NANJING UNIVERSITY

ECT* EUROPEAN CENTRE FOR THEORETICAL STUDIES IN NUCLEAR PHYSICS AND RELATED AREAS

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Orbital Angular Momentum in the Nucleon

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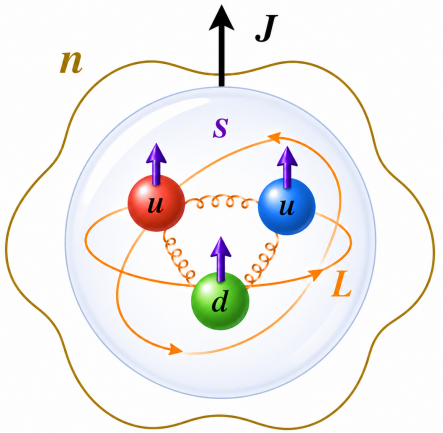
Background: Historical Development of Nucleon Structure

- In the early 20th century, nucleons were regarded as fundamental particles.
  - 1940s–1960s: Numerous hadrons were discovered experimentally, including π, K, Λ, and N(1440)…
  - It was apparent that not all hadrons could be fundamental particles.

Gell-Mann, Nobel Prize 1969



- Quark model (1964), A crucial step forward: nucleons are composed of three valence quarks bound together by the strong interaction.
- Hadrons are specified using the spectroscopic notation $n^{2S+1}L_J$.
  - Does not provide a dynamical description of confinement,
  - Often relies on a nonrelativistic treatment.



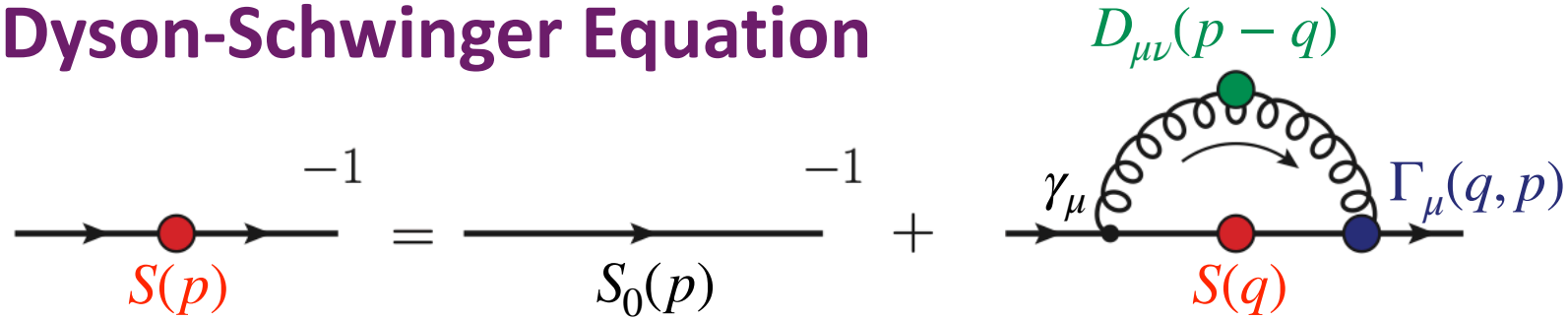
QCD and Nucleon Orbital Angular Momentum

- Quantum chromodynamics (QCD) is the fundamental theory of the strong interaction, with quarks and gluons as its dynamical degrees of freedom.
- ☑ QCD provides a relativistic and Poincaré-covariant framework that incorporates color dynamics, quark dressing, and the mechanisms associated with confinement and emergent hadron mass.
- While total angular momentum is associated with a Poincaré Casimir operator, orbital angular momentum (OAM) alone is frame dependent and not separately Poincaré invariant.
- We therefore study the nucleon's rest-frame OAM decomposition to quantify the S-, P-, and D-wave content generated by relativistic dynamics.

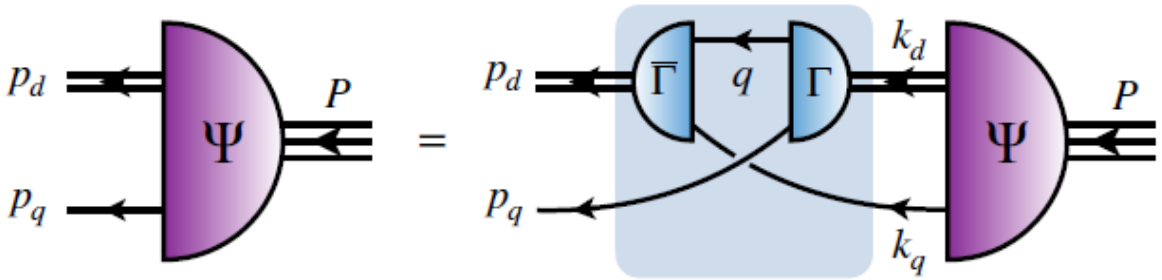
Continuum Schwinger Methods

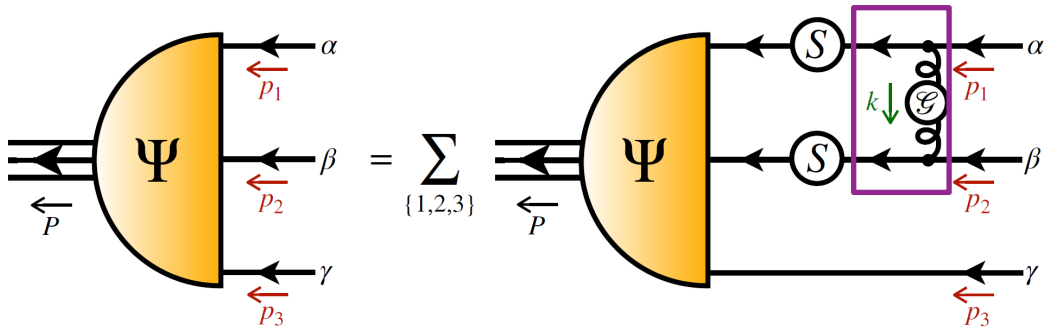
☑ At hadronic scales, QCD is strongly coupled and requires a nonperturbative treatment.

➢ Quark Dyson-Schwinger Equation



➢ Covariant Faddeev Equation





Quark-Diquark Approximation[1]

This Work: Full three-body Faddeev equation[2]

[1] R. T. Cahill, C. D. Roberts, J. Praschifka, Baryon structure and QCD, Austral. J. Phys. 42 (1989) 129–145.

[2] G. Eichmann et al., Covariant solution of the three-quark problem in quantum field theory: The Nucleon, EPJ Web Conf., 03028 (2010).

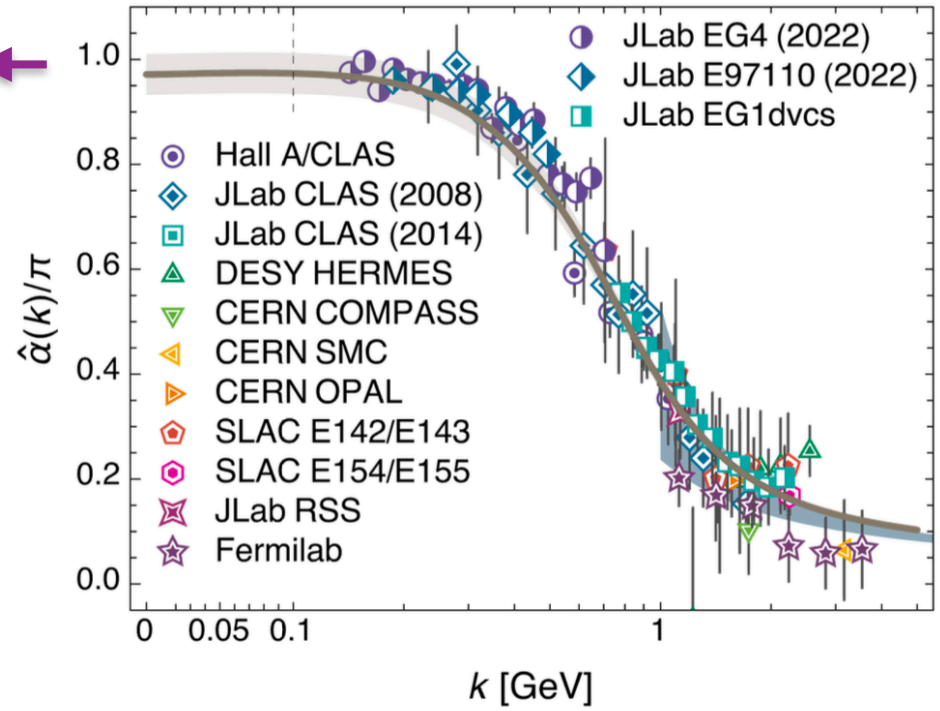
Models and Approximation

➢ Rainbow-ladder (RL) Truncation

$$\mathcal{G}^{CD}_{AB}(k) = \mathcal{G}(k) T_{\mu\nu}(k) [i\gamma_\mu \frac{\lambda^a}{2}]_{CB} [i\gamma_\nu \frac{\lambda^a}{2}]_{DA} \qquad T_{\mu\nu}(k) = \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}$$

➢ Relativistic Gauge Sector[1]

$$\mathcal{G}(q^2) \simeq \frac{4\pi \hat{\alpha}(q^2)}{q^2 + m_g^2(q^2)}$$

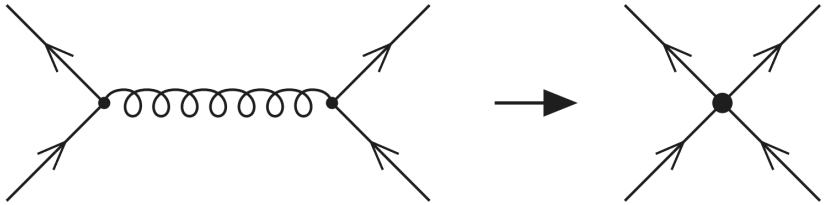


- **Infrared enhancement:** Gluon self-interactions produce anti-screening, causing the effective coupling to increase as k decreases.
- **Infrared saturation:** Dynamical gluon mass generation suppresses long-wavelength modes, so the effective coupling approaches a finite infrared limit.

[1] M. Ding, C. D. Roberts and S. M. Schmidt, Emergence of Hadron Mass and Structure, Particles no.1, 57-120

Models and Approximation

➢ Symmetry-preserving contact interaction [1]



$$\mathscr{G}_{\mu\nu} = \delta_{\mu\nu}\frac{4\pi\alpha_{IR}}{m_g^2}$$

$m_g = 0.5$ GeV: dynamically generated gluon mass scale

$\alpha_{\mathrm{IR}} = 0.36\pi$: controls the strength of the quark–quark interaction

➢ Schwinger Proper-time Regularization[2]

$$\frac{1}{\mathscr{T}} = \int_0^{+\infty} dx e^{-x\mathscr{T}} \rightarrow \int_{\tau_{uv}^2}^{\tau_{ir}^2} dx e^{-x\mathscr{T}}$$

➢ Simplified Hard-cutoff Regularization

$$\int_0^{+\infty} f(q^2)dq^2 \rightarrow \int_0^{\Lambda_{uv}^2} f(q^2)dq^2$$

Both regularizations are calibrated to the same dressed-quark mass,

$M_q = 0.36$ GeV

[1] X. Y. Bai et al., Contact interaction treatment of the nucleon Faddeev equation, Eur. Phys. J. A 62 (2026) no.7, 141
[2] Y. Yu et al., Contact interaction study of proton parton distributions, Eur. Phys. J. C 84 (2024) no.7, 739

Faddeev Amplitude

$$\Psi_{ABCD}(p,q,P) = \left(\sum_{n=1}^{2} \psi^{n}_{\alpha\beta;\gamma\delta}(p,q,P) \otimes F^{n}_{ab;cd}\right) \otimes \frac{\epsilon_{rst}}{\sqrt{6}}$$

➢ **Color part**

⇩

Antisymmetric color singlet

➢ **Flavor part**

Mixed-antisymmetric (MA) $\quad F^{1}_{ba;cd} = -F^{1}_{ab;cd}$

$$F^{1} = \frac{1}{\sqrt{2}} i\sigma_2 \otimes \mathbf{I}$$

Mixed-symmetric (MS) $\quad F^{2}_{ba;cd} = +F^{2}_{ab;cd}$

$$F^{2} = -\frac{1}{\sqrt{6}} \vec{\sigma} i\sigma_2 \otimes \vec{\sigma}$$

☑ **Proton/neutron:** contract d with $e_p = (1,0)$ or $e_n = (0,1)$.

➢ **Spin-momentum part**

$$\psi^{n}_{\alpha\beta;\gamma\delta}(p,q,P) = \sum_{g=1}^{64} c^{n}_{g}(p^2, q^2, z_1, z_2, z_0) X^{g}_{\alpha\beta;\gamma\delta}(\hat{p}_T, \hat{q}_t, \hat{P})$$

$$p^{\mu}_{T} := T^{\mu\nu}_{P} p^{\nu}$$

$$q^{\mu}_{t} := T^{\mu\nu}_{p_T} T^{\nu\lambda}_{P} q^{\lambda} = T^{\mu\nu}_{p_T} q^{\nu}_{T}$$

Covariant Dirac Basis $X^g_{\alpha\beta;\gamma\delta}(\hat{p}_T, \hat{q}_t, \hat{P})$

- The amplitude carries four Dirac indices, giving $4^4 = 256$ tensor structures.
- The indices $\alpha\beta\gamma$ label the three quarks, while δ labels the nucleon.
- Parity and energy projections select the positive-parity, positive-energy nucleon sector, leaving the $(1/2)^+$ nucleon $256/4 = 64$ independent covariants.
- Construct the angular momentum operators: [1]
  - Total: $W_j^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\hat{P}^\nu J^{\alpha\beta};\ W_j^2 \to j(j+1)$
  - Quark spin: $W_s^\mu = \frac{1}{4}\epsilon^{\mu\nu\alpha\beta}\hat{P}^\nu \sigma^{\alpha\beta};\ W_s^2 \to s(s+1)$
  - Relative orbital: $L^\mu_{(p)} = \frac{i}{2}\epsilon^{\mu\nu\alpha\beta}\hat{P}^\nu\left(p^\alpha \frac{\partial}{\partial p^\beta} - p^\beta \frac{\partial}{\partial p^\alpha}\right);\ (L_{(p)} + L_{(q)})^2 \to l(l+1)$

[1] G. Eichmann, Hadron properties from QCD bound-state equations, arXiv:0909.0703 [hep-ph].

Covariant Dirac Basis $X^{g}_{\alpha\beta;\gamma\delta}(\hat{p}_T, \hat{q}_t, \hat{P})$

- As an example: $X^{+}_{11,1} = S^{+}_{11} = \Lambda^{+}(P)\gamma_5 C \otimes \Lambda^{+}(P),\ \Lambda^{+} = (I + \hat{\not{P}})/2$
- Relative orbital: No dependence on p or q

$$\partial_{p^\alpha} X^{+}_{11,1} = \partial_{q^\alpha} X^{+}_{11,1} = 0 \rightarrow L^{\mu}_{(p)} X^{+}_{11,1} = L^{\mu}_{(q)} X^{+}_{11,1} = 0 \rightarrow l = 0$$

- Quark spin: The first two quarks form a spin singlet, $s_{12} = 0$

$$\Lambda^{+}\gamma_5 C \propto 1/\sqrt{2}(U^{\downarrow}U^{\uparrow} - U^{\uparrow}U^{\downarrow})\,[1]$$

- Coupling the spin-singlet pair to the third quark $s_3 = 1/2$, necessarily gives a total quark spin $s = 1/2$.
- Since $l = 0$ and $s = 1/2$, angular-momentum coupling yields $j = 1/2$.

g	s	l	$X^{r}_{ij,k}$
1/2	1/2	0	$X^{\pm}_{11,1} = S^{\pm}_{11}$
3/4	1/2	0	$X^{\pm}_{11,2} = P^{\pm}_{11}$
5/6	1/2	1	$X^{\pm}_{12,1} = S^{\pm}_{12}$
7/8	1/2	1	$X^{\pm}_{12,2} = P^{\pm}_{12}$
9/10	1/2	1	$X^{\pm}_{13,1} = S^{\pm}_{13}$
11/12	1/2	1	$X^{\pm}_{13,2} = P^{\pm}_{13}$
13/14	1/2	1	$X^{\pm}_{14,1} = S^{\pm}_{14}$
15/16	1/2	1	$X^{\pm}_{14,2} = P^{\pm}_{14}$
17/18	1/2	0	$\sqrt{3}X^{\pm}_{21,1} = V^{\pm}_{11}$
19/20	1/2	0	$\sqrt{3}X^{\pm}_{21,2} = A^{\pm}_{11}$
21/22	1/2	1	$\sqrt{3}X^{\pm}_{22,1} = V^{\pm}_{12}$
23/24	1/2	1	$\sqrt{3}X^{\pm}_{22,2} = A^{\pm}_{12}$
25/26	1/2	1	$\sqrt{3}X^{\pm}_{23,1} = V^{\pm}_{13}$
27/28	1/2	1	$\sqrt{3}X^{\pm}_{23,2} = A^{\pm}_{13}$
29/30	1/2	1	$\sqrt{3}X^{\pm}_{24,1} = V^{\pm}_{14}$
31/32	1/2	1	$\sqrt{3}X^{\pm}_{24,2} = A^{\pm}_{14}$
33/34	3/2	2	$\sqrt{6}X^{\pm}_{31,1} = (3S_{33} - V_{11})$
35/36	3/2	2	$\sqrt{6}X^{\pm}_{31,2} = 3P_{33} - A_{11}$
37/38	3/2	1	$\sqrt{6}X^{\pm}_{32,1} = 3S_{34} - 3S_{43} - 2V_{12}$
39/40	3/2	1	$\sqrt{6}X^{\pm}_{32,2} = 3P_{34} - 3P_{43} - 2A_{12}$
41/42	3/2	1	$\sqrt{6}X^{\pm}_{33,1} = 3S_{31} - V_{13}$
43/44	3/2	1	$\sqrt{6}X^{\pm}_{33,2} = 3P_{31} - A_{13}$
45/46	3/2	1	$\sqrt{6}X^{\pm}_{34,1} = 3S_{41} - V_{14}$
47/48	3/2	1	$\sqrt{6}X^{\pm}_{34,2} = 3P_{41} - A_{14}$
49/50	3/2	2	$\sqrt{2}X^{\pm}_{41,1} = 2S_{44} + S_{33} - V_{11}$
51/52	3/2	2	$\sqrt{2}X^{\pm}_{41,2} = 2P_{44} + P_{33} - A_{11}$
53/54	3/2	2	$\sqrt{2}X^{\pm}_{42,1} = S_{34} + S_{43}$
55/56	3/2	2	$\sqrt{2}X^{\pm}_{42,2} = P_{34} + P_{43}$
57/58	3/2	2	$\sqrt{2}X^{\pm}_{43,1} = -2S_{42} + S_{31} - V_{13}$
59/60	3/2	2	$\sqrt{2}X^{\pm}_{43,2} = -2P_{42} + P_{31} - A_{13}$
61/62	3/2	2	$\sqrt{2}X^{\pm}_{44,1} = 2S_{32} + S_{41} - V_{14}$
63/64	3/2	2	$\sqrt{2}X^{\pm}_{44,2} = 2P_{32} + P_{41} - A_{14}$

S

P

D

[1] G. Eichmann, Hadron properties from QCD bound-state equations, arXiv:0909.0703 [hep-ph].

Faddeev Wave Function

- The Faddeev wave function can be decomposed into spin-momentum, flavor and color parts.

$$\Theta_{ABCD}(p,q,P) = (\sum_{n=1}^{2} \theta^{n}_{\alpha\beta;\gamma\delta}(p,q,P) \otimes F^{n}_{ab;cd}) \otimes \frac{\epsilon_{rst}}{\sqrt{6}}$$

- ☑ The proton Faddeev wave function: $\Theta^{P}_{ABCD}(p,q,P) = (\sum_{n=1}^{2} \theta^{n}_{\alpha\beta;\gamma\delta}(p,q,P) \otimes F^{n}_{ab;cd} e_{pd}) \otimes \frac{\epsilon_{rst}}{\sqrt{6}}$
- ☑ The neutron Faddeev wave function: $\Theta^{N}_{ABCD}(p,q,P) = (\sum_{n=1}^{2} \theta^{n}_{\alpha\beta;\gamma\delta}(p,q,P) \otimes F^{n}_{ab;cd} e_{nd}) \otimes \frac{\epsilon_{rst}}{\sqrt{6}}$

- The Faddeev wave function can be constructed by attaching the three dressed-quark propagators to the Faddeev amplitude. In particular, **spin-momentum part**:

$$\theta^{n}_{\alpha_1\alpha_2;\alpha_3\delta}(p_1,p_2,p_3) = S_{\alpha_1\alpha_{1'}}(p_1) S_{\alpha_2\alpha_{2'}}(p_2) S_{\alpha_3\alpha_{3'}}(p_3) \psi^{n}_{\alpha_{1'}\alpha_{2'};\alpha_{3'}\delta}(p_1,p_2,p_3)$$

Rest-Frame OAM: Wave-Function Strengths

- For a proton with $J = 1/2$. To satisfy $J = |L - S|, \ldots, L + S$, the angular-momentum coupling allows four rest-frame channels:

$$J = 1/2,\ S = 1/2 \to L = 0,\ L = 1 \to {}^2S, {}^2P$$

$$J = 1/2,\ S = 3/2 \to L = 1,\ L = 2 \to {}^4P, {}^4D$$

- Expanding in terms of 64 basis elements:

$$\theta^n_{\alpha\beta;\gamma\delta}(p_1, p_2, p_3) = \sum_{g=1}^{64} C^n_g(p^2, q^2, z_1, z_2, z_0) X^g_{\alpha\beta;\gamma\delta}(\hat{p}_T, \hat{q}_t, \hat{P})$$

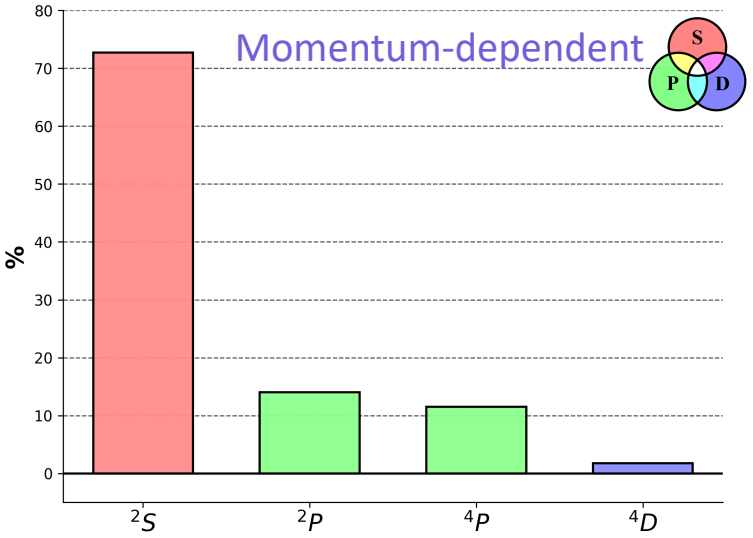
- Define the following set of nucleon angular momentum strengths $\mathcal{J} \in \{{}^2S, {}^2P, {}^4P, {}^4D\}$[1]:

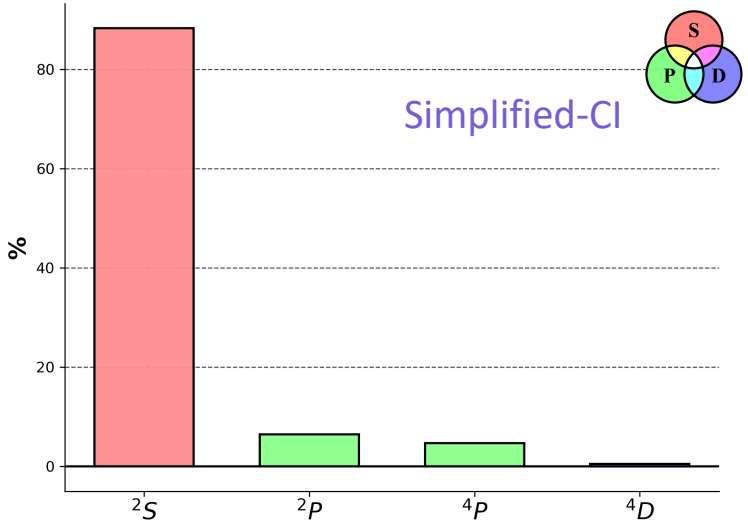
$$\mathcal{J} = \mathbb{N}^{-1} \sum_{n=1}^{2} \sum_{g \in \mathcal{J}} \int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} |\, C^n_g(p, q, P)\,|^2$$

☑ Normalization coefficient: $\mathbb{N} = \sum_{n=1}^{2} \sum_{g=1}^{64} \int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} |\, C^n_g(p, q, P)\,|^2$

[1] S. X. Qin, C. D. Roberts and S. M. Schmidt, Poincaré-covariant analysis of heavy-quark baryons, Phys. Rev. D 97 (2018) no.11, 114017

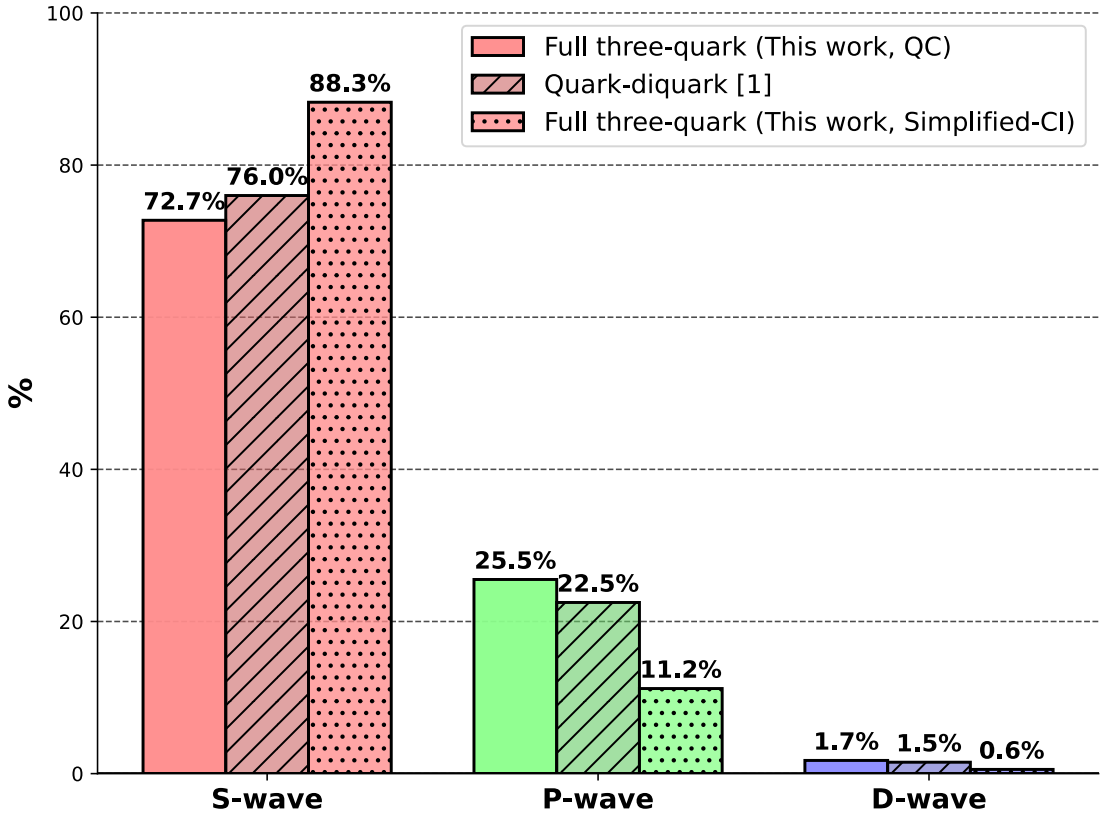
Results: Wave-Function Strengths





- **S-wave dominance (72.7%):** The nucleon favors the lowest relative-orbital configuration, which requires no orbital excitation.
- **Relativistic P waves (25.5%):** The sizable ^2P and ^4P strengths show that the nucleon is not a pure S-wave state, but contains substantial relativistic spin–momentum correlations.
- **Suppressed D wave (1.7%):** The ^4D channel requires both $l = 2$ and total quark spin $s = 3/2$, making it dynamically disfavored in the ground state.
- **Larger S-wave fraction in CI:** The pointlike, momentum-independent interaction suppresses OAM correlations and enhances $S-$wave component.

Results: Wave-Function Strengths



- **Support for the quark–diquark approximation:** Its close agreement with the full three-quark result indicates that dominant diquark correlations capture the leading rest-frame overall $S/P/D$ hierarchy.
- **No probability interpretation:** These normalized strengths measure the relative magnitudes of the dressing functions, but cannot be interpreted as partial-wave probabilities.

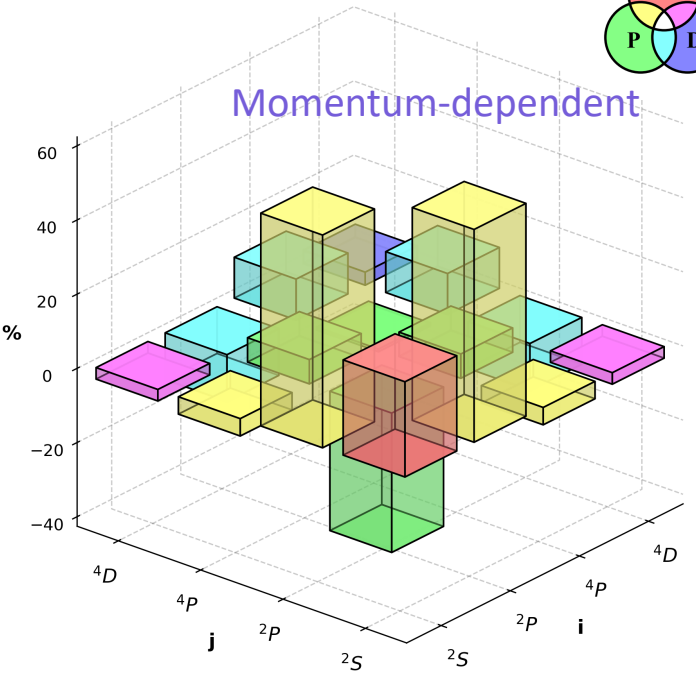
[1] C. Chen et al., Structure of the nucleon's low-lying excitations, Phys. Rev. D 97 (2018) no.3, 034016

Canonical Normalization

- The Faddeev equation determines the structure of the bound-state amplitude, but leaves its overall scale undetermined: $\Psi \to C\Psi$.
- Canonical normalization fixes C through the residue of the physical pole of the scattering matrix: $T(P^2) \xrightarrow{P^2 \to -M^2} \frac{\Psi\bar{\Psi}}{P^2+M^2}$ with $\overline{\Psi}(T^{-1})'\Psi = 1.$ $\qquad ' \equiv \frac{\partial}{\partial P^2}|_{P^2=-M^2}$
- This provides a physically grounded measure of partial-wave contributions, rather than from imposing $\int |\Psi|^2 = 1.$
- For an electromagnetic current satisfying the Ward–Takahashi identity, the same normalization guarantees the correct proton charge: $G_E^p(0) = 1.$
- Nakanishi Form: $\mathcal{N}_{fi} = -\lambda' \times \sum_{n_1=1}^{2}\sum_{n_2=1}^{2}\int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4} Tr[\bar{\theta}_f^{n_1}(p,q,P)G_0^{-1}\theta_i^{n_2}(p,q,P)]$

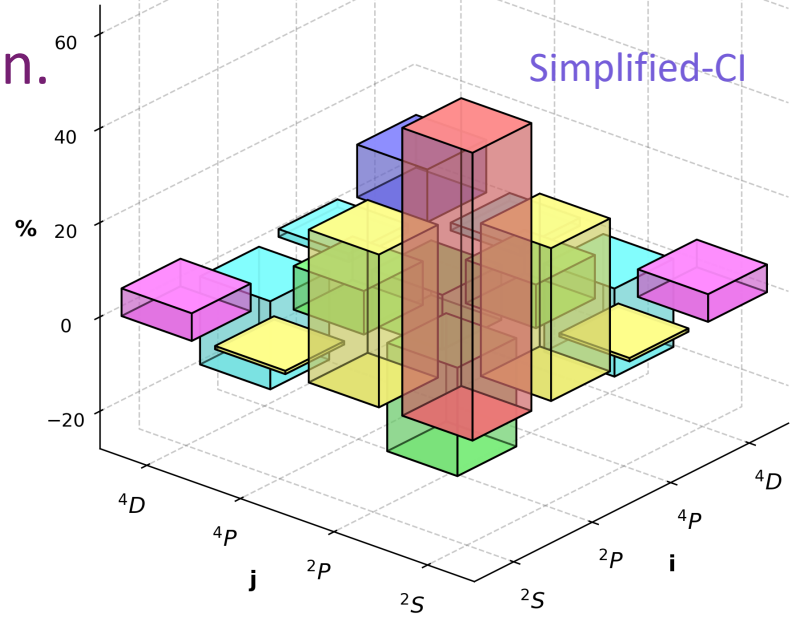
Results: Canonical Normalization





- Substantial $S-S$ contribution (25.7%): The lowest orbital configuration remains important.
- Dominant $S-P$ interference (105.4%): This signals strong relativistic spin–momentum correlations.
- Destructive $P-P$ contribution (-25.2%): This partially compensates the large positive $S-P$ contribution.



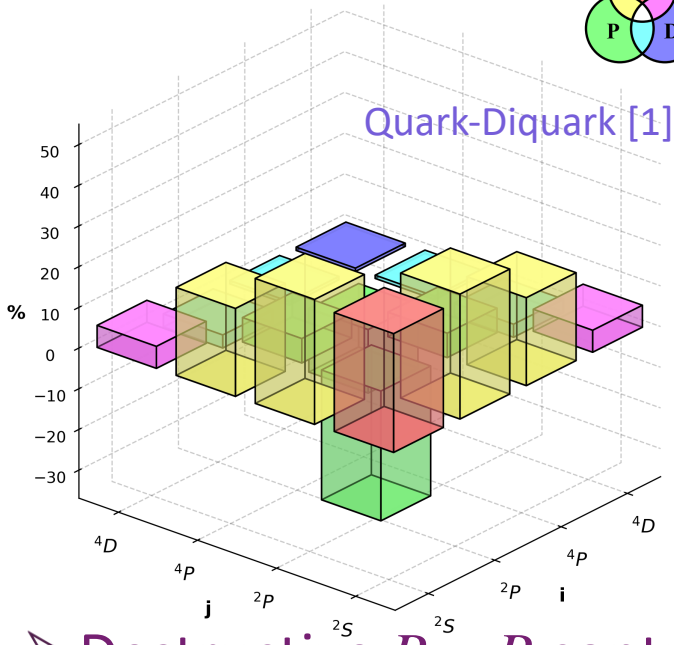


- Small D–wave effect (-5.9%): The $S-D$, $P-D$, and $D-D$ terms largely cancel.
- Larger S-wave fraction in CI: The pointlike, momentum-independent interaction enhances S–wave component.

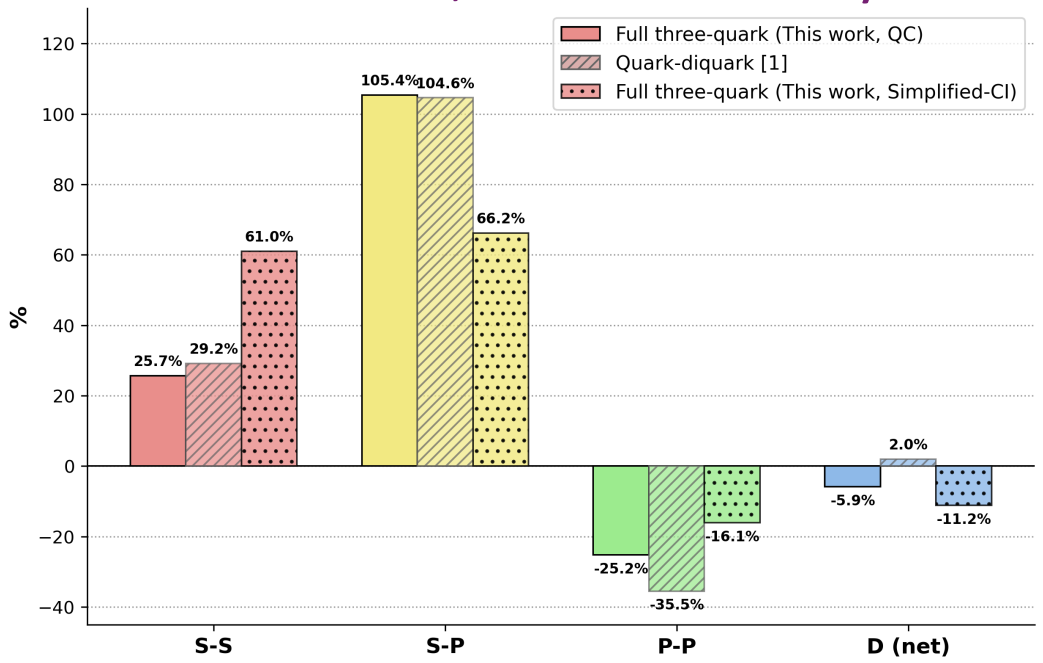
Results: Canonical Normalization

[1] P.Cheng et al., Insights into Nucleon Resonances via Continuum Schwinger Function Methods





- Substantial $S-S$: The lowest-orbital configuration remains an important part of the ground state.
- Robust $S-P$ interference: Momentum-dependent calculation and Quark-diquark produce nearly identical contributions. The robust result is the combined $S-P$ structure, rather than any individual element.
- Destructive $P-P$ contribution: All calculations produce a negative $P-P$ sector.
- Small $D-$wave sector: The $D-$wave effect remains small, while its detailed sign and distribution are more model dependent.



Summary and Outlook

Summary

- We calculate the Nakanishi form of **rest-frame nucleon OAM decomposition** using relativistic interactions and simplified contact interaction within the Poincaré-covariant full three-body Faddeev framework.
- In Poincaré invariant QCD, same as mesons, nucleons **cannot** be described as **pure single OAM states**, which differs from the quark model.
- **Agreement with the quark–diquark calculation** shows that dominant two-quark correlations capture the leading rest-frame OAM structure.

Outlook

- Other particles: **spin** $3/2$ **baryon** such as Δ.
- Nucleon OAM decomposition in **light-front frame**.


Thank You

Appendix A: Flavor Matrix

Flavor Matrix [1]

$$u=(1,0),\ d=(0,1)$$

$$\Lambda^{+}(P)\gamma_5 C=\begin{pmatrix}-i\sigma_2 & 0\\ 0 & 0\end{pmatrix}$$

The isospin singlet matrices (MA): $s^0=\frac{1}{\sqrt{2}}(ud^\dagger-du^\dagger)=\frac{1}{\sqrt{2}}i\sigma_2$

$$Tr\{s^{e\dagger}s^{e'}\}=\delta_{ee'}$$

The isospin tripletlet matrices (MS):

$$s^1=uu^\dagger=\frac{1}{2}(I+\sigma_3)\qquad s^2=\frac{1}{\sqrt{2}}(ud^\dagger+du^\dagger)=\frac{1}{\sqrt{2}}\sigma_1\qquad s^3=dd^\dagger=\frac{1}{2}(I-\sigma_3)$$

The Clebsch-Gordan construction of the corresponding flavor tensors reads:

$$F_{MA}=\frac{1}{\sqrt{2}}i\sigma_2\otimes\mathbf{I}=s^0\otimes(uu^\dagger+dd^\dagger)$$

$$F_{MS}=-\frac{1}{\sqrt{6}}\vec{\sigma}i\sigma_2\otimes\vec{\sigma}=\sqrt{\frac{2}{3}}s^1\otimes du^\dagger-\frac{1}{\sqrt{3}}s^2\otimes(uu^\dagger-dd^\dagger)-\sqrt{\frac{2}{3}}s^3\otimes ud^\dagger$$

E.g. Proton: $F_{MA}e_1=\frac{1}{\sqrt{2}}(udu-duu)\qquad F_{MS}e_1=\frac{1}{\sqrt{6}}(2uud-udu-duu)$

[1] G. Eichmann, Hadron properties from QCD bound-state equations, arXiv:0909.0703 [hep-ph].

Appendix B: Quark-Diquark Approximation

Quark-Diquark Approximation [1] $\psi_i = \sum_D \Gamma_D(k;K)\Delta_D(K)\left[\varphi_D(\ell;P)u(P)\right]$ $D = 0^+, 1^+$

- Diquark Amplitude

$$\Gamma_D(k;K) \sim g_D \mathscr{D}_D \mathscr{F}(k^2/\omega_D^2)$$

g_D: correlation strength; k: relative momentum

$\mathscr{D}_D$: spin, parity, flavor, and color structure

- Diquark propagator

$$\Delta_D(K) \sim \frac{1}{m_D^2}\mathscr{F}(K^2/\omega_D^2)$$

ω_D^{-1}: correlation length; m_D: mass scale

- Quark-Diquark faddeev amplitude

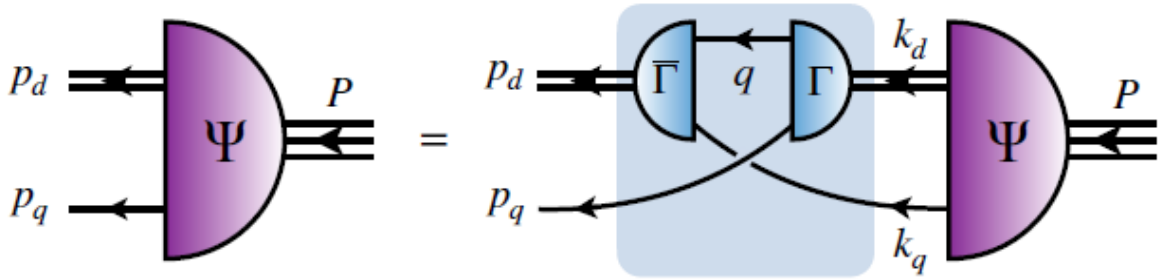
$$\varphi_D(\ell;P) = \sum_n f_n^D(\ell^2, \ell\cdot P)\tau_n^D(\ell;P)$$

ℓ: relative momentum between quark and diquark

f_n^D: scalar dressing function; τ_n^D: basis elements

- Binding through dressed-quark exchange

$$\psi_D = \sum_{D'} K_{DD'}\psi'_D \quad K_{DD'} \sim \Gamma_D S^T(q')\overline{\Gamma}_{D'} S(q)\Delta_{D'}$$



[1] C. Chen et al., Structure of the nucleons low-lying excitations, Phys. Rev. D 97 (2018) no.3, 034016

Appendix C: Covariant Dirac Basis

Covariant Dirac Basis $X^{g}_{\alpha\beta;\gamma\delta}(\hat{p}_T, \hat{q}_t, \hat{P})$

- 64 orthogonal basis elements:

$$\begin{pmatrix} S^r_{ij}(\hat{p}_T, \hat{q}_t, \hat{P}) \\ P^r_{ij}(\hat{p}_T, \hat{q}_t, \hat{P}) \end{pmatrix} := \begin{pmatrix} I \otimes I \\ \gamma^5 \otimes \gamma^5 \end{pmatrix} (\Gamma_i \otimes \Gamma_j)\Omega^r$$

64 = 2 x(4 x 4)x2

- ☑ Four independent elements: $\Gamma_i(\hat{p}_T, \hat{q}_t, \hat{P}) \in \{I, \frac{1}{2}[\hat{\not{p}}_T, \hat{\not{q}}_t], \hat{\not{p}}_T, \hat{\not{q}}_t\}$
- ☑ To keep the positive-parity and positive-energy nucleon: $\Omega^r(P) = \Lambda^r(P)\gamma_5 C \otimes \Lambda^+(P),\ r = \pm$

$$\Lambda^{\pm} = (I \pm \hat{\not{P}})/2$$

- Similarly to scalar and pseudoscalar covariants, we construct vector and axial-vector covariants.

$$\begin{pmatrix} V^r_{ij}(\hat{p}_T, \hat{q}_t, \hat{P}) \\ A^r_{ij}(\hat{p}_T, \hat{q}_t, \hat{P}) \end{pmatrix} := \begin{pmatrix} \gamma^\mu_T \otimes \gamma^\mu_T \\ \gamma^\mu_T\gamma^5 \otimes \gamma^\mu_T\gamma^5 \end{pmatrix} (\Gamma_i \otimes \Gamma_j)\Omega^r, \quad \gamma^\mu_T = T^{\mu\nu}_P \gamma^\nu$$

Combination → 64 eigenfunctions

Appendix D: Canonical Normalization

Nakanishi Form of Canonical Normalization

- The Faddeev equation can be written as the eigenvalue equation,

$$K G_0 \Psi_i = \lambda_i \Psi_i \Longleftrightarrow T^{-1}\Psi_i = (\lambda_i^{-1} - 1) G_0 \Psi_i \qquad \lambda_i(-M_i^2) = 1 \qquad G_0 = S \otimes S \otimes S$$

- Differentiating this expression with respect to P^2, we obtain

$$(T^{-1})'\Psi = [(\lambda^{-1} - 1)G_0]'\Psi = (\lambda^{-1})' G_0 \Psi + (\lambda^{-1} - 1) G_0' \Psi$$

- Using the canonical normalization condition and $\lambda = 1$ together with $(\lambda^{-1})' = -\lambda^{-2}\lambda' = -\lambda'$, we obtain the Nakanishi form canonical normalization $\overline{\Psi} G_0 \Psi = -\dfrac{1}{\lambda'}$.

- Since a canonically normalized Faddeev wave function is required for calculating physical observables, we express the normalization condition directly in terms of $\Theta = G_0 \Psi$,

$$\overline{\Theta} G_0^{-1} \Theta = -\frac{1}{\lambda'}$$

Canonical Normalization

- Define the scattering matrix by $G = G_0 + G_0 T G_0, \qquad G_0 = S(p_1) \otimes S(p_2) \otimes S(p_3),$
- The Dyson series for the Green function reads $G = G_0 + G_0 K G_0 + \cdots$, so $G = G_0 + G_0 K G$.
- Hence, the scattering matrix $T = K + K G_0 K + \cdots$, and $T = K + K G_0 T, K T^{-1} = I - K G_0$.
- The scattering matrix at the pole behaves as $T(P^2) \xrightarrow{P^2 \to -M^2} \dfrac{\Psi \bar{\Psi}}{P^2 + M^2}$.
- Treating P as the external total momentum, the Faddeev equation can be formulated as an eigenvalue problem: $K G_0 \Psi_i = \lambda_i \Psi_i$, using the equation above finally get $T^{-1} \Psi_i = (\lambda_i^{-1} - 1) G_0 \Psi_i$.

Canonical Normalization

$$
\begin{aligned}
&(TT^{-1})' = 0 \\
\rightarrow\ &T'T^{-1} + T(T^{-1})' = 0 \\
\rightarrow\ &T' = -T(T^{-1})'T \\
\rightarrow\ &-\frac{\Psi\bar{\Psi}}{(P^2+M^2)^2} = -\frac{\Psi\bar{\Psi}}{P^2+M^2}(T^{-1})'\frac{\Psi\bar{\Psi}}{P^2+M^2}
\end{aligned}
$$

Appendix: Older Version

Canonical Normalization

- The scattering matrix at the pole behaves as $T(P^2 = -M^2) \to \dfrac{\Psi\overline{\Psi}}{P^2 + M^2}$
- Evaluating the relation $T' = -T(T^{-1})'T$ at the bound-state pole yields the on-shell canonical normalization condition:

$$\overline{\Psi}(T^{-1})'\Psi = 1$$

- The nucleon electromagnetic current is given by $J^\mu = \overline{\Psi}_f(T^{-1})^\mu\Psi_i$. Current conservation then requires $Q^\mu J^\mu = \overline{\Psi}_f Q^\mu(T^{-1})^\mu\Psi_i = 0$, where Q^μ denotes the photon momentum.
- In the limit $Q \to 0$, the electromagnetic current condition degenerates into the canonical normalization condition, i.e. after canonical normalization, the charge conservation condition $G_E^p(0) = 1$ is automatically satisfied.