The Physics of Phase Transitions is Universal, Numerical Simulations aren't

[Hands & JO *PRB* **114** (2026)]

Simon Hands, **Johann Ostmeyer**

Helmholtz-Institut für Strahlen- und Kernphysik, University of Bonn, Germany

Relativistic fermions in Flatland, ECT*, Trento, Oct. 2, 2026

Gefördert durch

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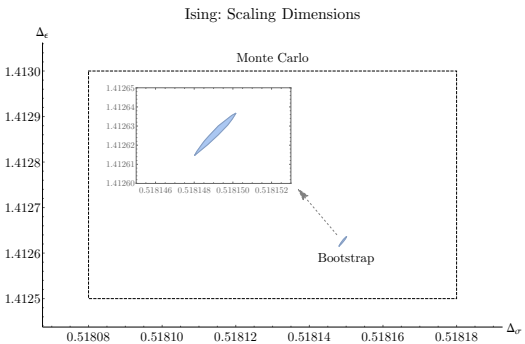
The 3D Ising model

[Hasenbusch *PRB* **17** (2010); Kos + *JHEP* **08** (2016); Chang + *JHEP* **03** (2025)]



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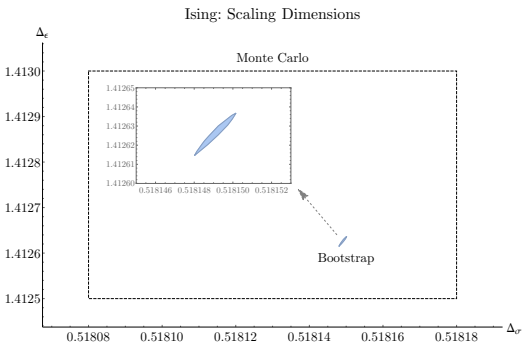

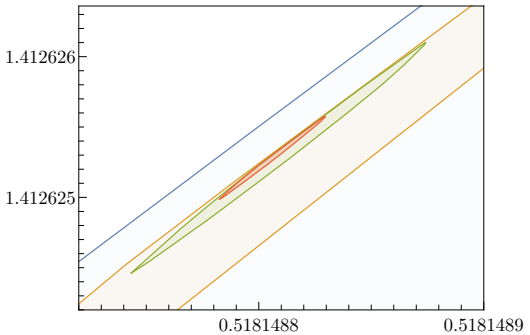

The SU(2) Gross-Neveu-Yukawa "chiral Heisenberg" universality class

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[Du + *cond-mat.str-el*/**2609**.14668]
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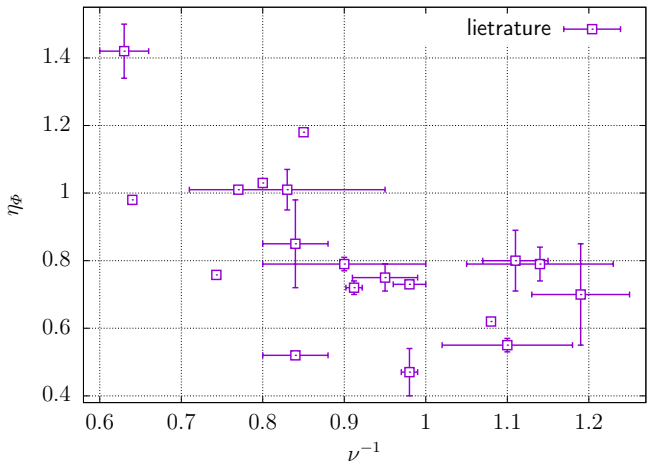

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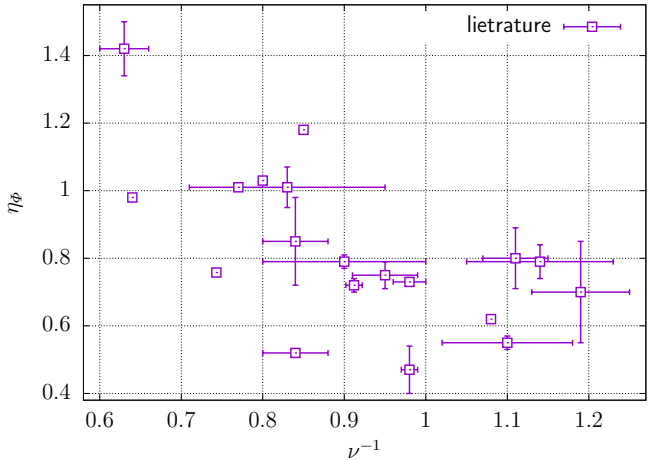

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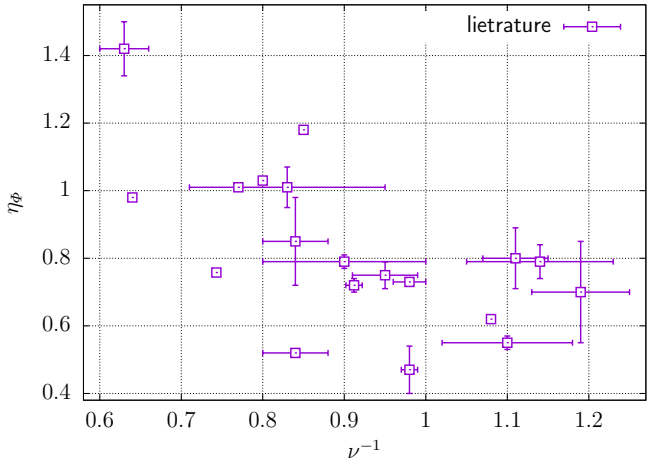

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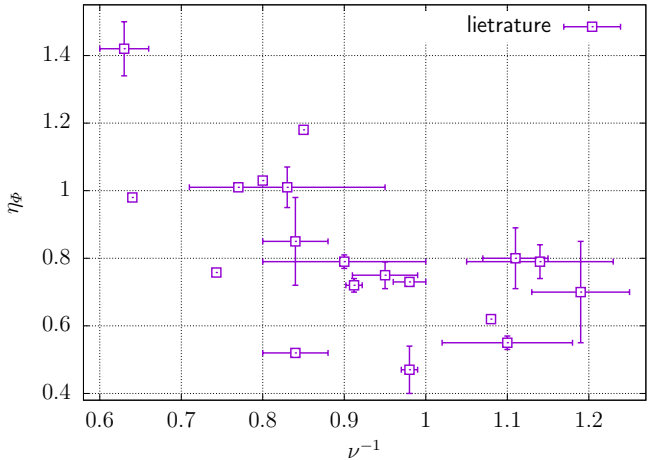

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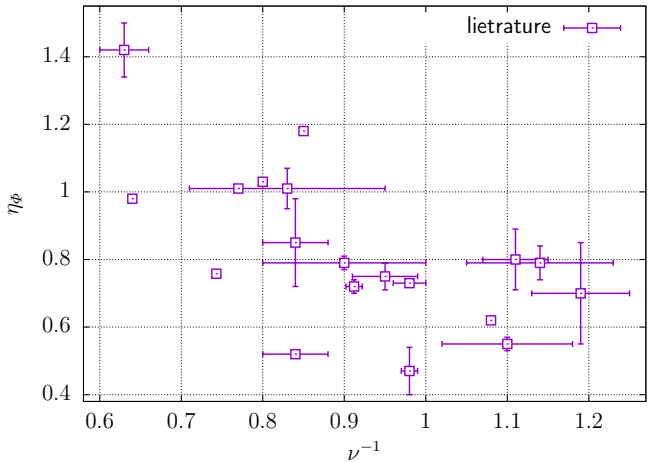

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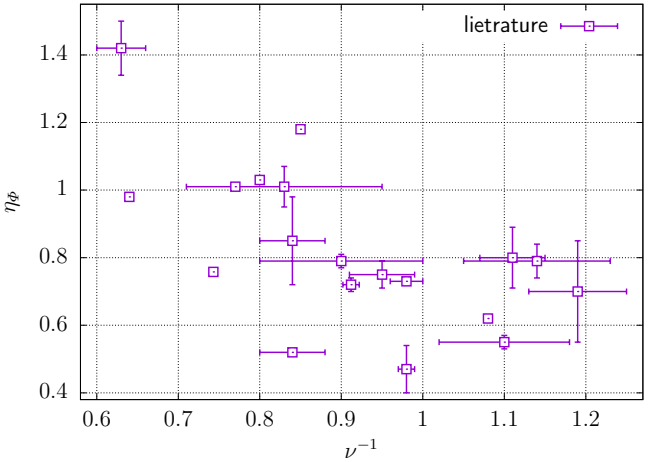

How did we make such a mess?

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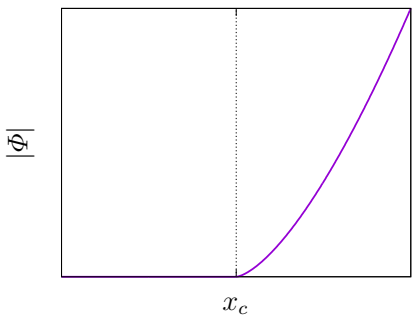
$$|\Phi| \propto \begin{cases} 0 & \text{if } x \leq x_c \\ (x - x_c)^\beta & \text{if } x > x_c \end{cases}$$



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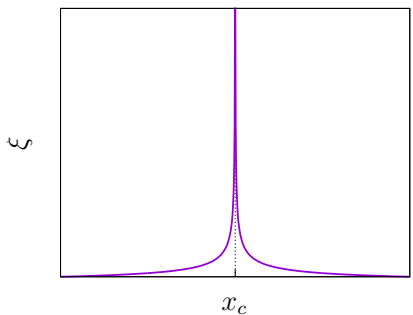

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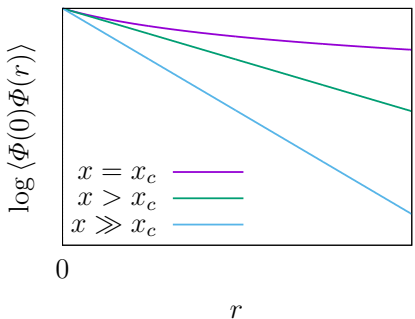
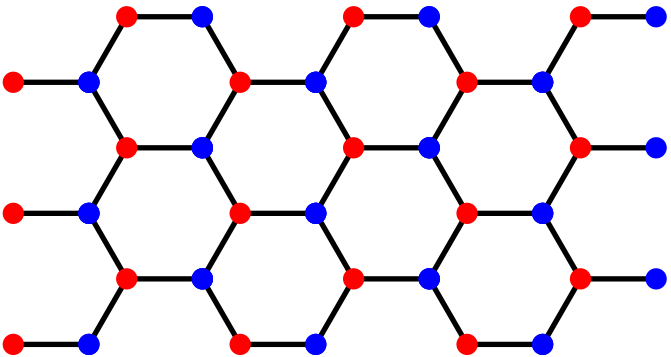
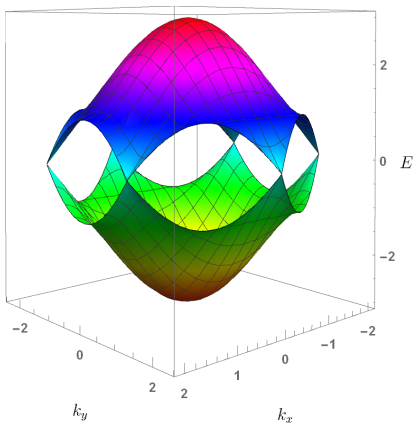


Hubbard model

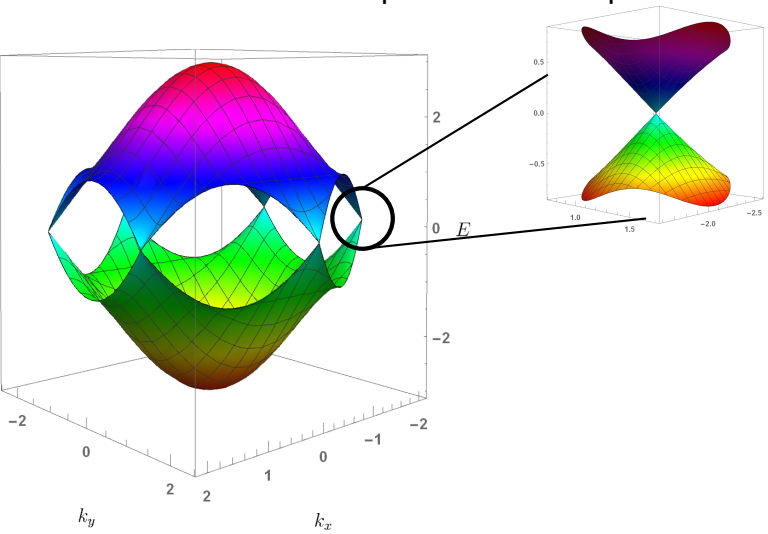
$$H = -\sum_{s=\uparrow\downarrow}\sum_{\langle x,y\rangle} c^{\dagger}_{x,s}c_{y,s} + \frac{1}{2}U\sum_{x} q_x^2\,, \qquad q_x = c^{\dagger}_{x,\uparrow}c_{x,\uparrow} + c^{\dagger}_{x,\downarrow}c_{x,\downarrow} - 1$$





Phase transition: chiral Heisenberg universality class [Janssen & Herbut *PRB* **89** (2014)]

2 bands $\times$ 2 Dirac points $\times$ 2 spins $=$ 2 flavors $\times$ 4 spinor



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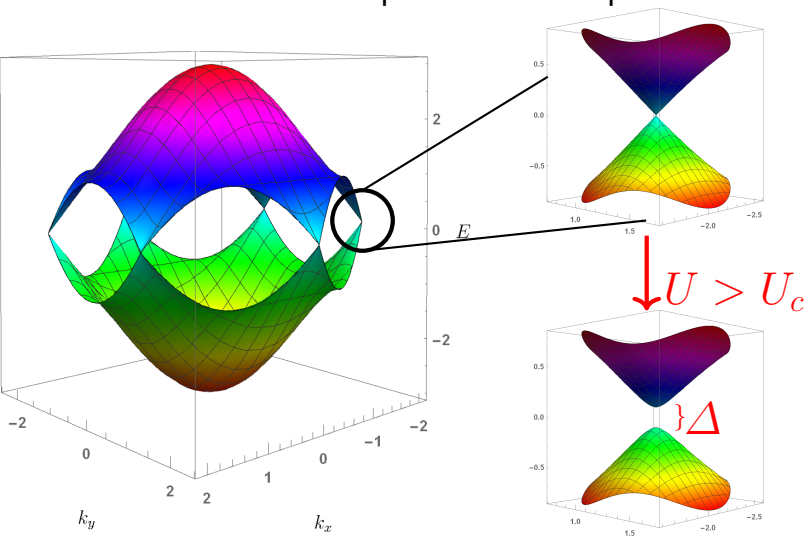

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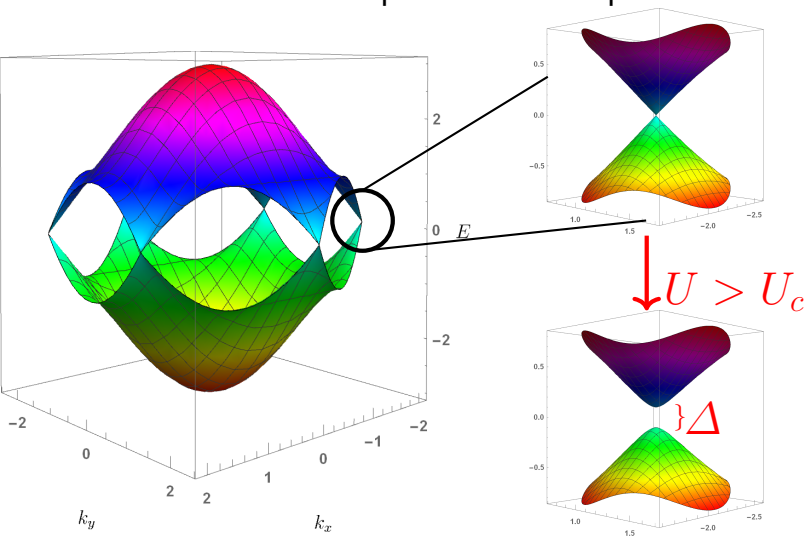

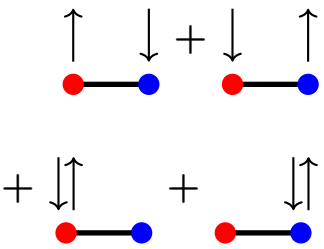

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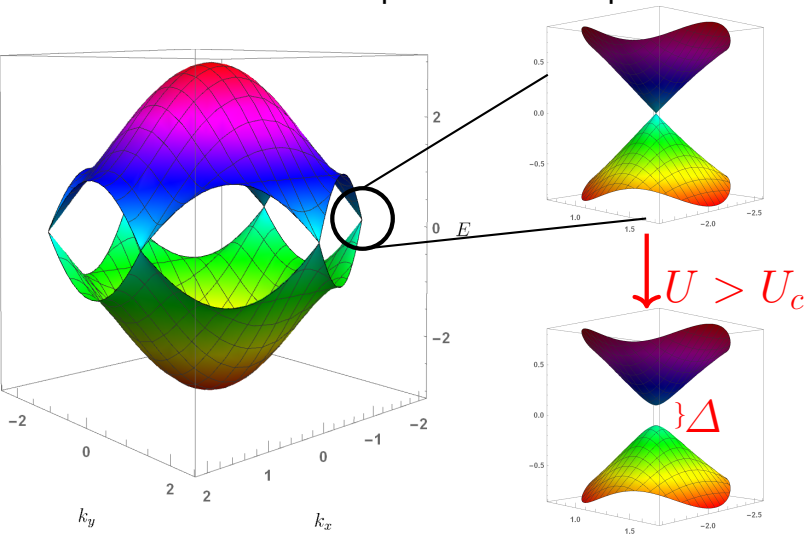

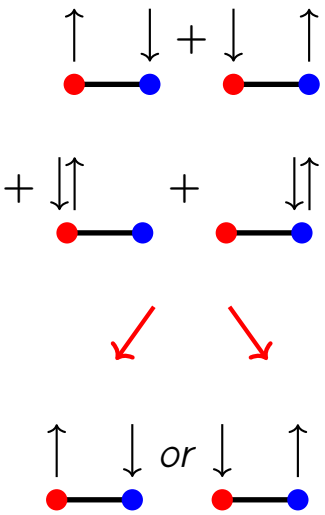

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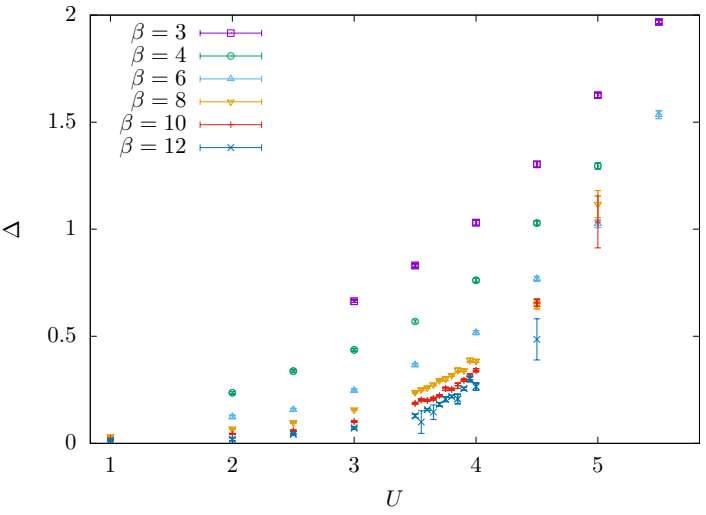

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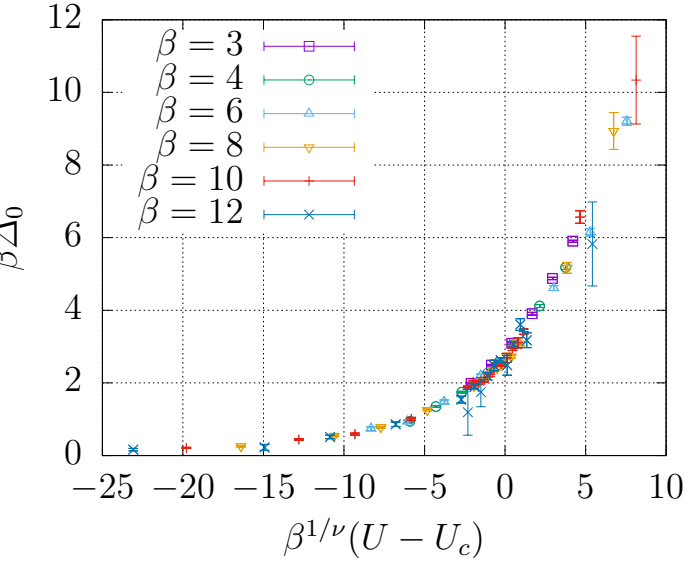

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The simulation [Kaplan *PLB* **288** (1992); Kennedy + *NuclPhysB* **73** (1999)]

[Hands *PRD* **99** (2019), *PLB* **754** (2016); Hands & JO *PRD* **107** (2023); Hands & Worthy *PRD* **111** (2025)]

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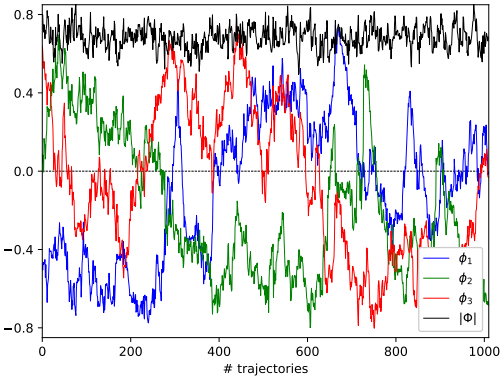
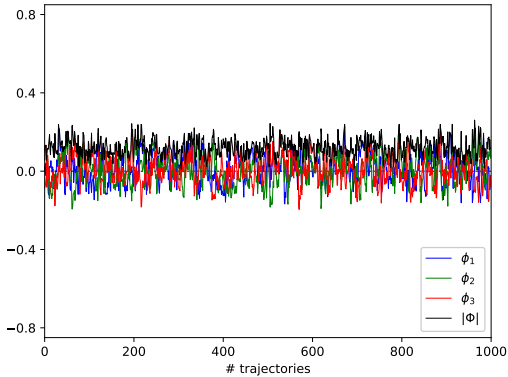
- Rational Hybrid Monte Carlo $\phi \sim \sqrt{\det \mathcal{M}^\dagger_{\mathsf{DW}}\mathcal{M}_{\mathsf{DW}}}$

The order parameter

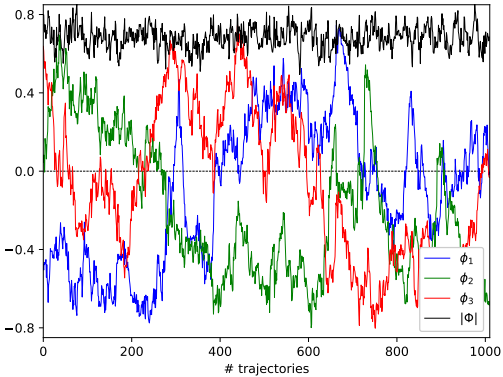


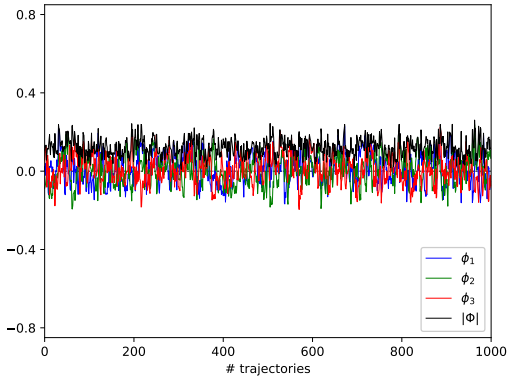
$\beta = 0.25$



$\beta = 0.45$

The order parameter



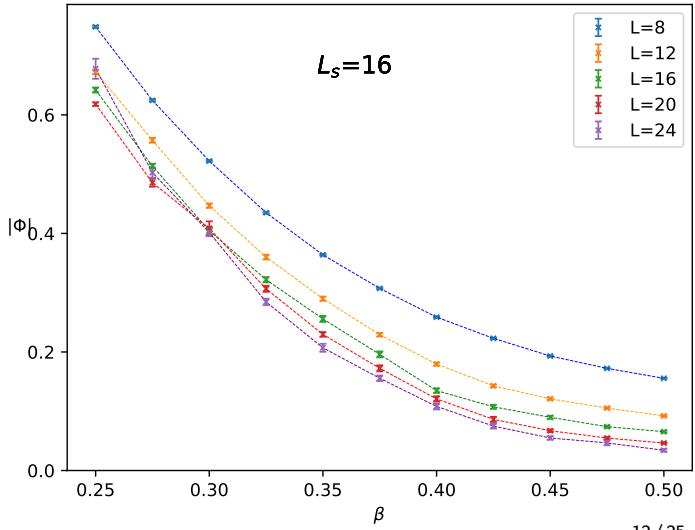


$\beta = 0.25$ $\beta = 0.45$

No explicit symmetry breaking $\Rightarrow \left\langle \vec{\phi} \right\rangle = 0$

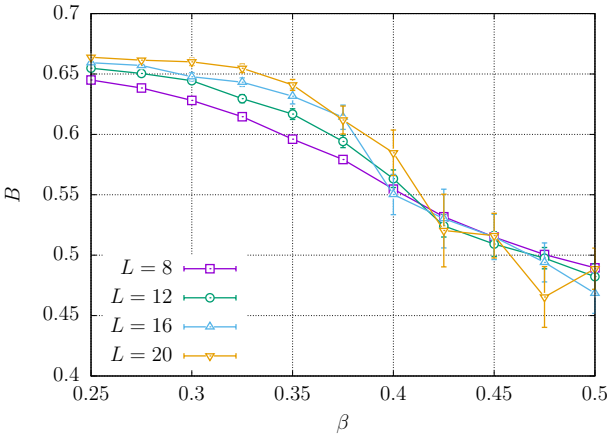
The pseudo-order parameter

$$|\Phi| = \frac{1}{L^3}\sqrt{\sum_{a=1}^{3}\left(\sum_{x}\phi_a(x)\right)^2}$$



Binder cumulant

$$B = 1 - \frac{\left\langle |\Phi|^4 \right\rangle}{3 \left\langle |\Phi|^2 \right\rangle^2}$$



Critical exponents

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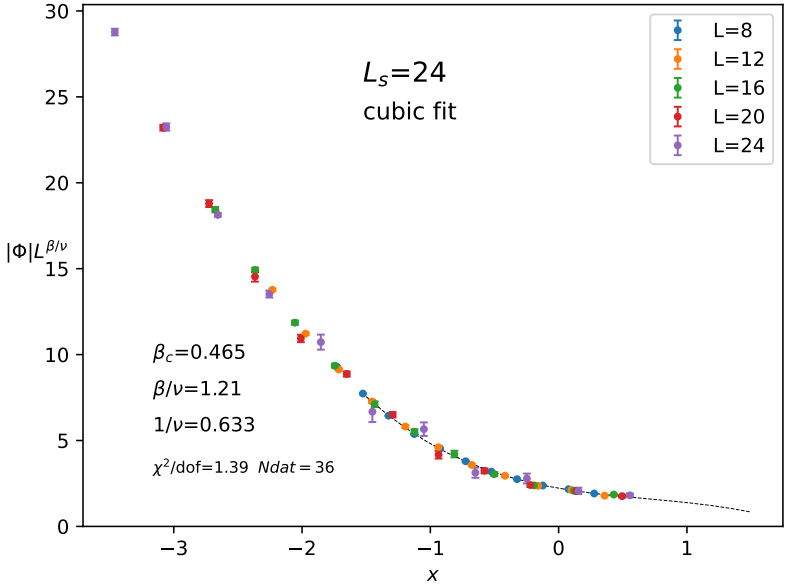
- Universal function:

$$|\Phi| L^{\frac{\beta_\Phi}{\nu}} = f(tL^{\frac{1}{\nu}}),$$
$$t \equiv (\beta - \beta_c)/\beta_c$$

Data collapse fit



The SU(2) Gross-Neveu-Yukawa "chiral Heisenberg" *universality* class

[Buividovich + *PRB* **98** (2018), *PRB* **99** (2019)]
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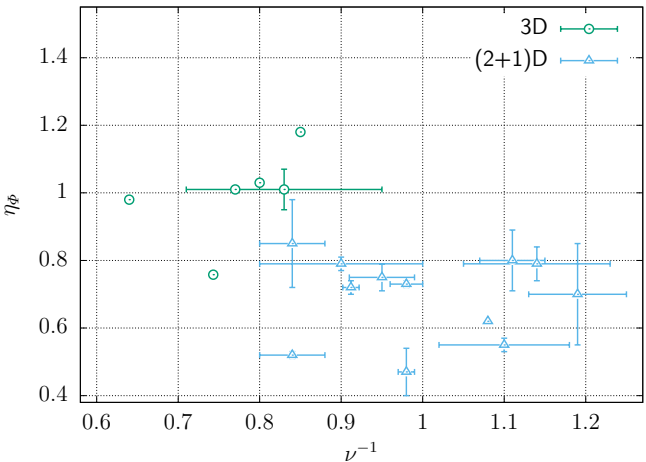

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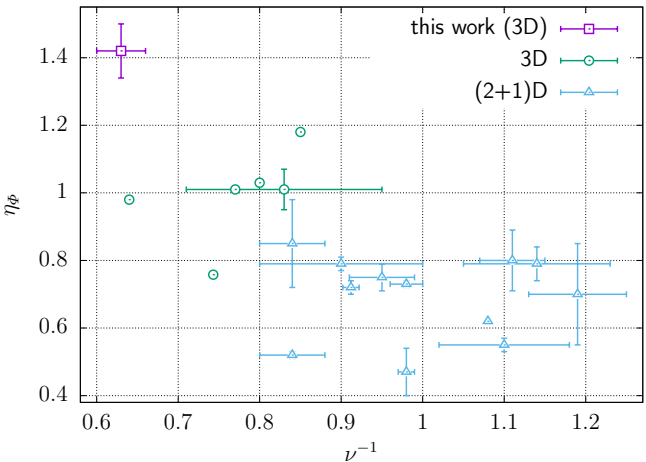

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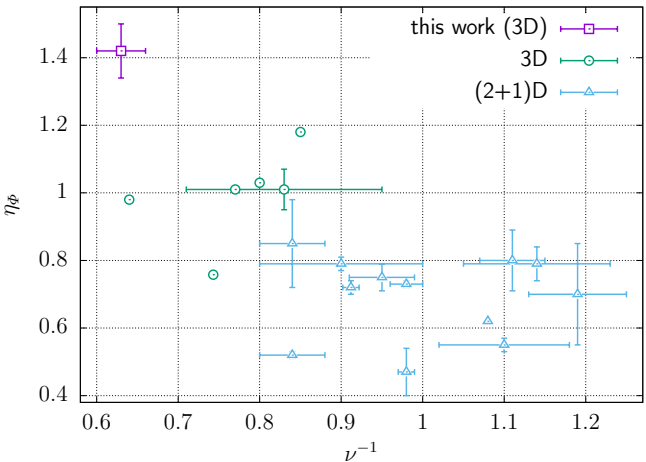

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Some brainstorming

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- Janssen conjecture: the phase transition is (close to) weakly 1st order.

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- Suggestions?

Fermion correlators

$$\left\langle \psi(0)\bar{\psi}(x_0)\right\rangle = \begin{pmatrix} z_3 & z_1 - \mathrm{i}\, z_2 \\ z_1 + \mathrm{i}\, z_2 & -z_3 \end{pmatrix} \equiv z_i \sigma_i \,,$$

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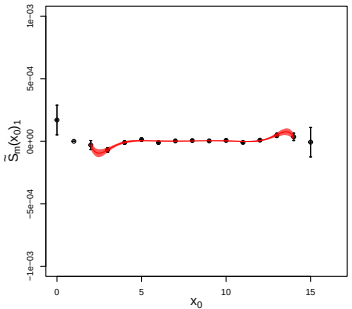
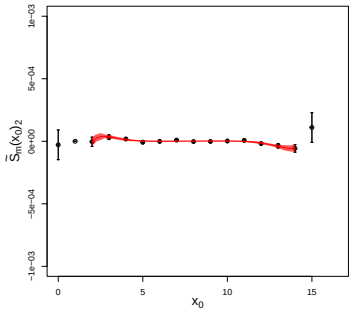
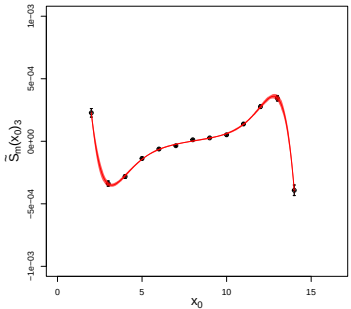
Use the Truncated Hankel Correlator (THC) method for fits.
[JO & Urbach *hep-lat*/**2510**.15500]

Anti-symmetric fermion correlators [JO & Urbach *hep-lat*/**2510**.15500]

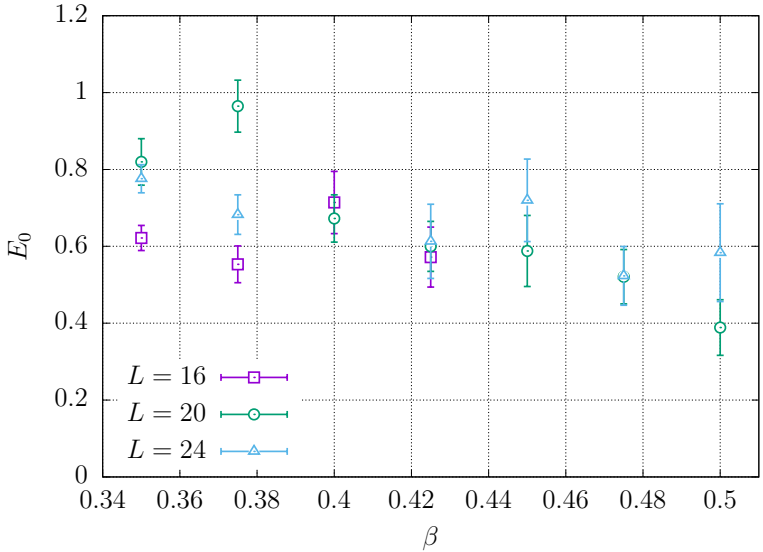
$$z_i = \sum_{l=1}^{k/2} \tilde{c}_{i,l} \, \sinh\left(E_l(x_0 - L/2)\right)$$



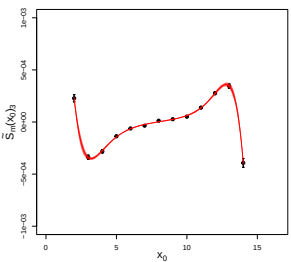


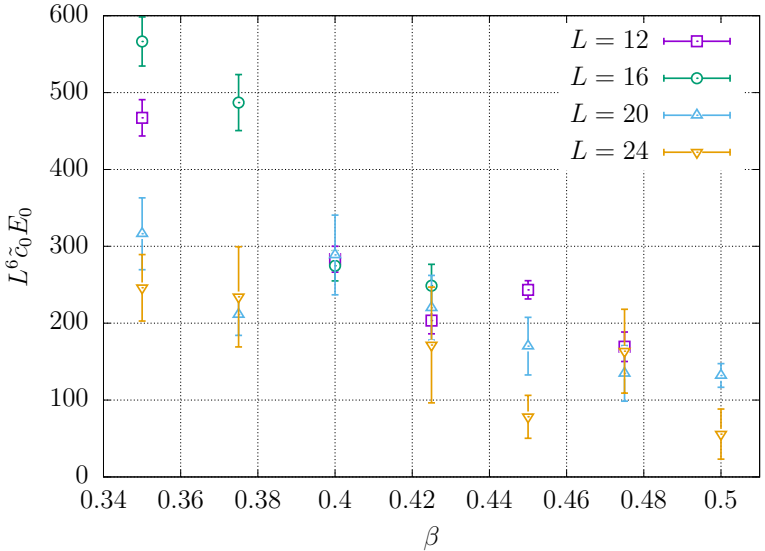


Fermion ground state energy



Another data collapse?





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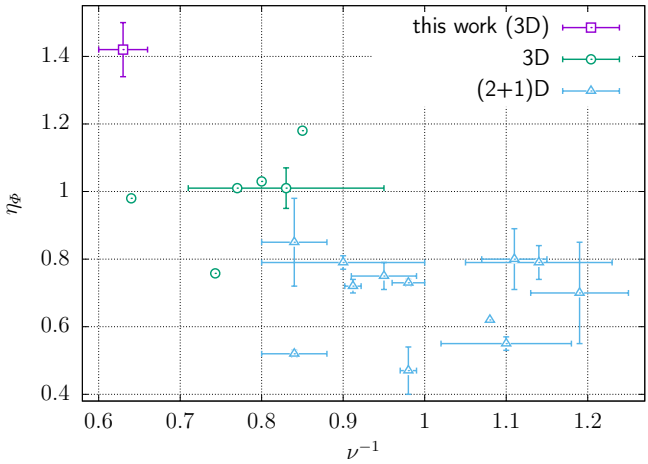

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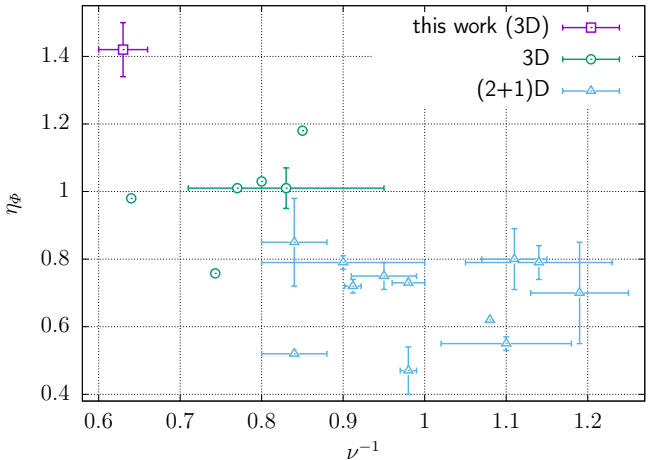
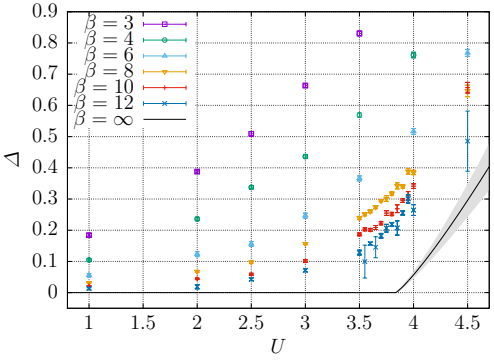


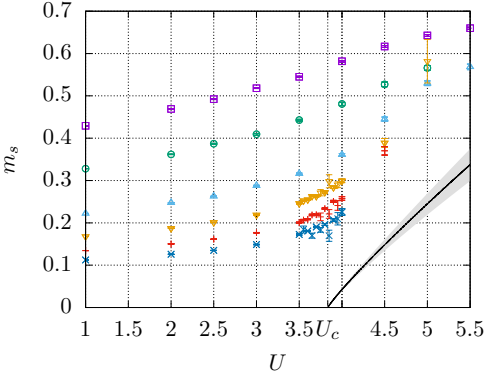
How did we make such a mess?

How can we clean it up?

$T = 0$ quantum phase transition [JO, Berkowitz + *PRB* **102** (2020), *PRB* **104** (2021)]

$$H = -\sum_{s=\uparrow\downarrow} \vec{c}_s^{\dagger} \kappa \vec{c}_s + \frac{1}{2} U \sum_x q_x^2$$





Domain wall fermion action

$$
\begin{aligned}
S &= S_{\text{kin}} + S_{\text{int}} + S_\phi = \bar{\Psi}\mathcal{M}_{\text{DW}}\Psi + S_\phi\,, \\
S_{\text{kin}} &= \sum_{x,y}\sum_{s,s'} \bar{\Psi}(x,s)[\delta_{s,s'}D_W(x|y) + \delta_{x,y}D_3(s|s')]\Psi(y,s'), \\
S_{\text{int}} &= \sum_x \vec{\phi}(x)\cdot[\bar{\Psi}(x,L_s)\mathcal{P}_-\vec{\tau}\Psi(x,1) + \bar{\Psi}(x,1)\mathcal{P}_+\vec{\tau}\Psi(x,L_s)], \\
S_\phi &= \beta\sum_x \vec{\phi}(x)\cdot\vec{\phi}(x)\,, \\
\psi(x) &= \mathcal{P}_-\Psi(x,1) + \mathcal{P}_+\Psi(x,L_s); \quad \bar{\psi}(x) = \bar{\Psi}(x,L_s)\mathcal{P}_- + \bar{\Psi}(x,1)\mathcal{P}_+, \\
D_W(M)_{x,y} &= -\frac{1}{2}\sum_\mu [(1-\gamma_\mu)\delta_{x+\hat{\mu},y} + (1+\gamma_\mu)\delta_{x-\hat{\mu},y}] + (3-M)\delta_{x,y}
\end{aligned}
$$

Fermion Correlator

$$\mathcal{S}(x,s;y,s') = \langle \Psi(x,s)\bar{\Psi}(y,s')\rangle$$

$$S(x_0) = \frac{1}{4}\sum_{\vec{x}} \mathrm{tr}[\mathcal{P}_{-}\mathcal{S}(0,1;x,L_s) + \mathcal{P}_{+}\mathcal{S}(0,L_s;x,1)]$$

Fit results

L_s	β_c	β_Φ/ν	$1/\nu$	χ^2/dof
8	0.453(9)	1.17(4)	0.634(28)	1.2
16	0.461(5)	1.18(4)	0.629(22)	0.8
24	0.465(11)	1.21(4)	0.633(26)	1.4

$$\eta_\Phi = 2\frac{\beta_\Phi}{\nu} - 1 = 1.42(8)$$

The Truncated Hankel Correlator (THC) algorithm [JO & Urbach *hep-lat*/**2510**.15500]

$$\hat{H} \leftarrow \begin{pmatrix} C(0) & C(1) & \cdots & C(T/2) \\ C(1) & C(2) & \cdots & C(T/2+1) \\ \vdots & \vdots & \ddots & \vdots \\ C(T/2) & C(T/2+1) & \cdots & C(T) \end{pmatrix};$$

$U \cdot D \cdot U^\dagger \leftarrow \hat{H};$

$U_k \leftarrow (u_1, \ldots, u_k)$; `// first` k `eigenvectors`

$M_0 \leftarrow P_0 U_k$; `//` $P_0 = \left(\mathbb{1}_{T/2-1}, 0\right)$ `removes last time slice`

$M_1 \leftarrow P_1 U_k$; `//` $P_1 = \left(0, \mathbb{1}_{T/2-1}\right)$ `removes first time slice`

$X \leftarrow \left(M_0^\dagger M_0\right)^{-1} M_0^\dagger M_1;$

$V \cdot \mathrm{diag}\left(\Lambda_1, \ldots, \Lambda_k\right) \cdot V^{-1} \leftarrow X;$

$E_l \leftarrow -\log \Lambda_l;$

Obtaining the coefficients/matrix elements with THC

$$\chi_{tl} \leftarrow \mathrm{e}^{-E_l t};$$
$$\chi^{-D} \leftarrow \mathrm{diag}\left(\max_t |\chi_{tl}|\right)^{-1}; \qquad \texttt{// split scales for stability}$$
$$\chi^{0} \leftarrow \chi \cdot \chi^{-D};$$
$$\tilde{c}_{\alpha\beta} \leftarrow \chi^{-D}\left(\chi^{0\dagger}\sigma_{\alpha\beta}^{-2}\chi^{0}\right)^{-1}\chi^{0\dagger}\sigma_{\alpha\beta}^{-2}C_{\alpha\beta};$$
$$\tilde{c}_{\alpha,l} \leftarrow \sqrt{\tilde{c}_{\alpha\alpha,l}}\,\frac{\tilde{c}_{\alpha 1,l}}{|\tilde{c}_{\alpha 1,l}|}; \qquad \texttt{// main contribution vector}$$

Bibliography I

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