Quantum electrodynamics in two space and one euclidean time dimensions

A focus on monopoles (arXiv 2608.25143)

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Monopoles

T.T. Wu and C.N. Yang, Nucl. Phys. B107, 365 (1976)

Vector potential

$$A_\phi = \begin{cases} A_\phi^+ = \frac{g}{r}\frac{1-\cos\theta}{\sin\theta} = \frac{g}{r}\tan\frac{\theta}{2} & 0 \le \theta < \frac{\pi}{2} + \delta \\ A_\phi^- = -\frac{g}{r}\frac{1+\cos\theta}{\sin\theta} = -\frac{g}{r}\cot\frac{\theta}{2} & \frac{\pi}{2} - \delta < \theta \le \pi \end{cases}$$

Magnetic field

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{g}{r^2}\hat{r}$$

Y.Kazama, C.N. Yang and A.S. Goldhaber, Phys. Rev. D15, 2287 (1977)

Gauge equivalence in the overlapping region

$$A_\phi^+ - A_\phi^- = \frac{2g}{r\sin\theta} = e^{2ig\phi}\frac{i}{r\sin\theta}\partial_\phi e^{-2ig\phi}$$

Quantization: g is a multiple of a half-integer

Spectrum of the massless two-component Dirac operator in the monopole background

Zero modes have an angular momentum of $|g| - \frac{1}{2}$ and hence a degeneracy of $2|g|$

Non-zero modes have an angular momentum of $p + |g| - \frac{1}{2}$ with p a integer larger than zero.

The energies of the non-zero modes are $\pm\sqrt{p^2 + 2p|g|}$ with a degeneracy of $2(|g| + p)$

Monopoles on the lattice — Compact gauge actions

A.M. Polyakov, Phys. Lett. B59, 82 (1975); Nucl. Phys. B120, 429 (1977)

M. Gopfert and G. Mack, Commun. Math. Phys. 82, 545 (1981) M.Karliner and G. Mack, Nucl. Phys. B225, 371 (1983)

V.F. Muller and W. Ruhl, Z. Phys. C9, 261 (1981)

Compact gauge fields

$$U_\mu(\mathbf{n}) = e^{i\theta_\mu(\mathbf{n})}; \quad -\pi < \theta_\mu(\mathbf{n}) \le \pi$$

Compact plaquettes

$$P_{\mu\nu}(\mathbf{n}) \equiv e^{i\theta_{\mu\nu}(\mathbf{n})} = U_\mu(\mathbf{n})U_\nu(\mathbf{n}+\hat{\mu})U^*_\mu(\mathbf{n}+\hat{\nu})U^*_\nu(\mathbf{n}); \quad -\pi < \theta_{\mu\nu}(\mathbf{n}) \le \pi$$

Wilson gauge action

$$S = \beta \sum_{\mathbf{n}} \sum_{\mu<\nu} \Big[1 - \cos\theta_{\mu\nu}(\mathbf{n})\Big]$$

A theory of integer valued scalar field

Fourier decomposition: $\theta_{\mu\nu}(\mathbf{n}) \to k_{\mu\nu}(\mathbf{n}) \in \mathbf{I}$

Integrals over $\theta_\mu(\mathbf{n})$ results in the constraints $\sum_\nu \Big(k_{\mu\nu}(\mathbf{n}) - k_{\mu\nu}(\mathbf{n}-\hat{\nu})\Big) = 0$

The constraints can be solved by $k_{\mu\nu}(\mathbf{n}) = \sum_\lambda \epsilon_{\mu\nu\lambda}\Big[h(\mathbf{n}) - h(\mathbf{n}-\hat{\lambda})\Big]; \quad h(\mathbf{n}) \in \mathbf{I}\,.$

In the weak coupling limit ($\beta \to \infty$) the action reduces to $S = \dfrac{1}{2\beta}\sum_{\mathbf{n},\mu}\Big(h(\mathbf{n}) - h(\mathbf{n}-\hat{\mu})\Big)^2$

The fields being integers results in a mass gap given by $M = am = \sqrt{8\pi^2\beta}e^{-c\beta}$

Numerical value for c is $0.2527\pi^2$

Analytical methods work better with the Villain action and results in a bound for the string tension that can be written as an equality in the semi-classical limit as $\Sigma = a^2\alpha = \dfrac{8}{\sqrt{2\pi^2\beta}}e^{-c\beta}$

Continuum limit(s)

A. Athenodorou and M. Teper, JHEP 01, 063 (2019), 1811.06280

Mass gap: $M = am = \sqrt{8\pi^2\beta}e^{-c\beta}$ String tension: $\Sigma = a^2\alpha = \frac{8}{\sqrt{2\pi^2\beta}}e^{-c\beta}$ Density of monopoles: $\frac{1}{L_M^3} \sim \Sigma M \sim 16\ e^{-2c\beta}$ $c = 0.2527\pi^2$

Let $\ell = aL$ be the linear size of the periodic box in the continuum

If we wish to keep the mass gap, $m\ell$, fixed, then we see that

$$\lim_{L\to\infty}\Big|_{m\ell} \alpha\ell^2 = \lim_{L\to\infty}\Big|_{m\ell} \frac{\ell^3}{\ell_M^3} = \infty.$$

If we wish to keep the number of monopoles, $\frac{\ell^3}{\ell_M^3}$, fixed, we have

$$\lim_{\beta\to\infty}\Big|_{\frac{\ell^3}{\ell_M^3}} m\ell = 0; \qquad \lim_{\beta\to\infty}\Big|_{\frac{\ell^3}{\ell_M^3}} \alpha^2\ell = \infty.$$

If we wish to keep the string tension, $\alpha\ell^2$, fixed, then we see that

$$\lim_{L\to\infty}\Big|_{\alpha\ell^2} m\ell = \lim_{L\to\infty}\Big|_{m\ell} \frac{\ell^3}{\ell_M^3} = 0.$$

The linear confining potential cannot really be observed since the mass scale is zero.

All we see is a Coulomb potential and the change in potential measured in units of string tension will be infinite.

$$\frac{\delta V_c}{E_\alpha} = \lim_{\beta\to\infty} \frac{\frac{1}{\beta}\ln(r_1/r_2)}{\pi\sqrt{\Sigma}} = \infty$$

We can take the continuum limit where we keep the coupling constant fixed, namely, $\beta = \frac{2L}{\ell}$

$$\lim_{L\to\infty} m\ell = \lim_{L\to\infty} \alpha\ell^2 = \lim_{L\to\infty} \frac{\ell^2}{\ell_M^3} = 0,$$ This is just the same as using a non-compact gauge action

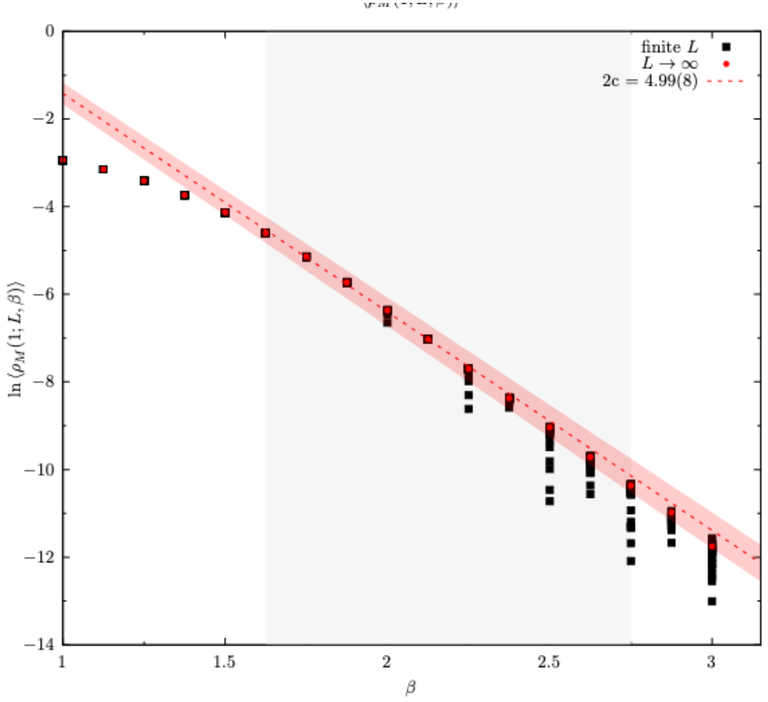
Monopole density

T.A. DeGrand and D. Toussaint, Phys. Rev. D22, 2478 (1980).

$$2\pi M(\mathbf{n}) = \sum_f \theta_f(\mathbf{n}); \qquad -\pi < \theta_f(\mathbf{n}) \leq \pi$$

The sum is over the six faces of a cube and θ_f is the value of the plaquette angle on a face

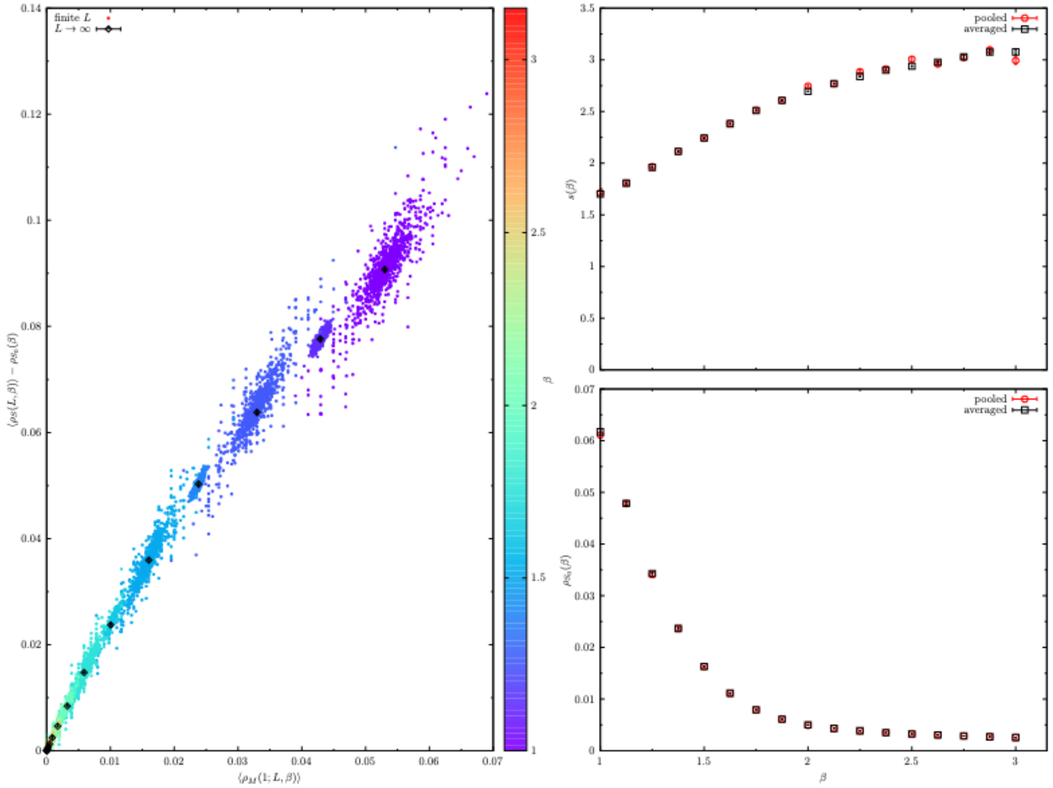
We used one step HYP smearing on the gauge fields before performing the measurement



Density of monopoles with unit charge

Action density versus monopole density

$$\rho_S(L,\beta) = s(\beta)\rho_M(1;L,\beta) + \rho_{S_0}(\beta)$$

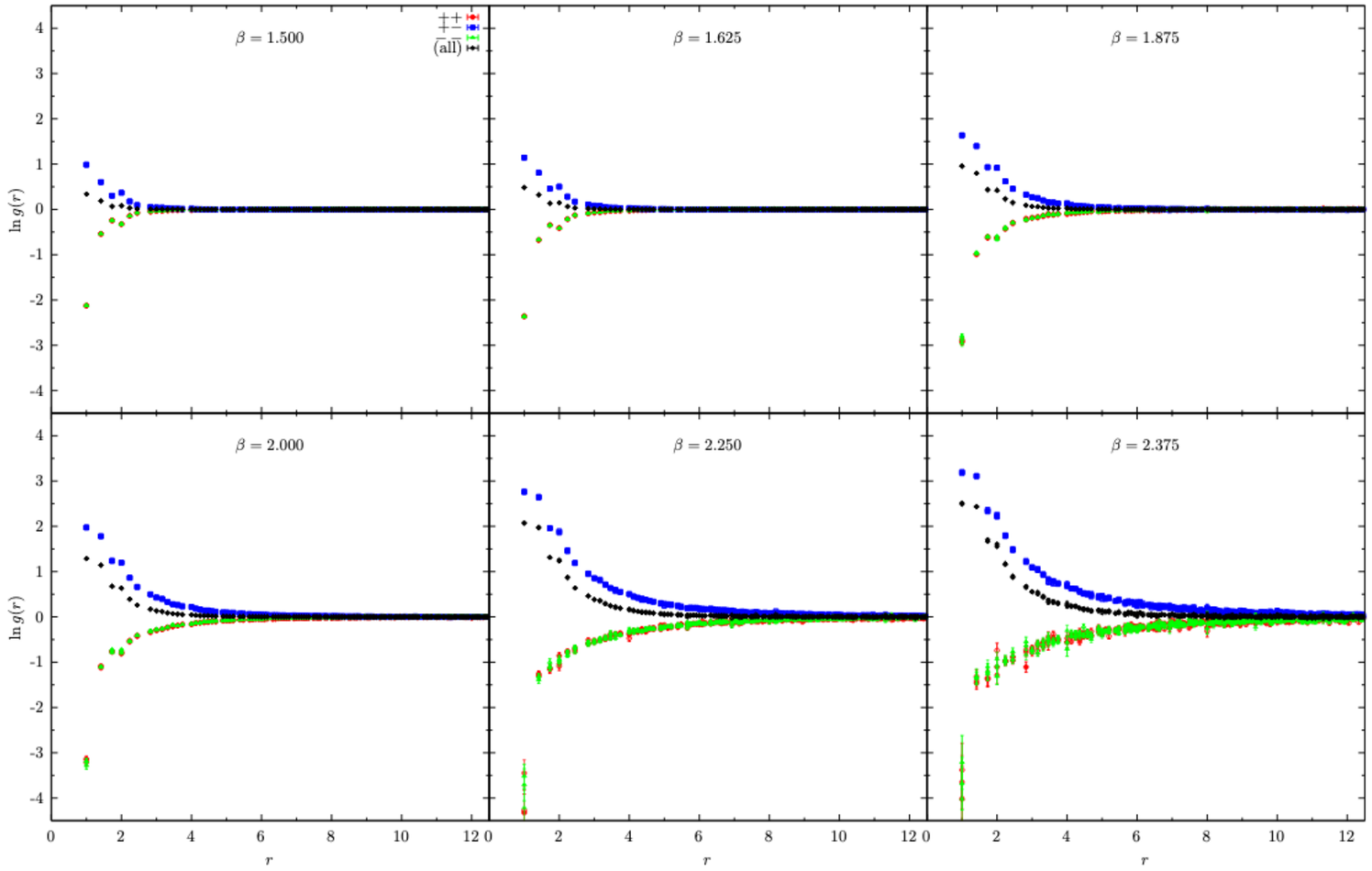


Evidence for short range attraction and repulsion between monopoles

Let $f(r)$ be the fraction of monopoles separated by a distance r

If we assume that monopoles and anti-monopoles form a random gas with no interactions, we can calculate $f_d(r)$.

$$g(r) = \frac{f(r)}{f_d(r)}$$



Wilson fermions and parity

Wilson Dirac operator for two-component fermions $$D_w(U) = 3 + m_w - \sum_{\mu=1}^{3} \Big[P_{-\mu} T_\mu(U) + P_{+\mu} T_\mu^\dagger(U) \Big]; \quad P_{\pm\mu} = \frac{1 \pm \sigma_\mu}{2}$$

Parity: $$(P\psi)(\mathbf{n}) = \psi^p(\mathbf{n}) = \psi(\mathbf{L} - \mathbf{n}); \qquad \mathbf{L} = (L, L, L) \qquad P T_\mu(U) P = T_\mu^\dagger(U^p) \quad \Rightarrow \quad U_\mu^p(\mathbf{n}) = e^{i\theta^p(\mathbf{n})}; \qquad \theta_\mu^p(\mathbf{n}) = -\theta_\mu(\mathbf{L} - \mathbf{n} - \hat{\mu})$$

$$P D_w(U) P = D_w^\dagger(U^p)\,.$$

A pair of fermions can be made parity invariant $$W(U) = \begin{pmatrix} 0 & D_w(U) \\ -D_w^\dagger(U) & 0 \end{pmatrix} \quad \Rightarrow \quad W(U^p) = \begin{pmatrix} 0 & P D_w^\dagger(U) P \\ -P D_w(U) P & 0 \end{pmatrix}.$$

The entire Wilson term goes to its negative under $D_w \to -D_w^\dagger$

$$W \begin{pmatrix} u_k \\ d_k \end{pmatrix} = i\Lambda_k \begin{pmatrix} u_k \\ d_k \end{pmatrix} \quad \Rightarrow \quad D_w D_w^\dagger u_k = \Lambda_k^2 u_k; \quad D_w^\dagger D_w d_k = \Lambda_k^2 d_k; \quad \Rightarrow \quad W \begin{pmatrix} \frac{D_i d_k}{\pm i\Lambda_k} \\ d_k \end{pmatrix} = \pm\, i\Lambda_k \begin{pmatrix} \frac{D_i d_k}{\pm i\Lambda_k} \\ d_k \end{pmatrix}; \quad \Lambda_k > 0$$

We expect the spectrum of W to be symmetric under $m_w \to -m_w$ for continuum like gauge fields

Background gauge field with a single well separated monopole and anti-monopole

$$B_{23}(\mathbf{n}) = B_{31}(\mathbf{n}) = 0; \qquad B_{12}(\mathbf{n}) = \begin{cases} 2\pi Q & n_1 = n_2 = \frac{L}{2}; \quad 1 \le n_3 \le s \\ 0 & \text{otherwise,} \end{cases}; \qquad \mathbf{n} \in [1, L].$$

Monopole charge: $Q(\mathbf{n}) = \frac{1}{4\pi} \sum_{\mu\nu\rho} \epsilon_{\mu\nu\rho} \left[B_{\mu\nu}(\mathbf{n} + \hat{\rho}) - B_{\mu\nu}(\mathbf{n}) \right] = Q \delta_{n_1, \frac{L}{2}} \delta_{n_2, \frac{L}{2}} \left[\delta_{n_3, 0} - \delta_{n_3, s} \right]$

Cannot find a gauge field that solves this background

Minimize: $S_g = \sum_{\mathbf{n}} \sum_{\mu<\nu=1}^{3} \left[F_{\mu\nu}(\mathbf{n}) - B_{\mu\nu}(\mathbf{n}) \right]^2; \qquad F_{\mu\nu}(\mathbf{n}) = \theta_\mu(\mathbf{n}) + \theta_\nu(\mathbf{n} + \hat{\mu}) - \theta_\mu(\mathbf{n} + \hat{\nu}) - \theta_\nu(\mathbf{n})$

$$\theta_\mu(\mathbf{n}) = \sum_{\mathbf{k}} \tilde{\theta}_\mu(\mathbf{k}) e^{i \frac{2\pi \mathbf{k} \cdot \mathbf{n}}{L}}; \qquad \tilde{\theta}_\mu(\mathbf{k}) = \frac{\tilde{J}_\mu(\mathbf{k})}{\hat{k}^2}; \qquad \hat{k}^2 = 4 \sum_{\mu=1}^{2} \sin^2 \frac{\pi k_\mu}{L}$$

Solution:

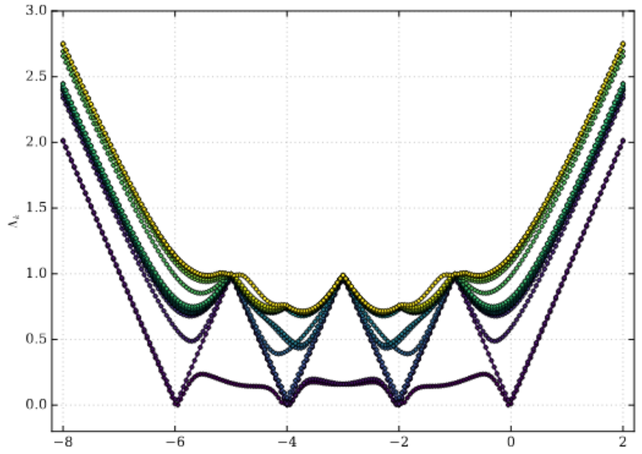
$$J_\mu(\mathbf{n}) = \sum_{\nu} \left[B_{\mu\nu}(\mathbf{n}) - B_{\mu\nu}(\mathbf{n} - \hat{\nu}) \right]; \qquad \tilde{J}_\mu(\mathbf{k}) = \frac{1}{L^3} \sum_{\mathbf{n}} J_\mu(\mathbf{n}) e^{-i \frac{2\pi \mathbf{k} \cdot \mathbf{n}}{L}}$$

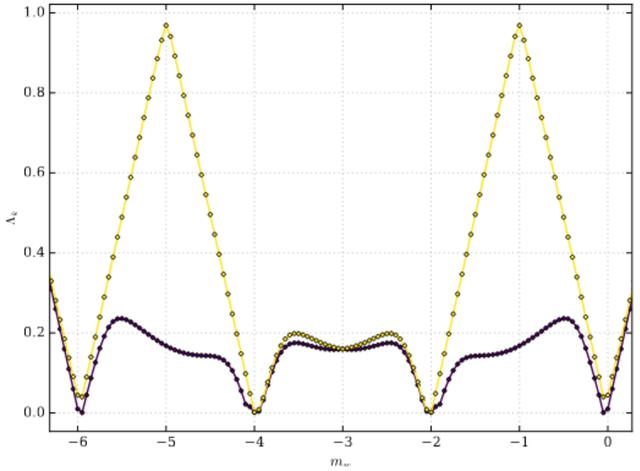
This background is the same under parity

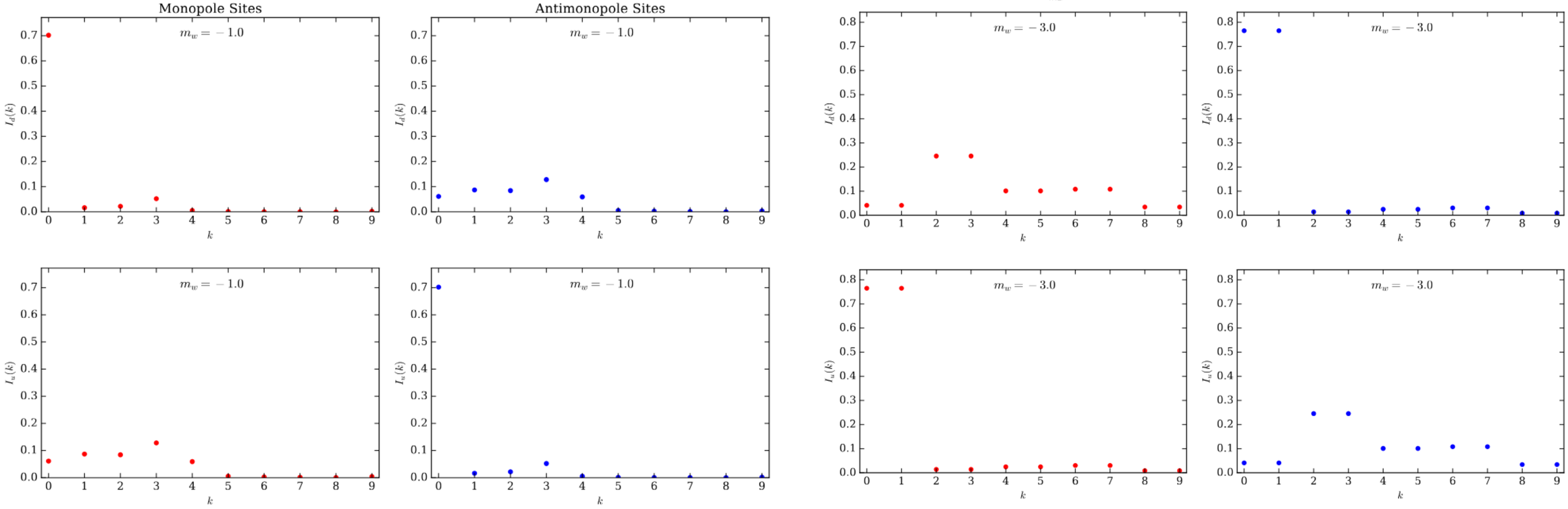
N. Karthik and R. Narayanan, Phys. Rev. D100 (2019) 9, 094501; 1908.05284

Wilson-Dirac spectrum in the background gauge field with a single well separated monopole and anti-monopole

$$W \begin{pmatrix} u_k \\ d_k \end{pmatrix} = i\Lambda_k \begin{pmatrix} u_k \\ d_k \end{pmatrix}$$

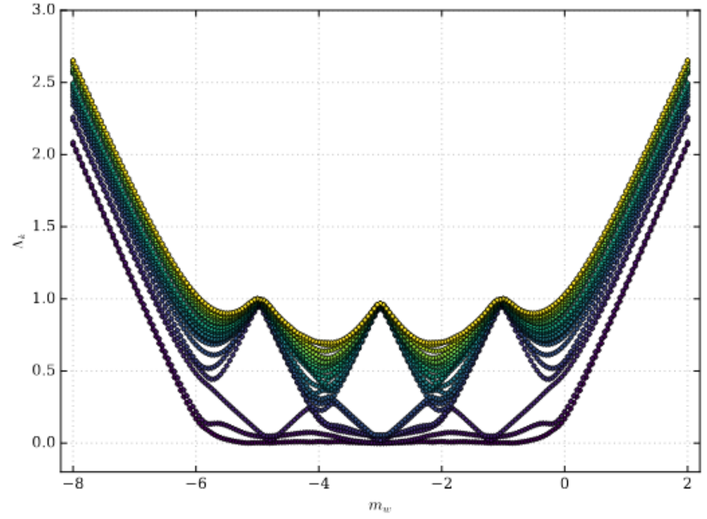


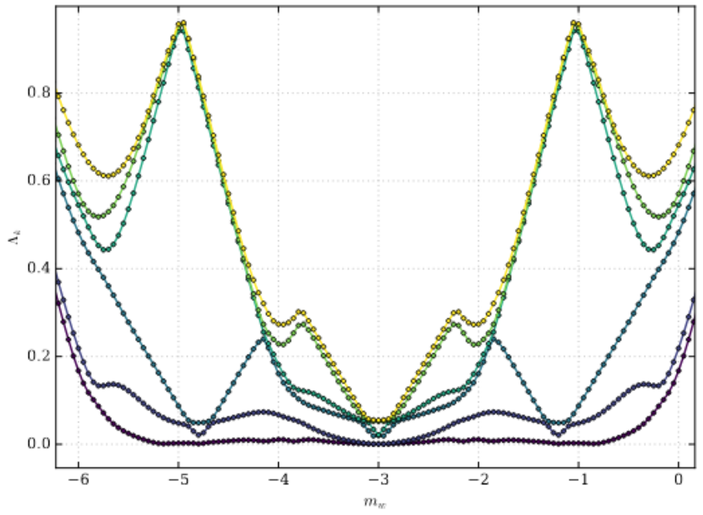


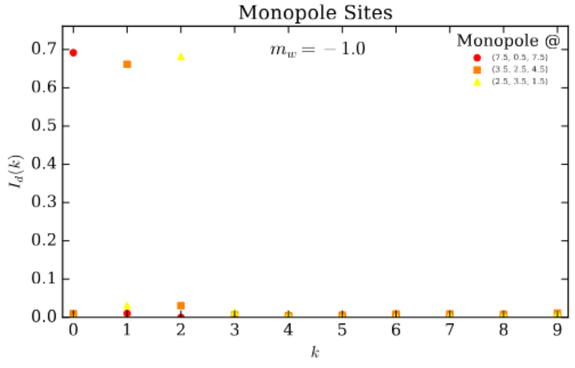


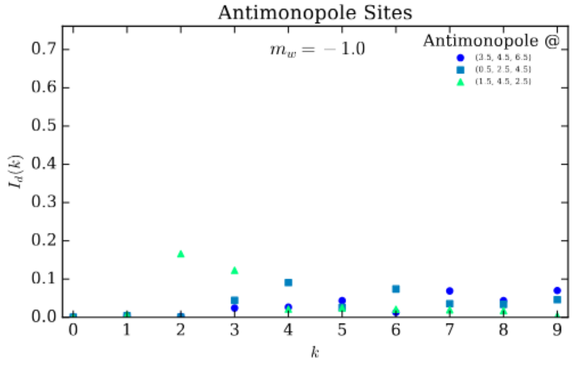
$$I_u(k) = \sum_{\mathbf{n}\in\text{cube}} u_k^\dagger(\mathbf{n}) u_k(\mathbf{n}); \qquad I_d(k) = \sum_{\mathbf{n}\in\text{cube}} d_k^\dagger(\mathbf{n}) d_k(\mathbf{n})$$

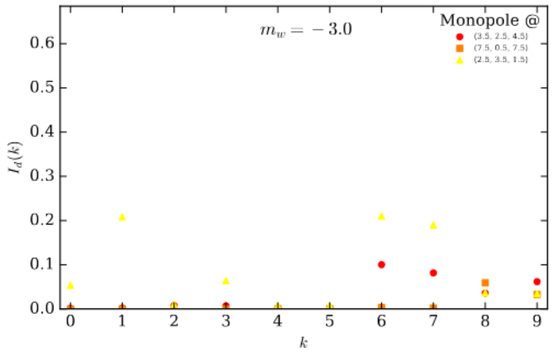
Wilson-Dirac spectrum in a thermalized background at $\beta = 1.7$ with three pairs of monopoles and anti-monopoles

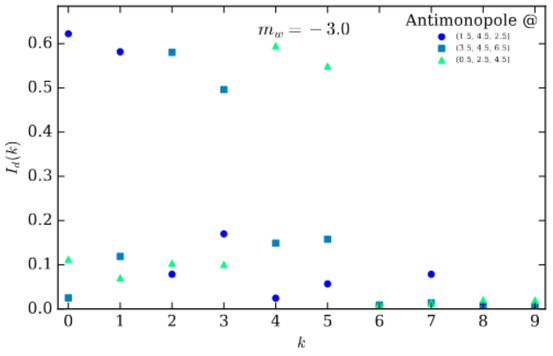


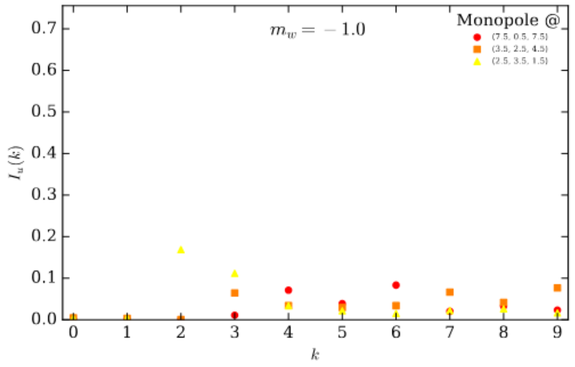


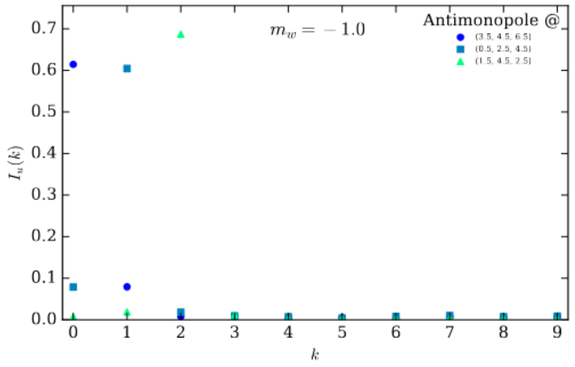


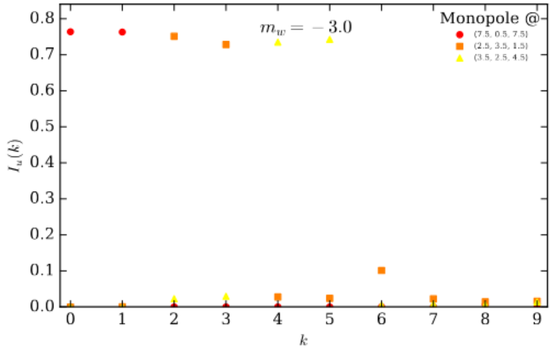


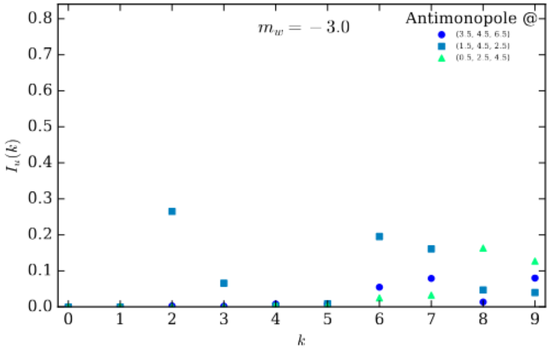




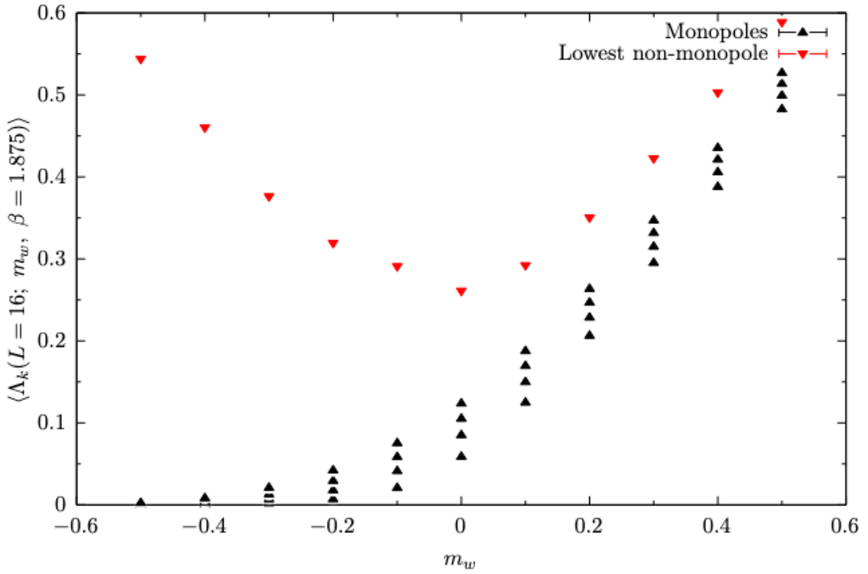




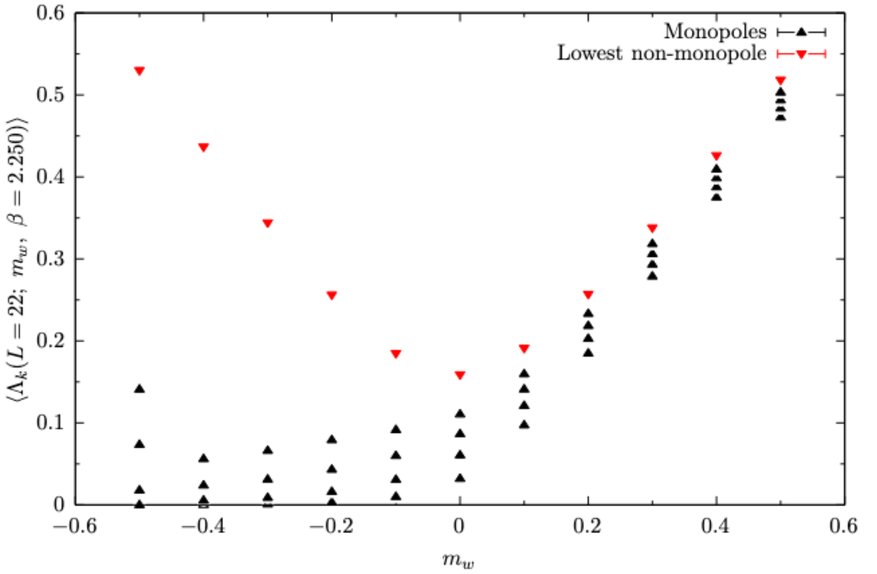




Anomalous behavior of the Wilson-Dirac spectrum

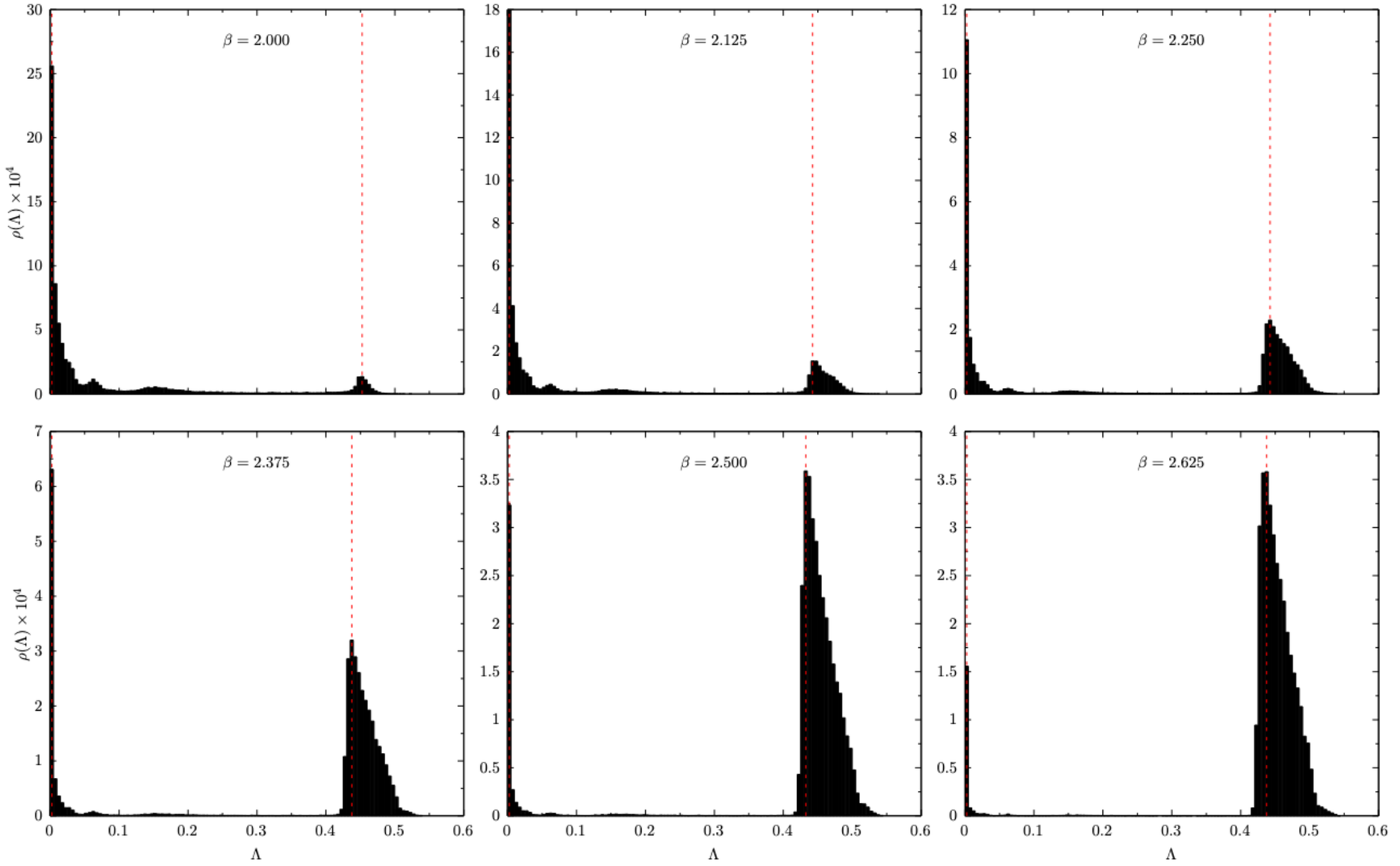


(a) $\beta = 1.875, L = 16$



(b) $\beta = 2.250, L = 22$

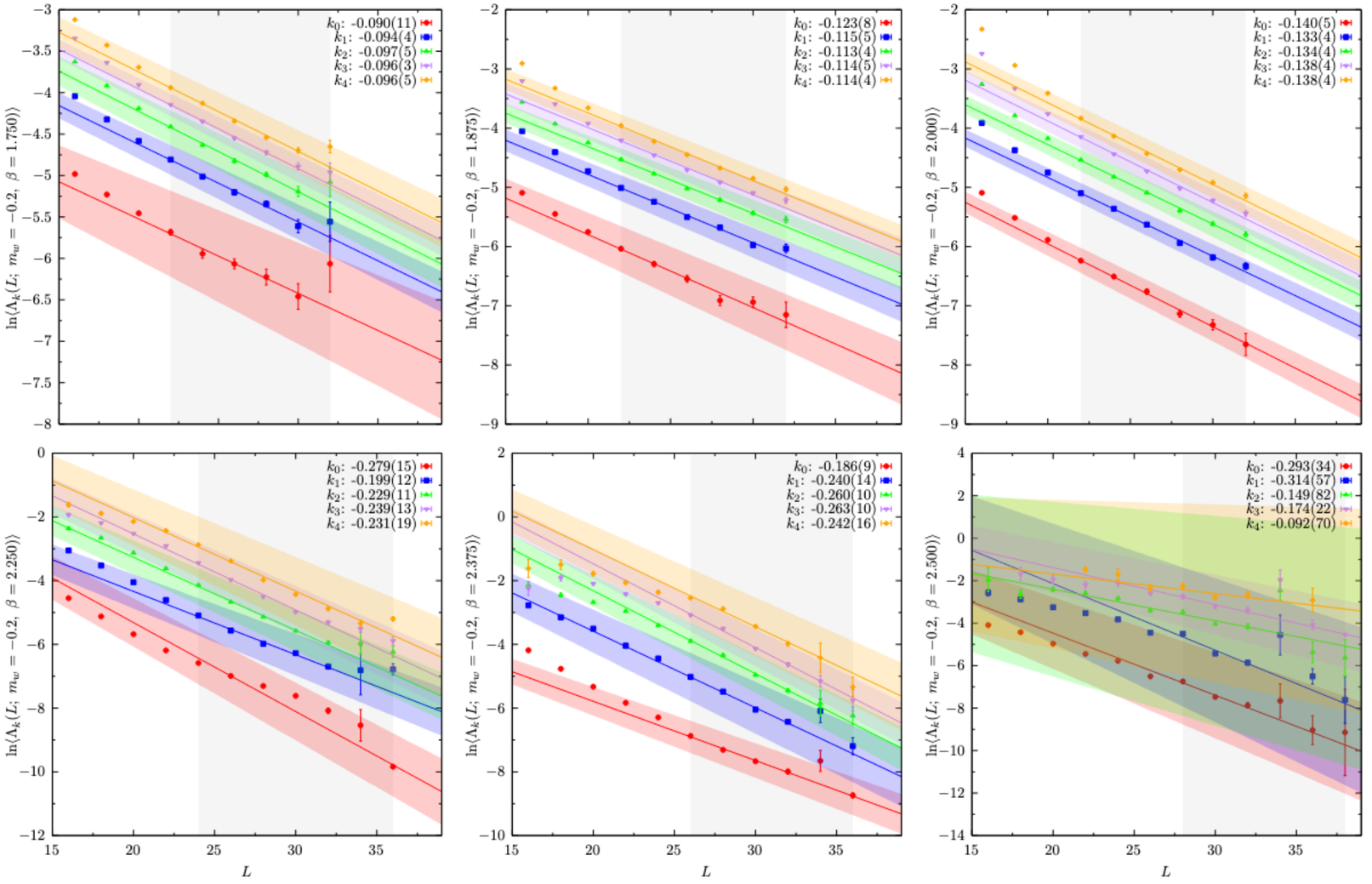
Anomalous behavior of the Wilson-Dirac spectrum



$$m_w = -0.4$$

Behavior of the almost zero eigenvalues in the Wilson-Dirac spectrum

$$\langle \Lambda_k(L; m_w, \beta) \rangle = c_{k,\Lambda}(m_w, \beta) e^{-M_\Lambda(m_w,\beta)L}$$

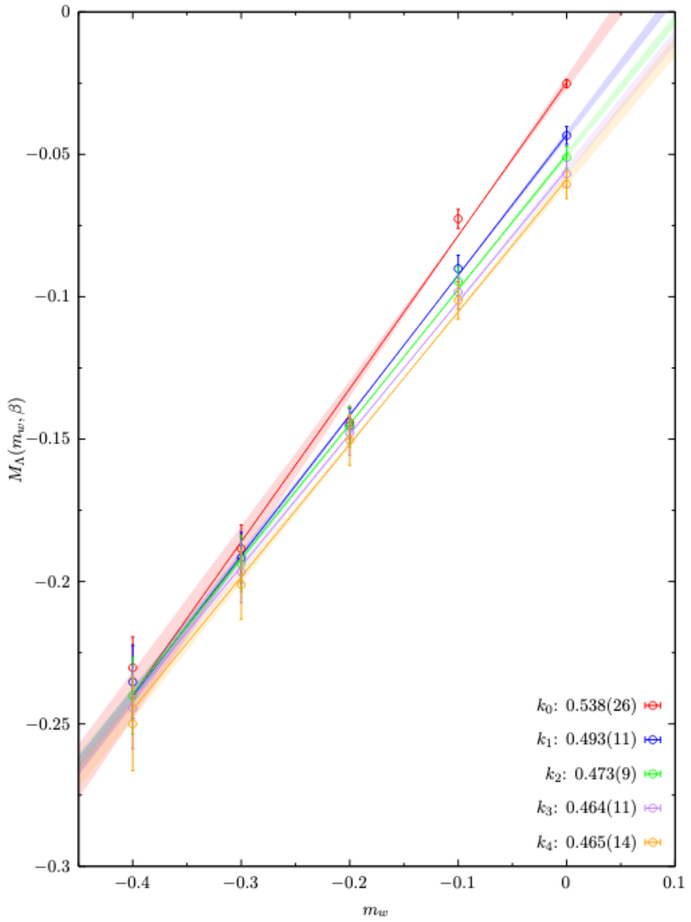


Five smallest eigenvalues at eight different couplings with $m_w = -0.2$

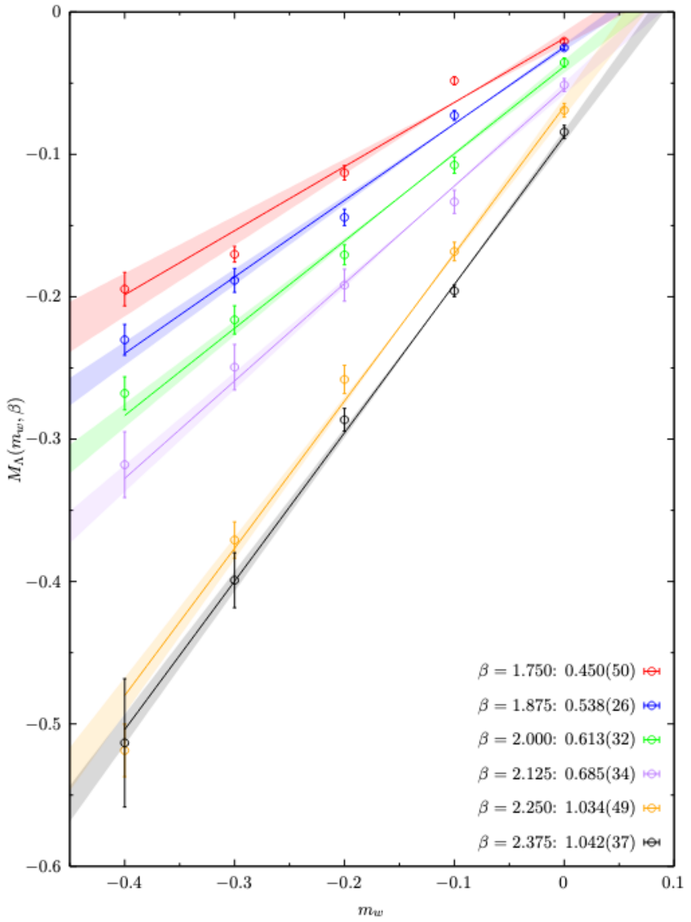
More on the behavior of the almost zero eigenvalues in the Wilson-Dirac spectrum

$$\langle \Lambda_k(L; m_w, \beta) \rangle = c_{k,\Lambda}(m_w, \beta) e^{-M_\Lambda(m_w,\beta)L}$$

$$M_\Lambda(m_w, \beta) = s_\Lambda(\beta) m_w + b_\Lambda(\beta)$$

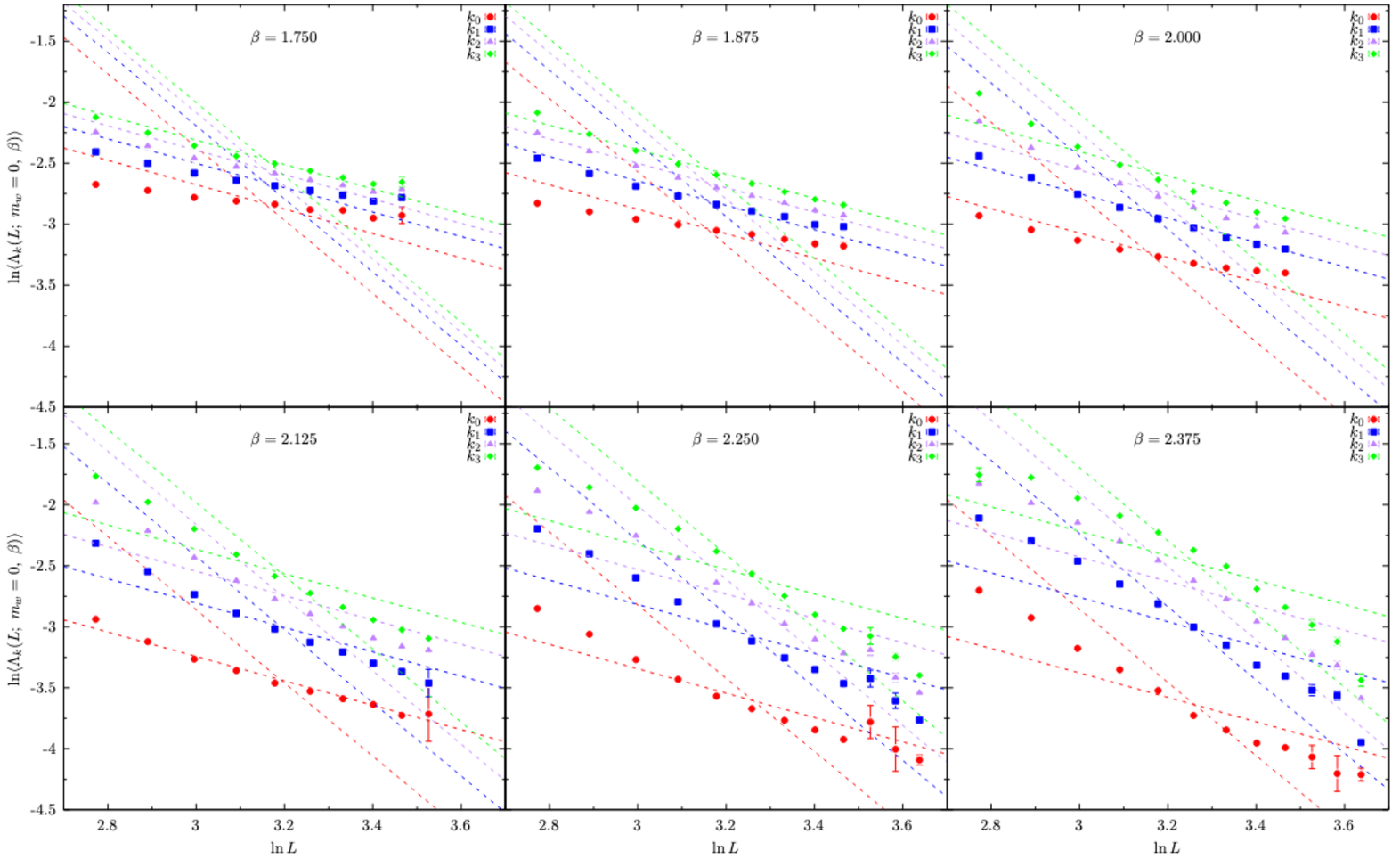


Mode independence



Linear dependence in m_w which increases with β

Behavior of the smallest eigenvalues at $m_w = 0$



The dashed lines correspond to $\frac{1}{L}$ (free field) and $\frac{1}{L^3}$ (condensate) behavior

Conclusions

Presence of monopoles in three dimensional gauge fields results in an anomalous behavior in the spectrum of the Wilson-Dirac operator for a pair of fermions that cancel the parity anomaly

The expected symmetry when the sign of the fermion mass is switched is violated and results in a violation of flavor symmetry

The cause is the singular nature of the monopoles that result in lattice like behavior of the Wilson mass term whose effect is expected to vanish in the continuum limit

The spectrum of a pair of overlap fermion operators that cancels the parity anomaly will have a symmetry around the massless point

The presence of very small eigenvalues of the Wilson-Dirac operator will have serious technical implications for the overlap Dirac operator and even question the order of limits in β and L.

Thanks to the organizers for the invitation