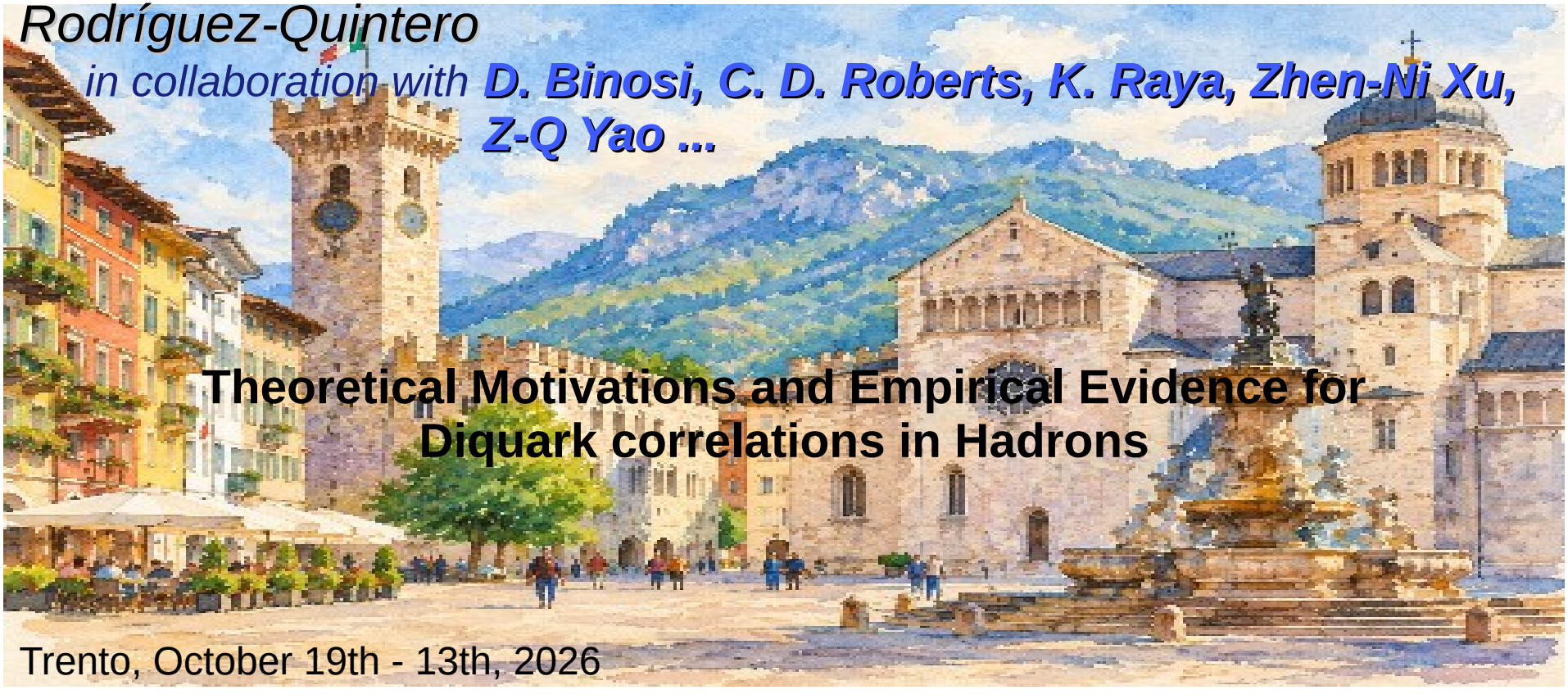
All-orders Evolution: Theory and Practice

J. Rodríguez-Quintero

in collaboration with ***D. Binosi, C. D. Roberts, K. Raya, Zhen-Ni Xu, Z-Q Yao ...***



Theoretical Motivations and Empirical Evidence for Diquark correlations in Hadrons

Trento, October 19th - 13th, 2026

Introduction: QCD running coupling

The field-theory lagrangians invariant under U(1) (**QED**) and SU(3) gauge transformations (**QCD**)

$$\mathcal{L}_{QCD} = -\frac{1}{4}F^{\mu\nu}_a F^a_{\mu\nu} - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 + \partial_\mu \bar{c}_a \partial^\mu c_a + g\, f_{abc} A^a_\mu (\partial^\mu \bar{c}_b) c_c + \bar{\psi}\,(i\gamma_\mu D_\mu - m)\,\psi$$

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Superfluous gauge degrees of freedom can be suppressed by constraining the lagrangian, I.e. by introducing a **gauge-fixing** term. This implies a new vertex in **QCD**, owing to its **non-abelian** character, while no ghost couples to photon field in **QED**.

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The quantization of the theory plagues the theory with divergencies that should be subsequently absorbed when relating the lagrangian and physical quantities (**renormalization**)

$$\begin{aligned} A_R &= Z_A^{-1/2} A_0 \\ \psi_R &= Z_\psi^{-1/2} \psi_0 \\ \Gamma_R &= Z_\Gamma \Gamma_0 \longleftarrow (x1) \\ e_R &= Z_e^{-1} e_0 \end{aligned} \qquad (x4) \longrightarrow \qquad \begin{aligned} A_R &= Z_A^{-1/2} A_0 \\ \psi_R &= Z_\psi^{-1/2} \psi_0 \\ \Gamma_R &= Z_\Gamma \Gamma_0 \\ c_R &= Z_c^{-1/2} c_0 \\ g_R &= Z_g^{-1} g_0 \end{aligned}$$

$$Z_e = Z_e(Z_\Gamma, Z_\psi, Z_A) \qquad Z_g = Z_g(Z_\Gamma, Z_\psi, Z_A, Z_c)$$

$$Z_e = \frac{Z_\Gamma}{Z_\psi^1 Z_A^{1/2}}$$

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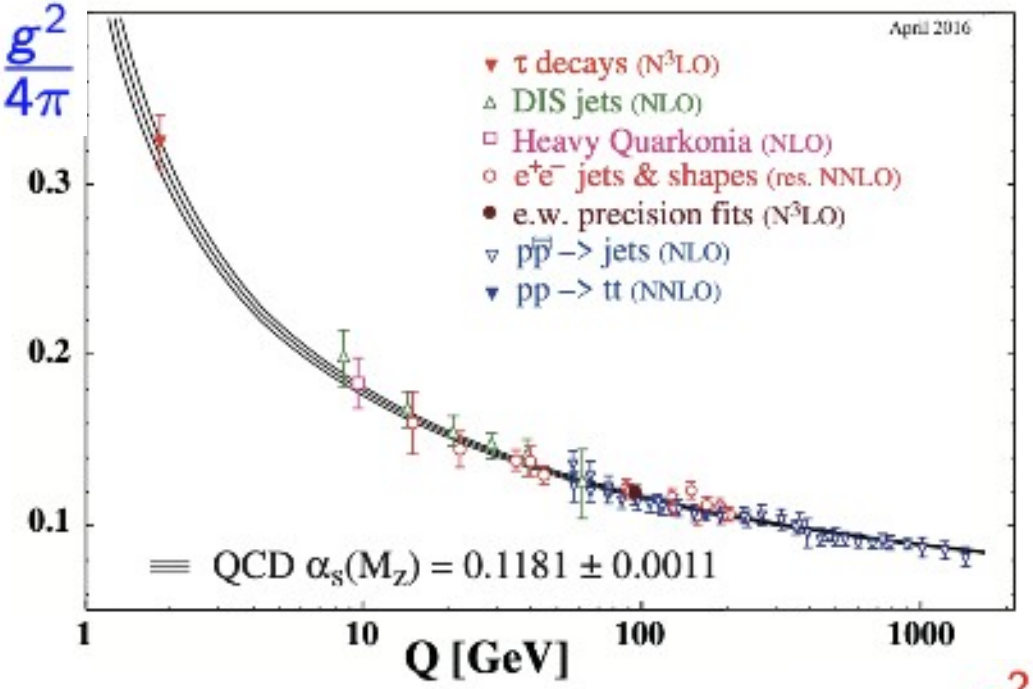
$$Z_e = \frac{Z_\Gamma}{Z_\psi^1 Z_A^{1/2}} \to Z_A^{-1/2} \qquad Z_\psi = Z_\Gamma$$

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Ward replaced by **Slavnov-Taylor** identities in **QCD**, preventing from simple, direct connections of vertices and propagators, thus requiring the choice of a vertex (and its kinematics) to fully define a **renormalized running coupling**.

Introduction: QCD running coupling

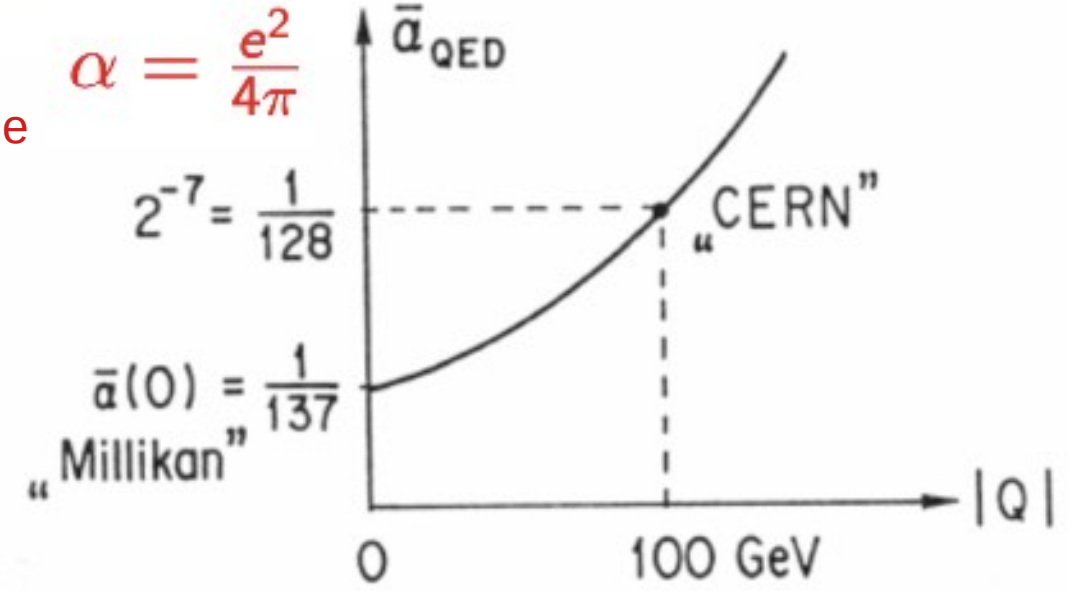
$$\alpha_s = \frac{g^2}{4\pi}$$



In QCD, the charge diminishes with the probing energy (anti-screening), due to its **non-abelian** character. **The theory is asymptotically free**.

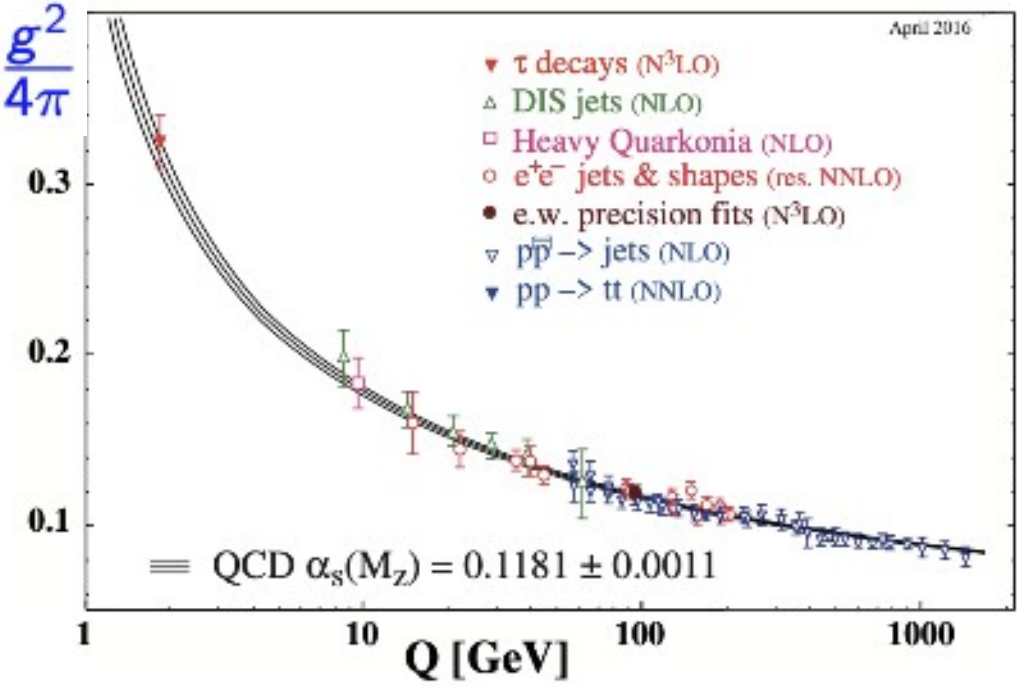
While, in QED, an abelian theory, the charge grows with the probing energy (screening).

$$\alpha = \frac{e^2}{4\pi}$$



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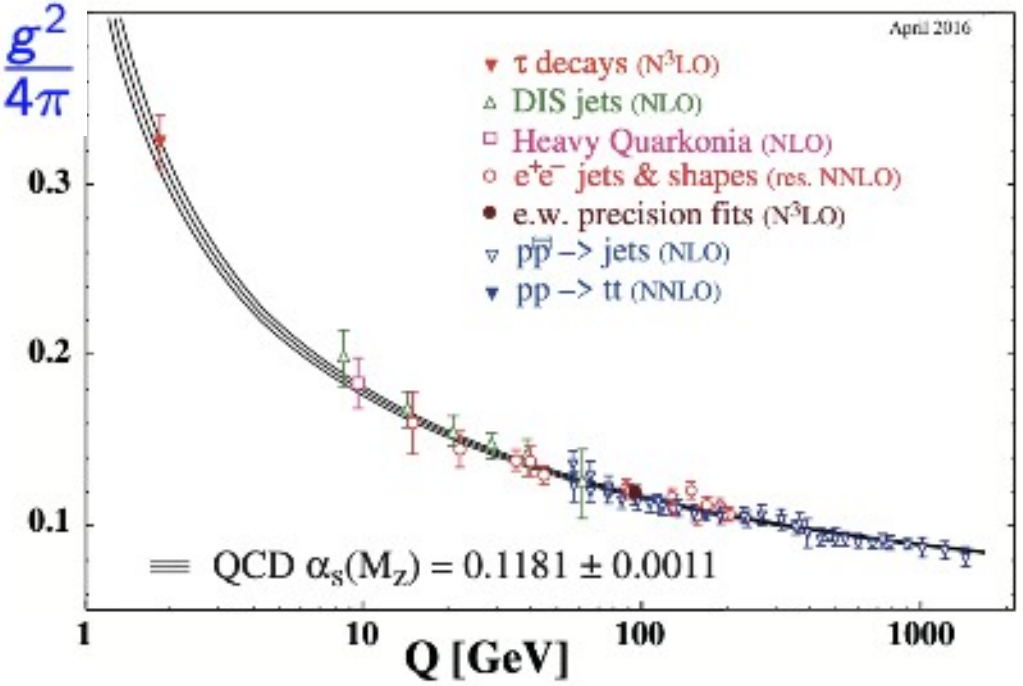


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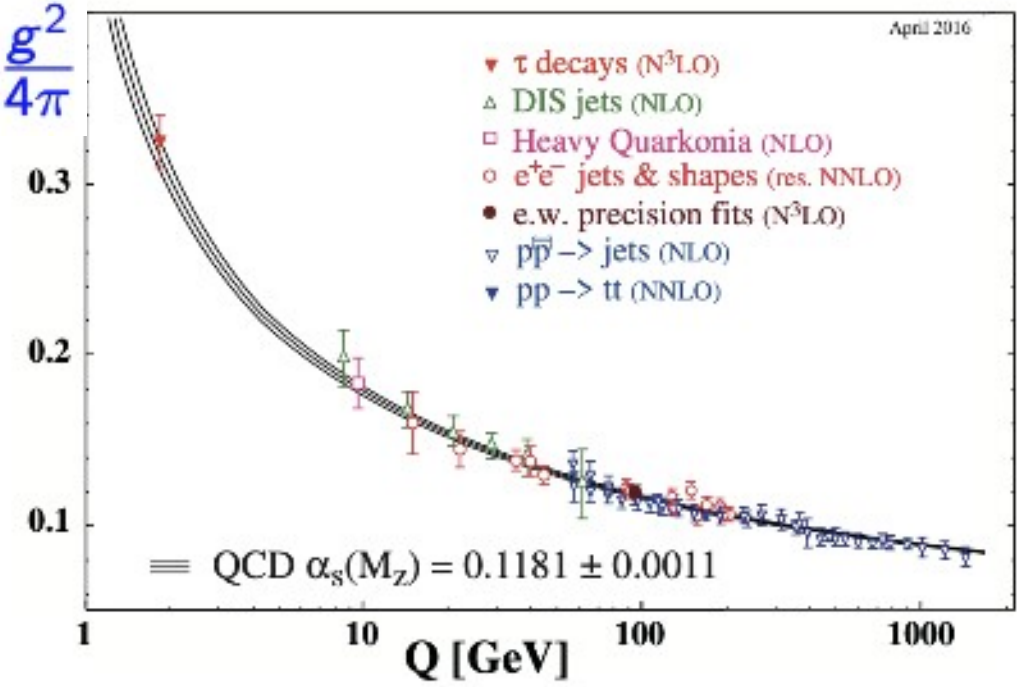
$$t = \ln\left(\frac{\zeta^2}{\Lambda^2}\right)$$

QCD fundamental scale

The perturbative running of couplings in different schemes is driven by their expansion, one in terms of each other, but the first coefficient of the expansion defines by itself how their QCD fundamental scales relate.

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The perturbative running of couplings in different schemes is driven by their expansion, one in terms of each other, but the first coefficient of the expansion defines by itself how their QCD fundamental scales relate. A new different coupling can be perturbatively defined by imposing an exact expansion up to this coefficient.

Effective charges: generalities

A perturbative scheme does not eliminate the **Landau pole**, and cannot hence cure the *fake infrared problem,* but **Grunberg's** notion of **effective charge** extended to the non-perturbative domain does.

G. Grunberg, Phys.Lett.B95, 70 (1980); Phys.Rev.D29, 2315 (1984)

Let's illustrate it with the one based on the **Bjorken sum rule**

J. D. Bjorken, Phys.Rev.D1, 1376(1970)

$$\Gamma_1^{p-n}(Q^2) := \int_0^1 dx \left[g_1^p(x,Q^2) - g_1^n(x,Q^2)\right] \overset{\Lambda_s^2/Q^2 \simeq 0}{=} \frac{g_A}{6}$$

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$$\Gamma_1^{\mathrm{p-n}}(Q^2) = \frac{g_{\mathrm{A}}}{6}\left[1 - \frac{\alpha_{\mathrm{s}}^{\mathrm{pQCD}}(Q^2)}{\pi} - 3.58\left(\frac{\alpha_{\mathrm{s}}^{\mathrm{pQCD}}(Q^2)}{\pi}\right)^2 - 20.21\left(\frac{\alpha_{\mathrm{s}}^{\mathrm{pQCD}}(Q^2)}{\pi}\right)^3 - 175.7\left(\frac{\alpha_{\mathrm{s}}^{\mathrm{pQCD}}(Q^2)}{\pi}\right)^4 + \sim -893.38\left(\frac{\alpha_{\mathrm{s}}^{\mathrm{pQCD}}(Q^2)}{\pi}\right)^5 + \mathcal{O}\left(\left(\alpha_{\mathrm{s}}^{\mathrm{pQCD}}\right)^6\right)\right] + \sum_{n>1}\frac{\mu_{2n}(Q^2)}{Q^{2n-2}}.$$

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A. Deur et al., Prog.Part.Nucl.Phys. 134(2024)

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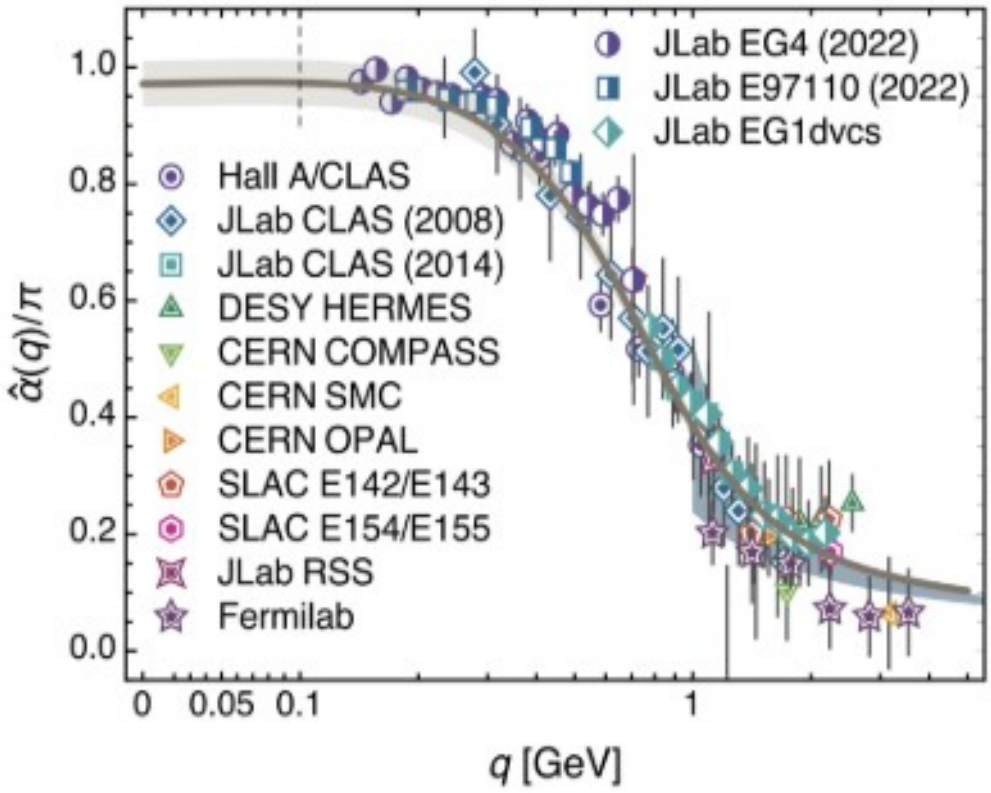
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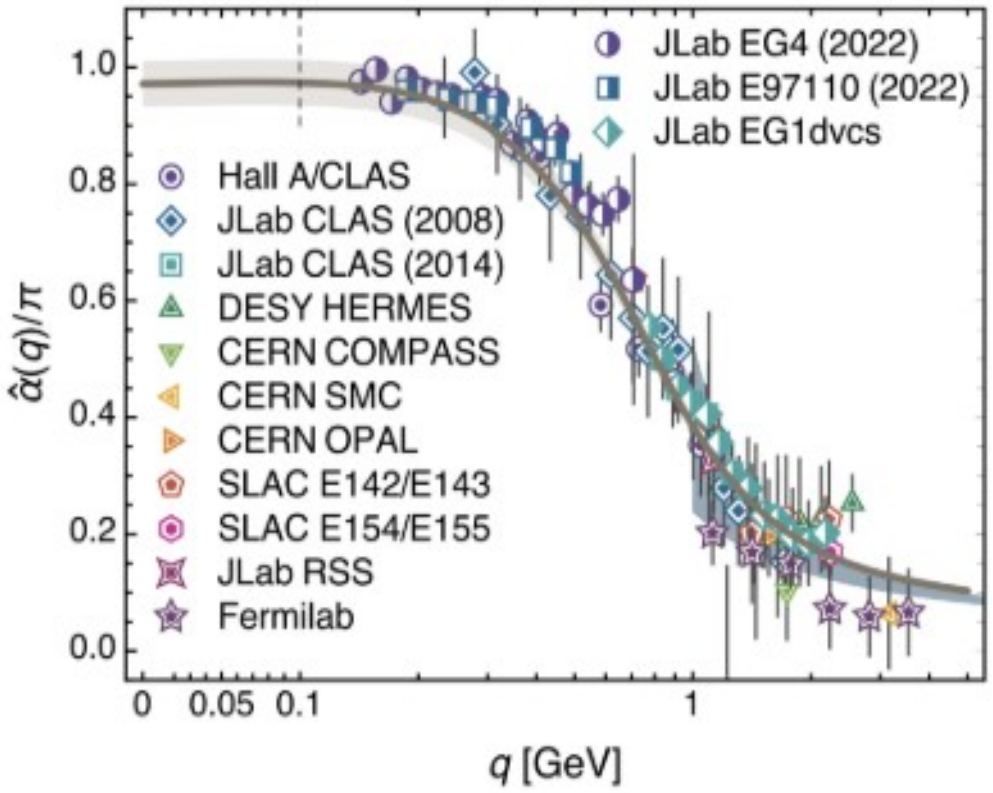
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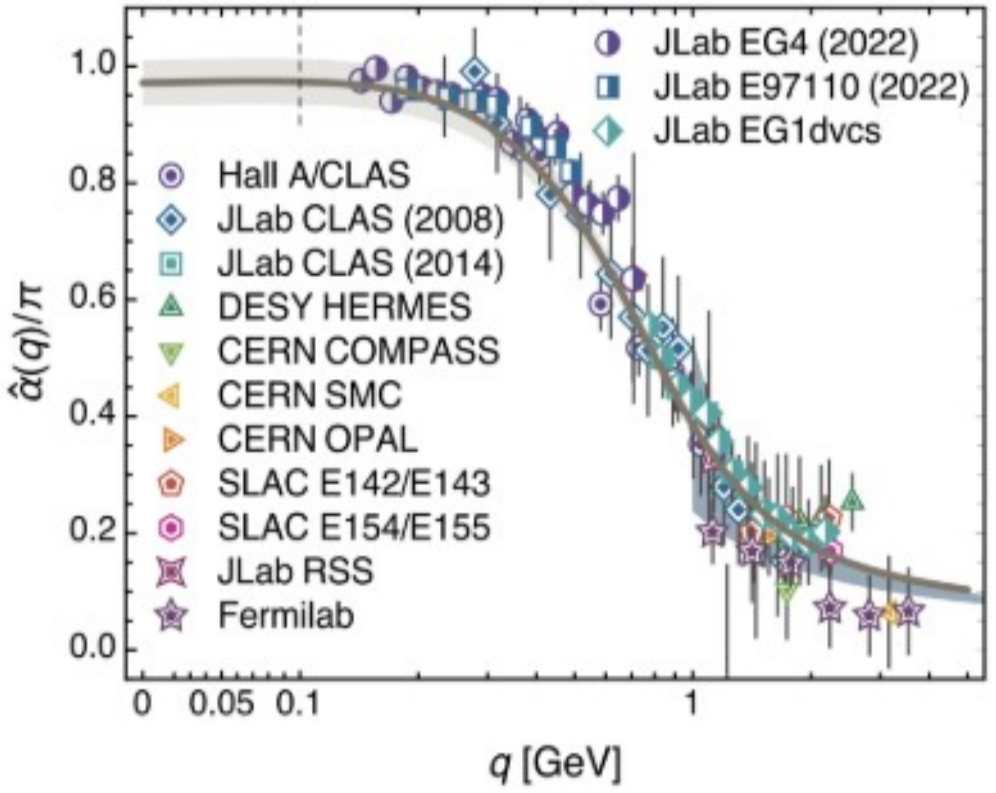
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It connects to other infrared non-perturbative calculations of the strong running coupling, as the one derived in the context of the **HLFQCD**.



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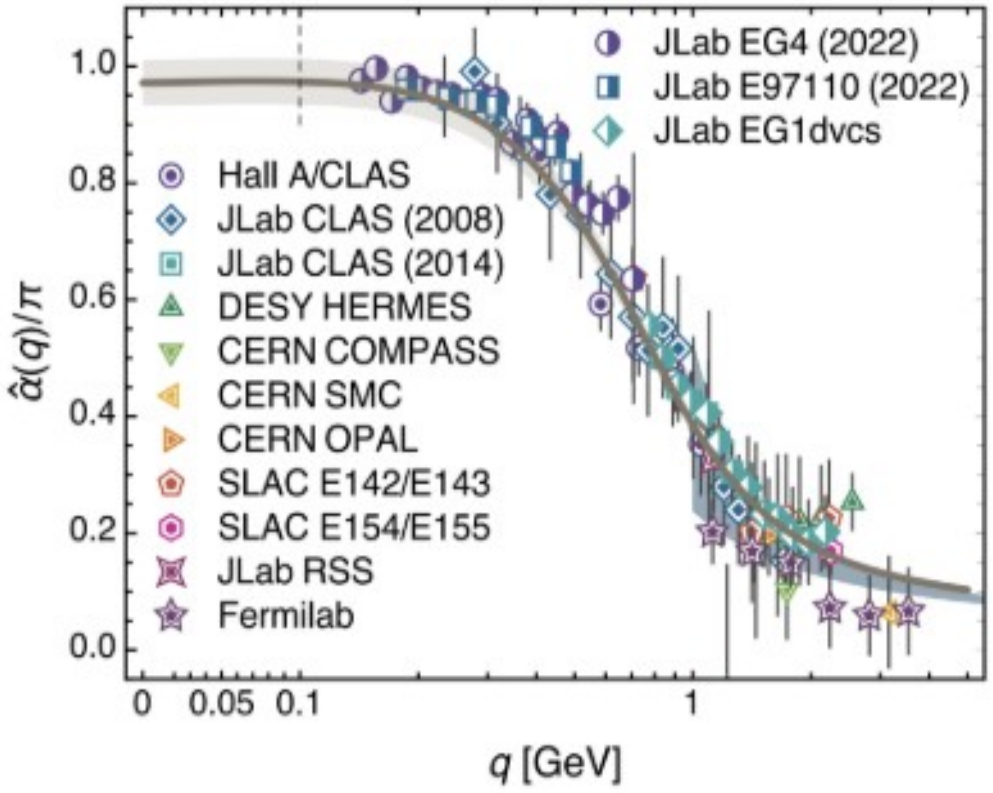
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A **process independent** effective charge can be also shown to connect to $\alpha_{g_1}(Q^2)$

$$I(k^2) := k^2 \widehat{d}(k^2) = \frac{\alpha_{\rm T}(k^2)}{[1 - L(k^2;\zeta^2)F(k^2;\zeta^2)]^2}$$

A **running interaction**, defined for the gap equation **kernel** within the ("abelianized") **PT-BFM** scheme for DSEs, results from the (massive) gluon vacuum polarization;



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Z-F Cui et al., Chin.Phys.C44 083102 (2020)

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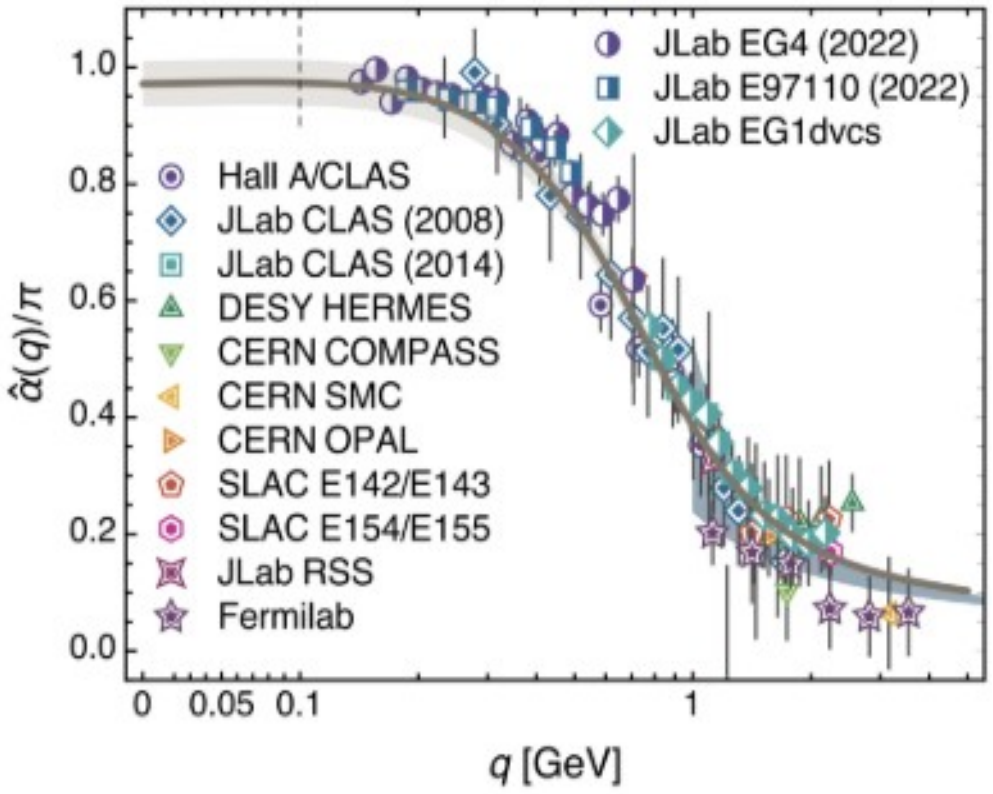
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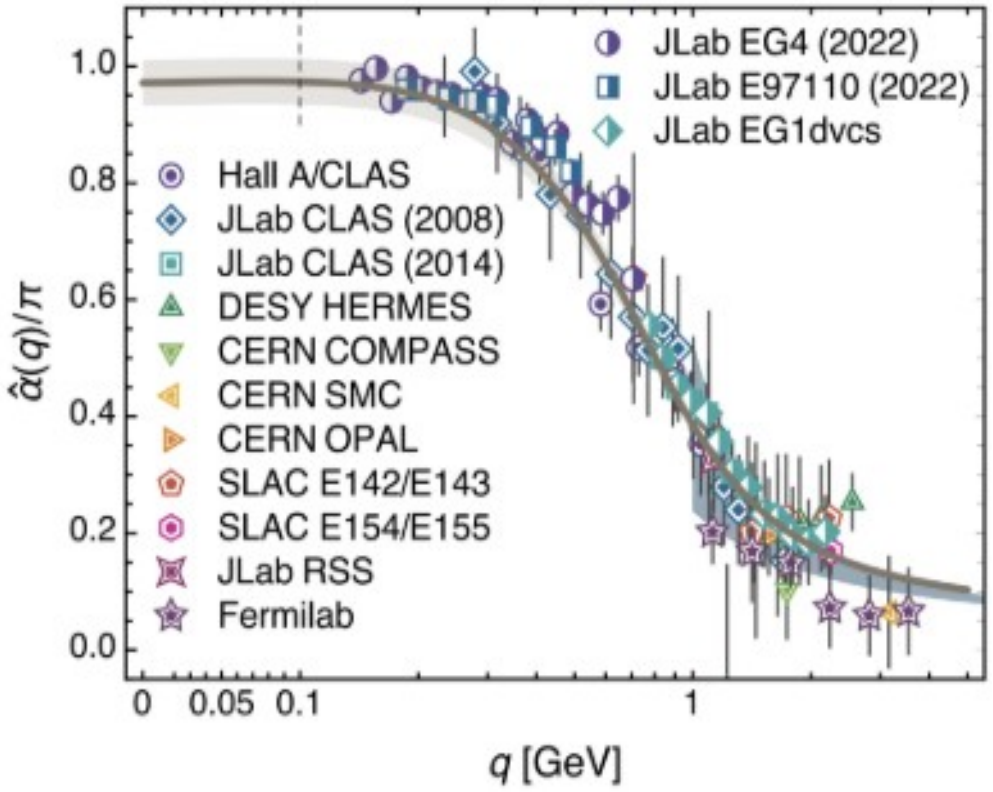
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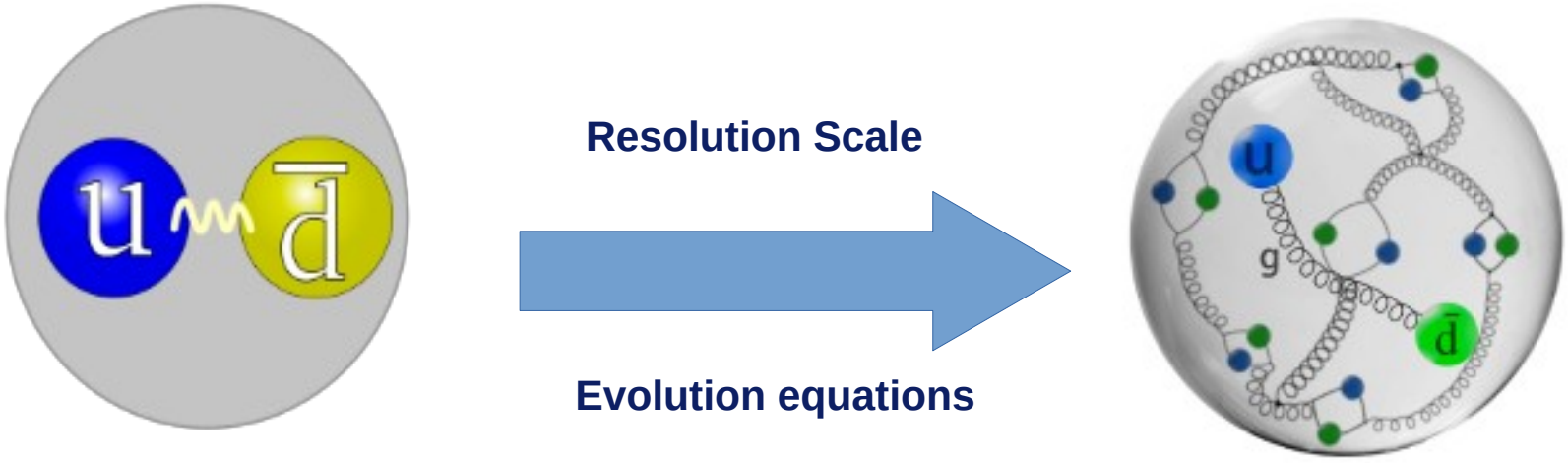
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Parton distributions: energy scales

ζ_H $\qquad$ $\zeta > \zeta_H$

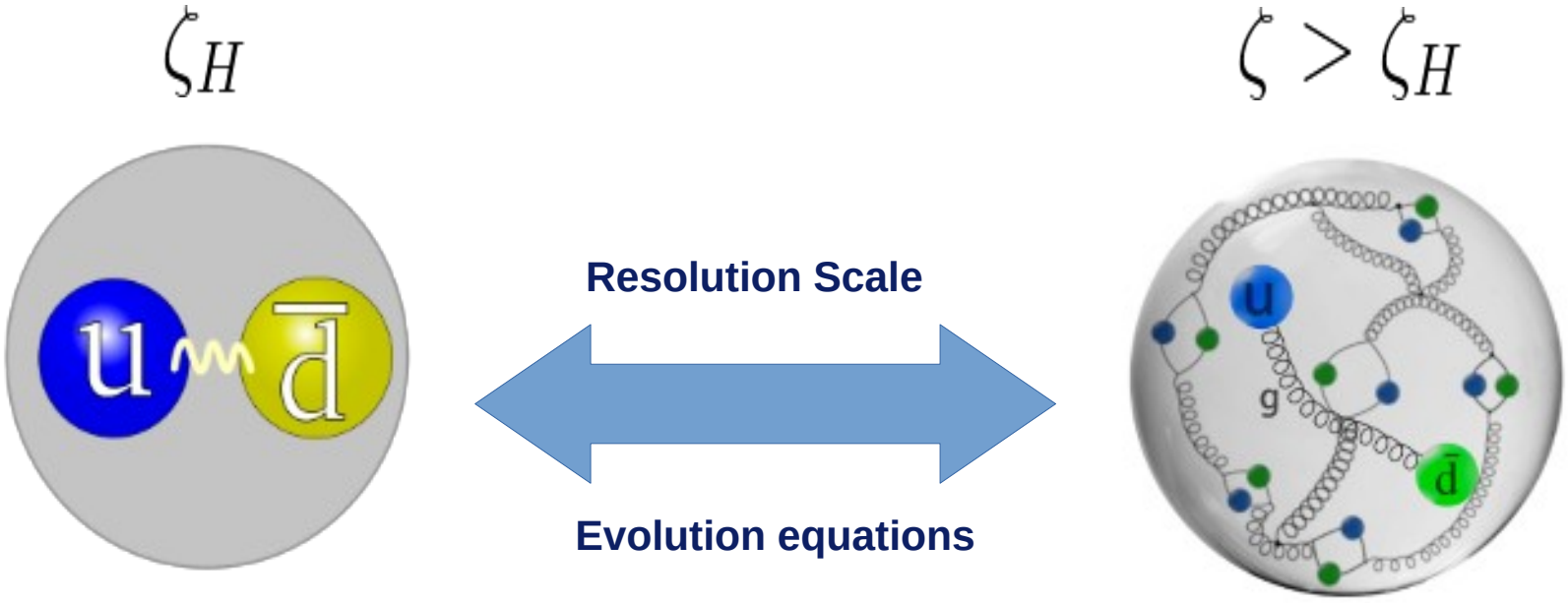


- **Fully-dressed valence quarks**
  - At this scale, **all properties** of the hadron are contained within their valence quarks.
  - **QCD constraints** are defined from here (e.g. large-x behavior of the PDF)

$$u^{\pi}(x;\zeta) \overset{x\simeq 1}{\sim} (1-x)^{\beta=2+\gamma(\zeta)}$$

- **Unveiling of glue and sea d.o.f.**
  - **Experimental** data is given **here**.
  - The interpretation of parton distributions from cross sections demands **special care**.
  - In addition, the synergy with **lattice QCD** and phenomenological approaches is welcome.

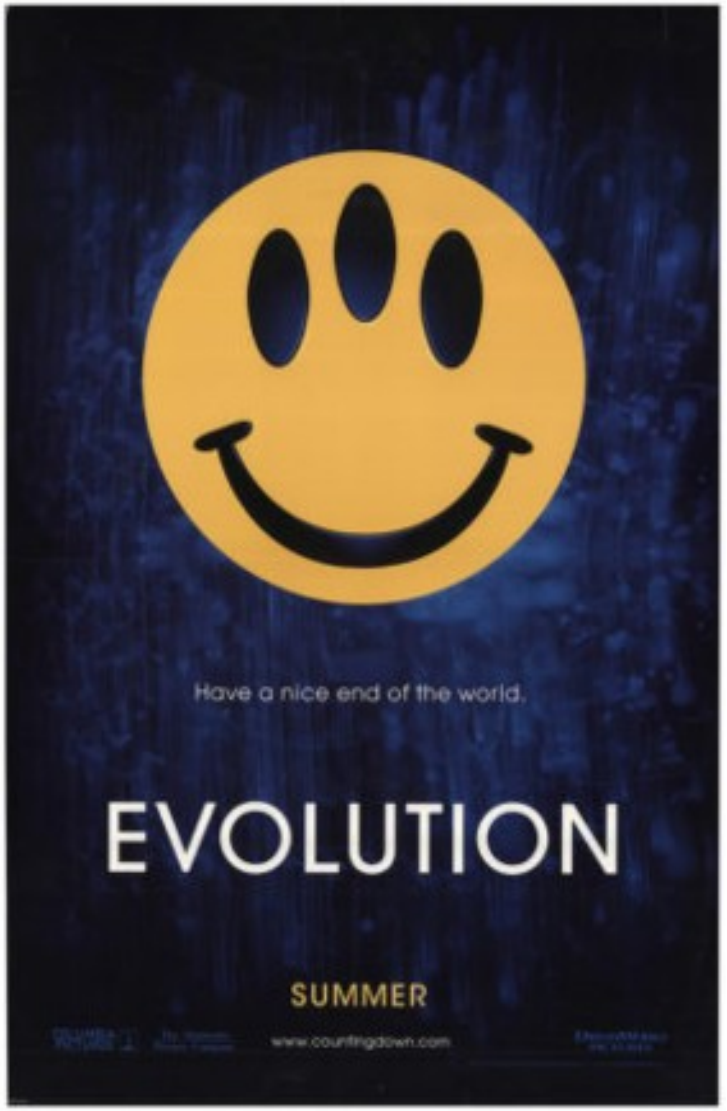
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- **Unveiling of glue and sea d.o.f.**
  - **Experimental** data is given **here**.
  - The interpretation of parton distributions from cross sections demands **special care**.
  - In addition, the synergy with **lattice QCD** and phenomenological approaches is welcome.

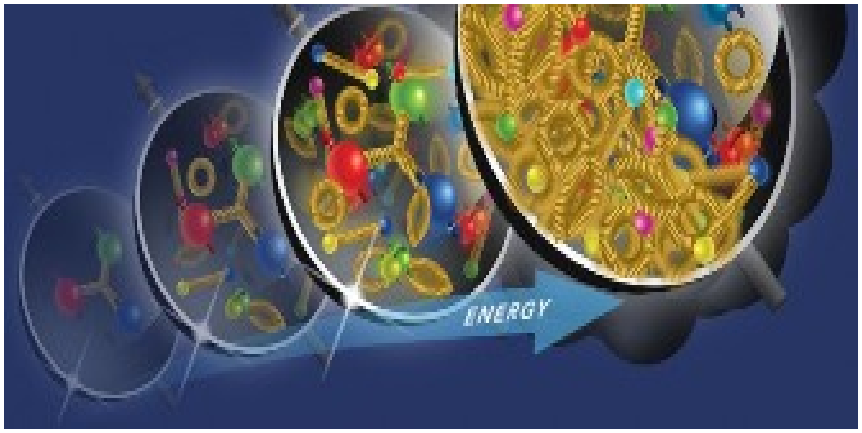

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DGLAP: All orders evolution

Z-F. Cui et al., Eur.Phys.J.C80 1064 (2020)
K. Raya et al. Chin.Phys.C46 013105(2022)

$$\left\{\zeta^2\frac{d}{d\zeta^2}\int_0^1 dy\delta(y-x) - \frac{\alpha(\zeta^2)}{4\pi}\int_x^1\frac{dy}{y}\begin{pmatrix} P_{qq}^{\rm NS}\left(\frac{x}{y}\right) & 0 \\ 0 & \mathbf{P}^{\rm S}\left(\frac{\mathbf{x}}{\mathbf{y}}\right)\end{pmatrix}\right\}\begin{pmatrix} H_\pi^{\rm NS,+}(y,t;\zeta) \\ \mathbf{H}_\pi^{\rm S}(y,t;\zeta)\end{pmatrix} = 0$$

DGLAP leading-order evolution equations



DGLAP: All orders evolution

Assumption: define an **effective** charge such that

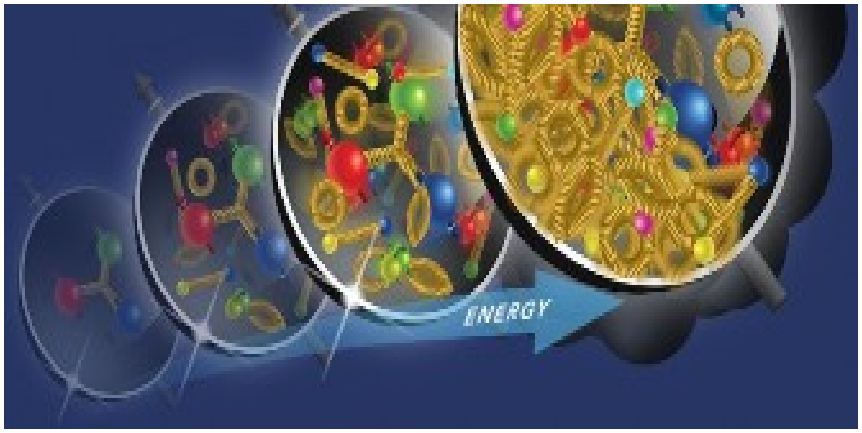
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Starting from fully-dressed **quasiparticles**, at ζ_H

→ **Sea** and **Gluon** content unveils, as prescribed by **QCD**

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DGLAP ~~leading order~~ evolution equations



- **Not** the **LO** QCD coupling but an **effective** one.
- Making this equation **exact**.
- Connecting with the **hadron scale**, at which the **fully-dressed** valence-**quarks** express **all** of the hadron's properties.

(thus carrying all the momentum)

DGLAP: All orders evolution

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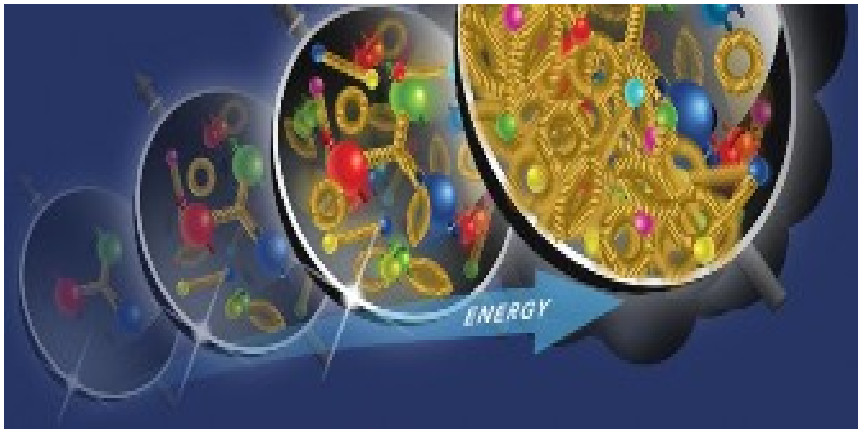
Starting from fully-dressed **quasiparticles**, at ζ_H

Sea and **Gluon** content unveils, as prescribed by **QCD**

$$\left\{ \zeta^2 \frac{d}{d\zeta^2} \mathbb{1} + \frac{\alpha(\zeta^2)}{4\pi} \begin{pmatrix} \gamma_{qq}^{(n)} & 0 & 0 \\ 0 & \gamma_{qq}^{(n)} & 2n_f \gamma_{qg}^{(n)} \\ 0 & \gamma_{gq}^{(n)} & \gamma_{gg}^{(n)} \end{pmatrix} \right\} \begin{pmatrix} \langle x^n \rangle_{\rm NS}(\zeta) \\ \langle x^n \rangle_{\rm S}(\zeta) \\ \langle x^n \rangle_g(\zeta) \end{pmatrix} = 0$$

DGLAP ~~leading order~~ evolution equations

$$\gamma_{AB}^{(n)} = -\int_0^1 dx\, x^n P_{AB}^C(x)$$



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DGLAP: All orders evolution

Z-F. Cui et al., Eur.Phys.J.C80 1064 (2020)

PDFs DGLAP evolutions equations, expressed by the corresponding massless splitting functions:

$$\Sigma^q_{\mathsf{H}}(x) = q_{\mathsf{H}}(x) + \bar{q}_{\mathsf{H}}(x)$$

singlet combination

$$\zeta^2 \frac{d}{d\zeta^2} q_{\mathsf{H}}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} P_{q\leftarrow q}\left(\frac{x}{y}\right) q_{\mathsf{H}}(y)$$

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Valence-quark PDF in Mellin space

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{q_H} = -\frac{\alpha(\zeta^2)}{4\pi} \gamma^n_{qq} \langle x^n \rangle^{\zeta}_{q_H}$$

$$\langle x^n \rangle^{\zeta}_{q_{II}} = \langle x^n \rangle^{\zeta_H}_{q_{II}} \exp\left(-\frac{\gamma^n_{qq}}{2\pi} \int_{\zeta_{II}}^{\zeta} \frac{dz}{z} \alpha(z^2) \right)$$

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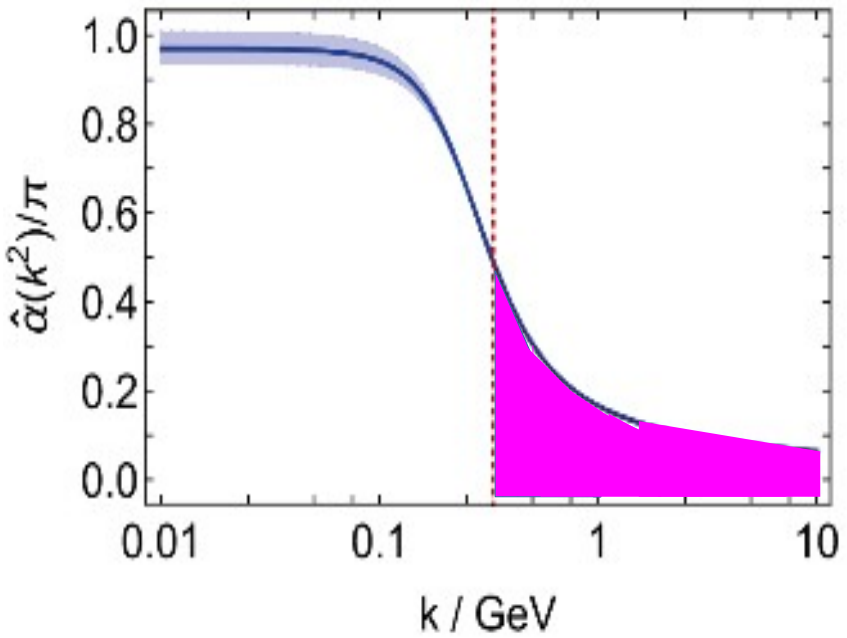
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Moments' evolution is controlled by the integrated "strength" of the coupling beyond the hadron scale

DGLAP: All orders evolution

Z-F. Cui et al., Eur.Phys.J.C80 1064 (2020)

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The ratio of lightcone momentum fractions encodes the required information of the charge

$$\frac{\langle x \rangle^{\zeta}_{q_H}}{\langle x \rangle^{\zeta_H}_{q_H}} = \exp\left(-\frac{\gamma_{qq}}{2\pi} \int_{\zeta_H}^{\zeta} \frac{dz}{z} \alpha(z^2) \right)$$

DGLAP: All orders evolution

Z-F. Cui et al., Eur.Phys.J.C80 1064 (2020)

Application 1: valence-quark PDF

$$\langle x^n\rangle^{\zeta}_{q_H} = \langle x^n\rangle^{\zeta_H}_{q_H} \exp\left(-\frac{\gamma^n_{qq}}{2\pi}\int_{\zeta_H}^{\zeta}\frac{dz}{z}\alpha(z^2)\right) = \langle x^n\rangle^{\zeta_H}_{q_H}\left[\frac{\langle x\rangle^{\zeta}_{q_H}}{\langle x\rangle^{\zeta_H}_{q_H}}\right]^{\gamma^n_{qq}/\gamma_{qq}}$$

This ratio encodes the information of the charge

Direct connection bridging from hadron to experimental scale: only one input is needed to evolve “all” the Mellin moments up and reconstruct the PDF.

DGLAP: All orders evolution

Z-F. Cui et al., Eur.Phys.J.C80 1064 (2020)

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This ratio encodes the information of the charge and use isospin symmetry (pion case)

$$\langle x \rangle^{\zeta_H}_{u_\pi} = \langle x \rangle^{\zeta_H}_{d_\pi} = \frac{1}{2}$$

DGLAP: All orders evolution

Z-F. Cui et al., Eur.Phys.J.C80 1064 (2020)

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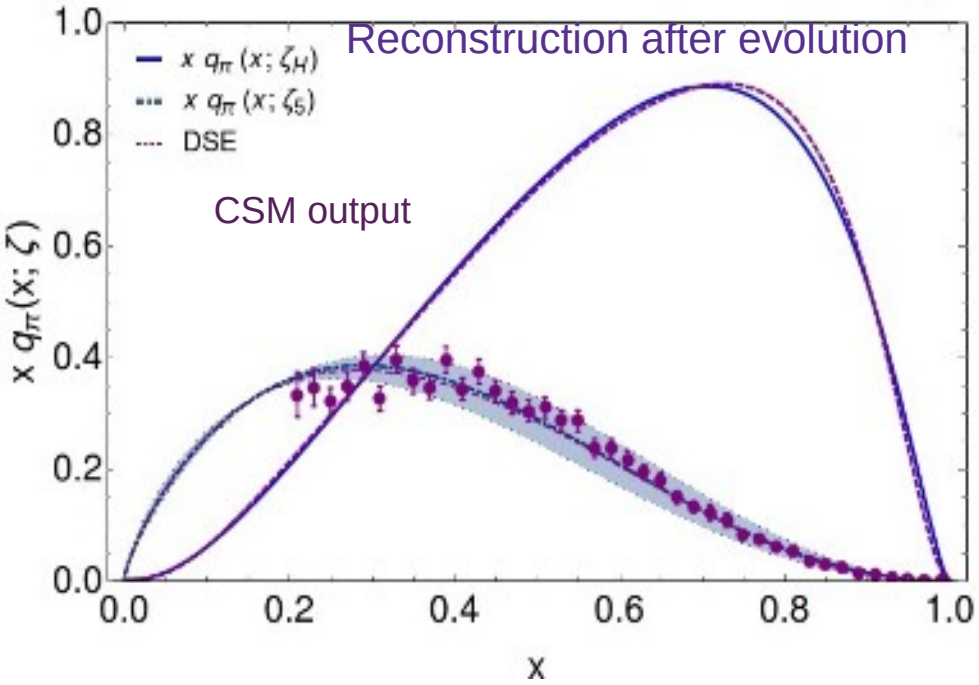
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Symmetry-preserving DSE computation of the valence-quark PDF:

[L. Chang et al., Phys.Lett.B737(2014)23]
[M. Ding et al., Phys.Rev.D101(2020)054014

$$q^\pi(x;\zeta) = N_c\,\mathrm{tr}\int_{dk}\delta^x_n(k_\eta)\Gamma^P_\pi(k_{\bar\eta\eta};\zeta)\,S(k_{\bar\eta};\zeta)$$
$$\times\left\{n\cdot\frac{\partial}{\partial k_\eta}\left[\Gamma^P_\pi(k_{\eta\bar\eta};\zeta)S(k_\eta;\zeta)\right]\right\}.$$

DGLAP: All orders evolution

Z-F. Cui et al., Eur.Phys.J.C80 1064 (2020)

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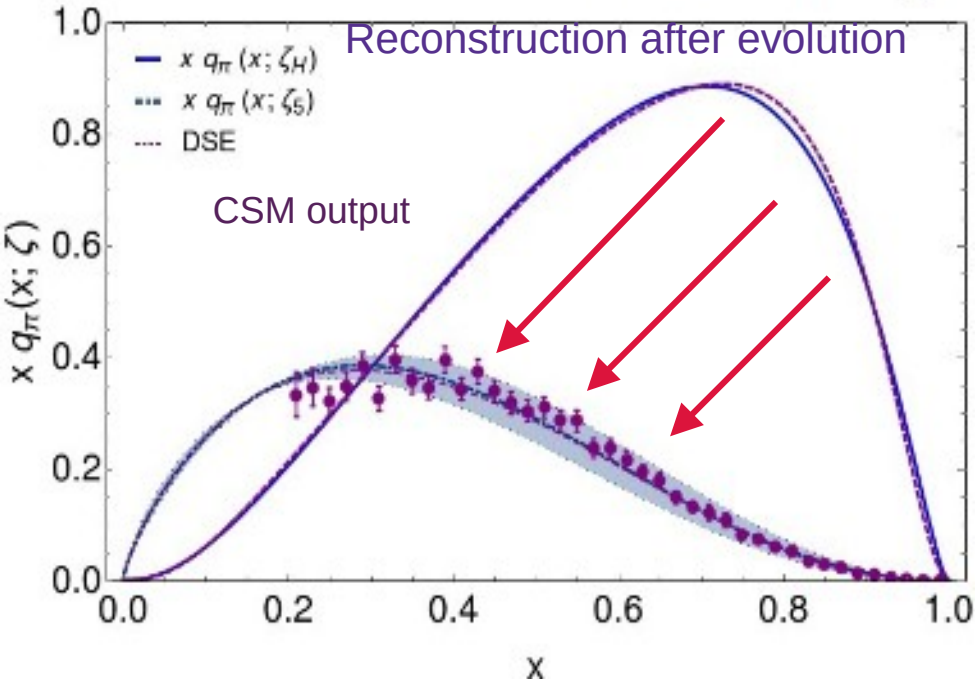
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DGLAP: All orders evolution

Z-F. Cui et al., Eur.Phys.J.A58 10 (2022)

Application 2: recursion of Mellin moments (pion case)

$$\langle x^{2n+1}\rangle^{\zeta}_{u_\pi} = \frac{(\langle 2x\rangle^{\zeta}_{u_\pi})^{\gamma_0^{2n+1}/\gamma_0^1}}{2(n+1)} \times \sum_{j=0,1,\ldots}^{2n} (-)^j \binom{2(n+1)}{j} \langle x^j\rangle^{\zeta}_{u_\pi} (\langle 2x\rangle^{\zeta}_{u_\pi})^{-\gamma_0^j/\gamma_0^1}.$$

- Since **isospin symmetry** limit implies:

$$q(x;\zeta_H) = q(1-x;\zeta_H)$$

- **Odd** moments can be expressed in terms of previous **even** moments.
- Thus arriving at the recurrence **relation** on the left which is satisfied if, and only if, the source distribution is related by evolution to a symmetric one at the initial scale .

DGLAP: All orders evolution

Z-F. Cui et al., Phys.Rev.D105 L091502 (2022)

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Reported lattice moments agree very well with the recursion formula

n	$\langle x^n\rangle^{\zeta_5}_{u_\pi}$ Ref. [99]	Eq. (17)
1	0.230(3)(7)	0.230
2	0.087(5)(8)	0.087
3	0.041(5)(9)	0.041
4	0.023(5)(6)	0.023
5	0.014(4)(5)	0.015
6	0.009(3)(3)	0.009
7		0.0078

[99] C. Alexandrou et al., PRD104(2021)054504

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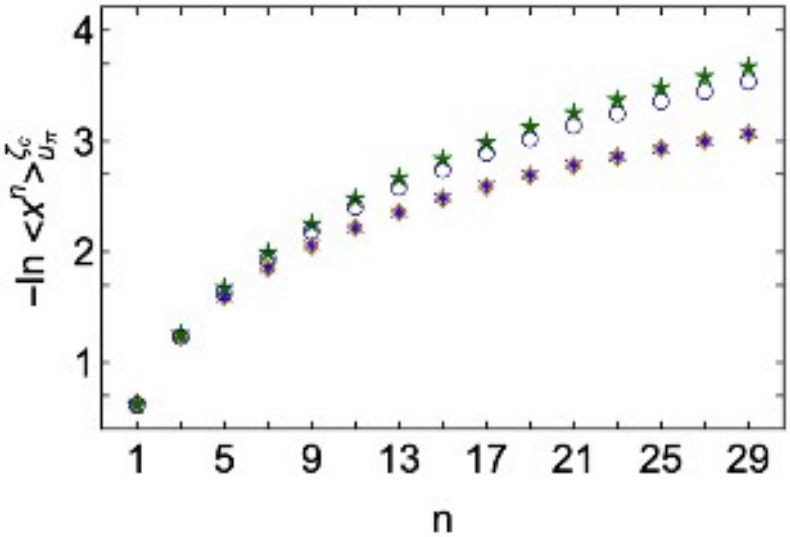
- **Odd** moments can be expressed in terms of previous **even** moments.
- Thus arriving at the recurrence **relation** on the left which is satisfied if, and only if, the source distribution is related by evolution to a symmetric one at the initial scale .

Reported lattice moments agree very well with the recursion formula and so also does and estimate for the 7-th moment from lattice reconstruction.

Moments from global fits can be also compared to the estimated from recursion !

n	$\langle x^n\rangle^{\zeta_5}_{u_\pi}$ Ref. [99]	Eq. (17)
1	0.230(3)(7)	0.230
2	0.087(5)(8)	0.087
3	0.041(5)(9)	0.041
4	0.023(5)(6)	0.023
5	0.014(4)(5)	0.015
6	0.009(3)(3)	0.009
7	0.0065(24)	0.0078

Moments computed from: P. Barry et al., PRL127(2021)232001



[99] C. Alexandrou et al., PRD104(2021)054504

DGLAP: All orders evolution

Z-F. Cui et al., Phys.Rev.D105 L091502 (2022)

Application 3: physical bounds (pion case)

$$\frac{1}{2^n} \leq \langle x^n \rangle^{\zeta}_{u_\pi} \left(\langle 2x \rangle^{\zeta}_{u_\pi} \right)^{-\gamma_0^n/\gamma_0^1} \leq \frac{1}{1+n}$$

$$q(x;\zeta_H) = \delta(x-1/2) \qquad q(x;\zeta_H) = 1$$

- Keeping **isospin symmetry,** implying:

$$q(x;\zeta_H) = q(1-x;\zeta_H)$$

- **Lower bound** is imposed by considering the limit of a system of two strongly massive and maximally correlated) partons: both carry half of the momentum.
- **Upper bound** comes out from considering the opposite limit of a weekly interacting system of two (then fully decorrelated) partons: all the momentum fractions are equally probable.

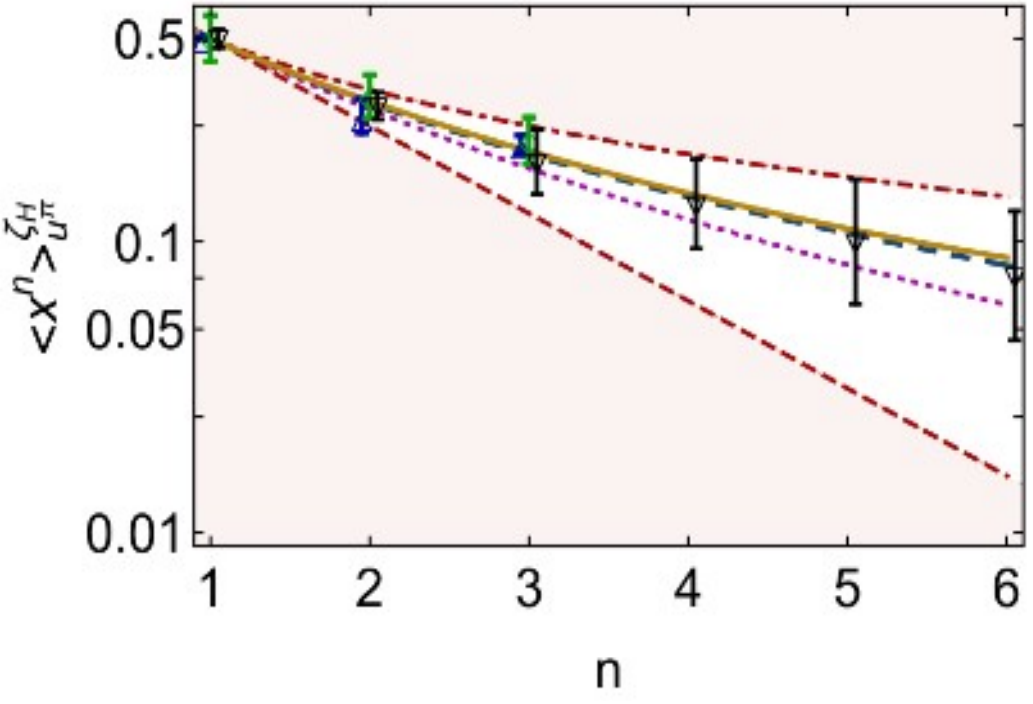
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Joo:2019bzr Sufian:2019bol Alexandrou:2021mmi

n	[61]	[62]	[63]
1	0.254(03)	0.18(3)	0.23(3)(7)
2	0.094(12)	0.064(10)	0.087(05)(08)
3	0.057(04)	0.030(05)	0.041(05)(09)
4			0.023(05)(06)
5			0.014(04)(05)
6			0.009(03)(03)

Lattice moments verifying the **recurrence relation** too.

DGLAP: All orders evolution

P-L. Yin et al., Chin.Phys.Lett.40 091201 (2022)

PDFs DGLAP evolutions equations, expressed by the corresponding massless splitting functions:

$$\Sigma^q_{\mathsf{H}}(x) = q_{\mathsf{H}}(x) + \bar{q}_{\mathsf{H}}(x)$$

singlet combination

$$\zeta^2 \frac{d}{d\zeta^2} q_{\mathsf{H}}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} P_{q\leftarrow q}\left(\frac{x}{y}\right) q_{\mathsf{H}}(y)$$

$$\zeta^2 \frac{d}{d\zeta^2} \Sigma^q_{\mathsf{H}}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} \left\{ P_{q\leftarrow q}\left(\frac{x}{y}\right) \Sigma^q_{\mathsf{H}}(y) + 2P^{\zeta}_{q\leftarrow g}\left(\frac{x}{y}\right) g_{\mathsf{H}}(y) \right\}$$

$$\zeta^2 \frac{d}{d\zeta^2} g_{\mathsf{H}}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} \left\{ P_{g\leftarrow q}\left(\frac{x}{y}\right) \Sigma^q_{\mathsf{H}}(y) + P_{g\leftarrow g}\left(\frac{x}{y}\right) g_{\mathsf{H}}(y) \right\}$$

Hard-wall threshold

$$\mathcal{P}^{\zeta}_q = \theta(\zeta - M_q)$$

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Hard-wall threshold

$$\mathcal{P}^{\zeta}_q = \theta(\zeta - M_q)$$

Quark singlet and glue PDFs in Mellin space

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{\Sigma_H} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma^n_{uu} \langle x^n \rangle^{\zeta}_{\Sigma_H} + 2n_f \mathcal{P}^{\zeta}_q \gamma^n_{ug} \langle x^n \rangle^{\zeta}_{g_H} \right\}$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{g_H} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma^n_{gu} \langle x^n \rangle^{\zeta}_{\Sigma_H} + \gamma^n_{gg} \langle x^n \rangle^{\zeta}_{g_H} \right\}$$

Sea-quark PDF

$$\langle x^n \rangle^{\zeta}_{S^q_H} = \langle x^n \rangle^{\zeta}_{\Sigma^q_H} - \langle x^n \rangle^{\zeta}_{q_H}$$

Full singlet and sea

$$\langle x^n \rangle^{\zeta}_{\Sigma_H} = \sum_q \langle x^n \rangle^{\zeta}_{\Sigma^q_H} \, , \langle x^n \rangle^{\zeta}_{S_H} = \sum_q \langle x^n \rangle^{\zeta}_{S^q_H}$$

DGLAP: All orders evolution

P-L. Yin et al., Chin.Phys.Lett.40 091201 (2022)

Application 4: glue and sea from valence

$$M_q = \zeta_H, \ \forall q$$

All quarks active

$$\zeta^2 \frac{d}{d\zeta^2} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \begin{pmatrix} \gamma^n_{uu} & 2n_f \gamma^n_{ug} \\ \gamma^n_{gu} & \gamma^n_{gg} \end{pmatrix} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix}$$

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$$\begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \begin{pmatrix} \alpha^n_+ S^n_- + \alpha^n_- S^n_+ & \beta^n_{\Sigma g}\left(S^n_- - S^n_+\right) \\ \beta^n_{g\Sigma}\left(S^n_- - S^n_+\right) & \alpha^n_- S^n_- + \alpha^n_+ S^n_+ \end{pmatrix} \begin{pmatrix} \langle x^n \rangle^{\zeta_H}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta_H}_{g_H} \end{pmatrix}$$

$$\alpha^n_\pm = \pm \frac{\lambda^n_\pm - \gamma^n_{uu}}{\lambda^n_+ - \lambda^n_-}$$

$$\lambda^n_\pm = \frac{1}{2}\mathrm{Tr}\left(\Gamma^n\right) \pm \sqrt{\frac{1}{4}\mathrm{Tr}^2\left(\Gamma^n\right) - \mathrm{Det}\left(\Gamma^n\right)}$$

$$\beta^n_{\Sigma g} = -\frac{2n_f\gamma^n_{ug}}{\lambda^n_+ - \lambda^n_-}$$

$$S^n_\pm = [S(\zeta_H, \zeta)]^{\lambda^n_\pm/\gamma_{uu}}$$

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In terms of the moments for the sum of all valence-quark distributions at hadronic scale

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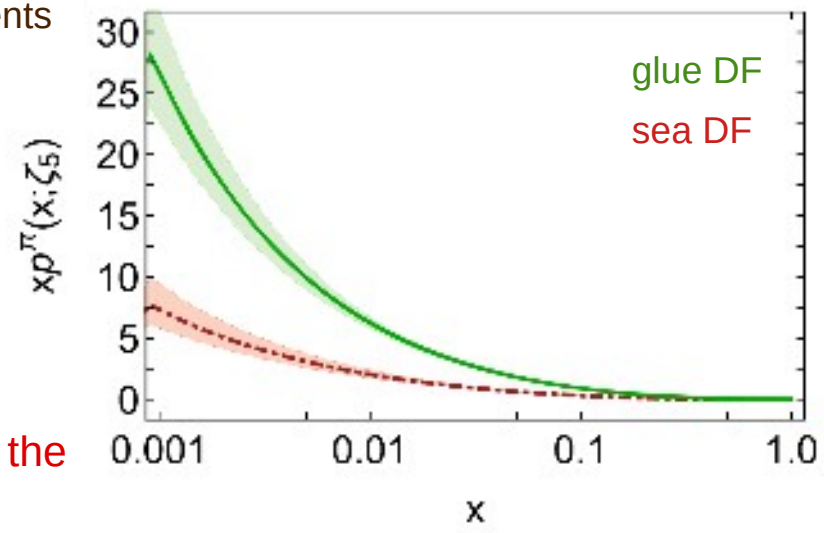
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Compute all the moments and reconstruct:

$$S^n_\pm = \left[S(\zeta_H, \zeta)\right]^{\lambda^n_\pm/\gamma_{uu}} \rightarrow = \left[\frac{\langle x \rangle^{\zeta}_{q_H}}{\langle x \rangle^{\zeta_H}_{q_H}}\right]^{\lambda^n_\pm/\gamma_{uu}}$$

$$\beta^n_{g\Sigma} = \frac{(\lambda^n_+ - \gamma^n_{uu})(\lambda^n_- - \gamma^n_{uu})}{2n_f\gamma^n_{ug}(\lambda^n_+ - \lambda^n_-)}$$



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n=1 case

$n_f = 4$

$$\langle x \rangle^{\zeta}_{\Sigma_H} = \sum_q \langle x \rangle^{\zeta}_{q_H} + \langle x \rangle^{\zeta}_{S_H} = \frac{3}{7} + \frac{4}{7}\left[S(\zeta_H, \zeta)\right]^{7/4}$$

$$S^n_\pm = \left[S(\zeta_H, \zeta)\right]^{\lambda^n_\pm/\gamma_{uu}}$$

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$$\langle x \rangle^{\zeta}_{g_\pi} = \frac{4}{7}\left(1 - [S(\zeta_H, \zeta)]^{7/4}\right)$$

ζ_5	$\langle 2x \rangle^{\pi}_q$	$\langle x \rangle^{\pi}_g$	$\langle x \rangle^{\pi}_{\mathrm{sea}}$
Ref.[55]	0.412(36)	0.449(19)	0.138(17)
Herein	0.40(4)	0.45(2)	0.14(2)

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Z-F. Cui et al., arXiv:2006.1465

R.S. Sufian et al., arXiv:2001.04960

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All n

$n_f = 4$

$$\Sigma_H(x) \underset{\zeta^2\to\infty}{=} \frac{3}{7}\frac{\delta(x)}{x}$$

$$g_H(x) \underset{\zeta^2\to\infty}{=} \frac{4}{7}\frac{\delta(x)}{x}$$

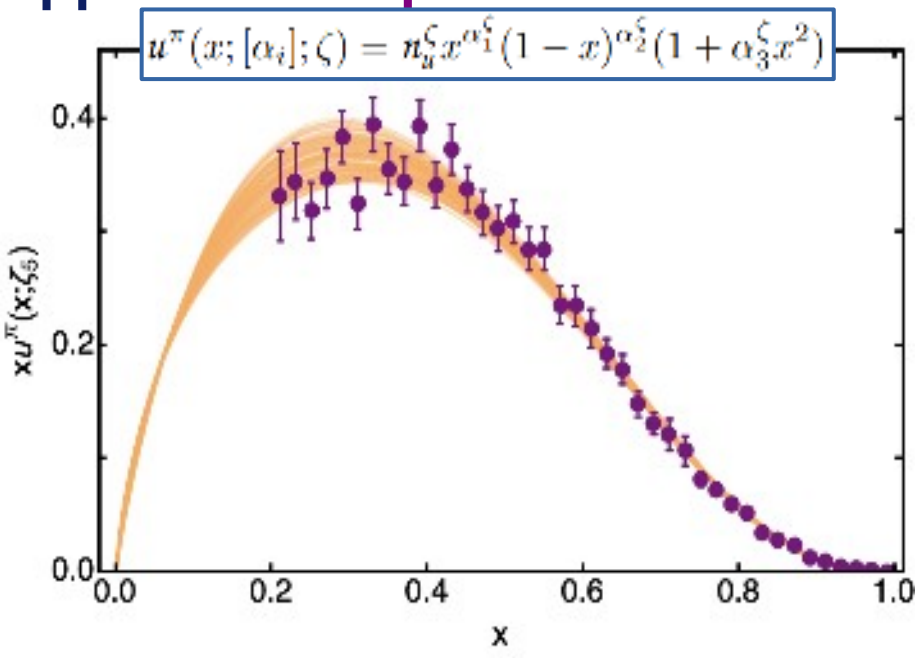
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G. Altarelli, Phys. Rep. 81, 1 (1982)

All-orders DGLAP: reverse engineering



Application 5: pion and kaon PDFs fully reconstructed from data



Z-F. Cui et al., Eur.Phys.J.A58 10 (2022)

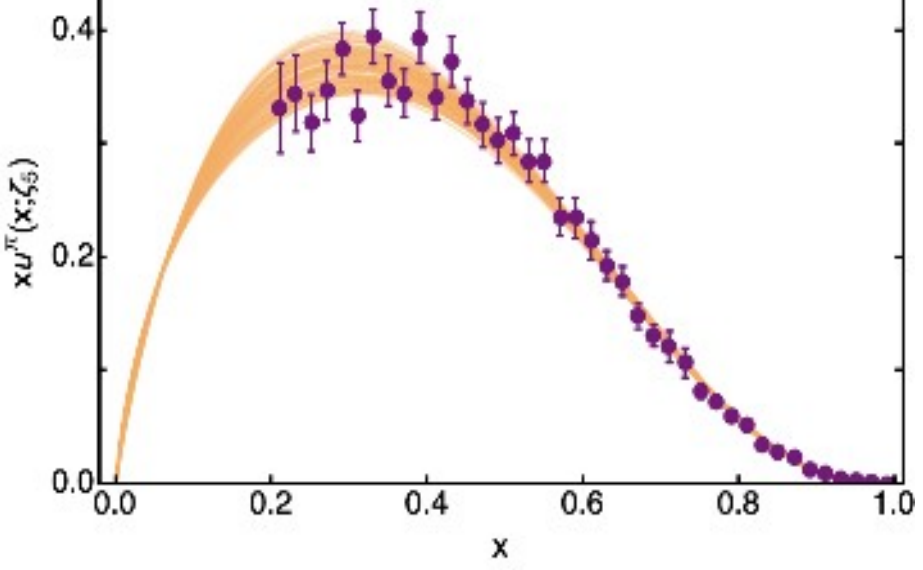
Data from [Aicher et al. Phys. Rev. Lett. 105, 252003 (2010)]

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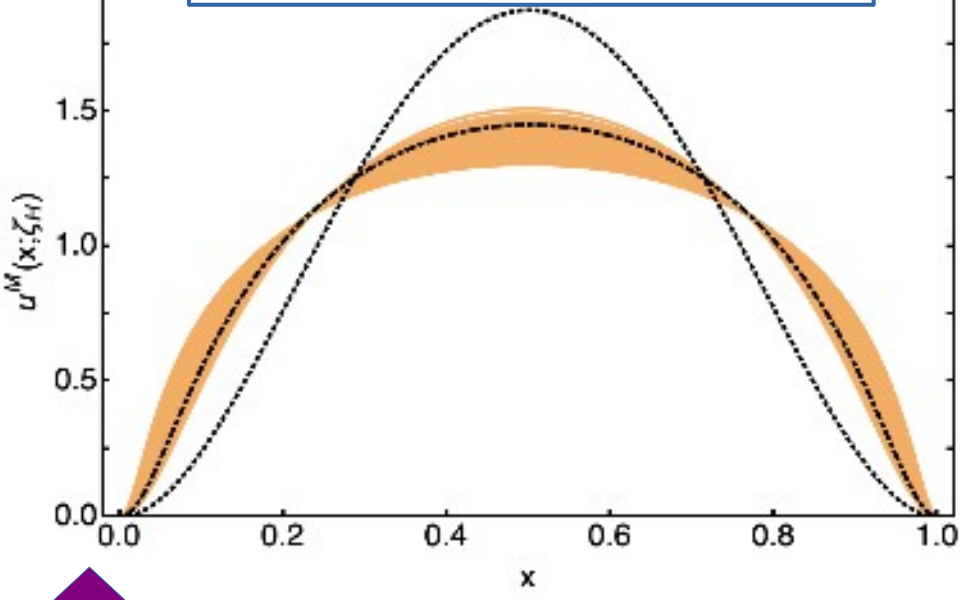


Application 5: pion and kaon PDFs fully reconstructed from data

$u^\pi(x;[\alpha_i];\zeta) = n_u^\zeta x^{\alpha_1^\zeta}(1-x)^{\alpha_2^\zeta}(1+\alpha_3^\zeta x^2)$



$u^\pi(x;\zeta_\mathcal{H}) = n_0 \ln(1+x^2(1-x)^2/\rho^2)$



All-orders back evolution from empirical to hadronic scale

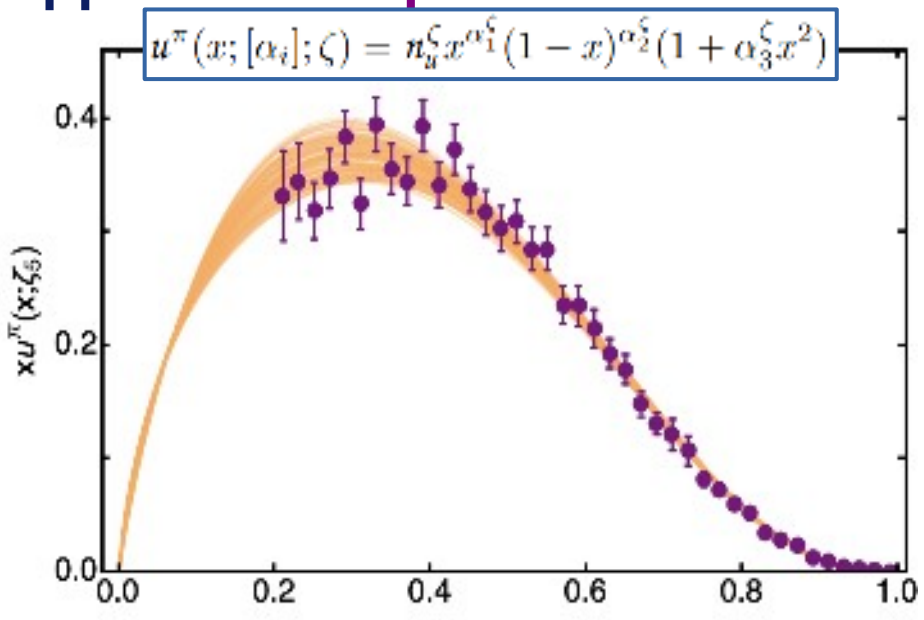
Z-F. Cui et al., Eur.Phys.J.A58 10 (2022)

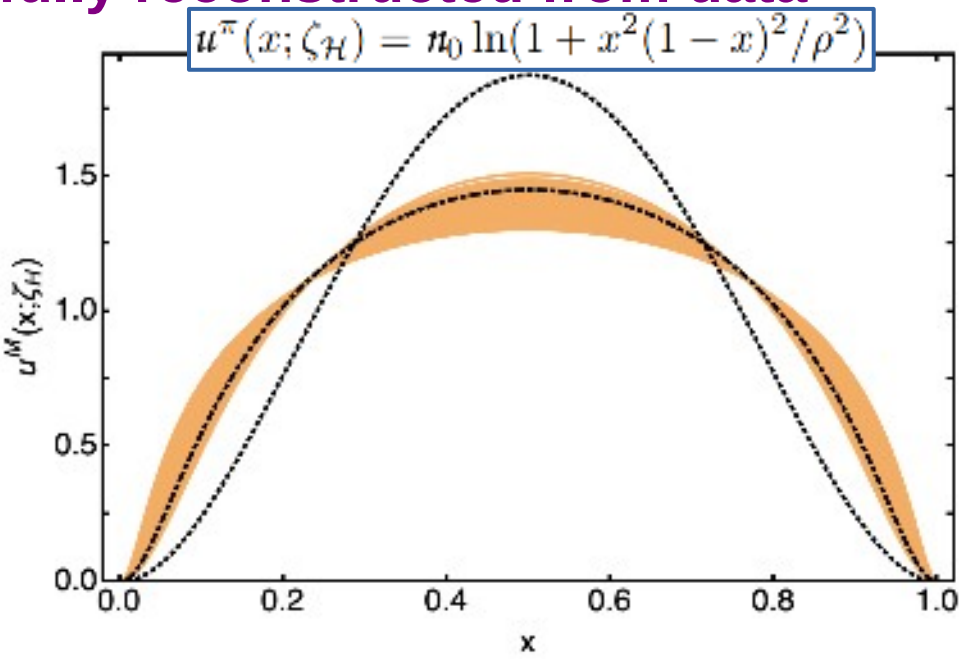
Data from [Aicher et al. Phys. Rev. Lett. 105, 252003 (2010)]

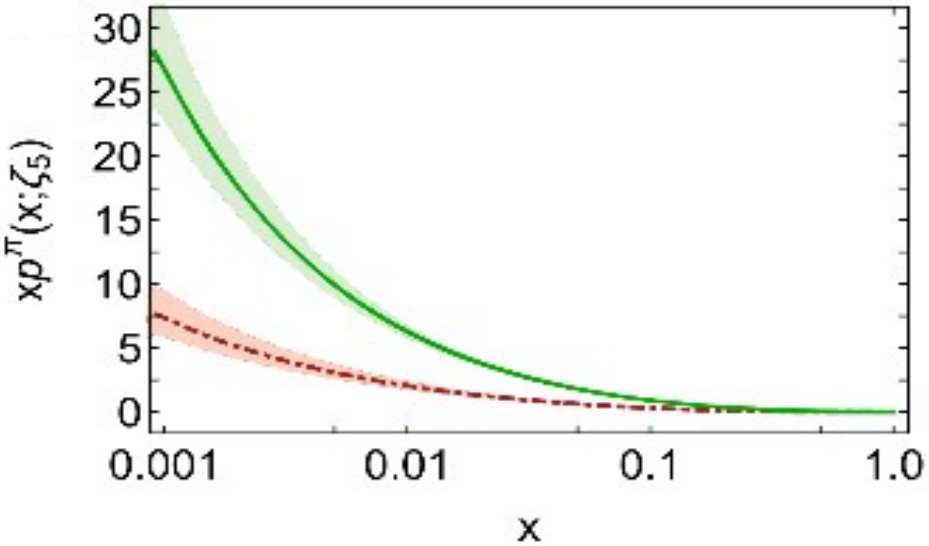
All-orders DGLAP: reverse engineering



Application 5: pion and kaon PDFs fully reconstructed from data







All-orders evolution from hadronic up to empirical scale, delivering glue and sea together with valence-quark PDFs

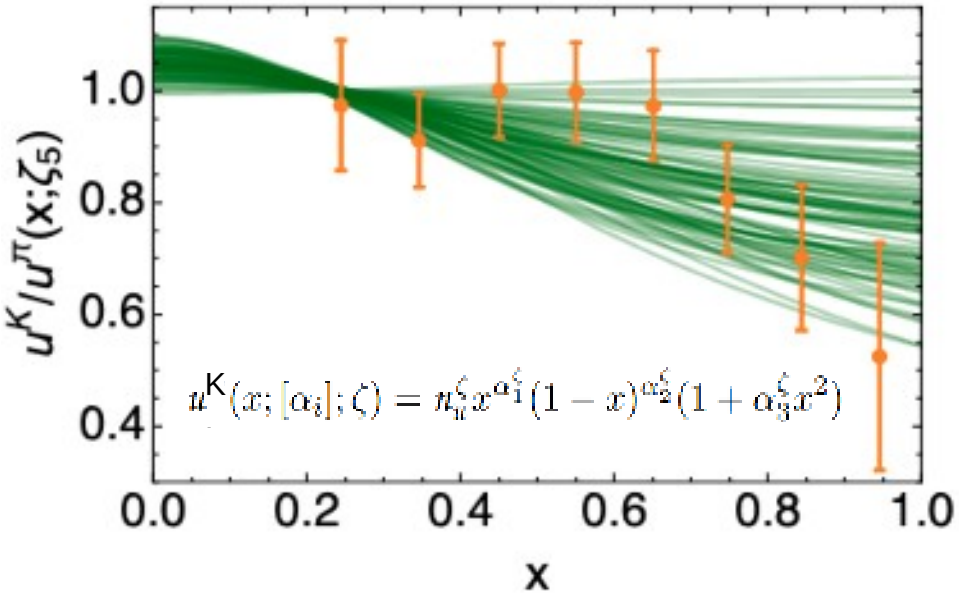
Z-F. Cui et al., Eur.Phys.J.A58 10 (2022)

Data from [Aicher et al. Phys. Rev. Lett. 105, 252003 (2010)]

All-orders DGLAP: reverse engineering



Application 5: pion and kaon PDFs fully reconstructed from data



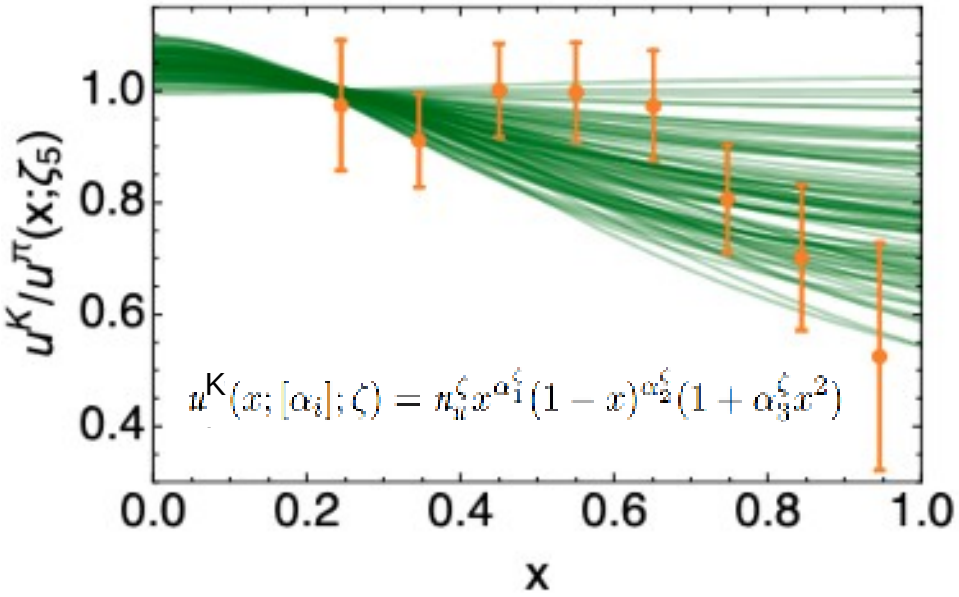
Z-F. Cui et al., Eur.Phys.J.A58 10 (2022)
Z-N. Xu et al., Phys.Lett.B685 139451(2025)

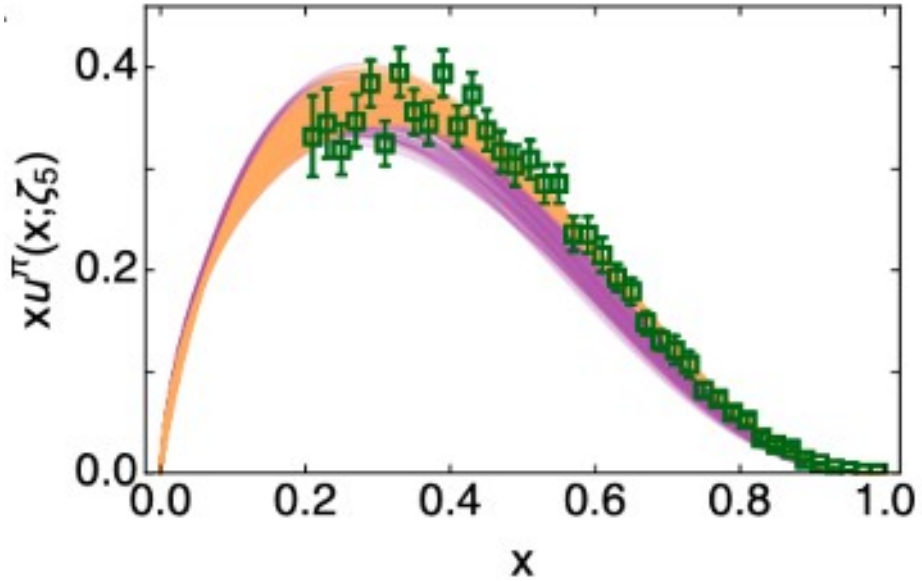
Kaon data from [Badier et al. Phys. Lett. B 94, 354 (1980)]

All-orders DGLAP: reverse engineering



Application 5: pion and kaon PDFs fully reconstructed from data





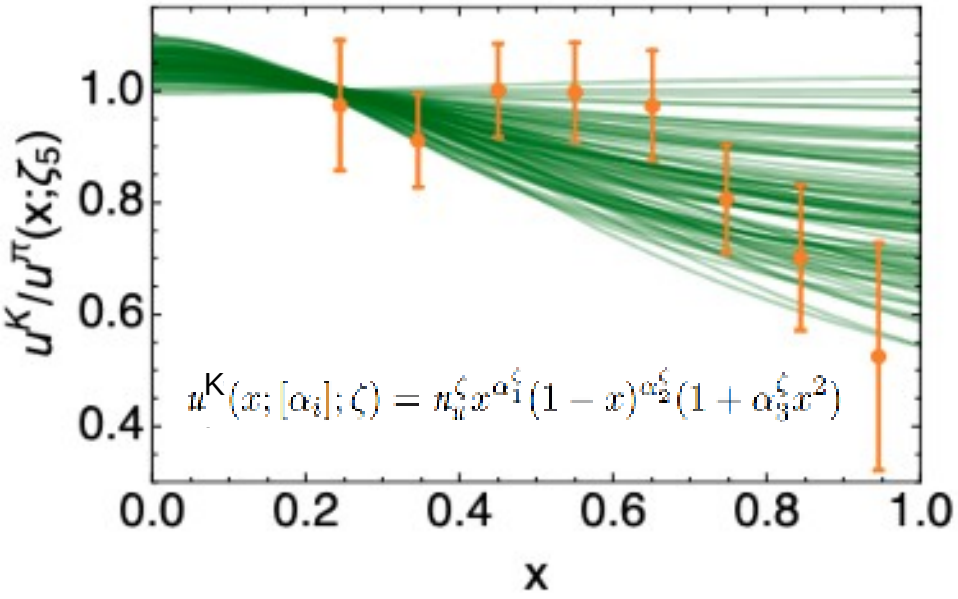
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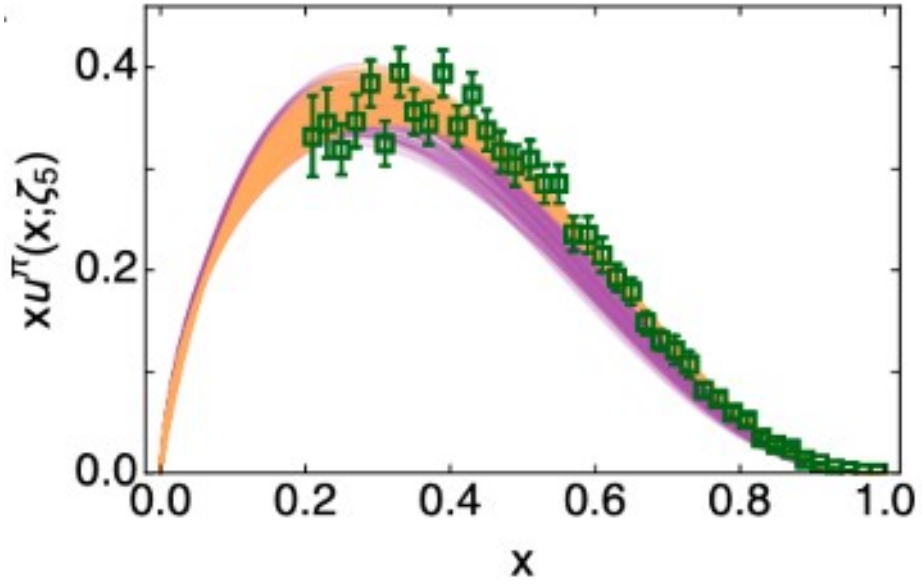
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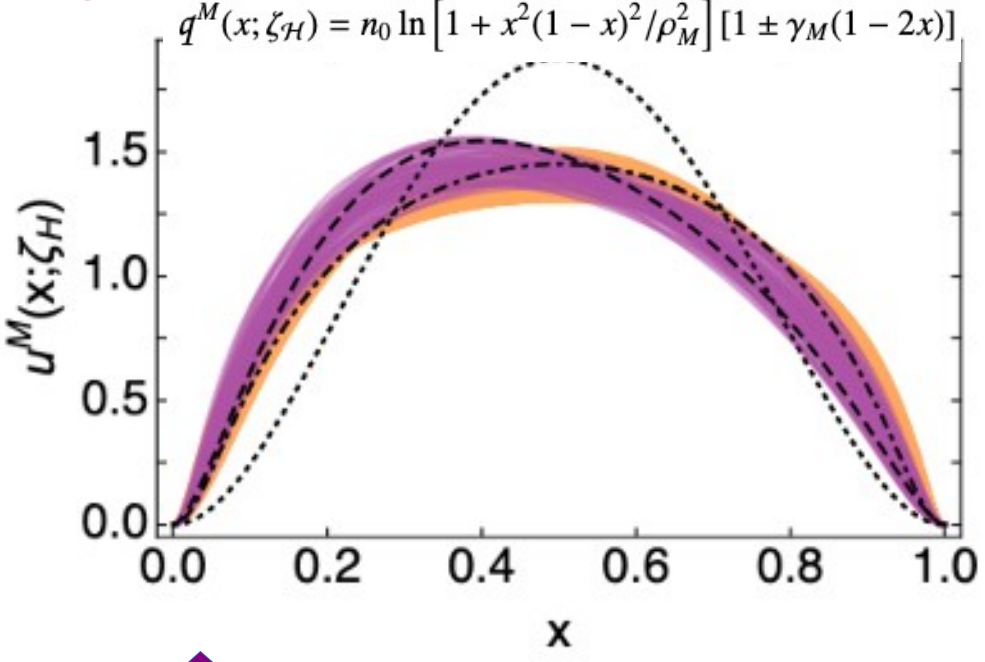
All-orders DGLAP: reverse engineering



Application 5: pion and kaon PDFs fully reconstructed from data







All-orders back evolution from empirical to hadronic scale

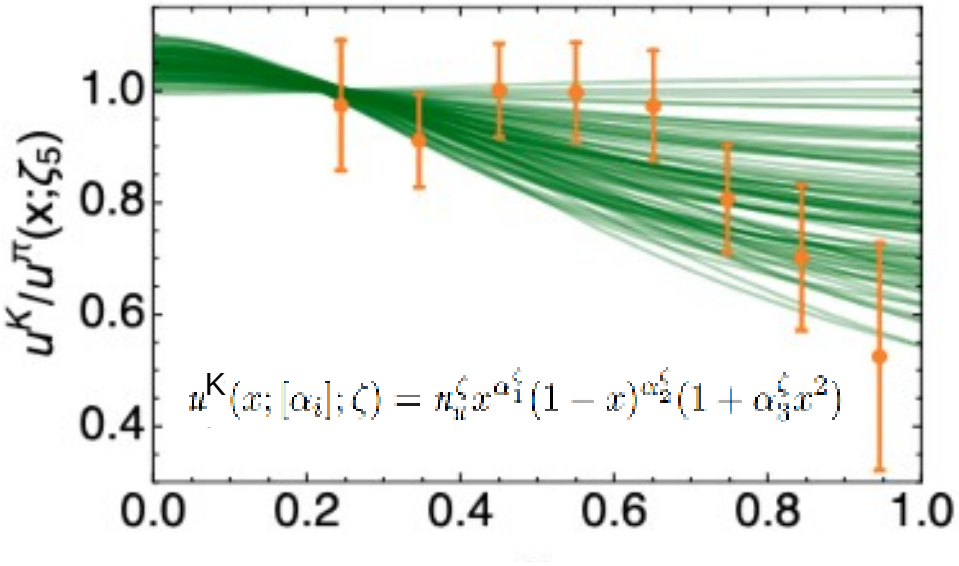
Z-F. Cui et al., Eur.Phys.J.A58 10 (2022)
Z-N. Xu et al., Phys.Lett.B685 139451(2025)

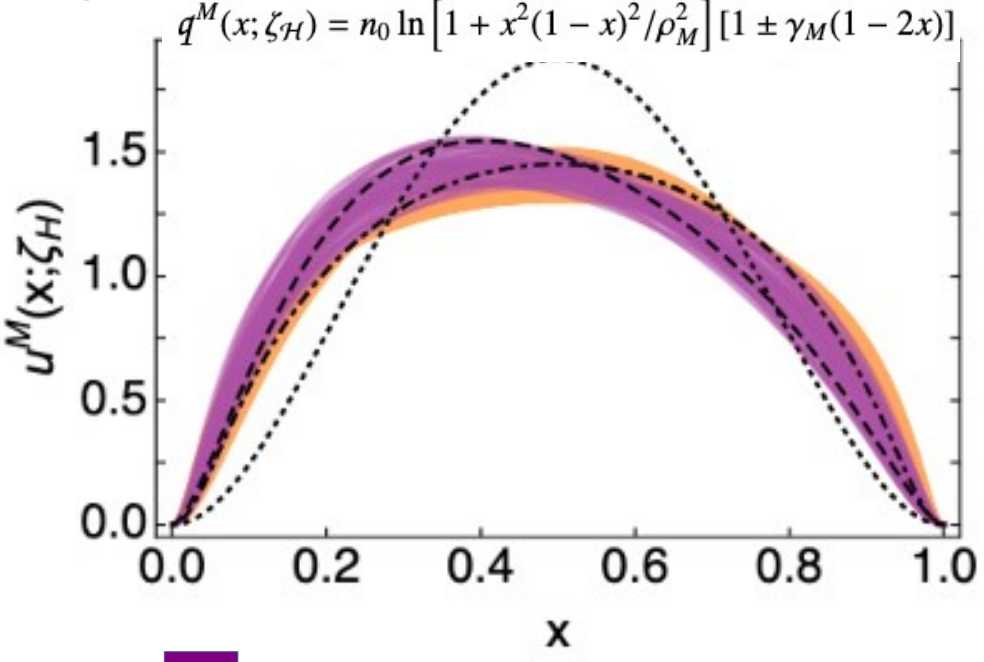
Kaon data from [Badier et al. Phys. Lett. B 94, 354 (1980)]

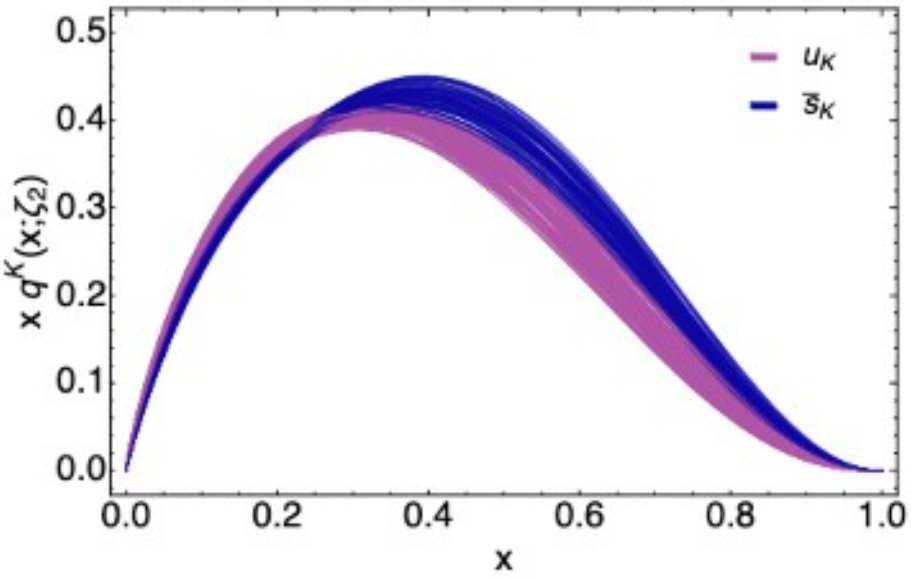
All-orders DGLAP: reverse engineering



Application 5: pion and kaon PDFs fully reconstructed from data







All-orders evolution from hadronic up to empirical scale, delivering valence-quark and -antiquark PDFs

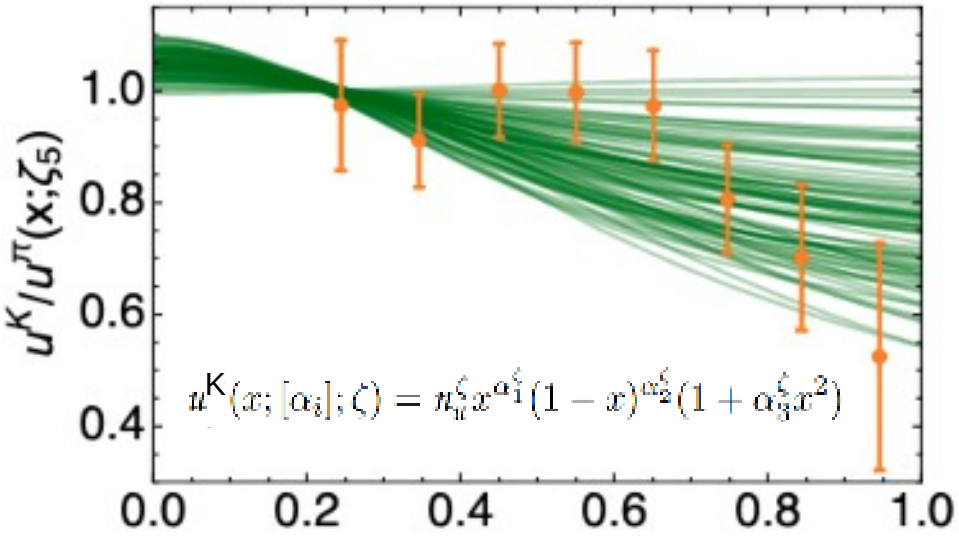
Z-F. Cui et al., Eur.Phys.J.A58 10 (2022)
Z-N. Xu et al., Phys.Lett.B685 139451(2025)

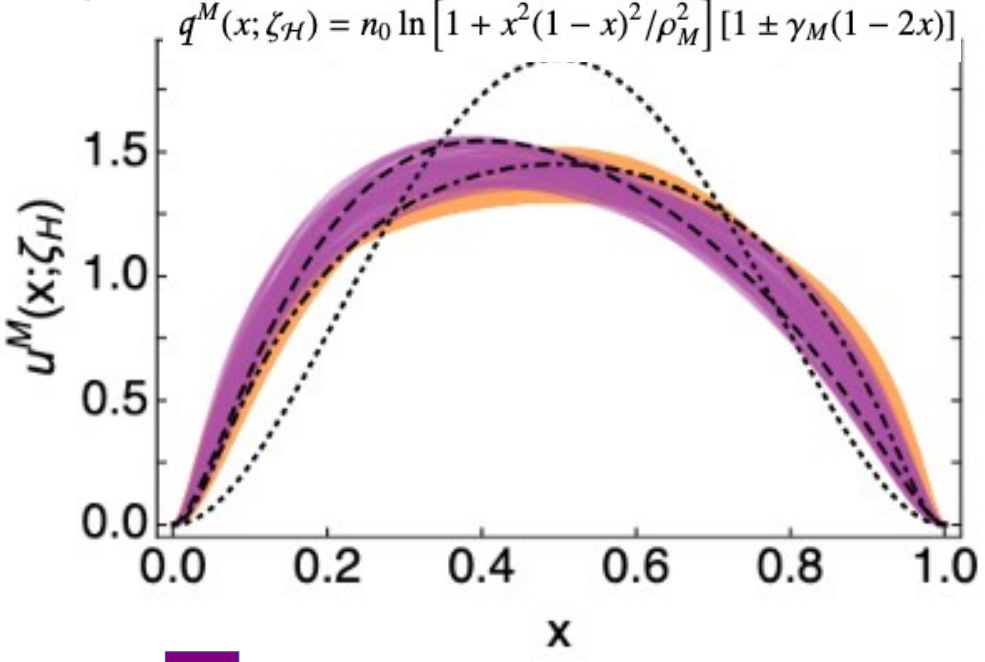
Kaon data from [Badier et al. Phys. Lett. B 94, 354 (1980)]

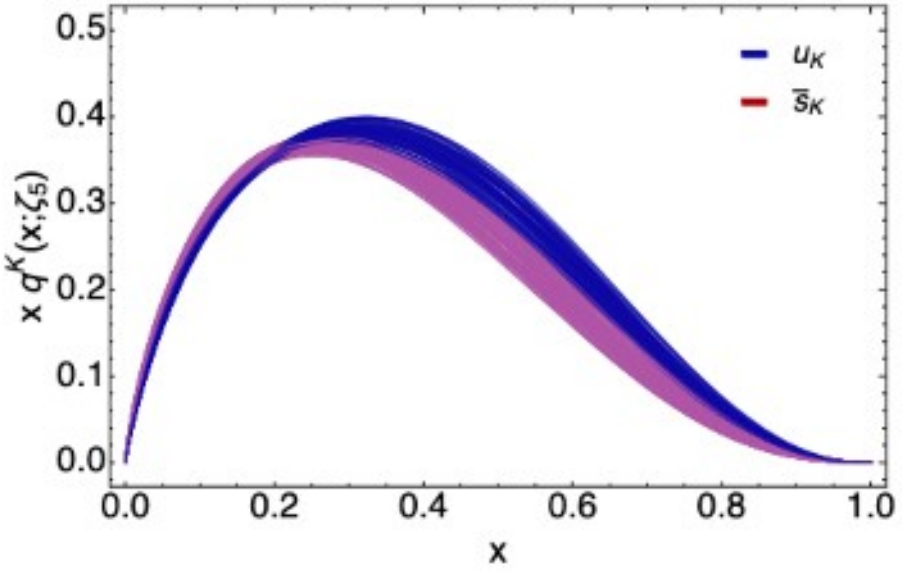
All-orders DGLAP: reverse engineering



Application 5: pion and kaon PDFs fully reconstructed from data







All-orders evolution from hadronic up to empirical scale, delivering valence-quark and -antiquark PDFs

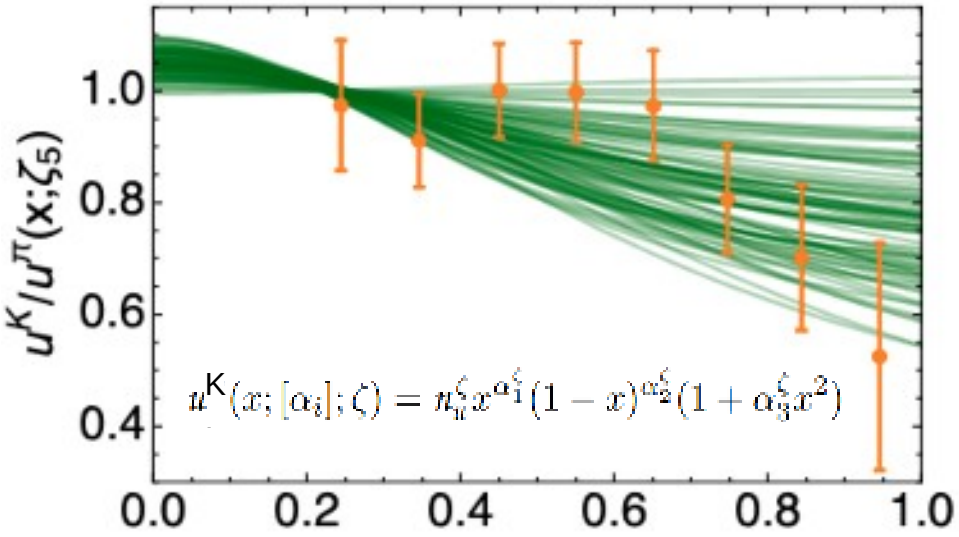
Z-F. Cui et al., Eur.Phys.J.A58 10 (2022)
Z-N. Xu et al., Phys.Lett.B685 139451(2025)

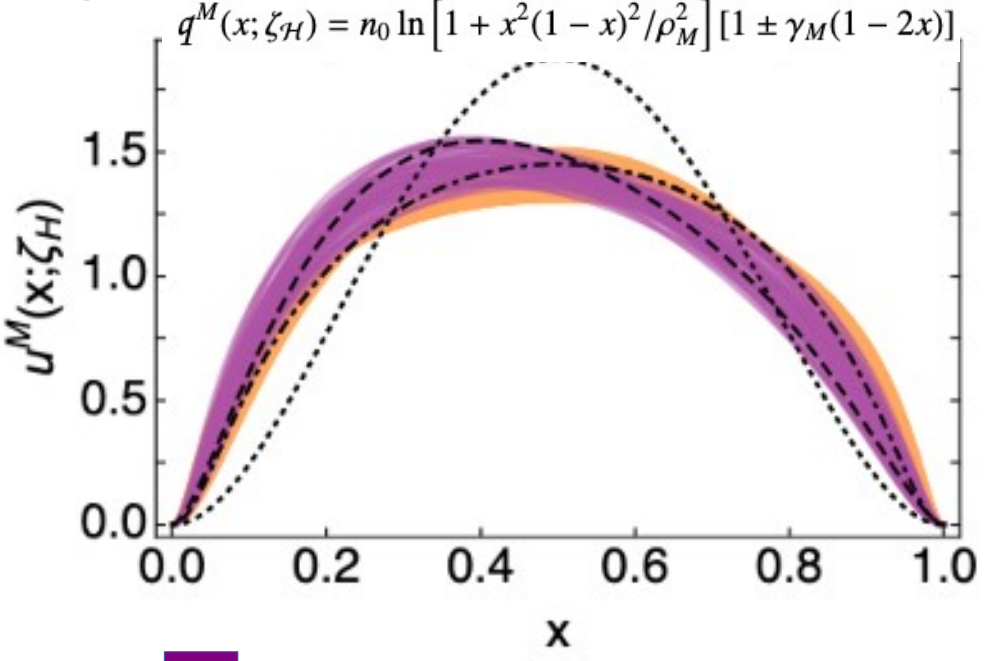
Kaon data from [Badier et al. Phys. Lett. B 94, 354 (1980)]

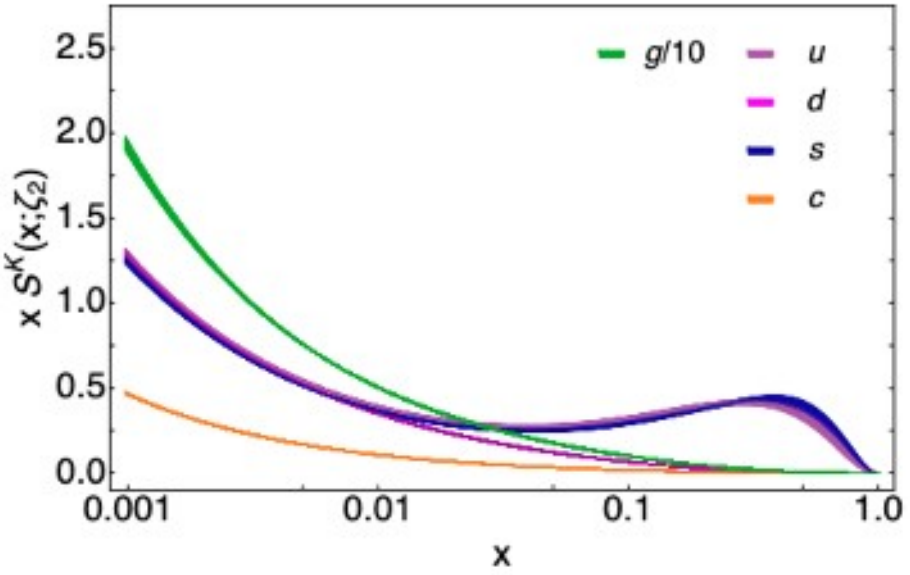
All-orders DGLAP: reverse engineering



Application 5: pion and kaon PDFs fully reconstructed from data







All-orders evolution from hadronic up to empirical scale, delivering valence-quark and -antiquark PDFs, and, again, glue and sea PDFs.

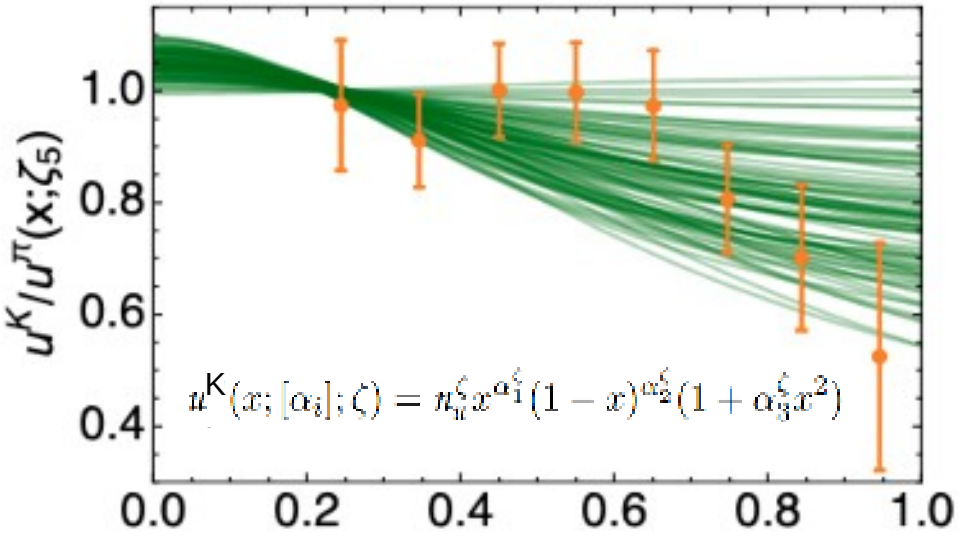
Z-F. Cui et al., Eur.Phys.J.A58 10 (2022)
Z-N. Xu et al., Phys.Lett.B685 139451(2025)

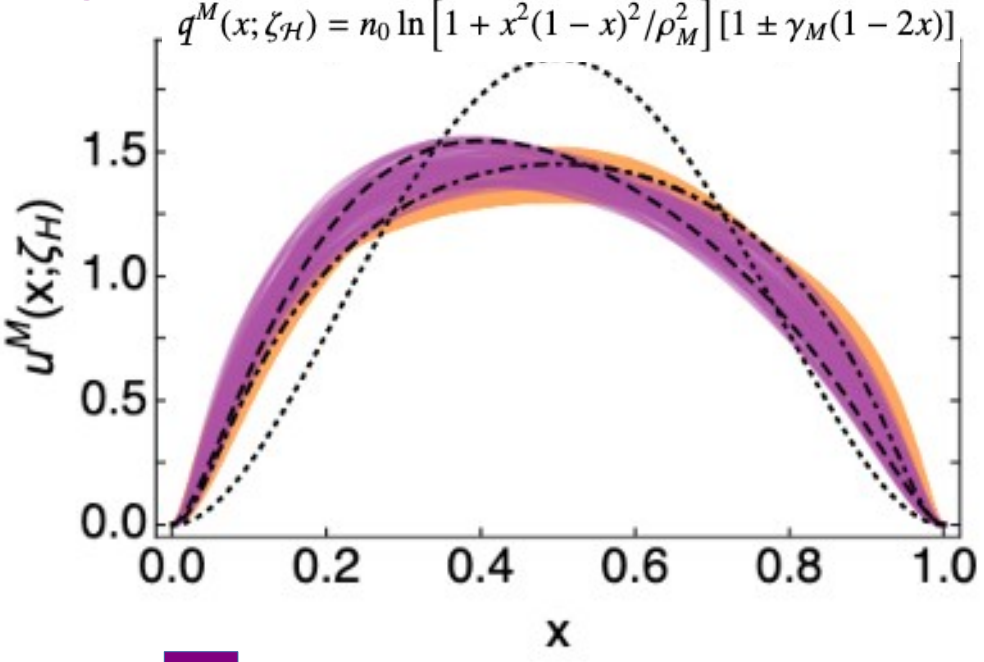
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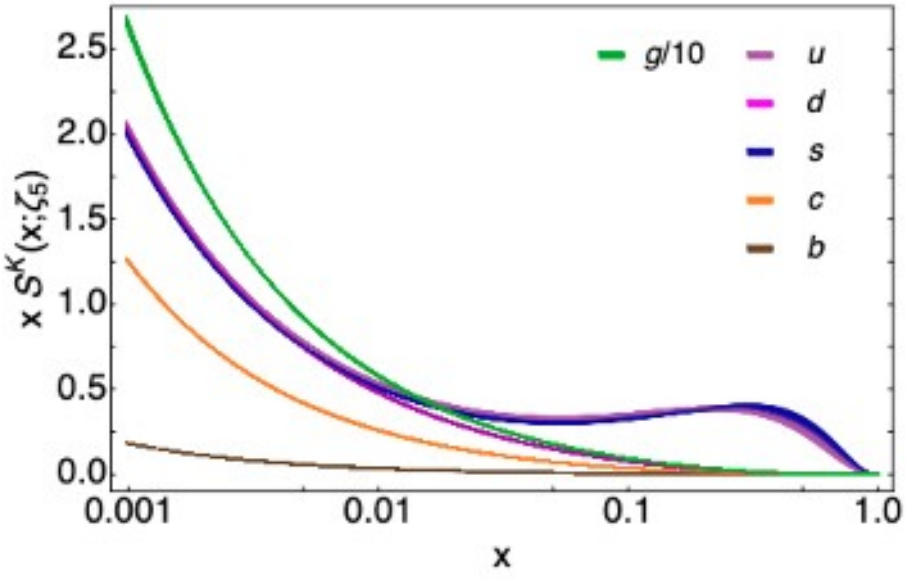
All-orders DGLAP: reverse engineering



Application 5: pion and kaon PDFs fully reconstructed from data







All-orders evolution from hadronic up to empirical scale, delivering valence-quark and -antiquark PDFs, and, again, glue and sea PDFs.

Z-F. Cui et al., Eur.Phys.J.A58 10 (2022)
Z-N. Xu et al., Phys.Lett.B685 139451(2025)

Kaon data from [Badier et al. Phys. Lett. B 94, 354 (1980)]

All-orders DGLAP: reverse engineering

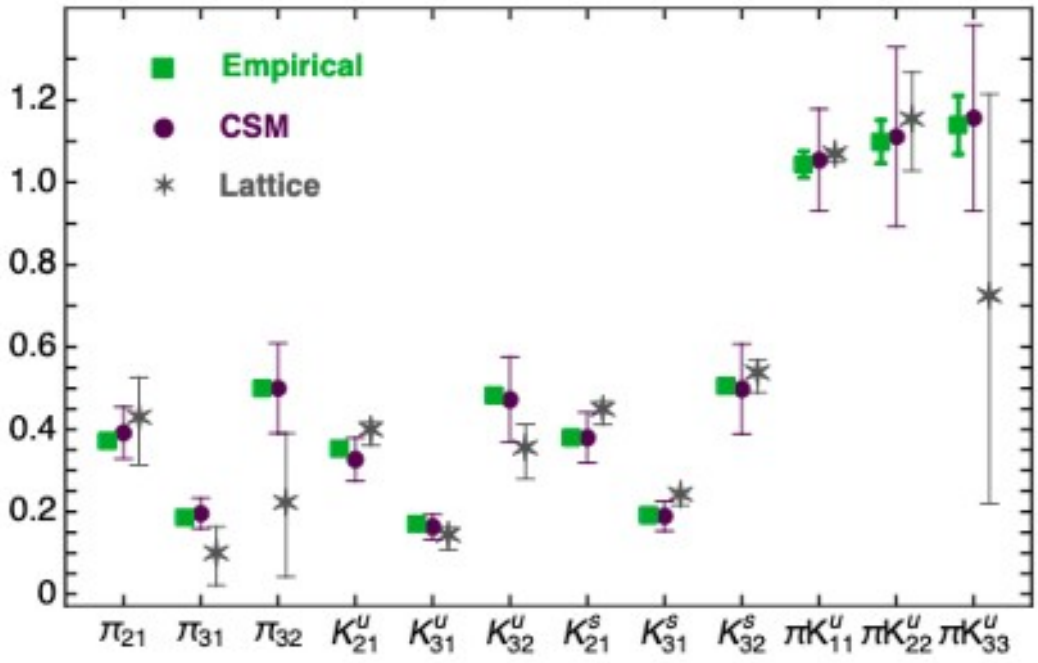


Application 5: pion and kaon PDFs fully reconstructed from data

	$\langle x\rangle^{\zeta}_{q^K}$	$\langle x\rangle^{\zeta}_{S^K_q}$	$\langle x\rangle^{\zeta}_{g^K}$	$\langle x\rangle^{\zeta}_{q^\pi}$
ζ_2				
u	0.230(6)(10)	0.028(2)		0.241(5)(10)
d	0	0.028(2)		0.241(5)(10)
s	0.252(6)(11)	0.026(1)		
c	0	0.008(1)		
b	0	0		
g			0.428(18)	
ζ_5				
u	0.197(5)(9)	0.036(2)		0.207(4)(9)
d	0	0.036(2)		0.207(4)(9)
s	0.216(5)(9)	0.034(2)		
c	0	0.019(1)		
b	0	0.003(1)		
g			0.461(20)	

	empirical		[32, lQCD]	
M	π	K	π	K
l	0.538(15)	0.286(12)	0.499(55)	0.317(19)
s	0.026(01)	0.278(13)	0.036(15)	0.339(11)
c	0.008(01)	0.008(01)	0.013(16)	0.028(21)
q	0.572(15)	0.572(18)	0.575(79)	0.683(50)
g	0.428(18)	0.428(18)	0.402(53)	0.422(67)

[32] Alexandrou, et al., arXiv:2405.08529 [hep-lat]



$$\pi_{ij} = \langle x^i\rangle_{u^\pi_S} / \langle x^j\rangle_{u^\pi_S}$$
$$K^q_{ij} = \langle x^i\rangle_{q^K_S} / \langle x^j\rangle_{q^K_S}$$
$$\pi K^u_{ij} = \langle x^i\rangle_{u^\pi_S} / \langle x^j\rangle_{u^K_S}$$

Z-F. Cui et al., Eur.Phys.J.A58 10 (2022)
Z-N. Xu et al., Phys.Lett.B685 139451(2025)

CSM = Z-F Cui, et al., Eur. Phys. J. C80 (2020) 1064.
Lattice = C. Alexandrou, et al., Phys. Rev. D 103 (1) (2021) 014508; Phys. Rev. D 104 (5) (2021) 054504.

All-orders DGLAP: fragmentation functions

Application 6: pion and kaon FFs

- Solve cascade equations at $\zeta_H \Rightarrow$ complete FFs
- Use all-orders (AO) scheme to evolve solutions to scales relevant for measurements, *e.g.*, $\zeta = \zeta_2 := 2$ GeV
  - AO scheme is nonperturbative extension of DGLAP
- Approach guarantees that particle and momentum sum rules are preserved

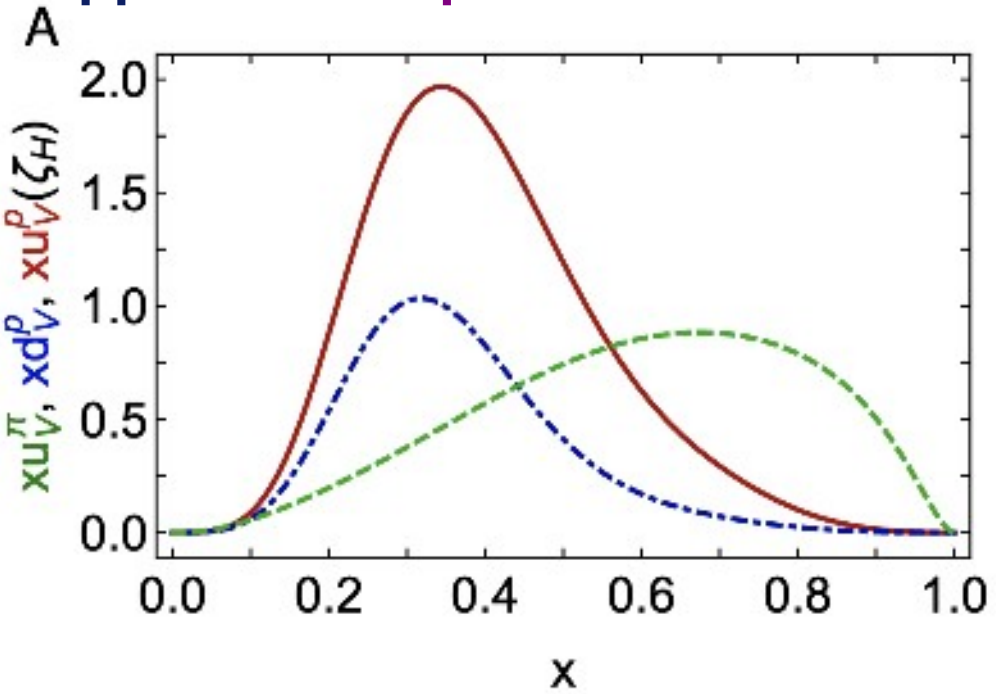
$$\sum_h \int_0^1 dz\, z D_p^h(z;\zeta) = 1$$

Table 2 SCI FF momentum fractions obtained from solutions of the cascade equations at the hadron scale and after evolution to $\zeta = \zeta_2 := 2\,\mathrm{GeV}$, following the prescription described in Sect. 4. (No entry means the fraction is zero. $c \to g \to h$ contributions are negligible in all cases.)

h	$\pi^+ + \pi^0 + \pi^-$		K^+	
	$\zeta_{\mathcal{H}}$	ζ_2	$\zeta_{\mathcal{H}}$	ζ_2
$\langle z\rangle^h_{D_{S_V^u}}$	0.664	0.433	0.182	0.119
$\langle z\rangle^h_{D_{S_V^d}}$		0.115		0.032
$\langle z\rangle^h_{D_{S_V^s}}$		0.085		0.023
$\langle z\rangle^h_{D_{S_V^c}}$		0.031		0.009
$\langle z\rangle^h_{D_{S_V^{\bar{u}}}}$		0.115		0.007
$\langle z\rangle^h_{D_{S_A^u}}$	0.664	0.443	0.042	0.028
$\langle z\rangle^h_{D_{S_A^d}}$		0.085		0.005
$\langle z\rangle^h_{D_{S_A^s}}$		0.031		0.002
$\langle z\rangle^h_{D_{S_A^c}}$		0.017		0.069
$\langle z\rangle^h_{D_{S_A^{\bar{u}}}}$		0.017		0.069
$\langle z\rangle^h_{D_{S_T^u}}$	0.098	0.059	0.396	0.239
$\langle z\rangle^h_{D_{S_T^c}}$		0.005		0.019
$\langle z\rangle^{\zeta}_{D^h_{g_u}}$	0.083	0.083	0.023	0.023
$\langle z\rangle^{\zeta}_{D^h_{g_d}}$	0.083	0.083	0.005	0.005
$\langle z\rangle^{\zeta}_{D^h_{g_s}}$	0.012	0.012	0.050	0.050

All-orders DGLAP: proton case

Application 6: proton PDFs



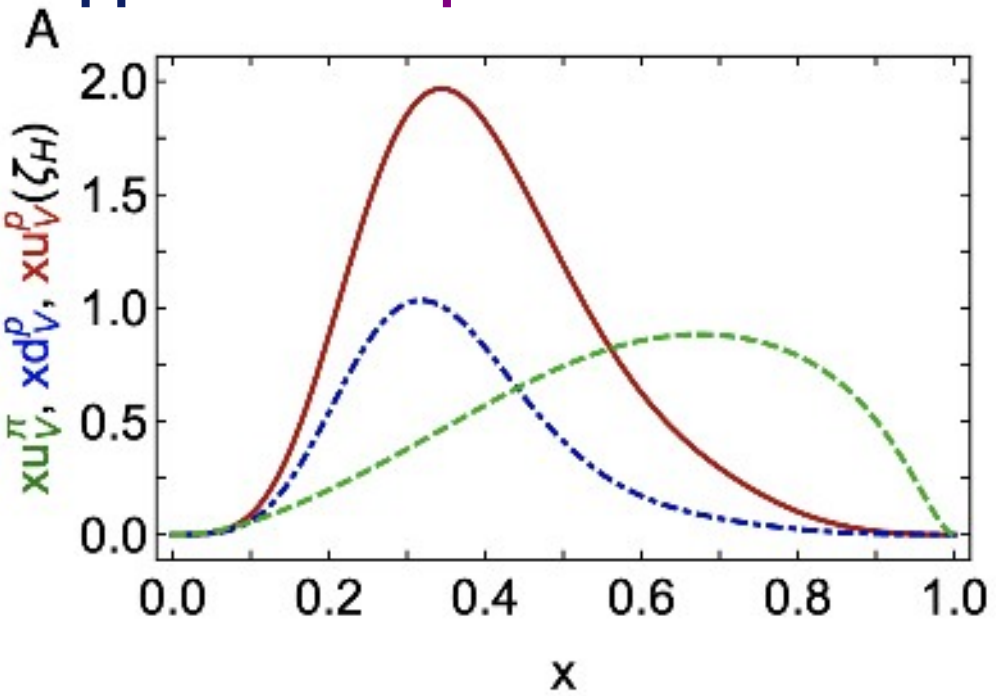
A symmetry-preserving DSE computation of the valence-quark PDFs within a proton, based on diquark-quark approach

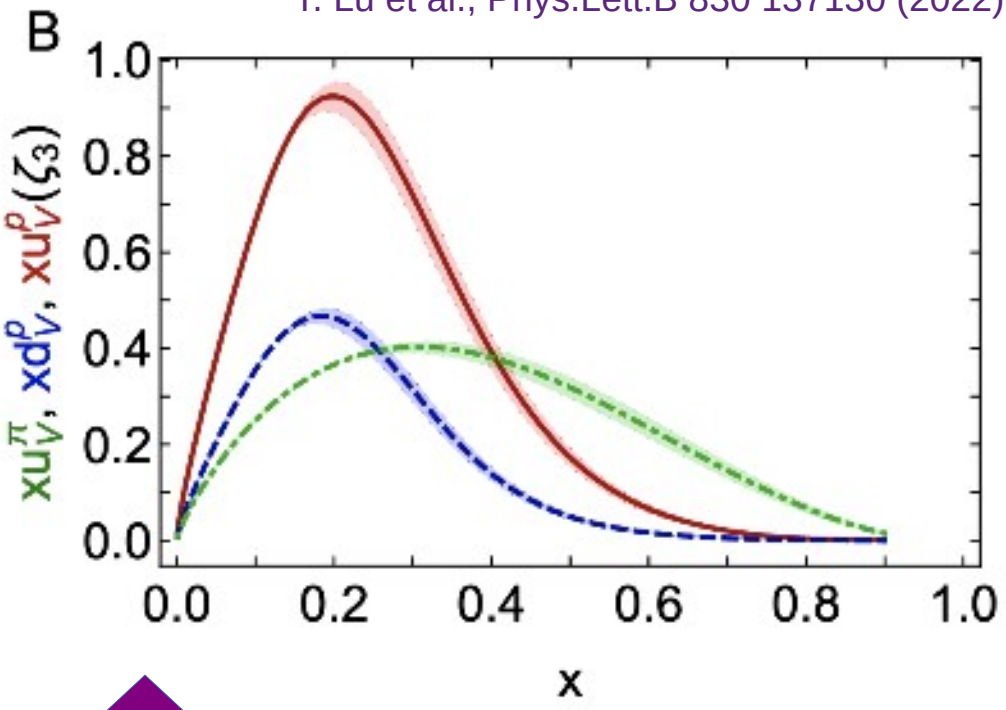
[L. Chang et al., Phys.Lett.B, arXiv:2201.07870]

All-orders DGLAP: proton case

Application 6: proton PDFs

Y. Lu et al., Phys.Lett.B 830 137130 (2022)



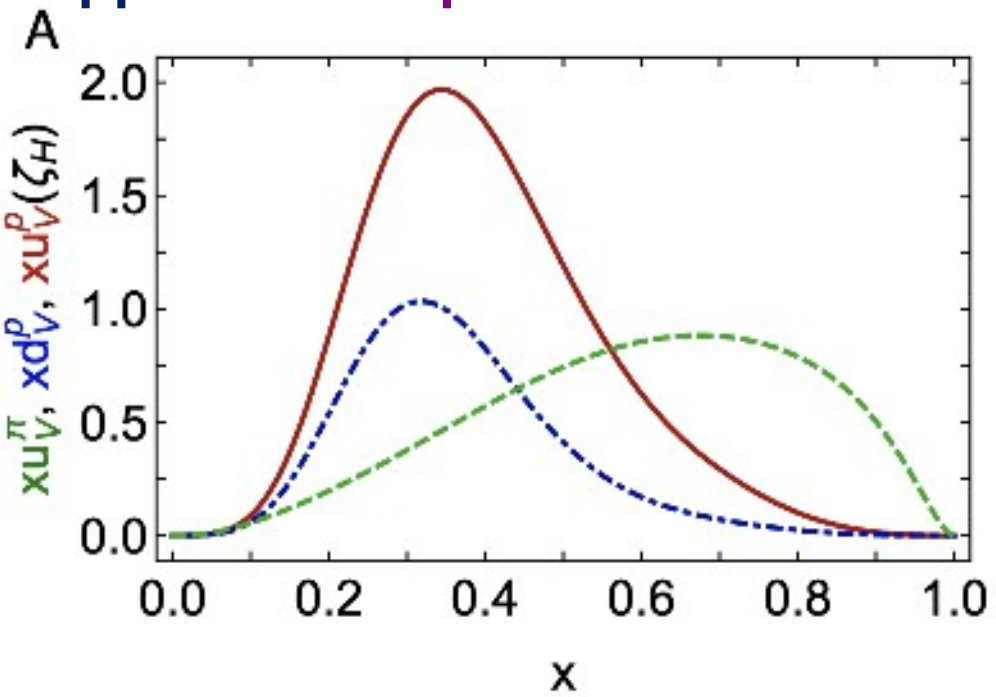


A symmetry-preserving DSE computation of the valence-quark PDFs within a proton, based on diquark-quark approach
[L. Chang et al., Phys.Lett.B, arXiv:2201.07870]

All-orders evolution from hadronic up to empirical scale.

All-orders DGLAP: proton case

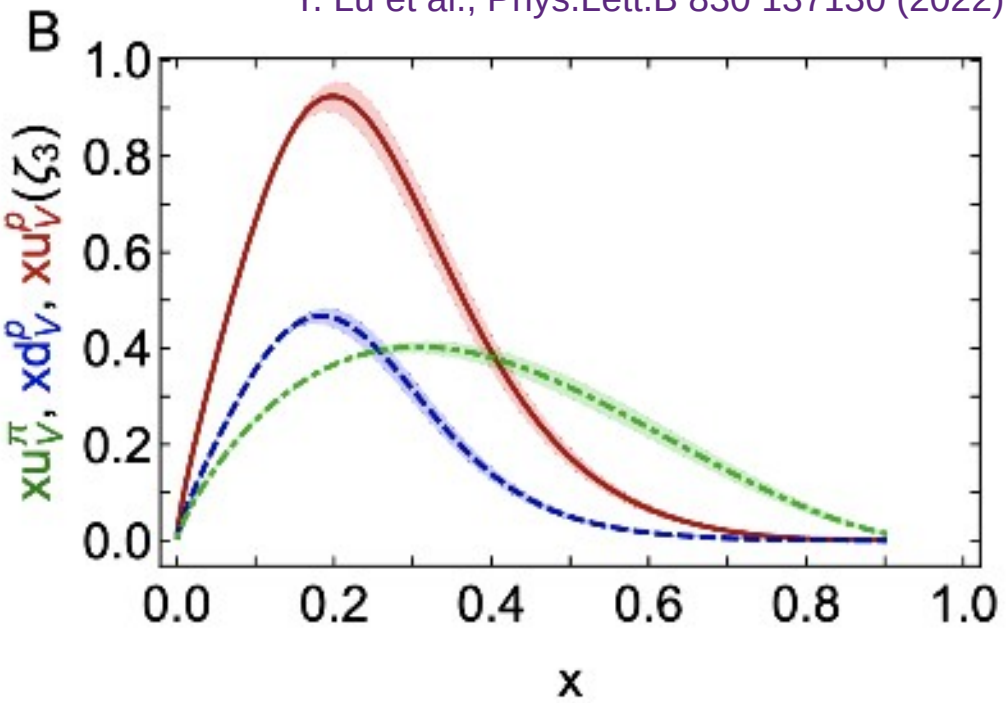
Application 6: proton PDFs

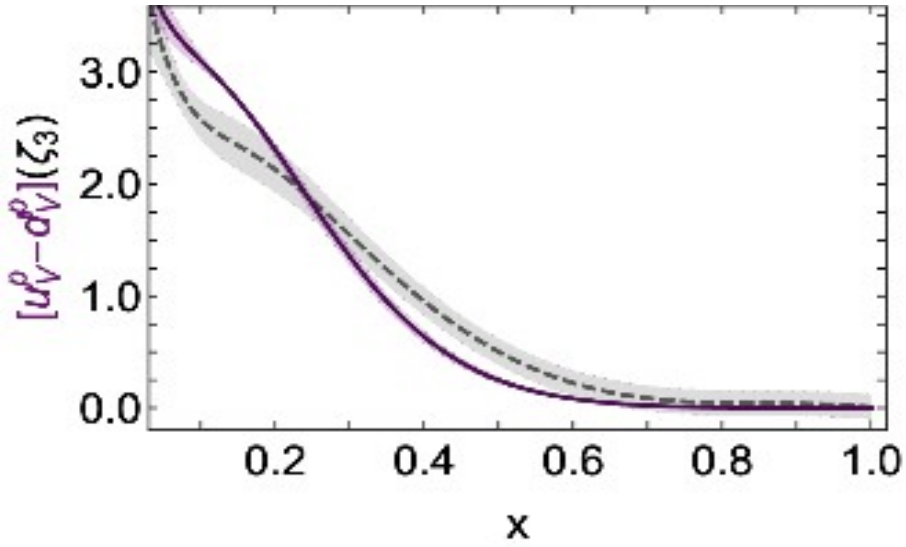


A symmetry-preserving DSE computation of the valence-quark PDFs within a proton, based on diquark-quark approach
[L. Chang et al., Phys.Lett.B, arXiv:2201.07870]

Producing an isovector distribution in fair agreement with lattice results
[H-W. Lin et al., arXiv:2011.14791]

Y. Lu et al., Phys.Lett.B 830 137130 (2022)

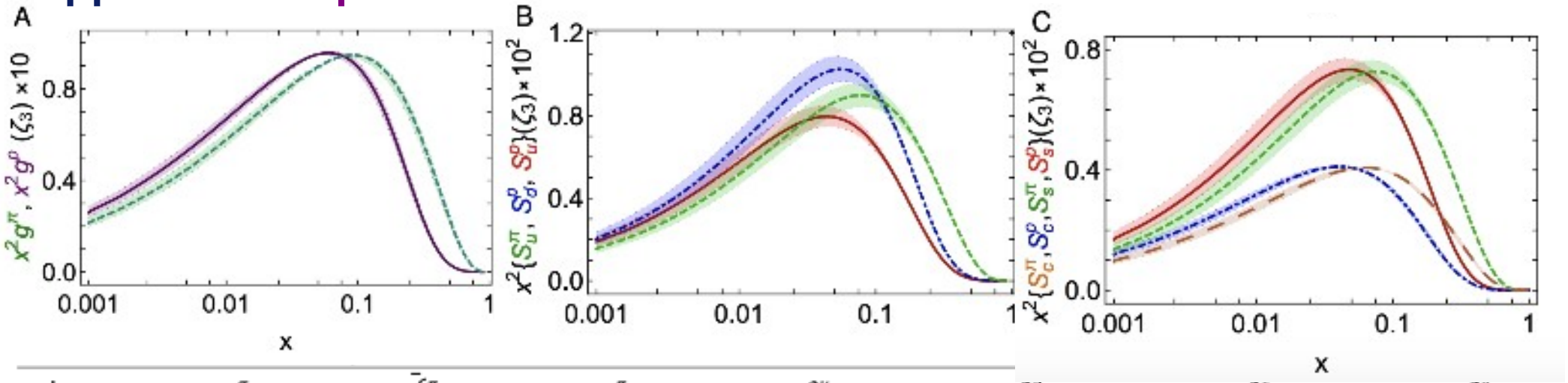




All-orders DGLAP: proton case

Application 6: proton PDFs

Y. Lu et al., Phys.Lett.B 830 137130 (2022)



pion	u^π	$\bar{d}^\pi$	g^π	S_π^u	S_π^a	S_π^s	S_π^c
$\langle x\rangle^{\zeta_2}$	24.0(1.1)	24.0(1.1)	41.0(1.2)	3.3(3)	3.3(3)	2.65(22)	1.33(5)
$\langle x^2\rangle^{\zeta_2}$	9.5(7)	9.5(7)	3.7(1)	0.27(1)	0.27(1)	0.21(1)	0.092(2)
$\langle x^3\rangle^{\zeta_2}$	4.7(4)	4.7(4)	0.92(6)	0.057(1)	0.057(1)	0.044(0)	0.018(1)
$\langle x\rangle^{\zeta_3}$	22.1(1.0)	22.1(1.0)	42.9(1.0)	3.7(3)	3.7(3)	3.0(2)	1.83(6)
$\langle x^2\rangle^{\zeta_3}$	8.4(6)	8.4(6)	3.5(1)	0.27(1)	0.27(1)	0.22(1)	0.120(3)
$\langle x^3\rangle^{\zeta_3}$	4.0(3)	4.0(3)	0.82(5)	0.056(0)	0.056(0)	0.044(0)	0.022(1)

proton	u^p	d^p	g^p	S_p^u	S_p^d	S_p^s	S_p^c
$\langle x\rangle^{\zeta_2}$	32.9(1.4)	15.0(0.7)	40.9(1.1)	2.9(2)	3.7(3)	2.64(22)	1.32(5)
$\langle x^2\rangle^{\zeta_2}$	8.7(6)	3.6(2)	2.4(1)	0.14(1)	0.21(1)	0.13(0)	0.059(2)
$\langle x^3\rangle^{\zeta_2}$	2.9(3)	1.1(1)	0.39(2)	0.019(0)	0.030(1)	0.019(0)	0.008(0)
$\langle x\rangle^{\zeta_3}$	30.4(1.3)	13.8(0.6)	42.8(1.0)	3.3(3)	4.1(3)	3.0(2)	1.82(6)
$\langle x^2\rangle^{\zeta_3}$	7.7(5)	3.2(2)	2.2(1)	0.15(1)	0.21(1)	0.14(0)	0.075(2)
$\langle x^3\rangle^{\zeta_3}$	2.5(2)	0.9(1)	0.35(2)	0.019(0)	0.028(0)	0.019(0)	0.010(1)

All-orders DGLAP: proton case

Application 7: polarized PDFs

Y. Lu et al., Phys.Lett.B 830 137130 (2022)

Polarized PDFs DGLAP evolutions equations, expressed by the corresponding massless splitting functions, after converting to Mellin space and specializing for 0-th order

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^0 \rangle^{\zeta}_{\tilde{\Sigma}_H} = 0$$

$$\left(\zeta^2 \frac{d}{d\zeta^2} + \tilde{\gamma}^0_{gg}(n_f) \frac{\alpha(\zeta^2)}{4\pi} \right) \langle x^0 \rangle^{\zeta}_{\tilde{g}_H} = 4 \frac{\alpha(\zeta^2)}{4\pi} \langle x^0 \rangle^{\zeta}_{\tilde{\Sigma}_H}$$

All-orders DGLAP: proton case

Application 7: polarized PDFs

Y. Lu et al., Phys.Lett.B 830 137130 (2022)

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$$\left(\zeta^2 \frac{d}{d\zeta^2} + \tilde{\gamma}^0_{gg}(n_f) \frac{\alpha(\zeta^2)}{4\pi} \right) \langle x^0 \rangle^{\zeta}_{\tilde{g}_H} = 4 \frac{\alpha(\zeta^2)}{4\pi} \langle x^0 \rangle^{\zeta}_{\tilde{\Sigma}_H}$$

In general, at any momentum scale $\zeta \geq M_c$:

$$a^{\zeta}_{0H} = \langle x^0 \rangle^{\zeta}_{\tilde{\Sigma}_H} = \langle x^0 \rangle^{\zeta_H}_{\tilde{\Sigma}_H} ,$$

$$\Delta G^{\zeta}_H = \langle x^0 \rangle^{\zeta}_{\tilde{g}^q_H} = \langle x^0 \rangle^{\zeta_H}_{\tilde{\Sigma}_H} \left\{ \frac{12}{29} \left([S(\zeta_H, M_s)]^{-87/32} - 1 \right) [S(M_s, M_c)]^{-81/32} [S(M_c, \zeta)]^{-75/32} \right.$$

$$\left. + \frac{4}{9} \left([S(M_s, M_c)]^{-81/32} - 1 \right) [S(M_c, \zeta)]^{-75/32} + \frac{12}{25} \left([S(M_c, \zeta)]^{-75/32} - 1 \right) \right\}$$

$$S(\zeta_H, \zeta) = \frac{\langle x \rangle^{\zeta}_{q_H}}{\langle x \rangle^{\zeta_H}_{q_H}} = \exp\left(-\frac{\gamma_{qq}}{2\pi} \int_{\zeta_H}^{\zeta} \frac{dz}{z} \alpha(z^2) \right)$$

All-orders DGLAP: proton case

Application 7: polarized PDFs

Y. Lu et al., Phys.Lett.B 830 137130 (2022)

Polarized PDFs DGLAP evolutions equations, expressed by the corresponding massless splitting functions, after converting to Mellin space and specializing for 0-th order

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^0 \rangle^{\zeta}_{\tilde{\Sigma}_H} = 0$$

$$\left(\zeta^2 \frac{d}{d\zeta^2} + \tilde{\gamma}^0_{gg}(n_f) \frac{\alpha(\zeta^2)}{4\pi} \right) \langle x^0 \rangle^{\zeta}_{\tilde{g}_H} = 4 \frac{\alpha(\zeta^2)}{4\pi} \langle x^0 \rangle^{\zeta}_{\tilde{\Sigma}_H}$$

In general, at any momentum scale $\zeta_H \leq \zeta$, and neglecting the mass thresholds:

$$a^{\zeta}_{0H} = \langle x^0 \rangle^{\zeta}_{\tilde{\Sigma}_H} = \langle x^0 \rangle^{\zeta_H}_{\tilde{\Sigma}_H} ,$$

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A [CT18]+ no thresholds

B [Ya2022]+no thresholds

C [Ya2022]+[Chen2022] +no thresholds

D [Ya2022]+ [Chen2022] + thresholds

a^{ζ}_{0p}	0.74(11)	0.74(11)	0.65(02)	0.65(02)
	A	B	C	D
ΔG^{ζ}_p	2.27(30)	1.50(25)	1.33(15)	1.41(16)
$\tilde{a}^{\zeta}_{0p}$	0.20(11)	0.38(11)	0.33(04)	0.32(04)

Abelian anomaly corrected:

$$\tilde{a}^{\zeta}_{0p} = a^{\zeta}_{0p} - n_f \frac{\hat{\alpha}(\zeta)}{2\pi} \Delta G^{\zeta}_p$$

All-orders DGLAP: proton case

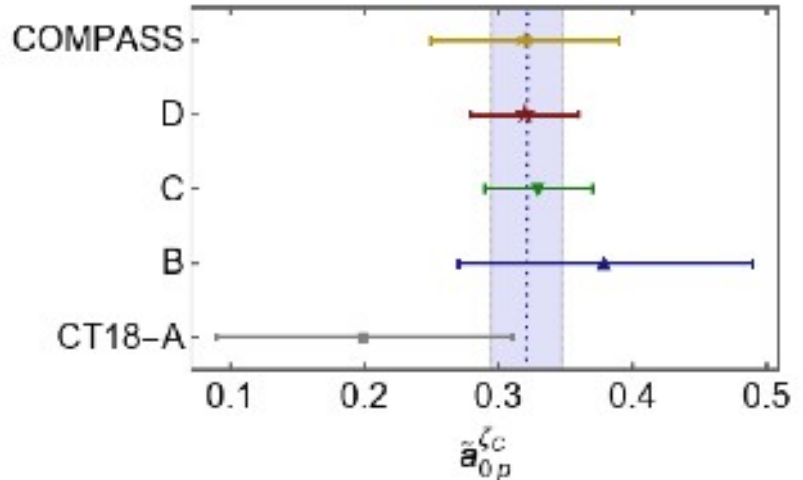
Application 7: polarized PDFs

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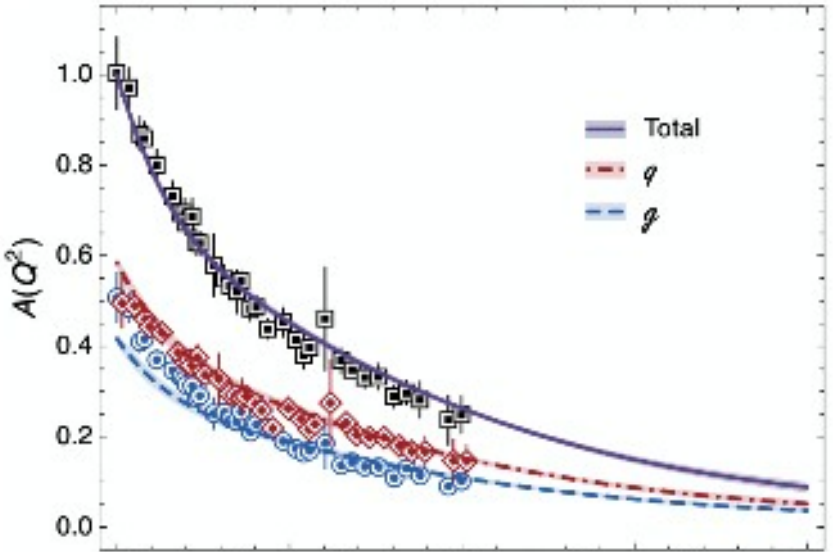
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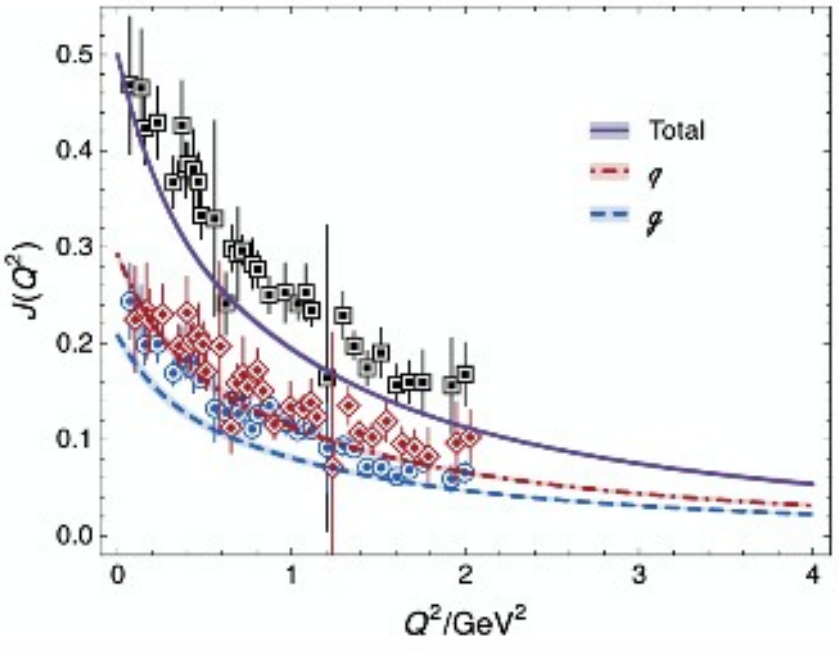
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All-orders DGLAP: proton case

Application 8: proton gravitational form factors

Z-Q. Yao et al., Eur.Phys.J.A 61 (2025) 5, 92



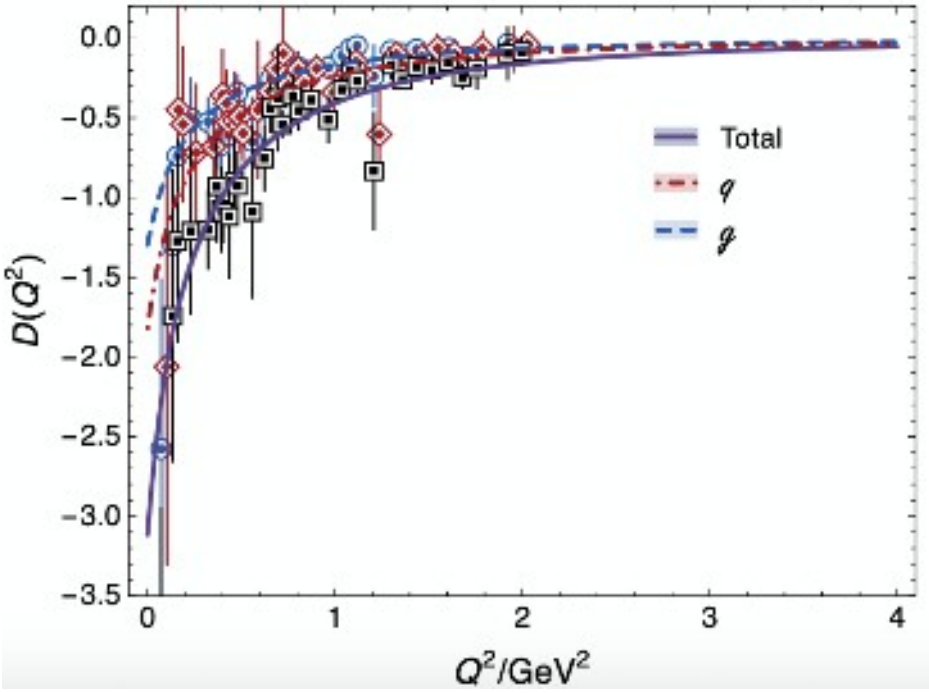


All-orders evolution implies:

$$F_p(t;\zeta) = \langle x \rangle_p^{\zeta} F(t)$$

$$F = A, J, D \text{ and } p = q, g, \Sigma_{\mathbf{p}}^q, \Sigma_{\mathbf{p}}.$$

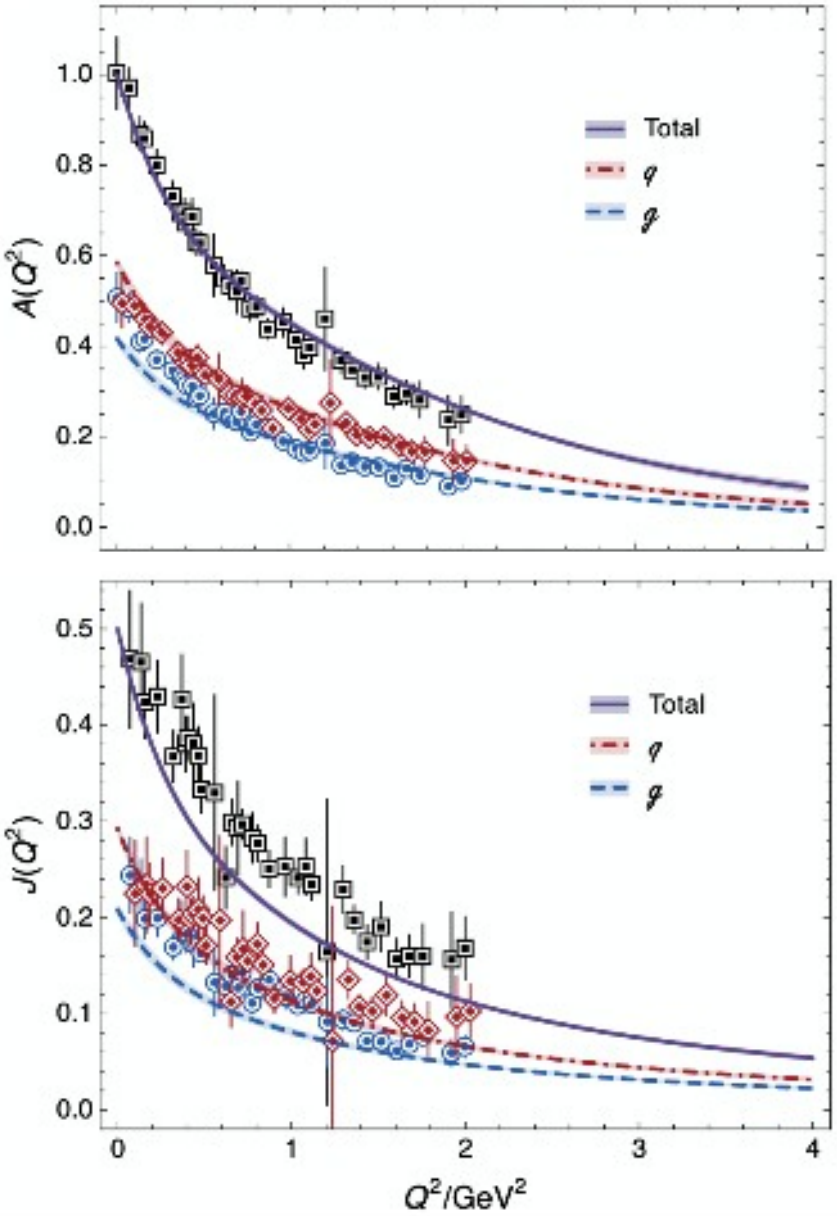
Lattice data from
D.C. Hacket et al., Phys.Rev.Lett.132 (2025)251904



All-orders DGLAP: proton case

Application 8: proton gravitational form factors

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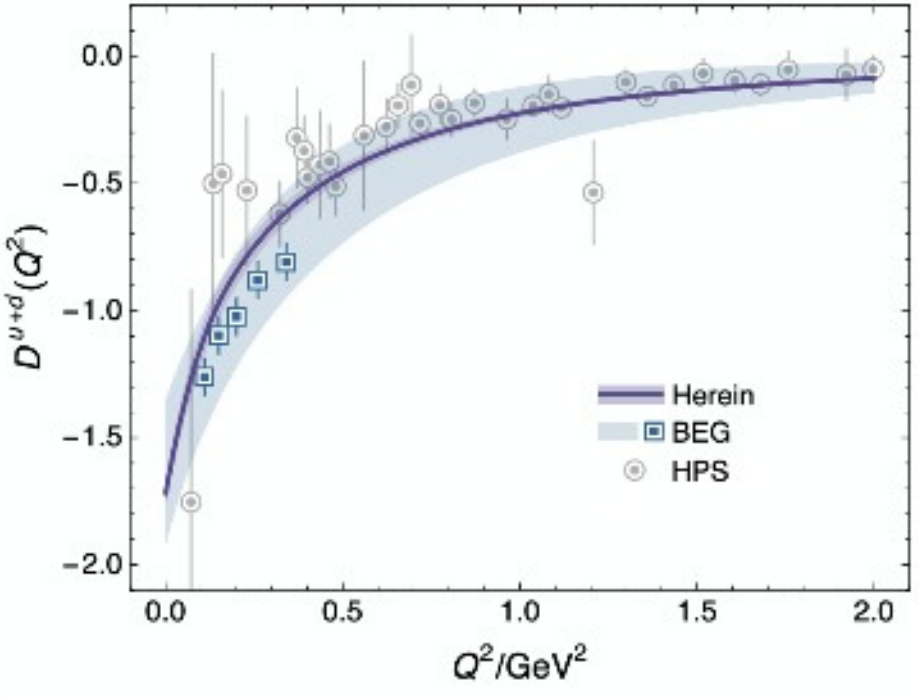


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Lattice data from
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BEG: V. D. Burkardt et al., Nature 557 (7705) 396-399 (2018)



Summary

- Grunberg's notion of effective charge, reinterpreted as a non-perturbative renormalization scheme for the running coupling extends naturally its domain to the IR and, extracted from an observable, eliminates the Landau pole. The one based on the Bjorken sum rule is shown to connect to other IR coupling definitions, specially to the **PI effective** charge. They are shaped by the expression of the **EHM** and become approximately conformal below a given momentum scale, where gluons acquiring a dynamical mass decouple from interaction.

- Capitalizing on the latter, two main ideas emerge:
  (i) the identification of that decoupling with a **hadronic scale** at which the structure of hadrons can be expressed only in terms of valence dressed partons;
  and (ii) the reliability of an **all-orders** evolution scheme, based on an effective charge defined to make exact one-loop DGLAP equations, describing the splitting of valence into more partons, generating thus the glue and sea, when the resolution scale increases.

- The robustness of the approach based on **all-orders** evolution of **CSM** results from **hadronic** to experimental scale has been proved with its application to the pion, kaon and proton cases, obtaining results for PDFs, unpolarized and polarized, FFs and GFFs which are show to be consistent with **Lattice QCD** and experimental data.

To be continued...

