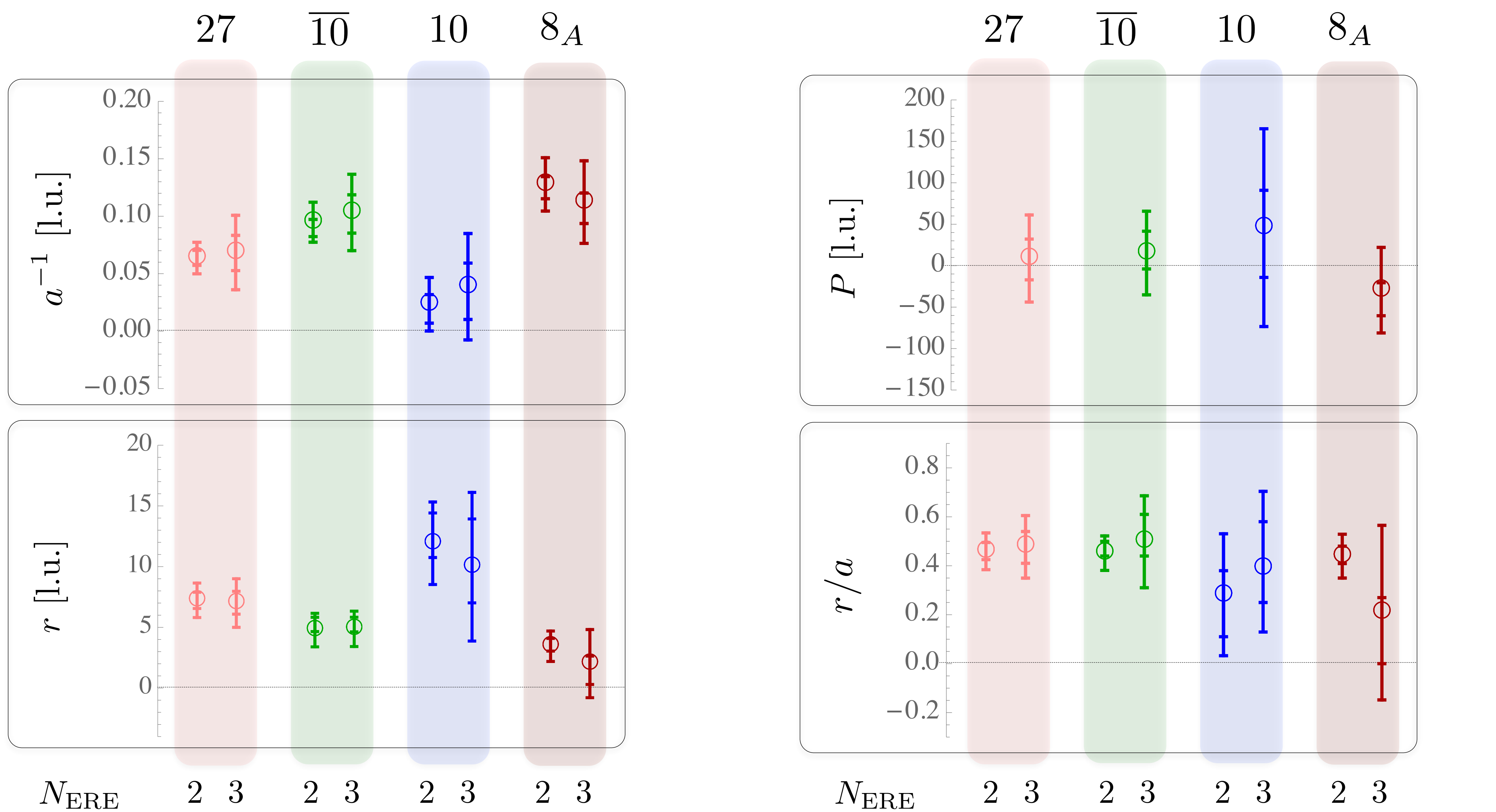
$N_f = 3,\ m_\pi = 0.806$ GeV, $a = 0.145(2)$ fm




$$B = 27.9^{(+3.1)(+2.2)}_{(-2.3)(-1.4)} \text{ MeV}$$

Step II: Feed the energies to the Luescher's equation and obtain the S-wave scattering phase shifts.

$N_f = 3,\ m_\pi = 0.806\ \text{GeV},\ a = 0.145(2)\ \text{fm}$




$$k^* \cot\delta(k^*) = -\frac{1}{a} + \frac{1}{2} r k^{*2} + \dots$$

This is a curious observation. Why are the scattering parameters so close to each other in different channels?

## (Step III): To impact studies of larger systems of baryons, match to a proper EFT:

i) There are only six independent interactions at leading order in chiral EFT:

$$\mathcal{L}_{BB}^{(0)} = -c_1 \mathrm{Tr}(B_i^\dagger B_i B_j^\dagger B_j) - c_2 \mathrm{Tr}(B_i^\dagger B_j B_j^\dagger B_i) - c_3 \mathrm{Tr}(B_i^\dagger B_j^\dagger B_i B_j)$$
$$-c_4 \mathrm{Tr}(B_i^\dagger B_j^\dagger B_j B_i) - c_5 \mathrm{Tr}(B_i^\dagger B_i)\mathrm{Tr}(B_j^\dagger B_j) - c_6 \mathrm{Tr}(B_i^\dagger B_j)\mathrm{Tr}(B_j^\dagger B_i).$$

$$B = \begin{bmatrix} \Sigma^0/\sqrt{2} + \Lambda/\sqrt{6} & \Sigma^+ & p \\ \Sigma^- & -\Sigma^0/\sqrt{2} + \Lambda/\sqrt{6} & n \\ \Xi^- & \Xi^0 & -\sqrt{\frac{2}{3}}\Lambda \end{bmatrix}$$

Spin indices

Savage and Wise (1996).

ii) Now if you solve for scattering lengths in different irreducible representations, you will find that:

$$\left[-\frac{1}{a^{(27)}} + \mu\right]^{-1} = \frac{M_B}{2\pi}\left(c_1 - c_2 + c_5 - c_6\right), \qquad \left[-\frac{1}{a^{(10)}} + \mu\right]^{-1} = \frac{M_B}{2\pi}\left(-c_1 - c_2 + c_5 + c_6\right),$$

$$\left[-\frac{1}{a^{(\overline{10})}} + \mu\right]^{-1} = \frac{M_B}{2\pi}\left(c_1 + c_2 + c_5 + c_6\right), \qquad \left[-\frac{1}{a^{(8_A)}} + \mu\right]^{-1} = \frac{M_B}{2\pi}\left(\frac{3c_3}{2} + \frac{3c_4}{2} + c_5 + c_6\right)$$

Renormalization scale for unnatural interactions

## EXERCISE 5

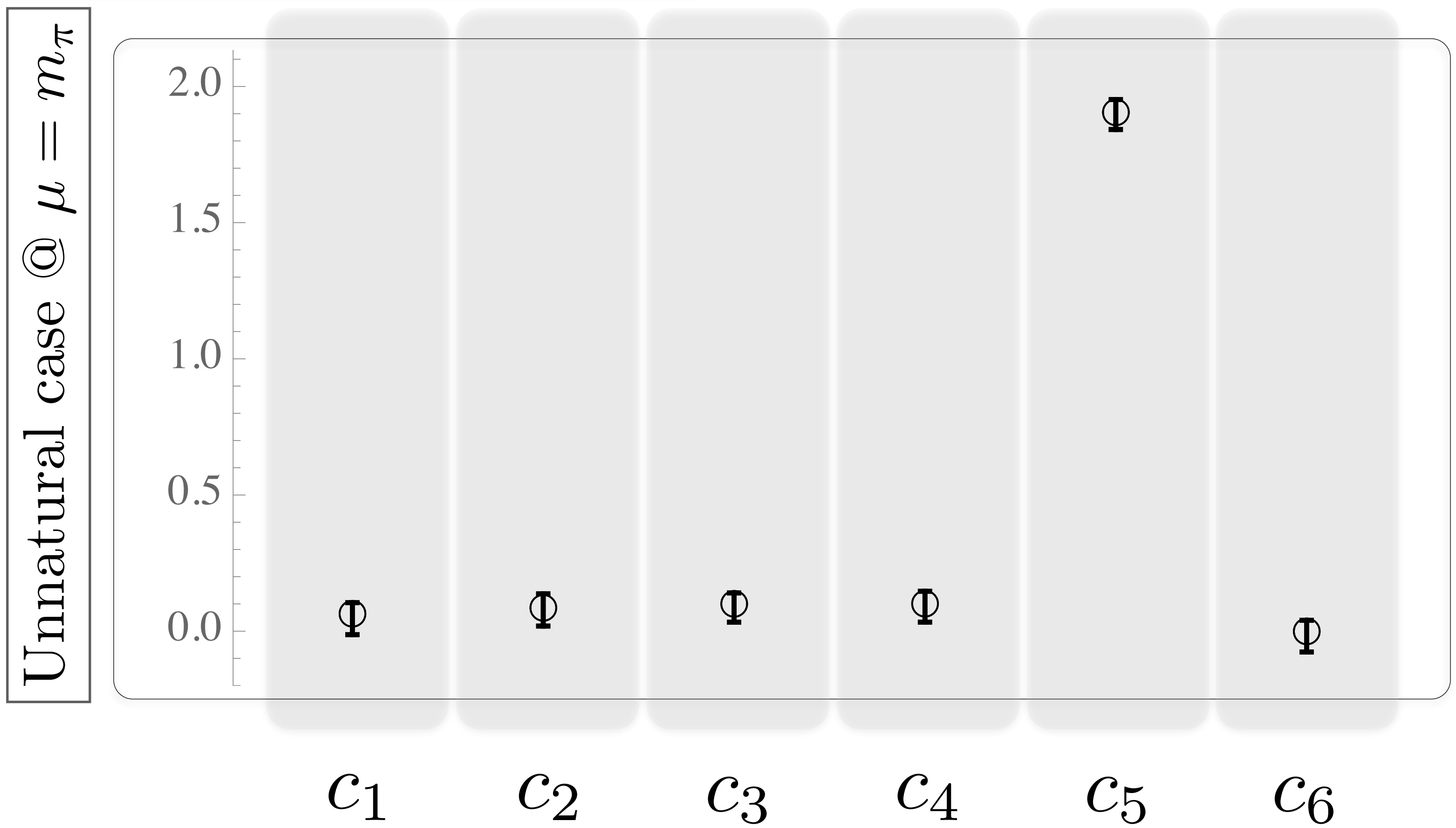
Starting from the leading-order SU(3) flavor-symmetric Lagrangian for interactions of two octet baryons, derive the relation between the scattering length and Savage-Wise coefficients in the 27 irreducible representation. You can express the scattering amplitude in terms of a leading-order effective range expansion and set $\mu$ equal to zero (assuming natural interactions).

## BONUS EXERCISE 3

Repeat the same exercise for all other irreducible representations of SU(3). If you have already automated this procedure using Mathematica or other programs in the above exercise, all relations can be obtained at the same time.

(Step III): To impact studies of larger systems of baryons, match to a proper EFT:

$N_f = 3, \; m_\pi = 0.806 \text{ GeV}, \; a = 0.145(2) \text{ fm}$



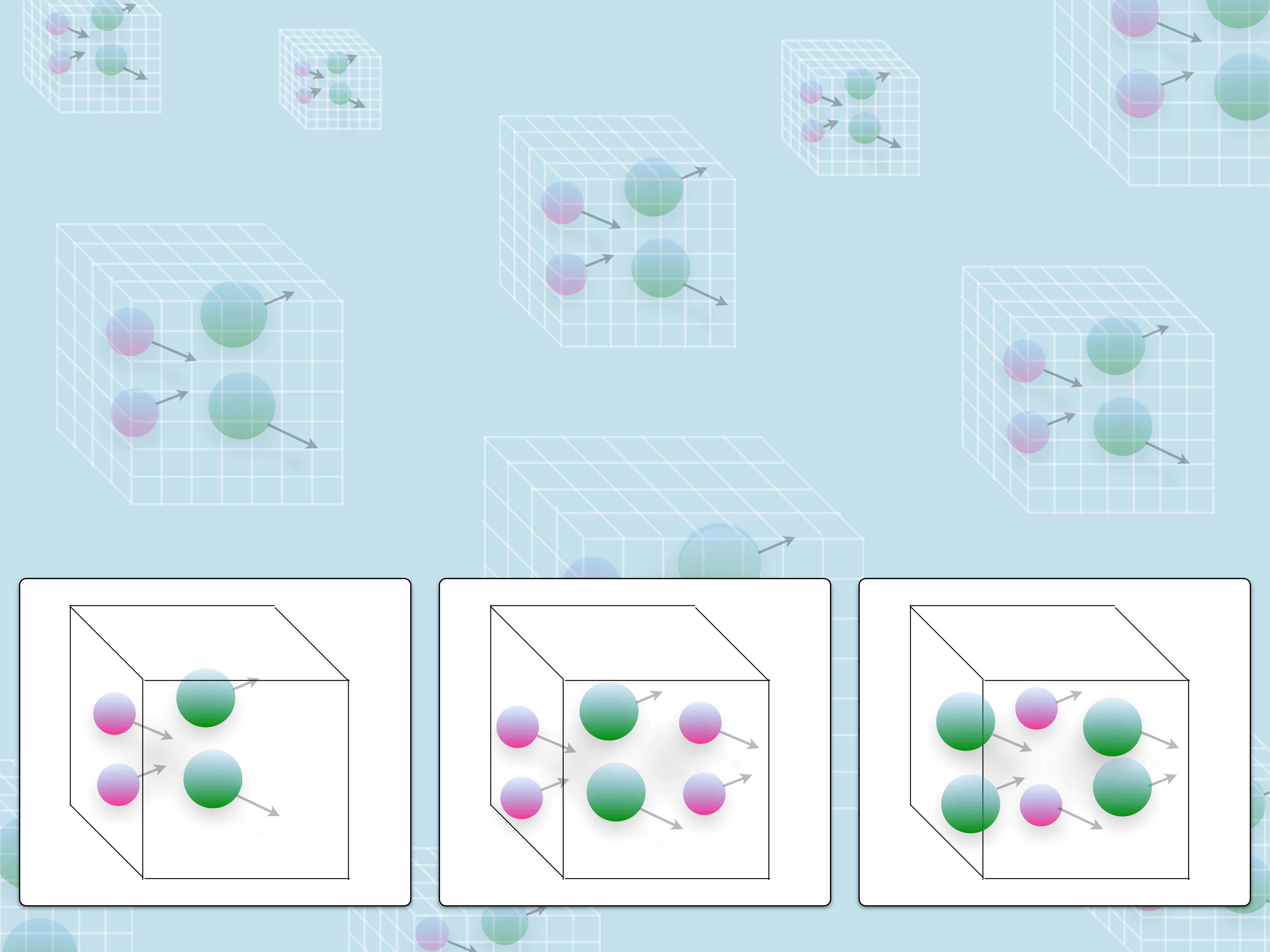

$$\mathcal{L}^{(0)}_{BB} = -c_1 \mathrm{Tr}(B_i^\dagger B_i B_j^\dagger B_j) - c_2 \mathrm{Tr}(B_i^\dagger B_j B_j^\dagger B_i) - c_3 \mathrm{Tr}(B_i^\dagger B_j^\dagger B_i B_j) - c_4 \mathrm{Tr}(B_i^\dagger B_j^\dagger B_j B_i) - c_5 \mathrm{Tr}(B_i^\dagger B_i)\mathrm{Tr}(B_j^\dagger B_j) - c_6 \mathrm{Tr}(B_i^\dagger B_j)\mathrm{Tr}(B_j^\dagger B_i).$$

Kaplan and Savage (1998).

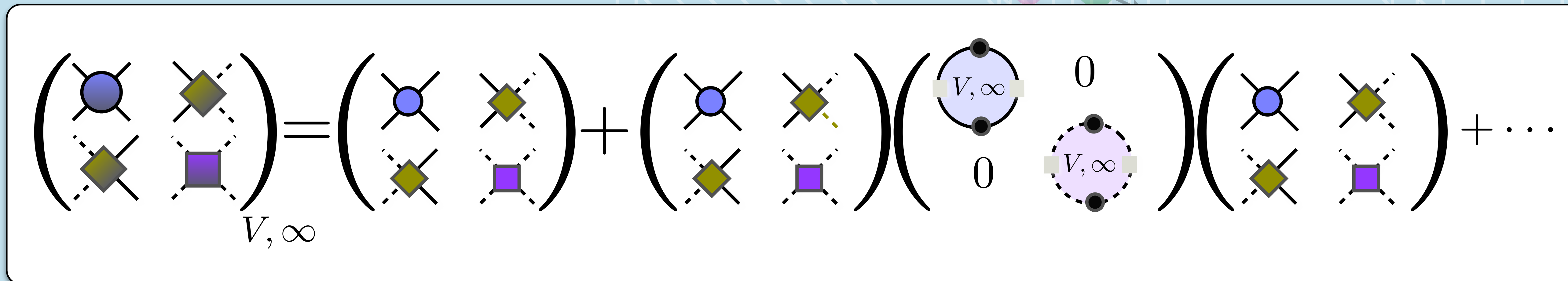
$SU(N_f = 3) \longrightarrow SU(2N_f = 6) \longrightarrow SU(16)$

This is in fact a prediction of QCD with a large number of colors for nuclear and hyper nuclear interactions.

# TWO-BODY INELASTIC SCATTERING

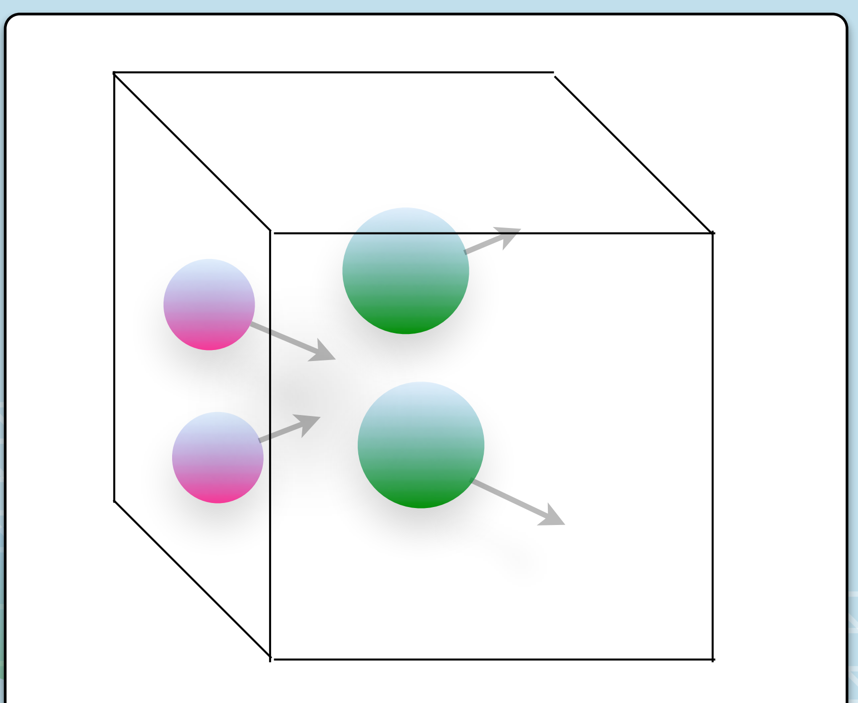


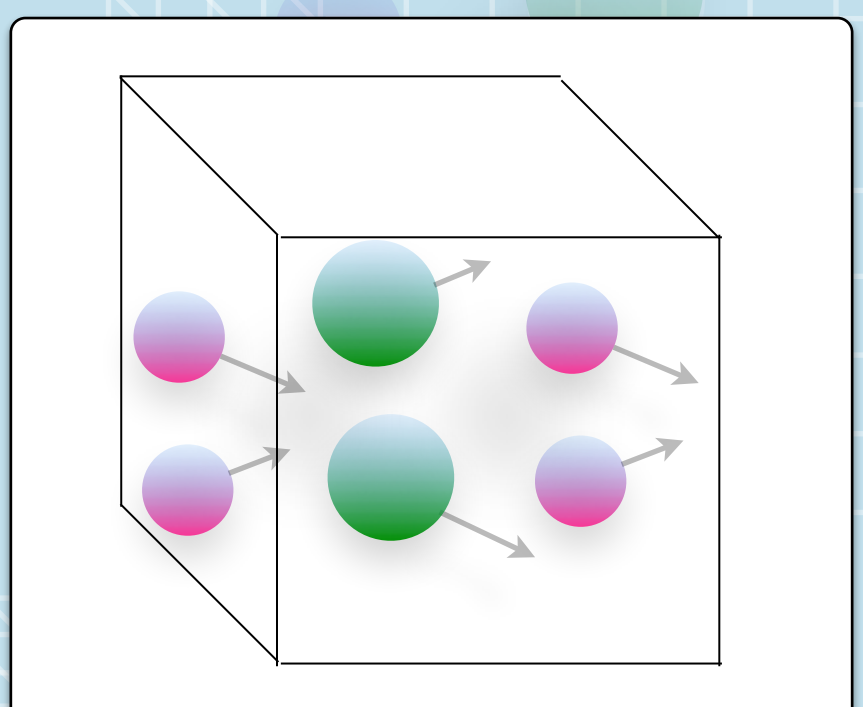
Coupled-channel generalization of Luescher's formula is straightforward. Requires upgrading amplitudes and finite-volume functions to matrices in the channel space:

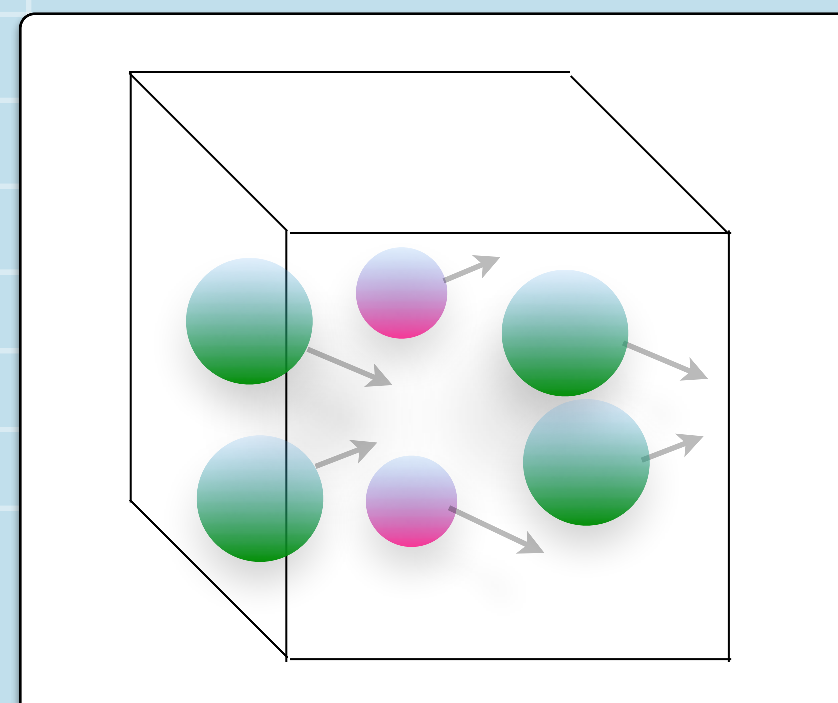



Briceno and ZD, Phys. Rev.D88,094507(2013).

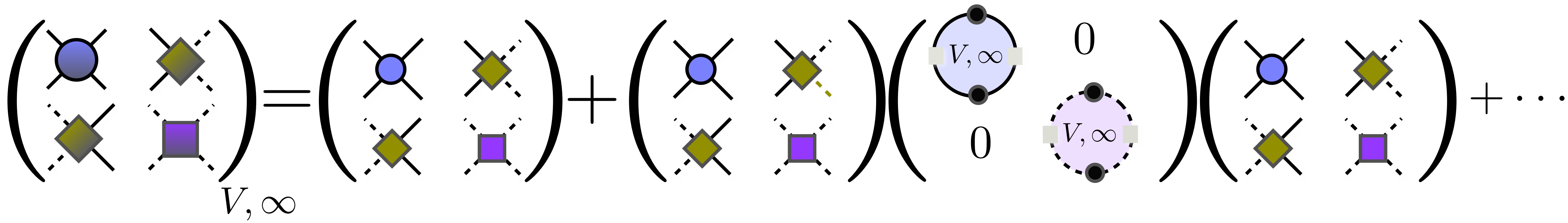
Hansen and Sharpe, Phys. Rev. D86, 016007(2012).







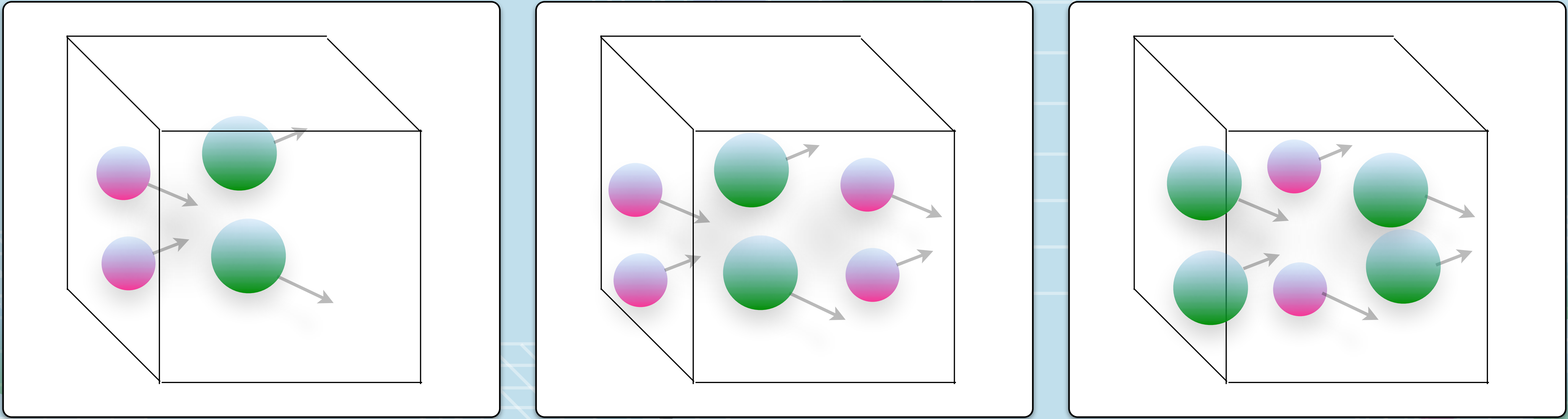
Coupled-channel generalization of Luescher's formula is straightforward. Requires upgrading amplitudes and finite-volume functions to matrices in the channel space:



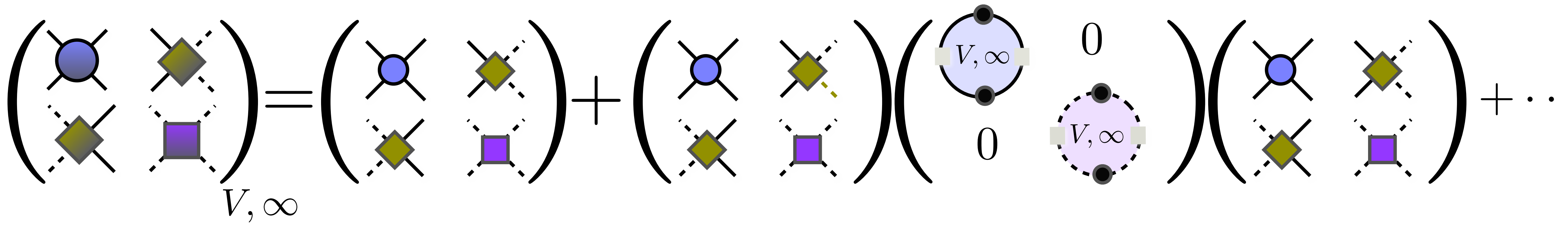

Briceno and ZD, Phys. Rev.D88,094507(2013).

Hansen and Sharpe, Phys. Rev. D86, 016007(2012).

$$\text{Det}\left[\delta\mathcal{G}^V(E^*) + \mathcal{M}^{-1}(E^*)\right] = 0$$



Coupled-channel generalization of Luescher's formula is straightforward. Requires upgrading amplitudes and finite-volume functions to matrices in the channel space:




Briceno and ZD, Phys. Rev.D88,094507(2013).

Hansen and Sharpe, Phys. Rev. D86, 016007(2012).

$$\text{Det}\left[\delta\mathcal{G}^V(E^*) + \mathcal{M}^{-1}(E^*)\right] = 0$$

$$(\mathcal{M}_{i,i})_{l_1,m_1;l_2,m_2} = \delta_{l_1,l_2}\delta_{m_1,m_2}\frac{8\pi E^*}{n_i q_i^*}\frac{\cos(2\bar{\epsilon})e^{2i\delta_i^{(l_1)}(q_i^*)} - 1}{2i},$$

Channel index I or II

$$(\mathcal{M}_{I,II})_{l_1,m_1;l_2,m_2} = \delta_{l_1,l_2}\delta_{m_1,m_2}\frac{8\pi E^*}{\sqrt{n_I n_{II} q_I^* q_{II}^*}}\sin(2\bar{\epsilon})\frac{e^{i(\delta_I^{(l_1)}(q_I^*)+\delta_{II}^{(l_1)}(q_{II}^*))}}{2}$$

Mixing angle between two channels

# EXERCISE 6

Derive the manifestly real form of a coupled two-channel scattering in the S-wave limit:

$$\cos 2\bar{\epsilon} \cos\left(\phi_1^P + \delta_1 - \phi_2^P - \delta_2\right) = \cos\left(\phi_1^P + \delta_1 + \phi_2^P + \delta_2\right)$$

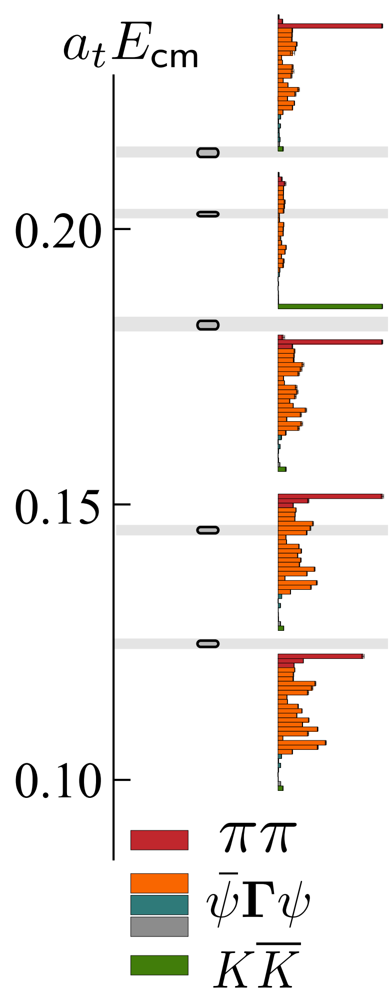
Here 1 and 2 indices refer to the two channels and superscript (0) is removed from the S-wave phase shifts for brevity. The finite-volume phase function $\phi_i^P$ is defined as:

$$q_i^* \cot(\phi_i^P) \equiv -4\pi c_{00}^P(q_i^{*2})$$

for *i*=1,2. This is a generic result: Luescher's "quantization condition" is a real condition.

Now let's see an application of the coupled-channel formalism: Hunting resonances using lattice QCD in the P-wave coupled $\pi\pi - K\overline{K}$ channel

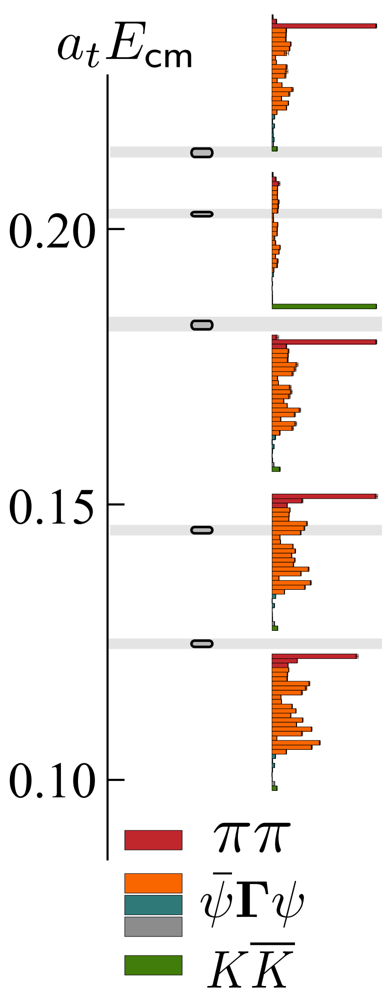
Wilson et al.(HadSpec), Phys.Rev. D92 (2015), 094502



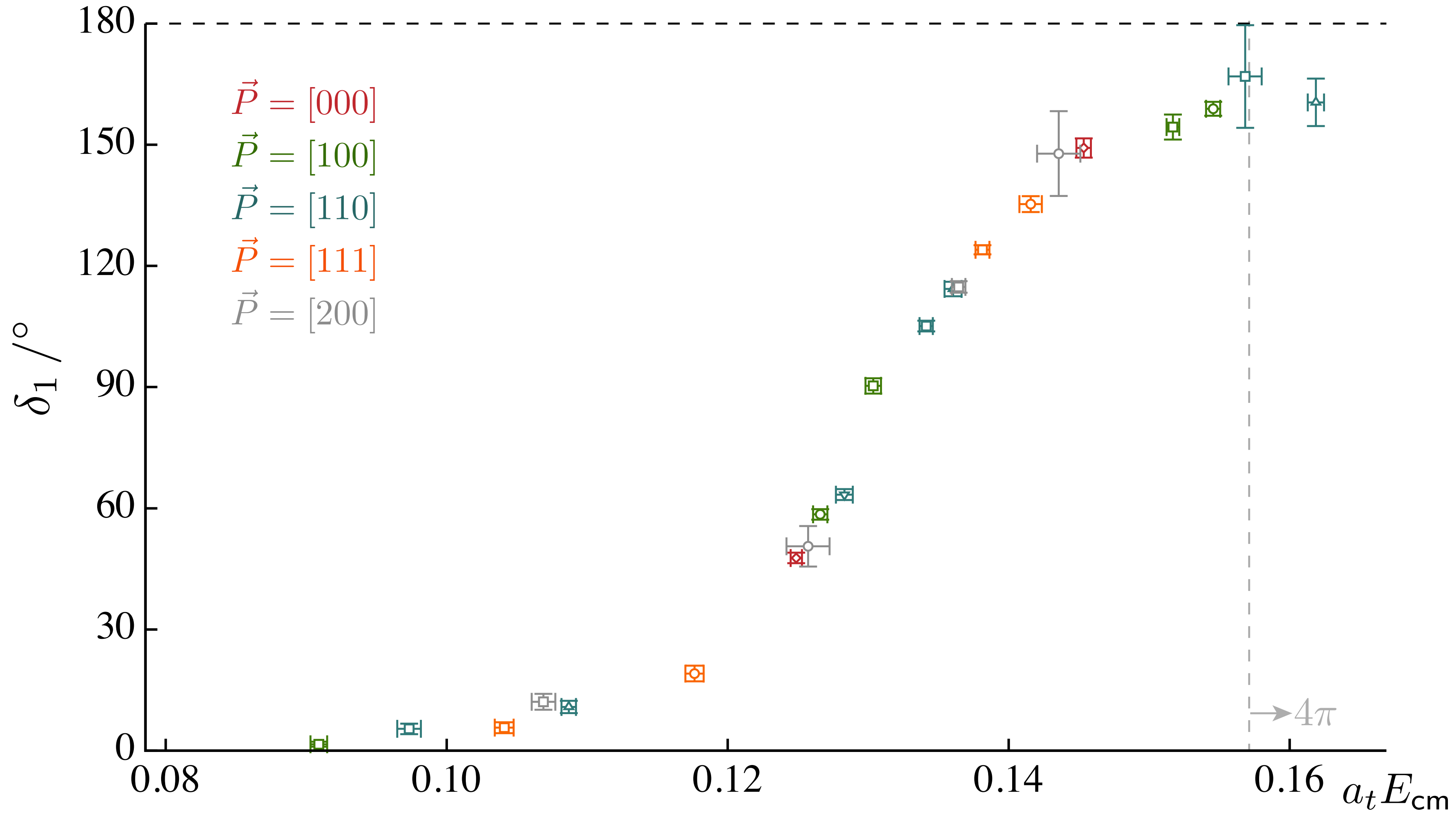

Example: T1 irrep energies

$N_f = 2+1, m_\pi = 236$ MeV, $V \approx (4\ \text{fm})^3$

Wilson et al.(HadSpec), Phys.Rev. D92 (2015), 094502



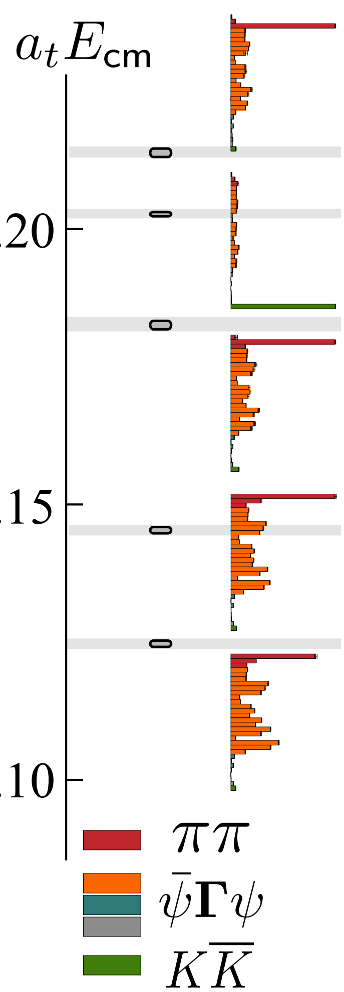

Example: T1 irrep energies



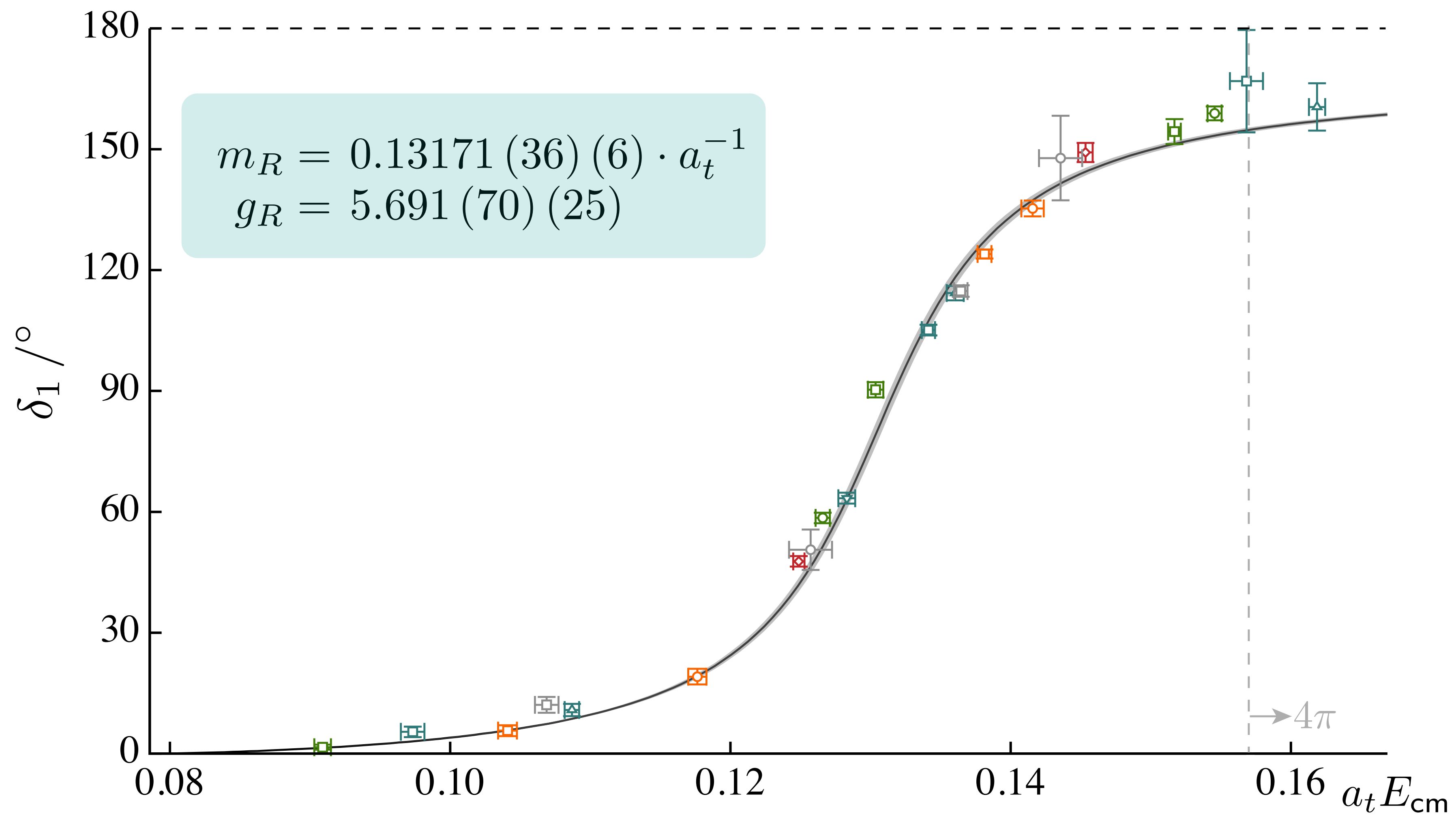

P-wave $\pi\pi$ phase shift as a function of energy

$N_f = 2+1, m_\pi = 236$ MeV, $V \approx (4 \text{ fm})^3$

Wilson et al.(HadSpec),
Phys.Rev. D92 (2015), 094502




Example: T1 irrep energies




Fit to a Breit-Wigner form

$$\mathcal{M}(s) = \frac{1}{\rho(s)} \frac{\sqrt{s}\,\Gamma(s)}{m_R^2 - s - i\sqrt{s}\,\Gamma(s)}$$

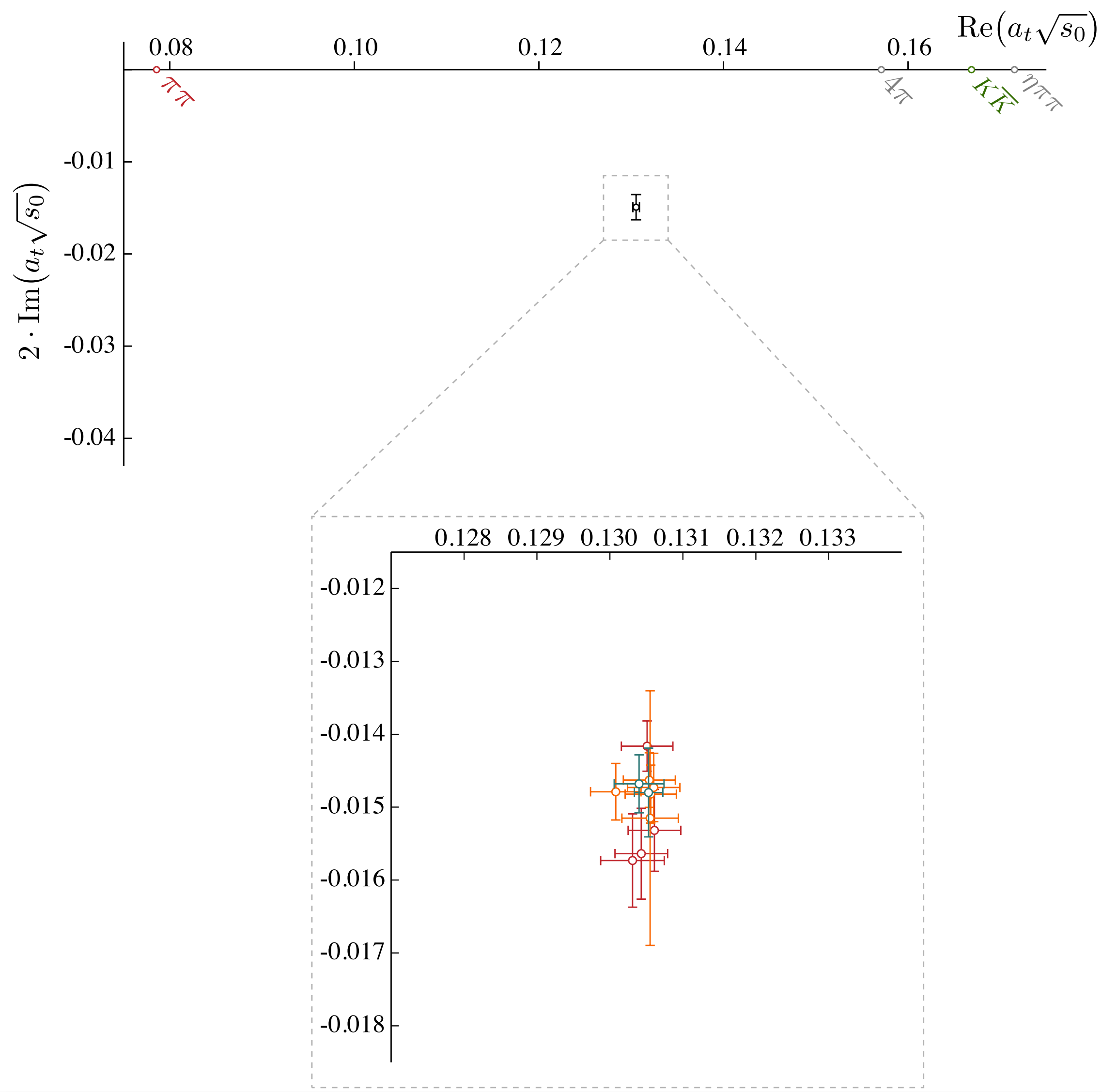
$$\rho_i(E_{cm}) = 2k_i/E_{cm}$$
$$s = E_{cm}^2$$
$$\Gamma(s) = \frac{g_R^2}{6\pi}\frac{k^3}{s}$$

$N_f = 2+1, m_\pi = 236 \text{ MeV}, \; V \approx (4 \text{ fm})^3$

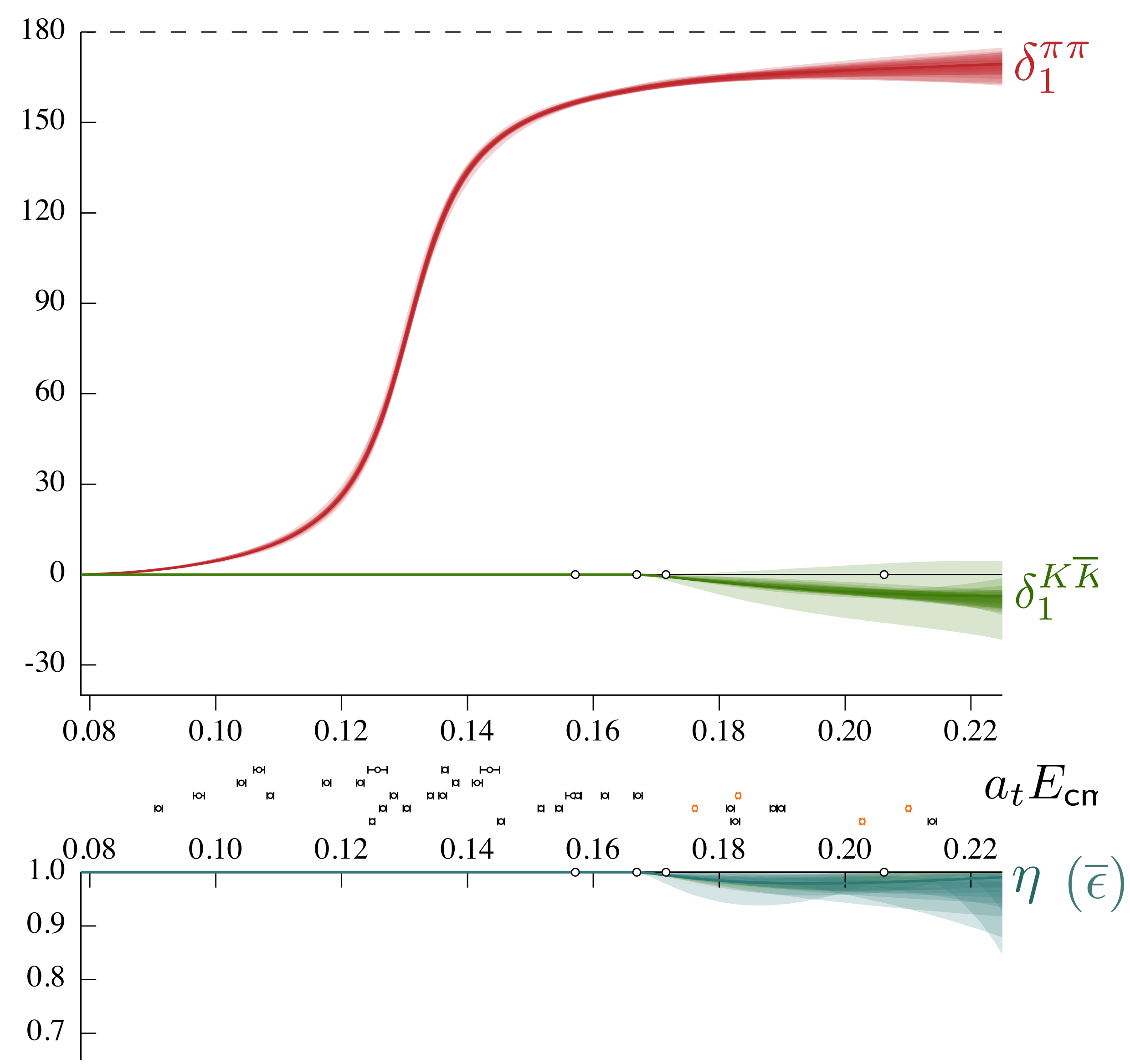
Wilson et al.(HadSpec), Phys.Rev. D92 (2015), 094502

# Using a range of parametrizations:

## Pole position:




## All three scattering parameters:




$N_f = 2+1, m_\pi = 236\ \mathrm{MeV},\ V \approx (4\ \mathrm{fm})^3$

# TWO-NUCLEON OBSERVABLES

# EXERCISE 7

Total angular momentum and parity can be used to classify two-nucleon states ($J^P$). Using the laws of addition of angular momentum as well as Pauli's exclusion principle, prove (justify) the following table. Each pair of numbers refers to *(L,S)* where *S* is the total spin and *L* is the total orbital angular momentum of the state.

| $J^P$ | $0^+$ | $0^-$ | $1^+$ | $1^-$ | $2^+$ | $2^-$ | $3^+$ | $3^-$ | $4^+$ | $4^-$ |
|---|---|---|---|---|---|---|---|---|---|---|
| I=0 | — | — | {(0,1),(2,1)} | (1,0) | (2,1) | — | {(2,1),(4,1)} | (3,0) | (4,1) | — |
| I=1 | (0,0) | (1,1) | — | (1,1) | (2,0) | {(1,1),(3,1)} | — | (3,1) | (4,0) | {(3,1),(5,1)} |

Which entry corresponds to the deuteron channel?

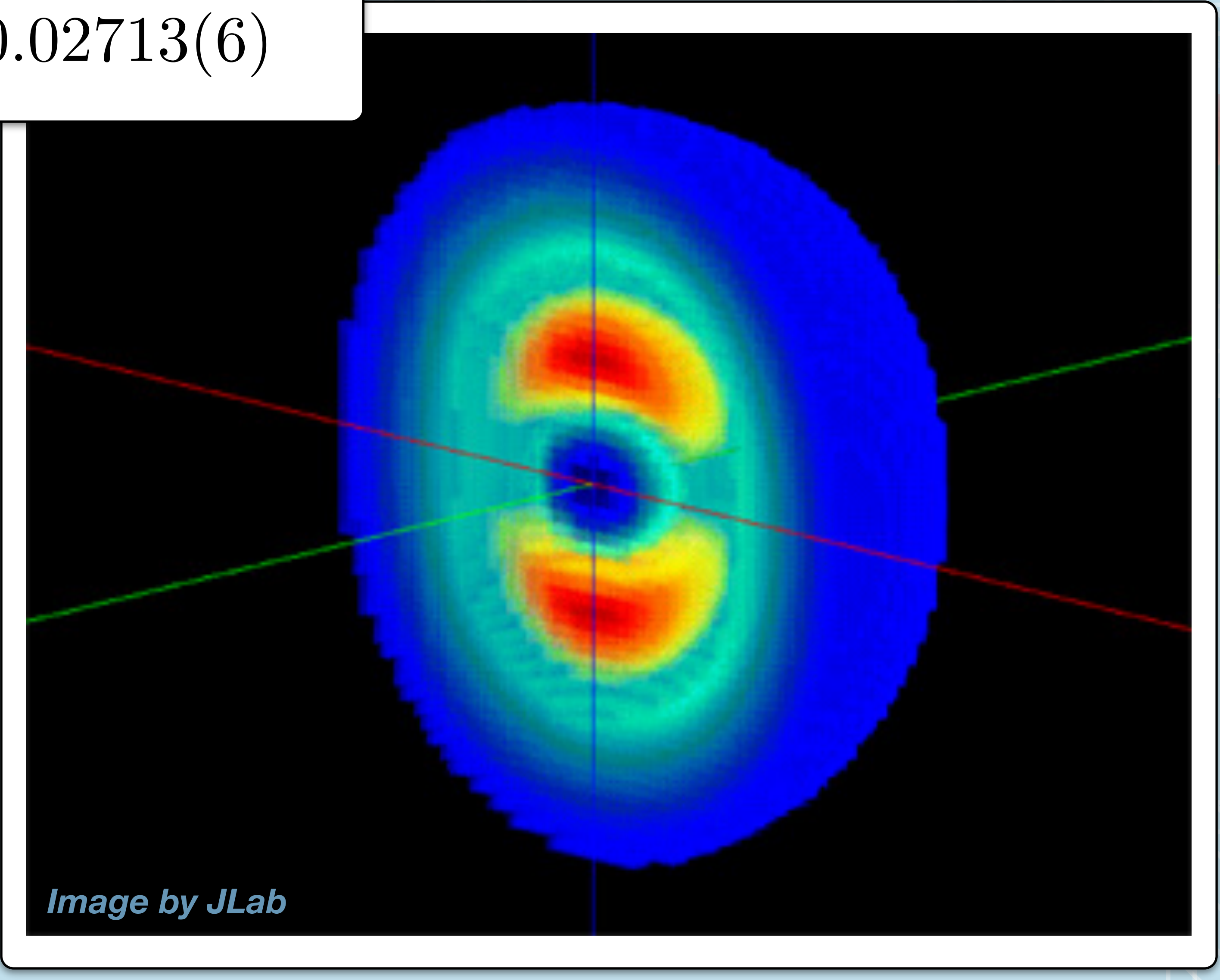
Small quantities are prevalent in the deuteron and in nuclear physics. Can we ever get to the precision to constrain them?

Deuteron is a shallow bound state of proton and neutron.

$$B_d = 2.224644(24)\ [\mathtt{MeV}]$$

$$\eta = -\tan \epsilon_1|_{B_d} \approx 0.02713(6)$$

Its wavefunction has a tiny D-wave admixture.



# Two nucleon systems are coupled channels in nature

Briceno, ZD and Luu,
Phys.Rev.D88, 034502(2013).

$$\left[\delta\mathcal{G}^V\right]_{JM_J,IM_I,LS;J'M'_J,I'M'_I,L'S'} = \frac{iMk^*}{4\pi}\delta_{II'}\delta_{M_IM'_I}\delta_{SS'}\Bigg[\delta_{JJ'}\delta_{M_JM'_J}\delta_{LL'} + i\sum_{l,m}\frac{(4\pi)^{3/2}}{k^{*l+1}}c^{\mathbf{P}}_{lm}(k^{*2})$$

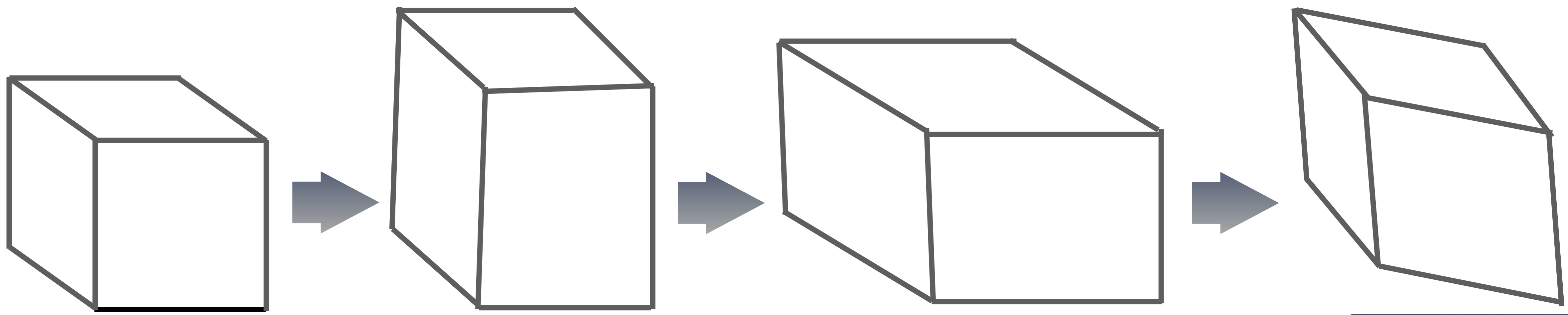
$$\times \sum_{M_L,M'_L,M_S}\langle JM_J|LM_L,SM_S\rangle\langle L'M'_L,SM_S|J'M'_J\rangle\int d\Omega\, Y^*_{L,M_L}Y^*_{l,m}Y_{L',M'_L}\Bigg]$$

# Two nucleon systems are coupled channels in nature

Briceno, ZD and Luu, Phys.Rev.D88, 034502(2013).

$$[\delta\mathcal{G}^V]_{JM_J,IM_I,LS;J'M'_J,I'M'_I,L'S'} = \frac{iMk^*}{4\pi}\delta_{II'}\delta_{M_IM'_I}\delta_{SS'}\Bigg[\delta_{JJ'}\delta_{M_JM'_J}\delta_{LL'} + i\sum_{l,m}\frac{(4\pi)^{3/2}}{k^{*l+1}}c^{\mathbf{P}}_{lm}(k^{*2})$$

$$\times \sum_{M_L,M'_L,M_S}\langle JM_J|LM_L,SM_S\rangle\langle L'M'_L,SM_S|J'M'_J\rangle\int d\Omega\, Y^*_{L,M_L}Y^*_{l,m}Y_{L',M'_L}\Bigg]$$



$\mathbf{P}=\mathbf{0}$ $\quad$ $\mathbf{P}=\frac{2\pi}{L}(0,0,1)$ $\quad$ $\mathbf{P}=\frac{2\pi}{L}(1,1,0)$ $\quad$ $\mathbf{P}=\frac{2\pi}{L}(1,1,1)$

| $\mathbf{d}$ | point group | classification | $N_{\text{elements}}$ | irreps (dimension) |
|---|---|---|---|---|
| $(0,0,0)$ | $O$ | cubic | 24 | $\mathbb{A}_1(1),\mathbb{A}_2(1),\mathbb{E}(2),\mathbb{T}_1(3),\mathbb{T}_2(3)$ |
| $(0,0,1)$ | $D_4$ | tetragonal | 8 | $\mathbb{A}_1(1),\mathbb{A}_2(1),\mathbb{E}(2),\mathbb{B}_1(1),\mathbb{B}_2(1)$ |
| $(1,1,0)$ | $D_2$ | orthorhombic | 4 | $\mathbb{A}(1),\mathbb{B}_1(1),\mathbb{B}_2(1),\mathbb{B}_3(1)$ |
| $(1,1,1)$ | $D_3$ | trigonal | 6 | $\mathbb{A}_1(1),\mathbb{A}_2(1),\mathbb{E}(2)$ |

Briceno, ZD and Luu, Phys.Rev.D88, 034502(2013).

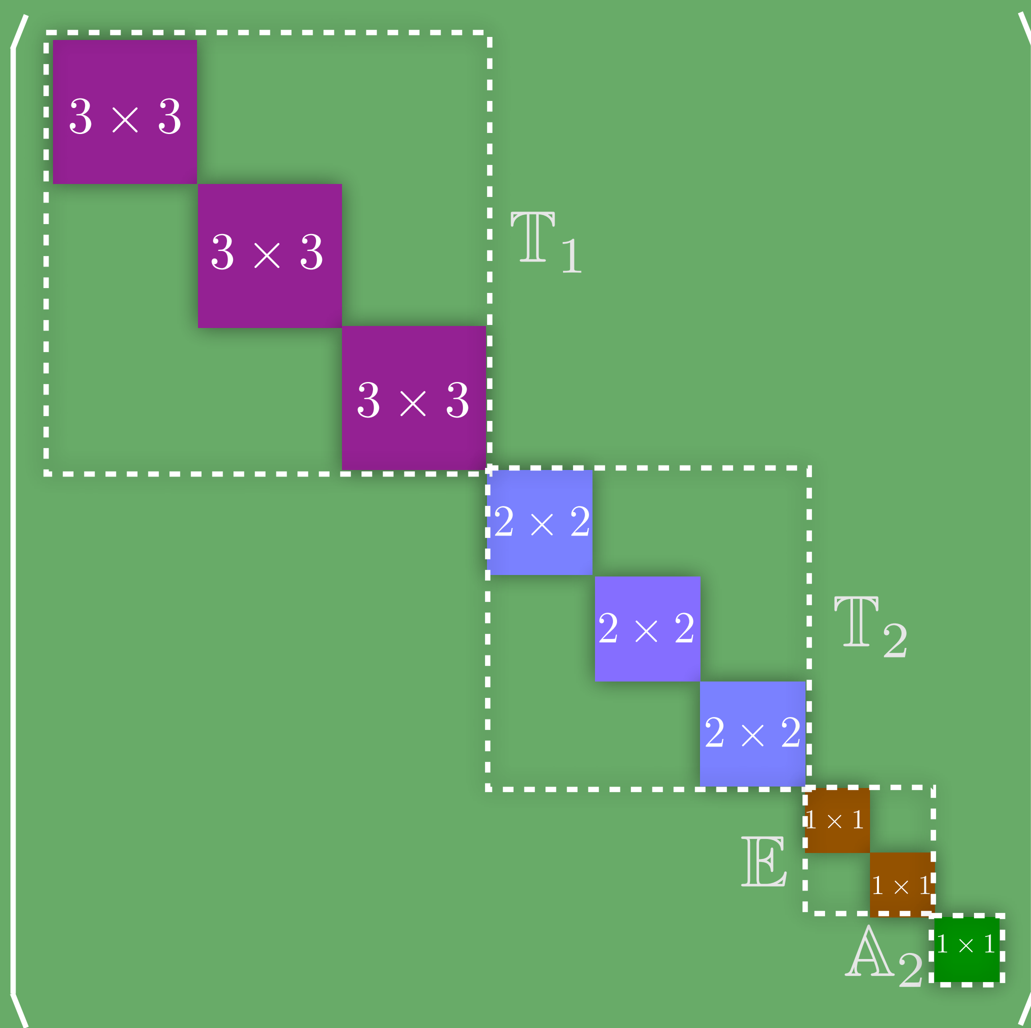
# Two nucleon systems are coupled channels in nature

$$[\delta\mathcal{G}^V]_{JM_J,IM_I,LS;J'M_J',I'M_I',L'S'} = \frac{iMk^*}{4\pi}\delta_{II'}\delta_{M_IM_I'}\delta_{SS'}\left[\delta_{JJ'}\delta_{M_JM_J'}\delta_{LL'} + i\sum_{l,m}\frac{(4\pi)^{3/2}}{k^{*l+1}}c^{\mathbf{P}}_{lm}(k^{*2})\right.$$

$$\left.\times\sum_{M_L,M_L',M_S}\langle JM_J|LM_L,SM_S\rangle\langle L'M_L',SM_S|J'M_J'\rangle\int d\Omega\; Y^*_{L,M_L}Y^*_{l,m}Y_{L',M_L'}\right]$$

$L \leq 3, P = +$

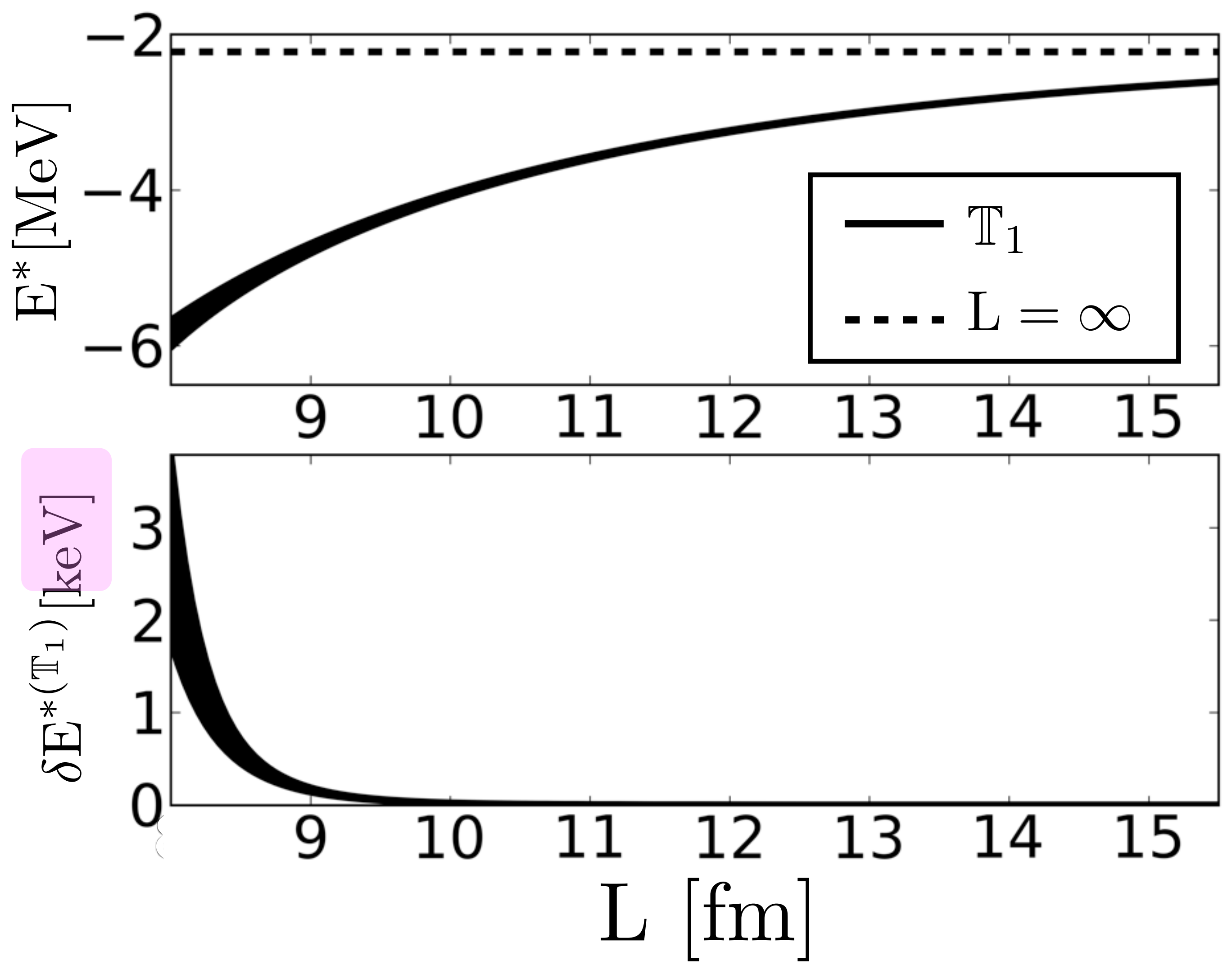
$$\mathcal{M} = \begin{pmatrix} \mathcal{M}_{1,S} & \mathcal{M}_{1,SD} & 0 & 0 \\ \mathcal{M}_{1,SD} & \mathcal{M}_{1,D} & 0 & 0 \\ 0 & 0 & \mathcal{M}_{2,D} & 0 \\ 0 & 0 & 0 & \mathcal{M}_{3,D} \end{pmatrix}, \text{ QC} \rightarrow$$

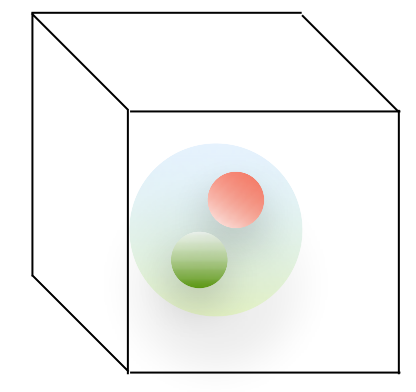


Now let's do a reverse exercise. Let's take the experimental values of scattering parameters and figure out how the spectrum in a finite volume should look like! In particular, let's see how sensitive it is to the S-D mixing.

**P = 0**

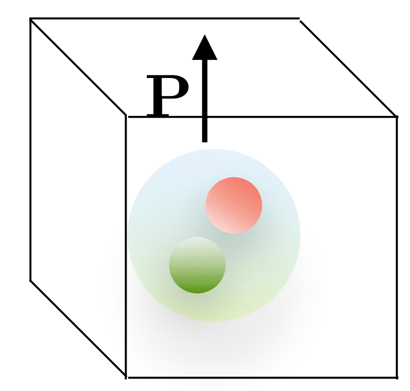
$$\delta E^{*(\Gamma)} = E^{*(\Gamma)} - E^{*(\Gamma)}(\epsilon_1 = 0)$$



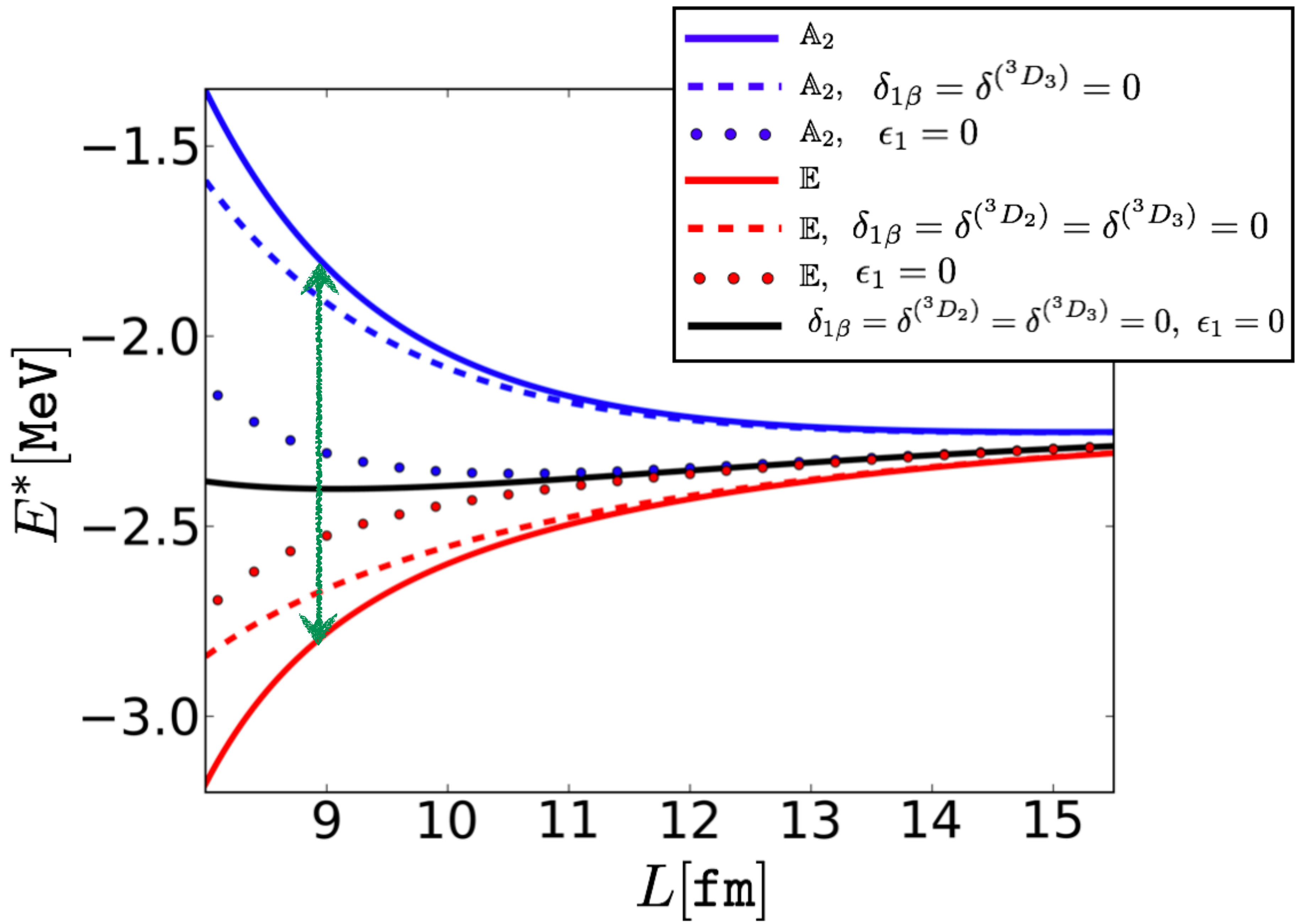



Briceno, ZD, Luu and Savage, Phys. Rev. D88,114507 (2013).

Discovering sensitivity to S-D mixing in a finite volume in boosted systems:




$$\mathbf{P} = \frac{2\pi}{L}(0,0,1)$$




Briceno, ZD, Luu and Savage, Phys. Rev. D88,114507 (2013).

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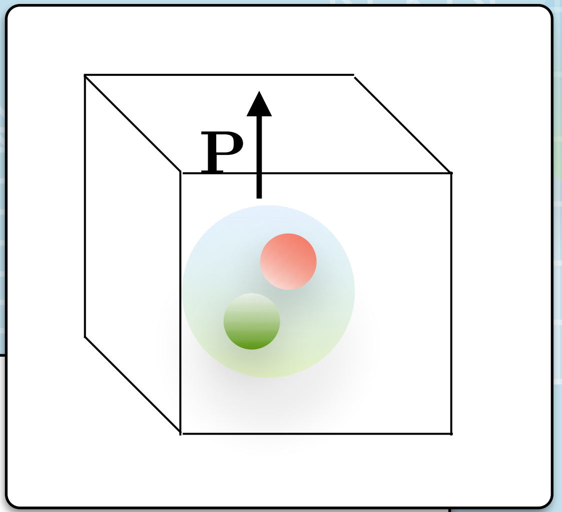
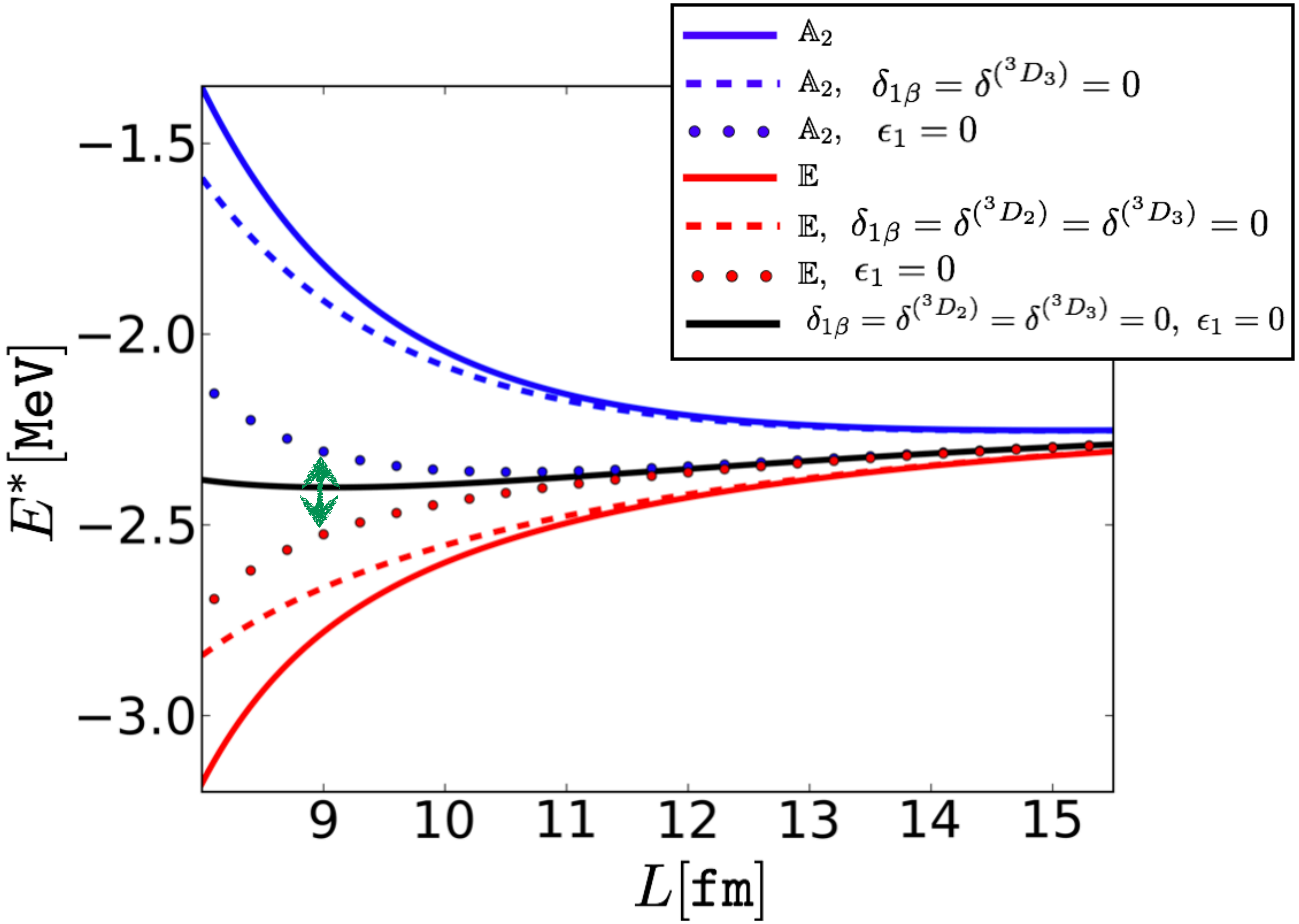
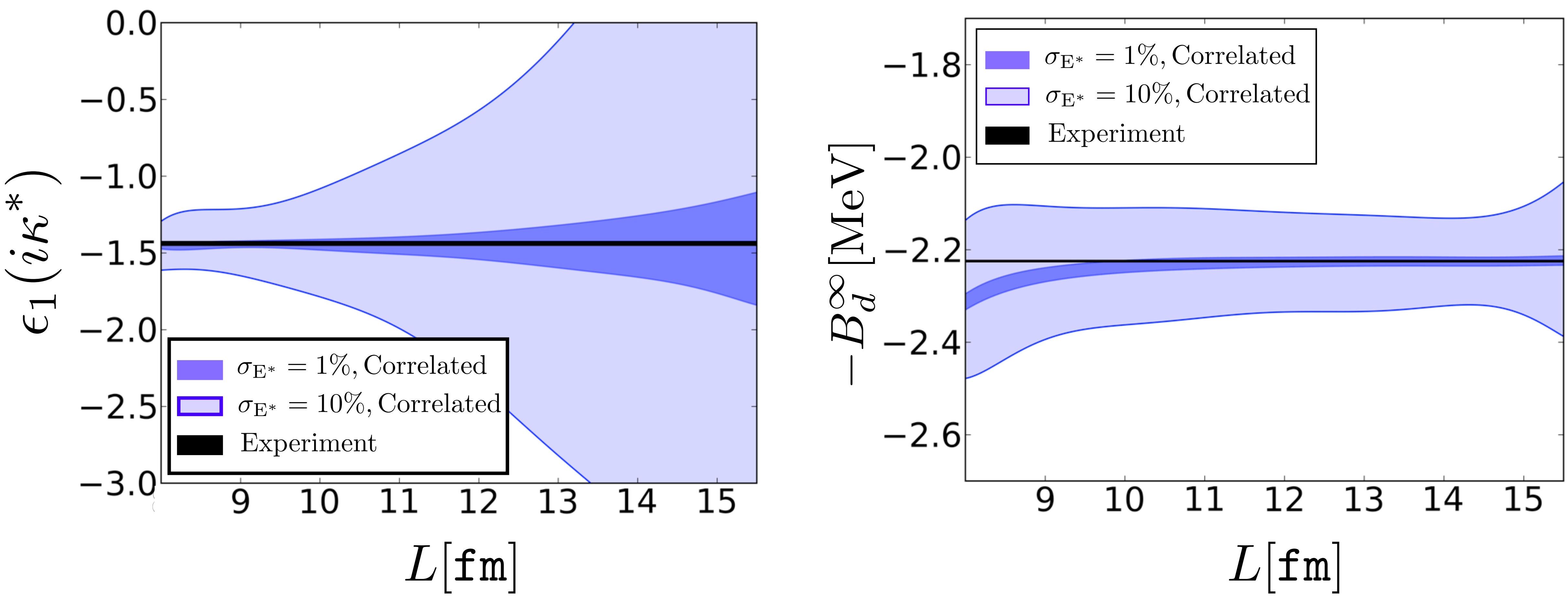



Briceno, ZD, Luu and Savage, Phys. Rev. D88,114507 (2013).

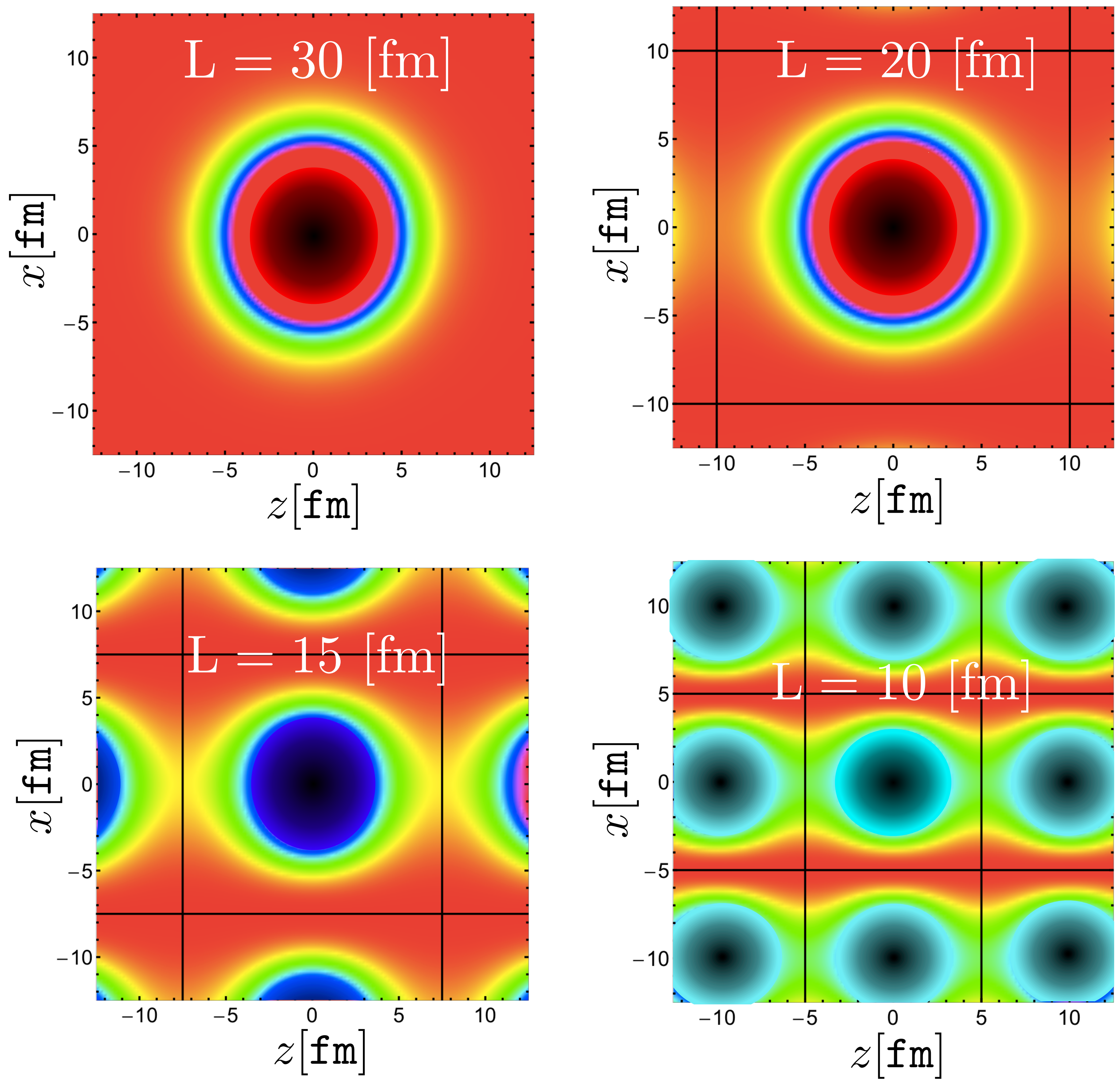
The required precision on energies at the physical point for an extraction of the mixing angle then is:




Briceno, ZD, Luu and Savage, Phys.Rev.D88, 114507(2013).

One can use the finite-volume formalism to deduce deuteron's asymptotic wavefunction in a finite volume (E irrep shown):

$\mathbf{P} = \frac{2\pi}{L}(0,0,1)$

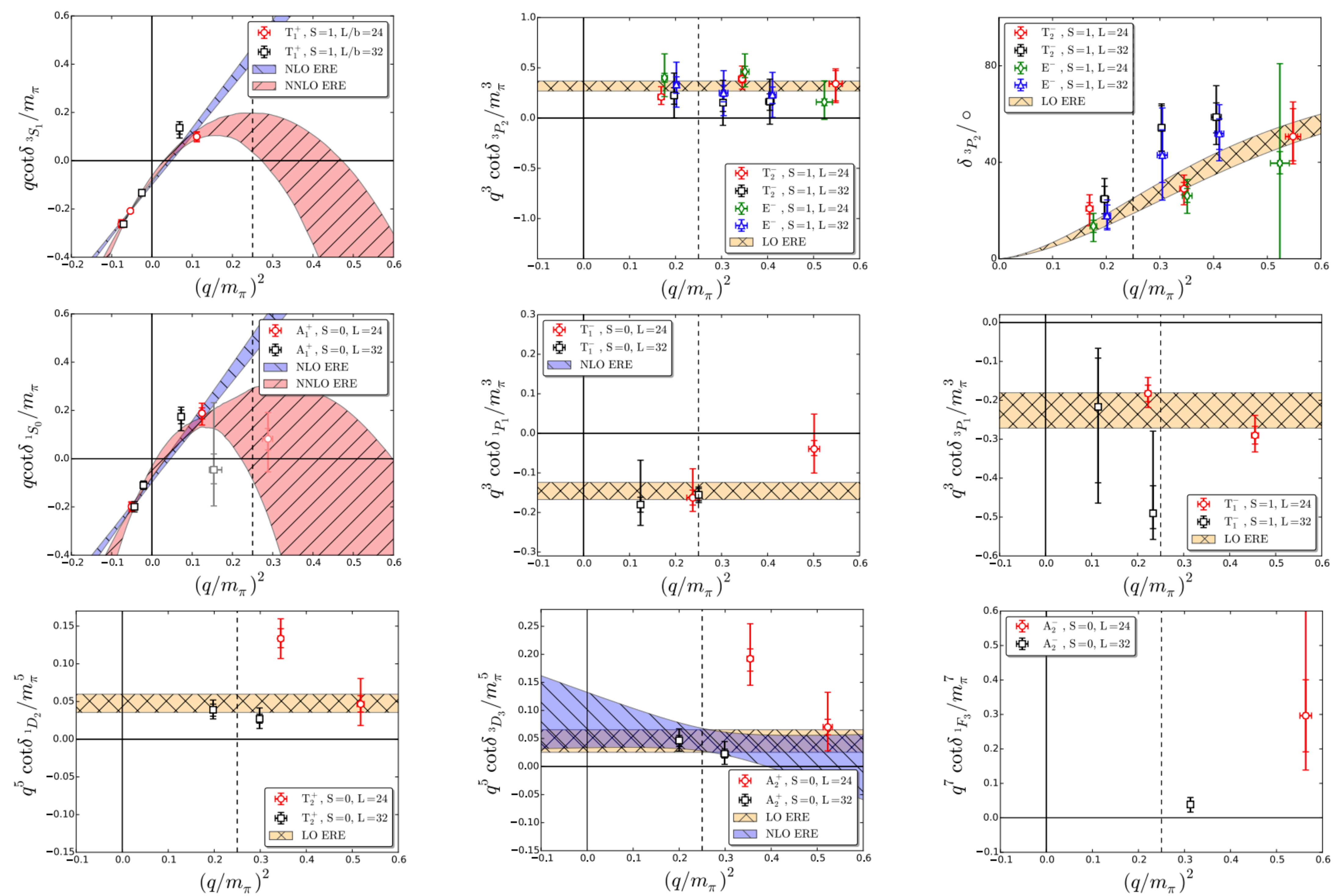



Briceno, ZD, Luu and Savage, Phys.Rev.D88, 114507(2013).

And here's an application of the *NN* quantization conditions in a lattice QCD study of scattering in higher partial waves:

$N_f = 3,\ m_\pi = 0.806\ \text{GeV},\ a = 0.145(2)\ \text{fm}$

Berkowitz et al (CalLatt), Phys.Let.B,765(2017).



What about bound states? It turned out that they are described by the same Luescher's QC if the CM momentum is analytically continued to an imaginary momentum. It can then be shown that for low-energy S-wave bound state:

Binding momentum

$$\alpha = \frac{1}{2}\left[1 + \frac{m_1^2 - m_2^2}{E^{*2}}\right]$$

$$k^* \cot\delta(k^*)|_{k^*=i\kappa} + \kappa = \frac{1}{L}\sum_{\mathbf{m}\neq\mathbf{0}} \frac{1}{|\hat{\gamma}\mathbf{m}|} e^{i2\pi\alpha\mathbf{m}\cdot\mathbf{d}} e^{-|\hat{\gamma}\mathbf{m}|\kappa L} \quad (1)$$

An integer vector

Momentum vector in units of $2\pi/L$

$$(\gamma - 1)\frac{\mathbf{m}\cdot\mathbf{d}}{|\mathbf{d}|^2}\mathbf{d} + \mathbf{m}$$

---

Which for a bound states comprised of two equal-mass hadrons at rest, becomes:

$$Z = \frac{1}{\sqrt{1 - 2\kappa_0 \frac{d}{dp^2} p\cot\delta|_{i\kappa_0}}}$$

$$\kappa = \kappa^{(\infty)} + \frac{Z^2}{L}\left[6e^{-\kappa^{(\infty)}L} + \frac{12}{\sqrt{2}}e^{-\sqrt{2}\kappa^{(\infty)}L} + \frac{8}{\sqrt{3}}e^{-\sqrt{3}\kappa^{(\infty)}L}\right] + \mathcal{O}\left(\frac{e^{-2\kappa^{(\infty)}L}}{L}\right) \quad (2)$$

Infinite-volume binding momentum

ZD and Savage, Phys. Rev. D84, 114502 (2011).

Bour, Konig, Lee, Hammer and Meissner, Phys. Rev. D84, 091503 (2011).

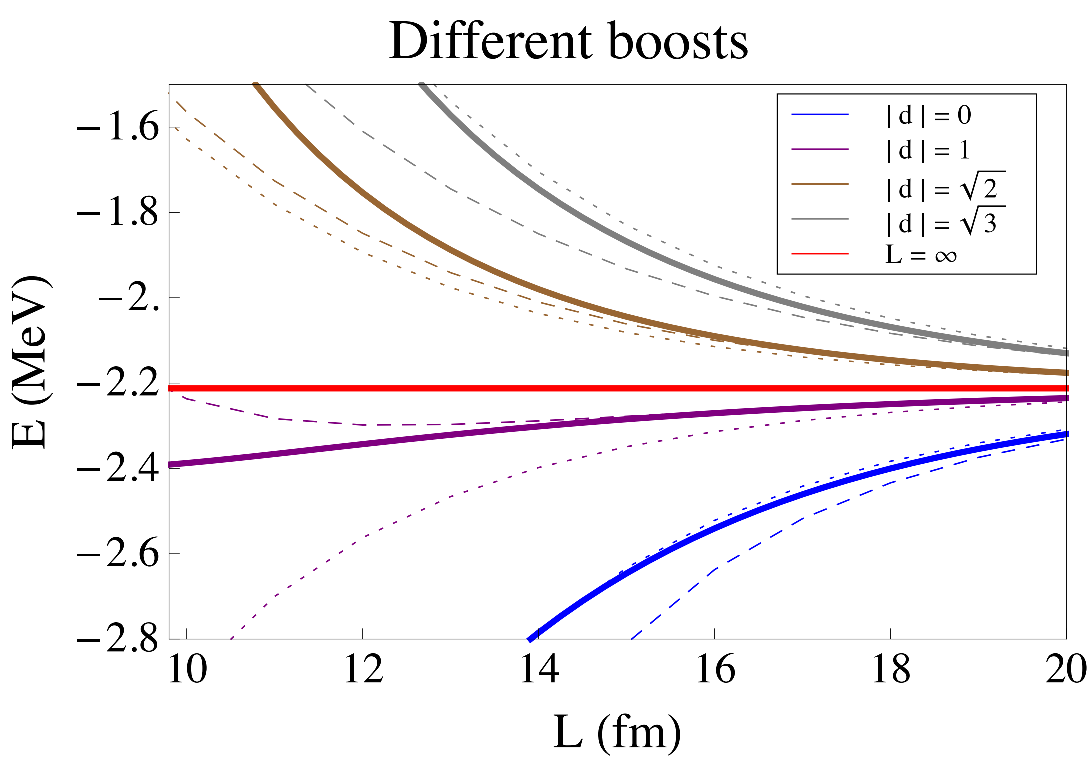
## EXERCISE 8

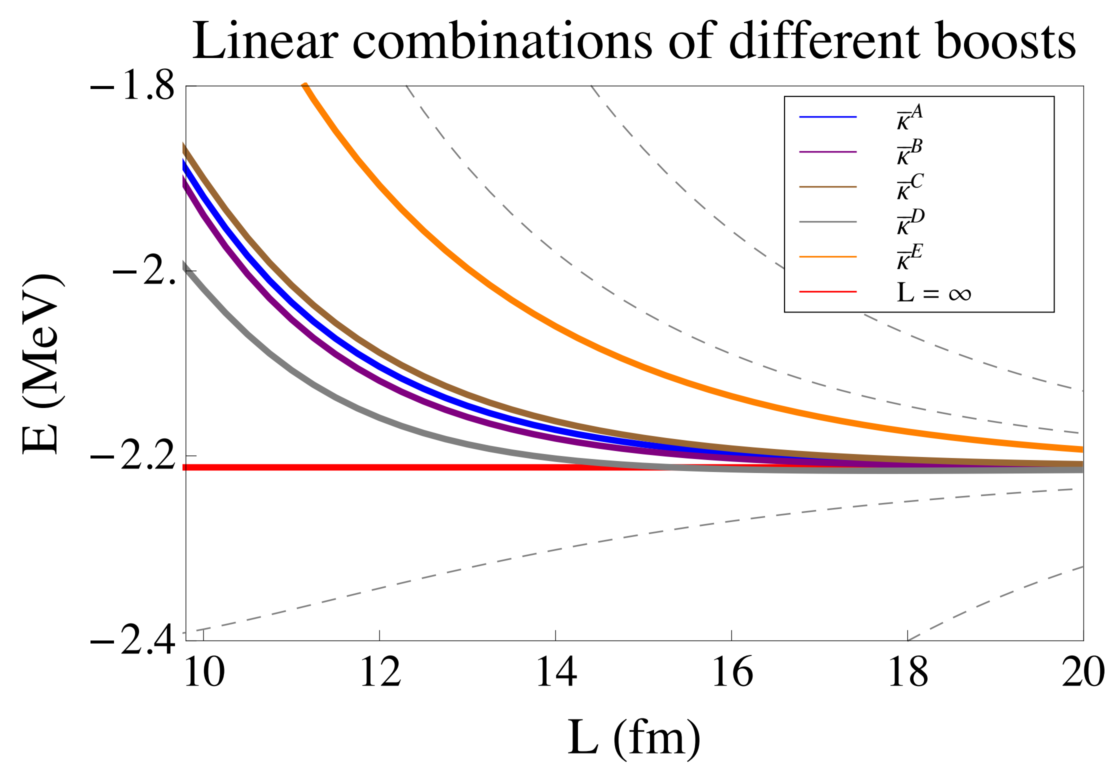
Starting from relation (1), prove relation (2) for the finite-volume dependence of the binding momentum of a S-wave bound state of two hadrons with equal masses.

## BONUS EXERCISE 8

Starting from the Luescher's S-wave quantization condition for boosted systems with equal masses, prove relation (1) for a bound state.

Again using physical information about the deuteron, we can predict it finite-volume spectrum and find tricks to improve it:




ZD and Savage, Phys. Rev. D84, 114502 (2011).

NEXT TIME WE WILL GO BEYOND TWO-BODY ELASTIC SCATTERING PROBLEMS AND WILL CONSIDER TRANSITION AMPLITUDES (IN ONE-BODY AND TWO-BODY SECTOR).

QUESTIONS?