Hadron masses calculations on the lattice

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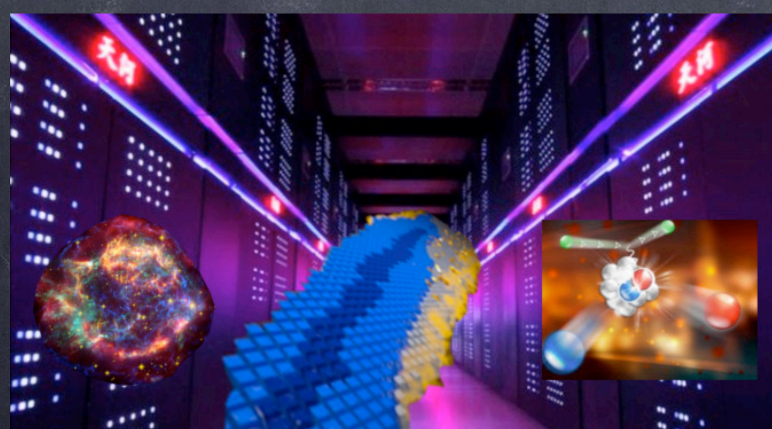
TALENT School - From quarks and gluons to nuclear forces and structures



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Spectroscopy calculations

Calculation of hadron masses

- Find hadron interpolating fields such that the corresponding Hilbert space operators annihilate and create the particles we want to analyze

$$O(x) \quad \overline{O}(x) \quad \longleftrightarrow \quad \widehat{O}(x) \quad \widehat{O}^{\dagger}(x)$$

- Hadron interpolators O(x) are functional of the lattice fields with the quantum numbers one is interested in

$$O[\psi, \overline{\psi}, U]$$

- Gauge invariant <--> color singlet

Spectroscopy calculations

Dirac matrix

- Meson interpolating fields $O_M(x) = \overline{\psi}(x)\Gamma\psi(x)$
- Baryon interpolating fields $O_B(x) \sim 3 \text{ quarks}$
- Pure gauge interpolating fields: plaquette, Wilson loop -> glueballs, certain mesons,...
- Interpolating fields with 3n quarks for nuclei (deuterium,...)

$$\left\langle O(\underline{x},t)\overline{O}(\underline{0},0)\right\rangle = \left\langle 0|\widehat{O}(\underline{x},t)\widehat{O}^{\dagger}(\underline{0},0)|0\right\rangle$$

Spectroscopy calculations

$$\widetilde{f}(p) = \int dx\ \mathrm{e}^{-ipx} f(x) \qquad \widetilde{f}(0) = \int dx\ f(x)$$

$\underline{p} = \underline{0}$ projection $a^3 \sum_{\underline{x}}$ $\qquad \widehat{H}|0\rangle = 0 \quad \widehat{\underline{p}}|0\rangle = 0$

$$a^3 \sum_{\underline{x}} \langle O(\underline{x}, t)\overline{O}(\underline{0}, 0)\rangle = \sum_n \left\langle 0|\mathrm{e}^{\hat{H}t}\widehat{O}\mathrm{e}^{-\hat{H}t}|n\right\rangle \left\langle n|\widehat{O}^\dagger|0\right\rangle = \sum_n \left\langle 0|\widehat{O}|n\right\rangle \left\langle n|\widehat{O}^\dagger|0\right\rangle \mathrm{e}^{-E_n t}$$

$$O \to h \quad \left\langle 0|\widehat{O}|h\right\rangle \neq 0 \quad \widehat{H}|h\rangle = E_h|h\rangle$$

$$a^3 \sum_{\underline{x}} \langle O(\underline{x}, t)\overline{O}(\underline{0}, 0)\rangle = \mathrm{e}^{-E_h t}\left|\left\langle 0|\widehat{O}|h\right\rangle\right|^2 + \mathrm{e}^{-E_h' t}\left|\left\langle 0|\widehat{O}|h'\right\rangle\right|^2 + \cdots$$

$$= A\mathrm{e}^{-M_h t}\left[1 + B\mathrm{e}^{-(E_h' - M_h)t}\right] + \cdots$$

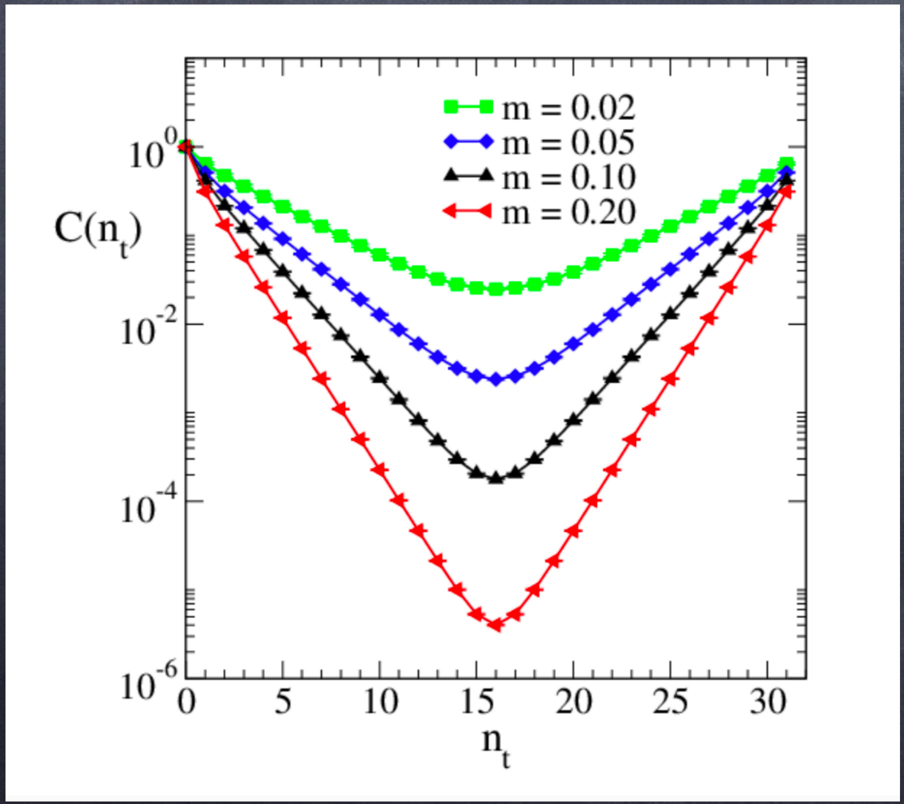
$T \to \infty$

Effective mass

$$C_0(t) = a^3 \sum_{\underline{x}} \langle O(\underline{x}, t)\overline{O}(\underline{0}, 0)\rangle = A\mathrm{e}^{-(aM_h)(t/a)} \left[1 + \mathrm{e}^{-a(E_h'-M_h)(t/a)}\right] + \cdots$$

$$(aM)_{\text{eff}}(t) = \ln\left[\frac{C_O(t)}{C_O(t+a)}\right] \qquad \frac{C_O(t)}{C_O(t+a)} = \frac{\cosh\left[(aM)_{\text{eff}}(t/a - T/2a)\right]}{\cosh\left[(aM)_{\text{eff}}(t/a + 1 - T/2a)\right]}$$

Finite T



Pion correlator

Symmetries

- Understand symmetry transformation to choose interpolating fields

$\mathcal{C} \rightarrow$ Charge conjugation

$\mathcal{P} \rightarrow$ Parity

Lattice breaks translation and rotational symmetry

$$\begin{cases} \psi(x) \xrightarrow{\mathcal{C}} \psi^c(x) = C^{-1}\overline{\psi}(x)^T \\ \overline{\psi}(x) \xrightarrow{\mathcal{C}} \overline{\psi}^c(x) = -\psi(x)^T C \end{cases}$$

$$C\gamma_\mu C^{-1} = -\gamma_\mu^T$$

$$C\gamma_5 C^{-1} = \gamma_5^T$$

$$C = C^{-1} = C^\dagger = -C^T$$

$$U(x;\mu) \xrightarrow{\mathcal{C}} U^c(x;\mu) = U(x;\mu)^* = \left(U(x;\mu)^\dagger\right)^T$$

$$\overline{\psi}(x)\psi(x) \xrightarrow{\mathcal{C}} ?$$

Symmetries

Careful with labeling

$$\begin{cases} \psi(\underline{x},t) \xrightarrow{\mathcal{P}} \psi^P(\underline{x},t) = \gamma_4 \psi(-\underline{x},t) \\ \overline{\psi}(\underline{x},t) \xrightarrow{\mathcal{P}} \overline{\psi}^P(\underline{x},t) = \overline{\psi}(-\underline{x},t)\gamma_4 \end{cases}$$

$$\begin{cases} U(\underline{x},t;k) \xrightarrow{\mathcal{P}} U^P(\underline{x},t;k) = U(-\underline{x} - a\hat{k},t;k)^\dagger \quad k = 1,2,3 \\ U(\underline{x},t;4) \xrightarrow{\mathcal{P}} U^P(\underline{x},t;4) = U(-\underline{x},t;4) \end{cases}$$

$$\begin{cases} \psi(x) \xrightarrow{\mathcal{P}_\mu} \psi^{P_\mu}(x) = \gamma_\mu \psi(P_\mu(x)) \\ \overline{\psi}(x) \xrightarrow{\mathcal{P}_\mu} \overline{\psi}^{P_\mu}(x) = \overline{\psi}(P_\mu(x))\gamma_\mu \end{cases} \qquad P_1(x) = (x_1, -x_2, -x_3, -x_4) \\ \vdots$$

$$\begin{cases} U(x;\nu) \xrightarrow{\mathcal{P}_\mu} U^{P_\mu}(x;\nu) = U(P_\mu(x) - a\hat{\nu};\nu)^\dagger \quad \nu \neq \mu \\ U(x;\mu) \xrightarrow{\mathcal{P}_\mu} U^{P_\mu}(x;\mu) = U(P_\mu(x);\mu) \end{cases}$$

Symmetries

Careful with labeling

$$\begin{cases} \psi(\underline{x},t) \xrightarrow{\mathcal{P}} \psi^P(\underline{x},t) = \gamma_4 \overline{\psi}(-\underline{x},t) \\ \overline{\psi}(\underline{x},t) \xrightarrow{\mathcal{P}} \overline{\psi}^P(\underline{x},t) = \psi(-\underline{x},t)\gamma_4 \end{cases}$$

$$\begin{cases} U(\underline{x},t;k) \xrightarrow{\mathcal{P}} U^P(\underline{x},t;k) = U(-\underline{x} - a\hat{k},t;k)^\dagger \quad k = 1,2,3 \\ U(\underline{x},t;4) \xrightarrow{\mathcal{P}} U^P(\underline{x},t;4) = U(-\underline{x},t;4) \end{cases}$$

In Euclidean space there is no distinction between space and time

$$\mathcal{P}_1 \cdot \mathcal{P}_2 \cdot \mathcal{P}_3 = \text{ time reflection}$$

Reconstruction Hilbert space for Minkowski

Fermions a.b.c. gauge p.b.c not relevant in infinit volume but finite temp.

Symmetries

$\gamma_5 -$ hermiticity $\qquad Q = \gamma_5 D$ **Hermitian**

$$(\gamma_5 D)^\dagger = \gamma_5 D \Rightarrow D^\dagger = \gamma_5 D \gamma_5 \qquad \Rightarrow D^{-1} \text{ is } \gamma_5\text{-Hermitian}$$

Eigenvalues are either real or come in complex conjugate pairs -> reality of determinant

Pions

u- and d-quarks

SU(2)

u: $I = \frac{1}{2} \quad I_3 = \frac{1}{2} \quad Q = \frac{2}{3}e \quad Q_P = e$

d: $I = \frac{1}{2} \quad I_3 = -\frac{1}{2} \quad Q = -\frac{1}{3}e \quad Q_P = e$

$$I = 1 \quad (\pi^+, \pi^0, \pi^-)$$

$$I = 0 \quad (\eta)$$

Pseudoscalar combinations are grouped in an isotriplet and isosinglet

$$M_{\pi^\pm} \simeq 140 \text{ MeV}$$

$$J = 0 \quad P = -1$$

$$I = 1 \quad I_3 = +1, 0, -1 \quad Q = \pm e, 0$$

Pions

$$O_{\pi^+}(x) = \overline{d}(x)\gamma_5 u(x) = \overline{d}(x)^C_\alpha \, (\gamma_5)_{\alpha\beta} \, u(x)^C_\beta$$

$$O_{\pi^-}(x) = \overline{u}(x)\gamma_5 d(x) = \overline{u}(x)^C_\alpha \, (\gamma_5)_{\alpha\beta} \, d(x)^C_\beta$$

Negative parity

$$O_{\pi^+}(x) = \overline{d}(x)\gamma_5 u(x) \xrightarrow{\mathcal{P}} \overline{d}(-\underline{x},t)\gamma_4\gamma_5\gamma_4 u(-\underline{x},t) = -O_{\pi^+}(-\underline{x},t)$$

Irrelevant when $\underline{p} = \underline{0}$

$$O_{\pi^+}(x) = \overline{d}(x)\gamma_5 u(x) \xrightarrow{\mathcal{C}} -d(x)^T C\gamma_5 C^{-1}\overline{u}(x)^T =$$

$$= -d(x)^T \gamma_5^T \overline{u}(x)^T = \overline{u}(x)\gamma_5 d(x) = O_{\pi^-}(x)$$

Mesons

$$\pi^0: \quad I_3 = 0 \quad P = -1 \quad C = +1$$

$$O_{\pi^0}(x) = \frac{1}{\sqrt{2}} \left[\overline{u}(x)\gamma_5 u(x) - \overline{d}(x)\gamma_5 d(x) \right]$$

$$\eta: \quad I = 0 \quad P = -1 \quad C = +1$$

$$O_{\eta}(x) = \frac{1}{\sqrt{2}} \left[\overline{u}(x)\gamma_5 u(x) + \overline{d}(x)\gamma_5 d(x) \right]$$

$$O_{K^+}(x) = \overline{s}(x)\gamma_5 u(x)$$

Different spin and parity correspond to different Γ matrices

$$O_M(x) = \overline{\psi}_{f_1}(x)\Gamma\psi_{f_2}(x) \qquad \overline{O}_M(x) = O^c_M(x) = \pm\overline{\psi}_{f_2}(x)\Gamma\psi_{f_1}(x)$$

Correlation functions

$$\langle O_M(x)\overline{O}_M(y)\rangle \qquad A = O_M(x)\overline{O}_M(y) \qquad \langle A\rangle = \langle [A]_F\rangle_G$$

$$[A]_F = \frac{1}{Z_F}\int \mathcal{D}[\psi,\overline{\psi}]\mathrm{e}^{-S_F[\psi,\overline{\psi},U]}A[\psi,\overline{\psi},U]$$

$$Z_F = \int \mathcal{D}[\psi,\overline{\psi}]\mathrm{e}^{-S_F[\psi,\overline{\psi},U]} = \det M[U]$$

$$S_F = a^8\sum_{x,y}\overline{\psi}(x)M(x,y)\psi(y)$$

$$\langle [A]_F\rangle_G = \frac{1}{Z}\int \mathcal{D}[U]\mathrm{e}^{-S_G[U]}Z_F[U]\,[A]_F$$

Correlation functions

$O_M(x) = \overline{d}(x)\Gamma u(x)$

$$\left[O_M(x)\overline{O}_M(y)\right]_F = \left[\overline{d}(x)\Gamma u(x)\overline{u}(y)\Gamma d(y)\right]_F =$$

$$= \Gamma_{\alpha_1\beta_1}\Gamma_{\alpha_2\beta_2}\left[\overline{d}(x)^{C_1}_{\alpha_1}u(x)^{C_1}_{\beta_1}\overline{u}(y)^{C_2}_{\alpha_2}d(y)^{C_2}_{\beta_2}\right]_F =$$

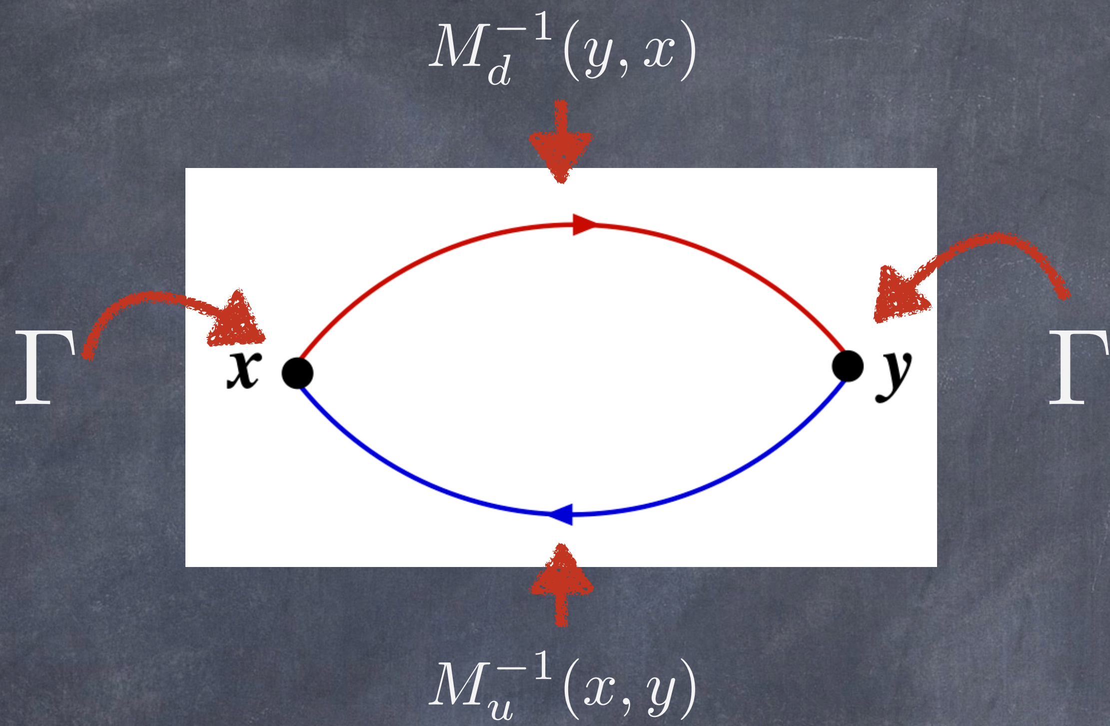
$$= -\Gamma_{\alpha_1\beta_1}\Gamma_{\alpha_2\beta_2}\left[d(y)^{C_2}_{\beta_2}\overline{d}(x)^{C_1}_{\alpha_1}u(x)^{C_1}_{\beta_1}\overline{u}(y)^{C_2}_{\alpha_2}\right]_F =$$

$$= -\Gamma_{\alpha_1\beta_1}\Gamma_{\alpha_2\beta_2}M_d^{-1}(y,x)^{C_2C_1}_{\beta_2\alpha_1}M_u^{-1}(x,y)^{C_1C_2}_{\beta_1\alpha_2} =$$

$$= -\Gamma_{\alpha_2\beta_2}M_d^{-1}(y,x)^{C_2C_1}_{\beta_2\alpha_1}\Gamma_{\alpha_1\beta_1}M_u^{-1}(x,y)^{C_1C_2}_{\beta_1\alpha_2} =$$

$$= -\mathrm{Tr}\left[\Gamma M_d^{-1}(y,x)\Gamma M_u^{-1}(x,y)\right]$$

Correlation functions



$$= -\mathrm{Tr}\left[\Gamma M_d^{-1}(y,x)\Gamma M_u^{-1}(x,y)\right]$$

Isosinglet correlator

$$O_S(x) = \overline{u}(x)\Gamma u(x)$$

$$\left[O_S(x)\overline{O}_S(y)\right]_F = \left[\overline{u}(x)\Gamma u(x)\overline{u}(y)\Gamma u(y)\right]_F =$$

$$= -\mathrm{Tr}\left[\Gamma M_u^{-1}(y,x)\Gamma M_u^{-1}(x,y)\right] +$$

$$+\Gamma_{\alpha_1\alpha_2}\Gamma_{\beta_1\beta_2}\left[u(x)^C_{\alpha_2}\overline{u}(x)^C_{\alpha_1}u(y)^D_{\beta_2}\overline{u}(y)^D_{\beta_1}\right]_F =$$

$$= -\mathrm{Tr}\left[\Gamma M_u^{-1}(y,x)\Gamma M_u^{-1}(x,y)\right] +$$

$$+\mathrm{Tr}\left[\Gamma M_u^{-1}(x,x)\right]\mathrm{Tr}\left[\Gamma M_u^{-1}(y,y)\right]$$