High order perturbative calculations for Gross-Neveu-Yukawa systems

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in collaboration with

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- apply available technology to critical properties of condensed matter models
- Generally interested in results in 3d but
  $\hookrightarrow$ perturbatively only accessible in 2d or 4d
- Calculate anomalous dimensions/critical exponents of the ren. group equations in 2d and/or 4d and extrapolate to 3d

Models

- Gross-Neveu-Yukawa [Zerf,Mihaila,PM,Herbut,Scherer '17]
  - chiral Ising model
  - chiral XY model
  - chiral Heisenberg model
- QED_3-Gross-Neveu-Yukawa [Zerf,PM,Boyack,Maciejko '18]
- Néel algebraic spin liquid [Zerf,Boyack,PM,Gracey,Maciejko '19]
- Abelian Higgs model [Ihrig,Zerf,PM,Herbut,Scherer '19]
- lattice quantum electrodynamics [Zerf,Boyack,PM,Gracey,Maciejko '20]

In short: Models with interactions between scalars, fermions and photons

Problem

Model: Gross-Neveu-Yukawa

renormalised Lagrangian

$$\mathcal{L} = Z_\psi \bar{\psi} \partial\!\!\!/ \psi + Z_{\phi\bar{\psi}\psi} y \mu^{\frac{\epsilon}{2}} \phi \bar{\psi} \psi + \frac{1}{2} \phi (Z_{\phi^2} m^2 - Z_\phi \partial_i^2) \phi + Z_{\phi^4} \lambda \mu^\epsilon \phi^4$$

with $O(N)$ symmetry

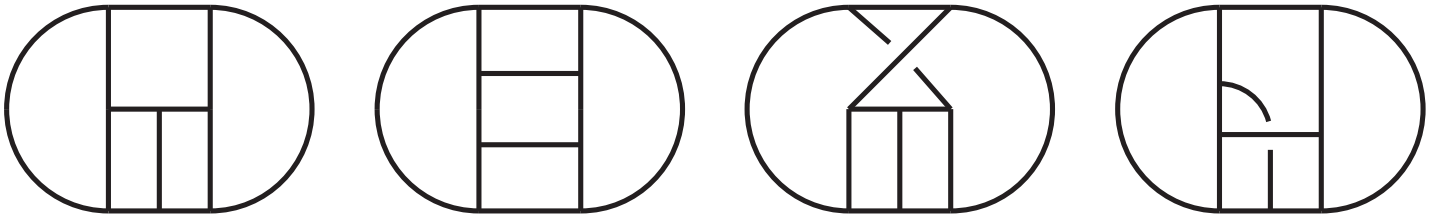
1. calculate corresponding β functions and anomalous dimensions in 4 dimensions ($d = 4 - \epsilon$)
2. extrapolate to 3 dimensions

Setup

Use computational setup similar to calculation of QCD β function @ five loops

[Luthe,Maier,Marquard,Schröder]

- isolate ultraviolet divergences by introducing an auxiliary mass
- expand required n-point functions in external momenta
- reduce to master integrals NEW using finite fields
- 5-loop fully massive tadpoles need to be calculated
  $\Rightarrow$ 4 top-level families, high-precision numerical results [Luthe '15]



- use PSQL at the end to obtain result in terms of zetas

Step-by-stop: Diagram and amplitude generation

- Gives the Lagrangian and the corresponding Feynman rules the required diagrams can easily be generated, e.g. using `qgraf` [Nogueira]

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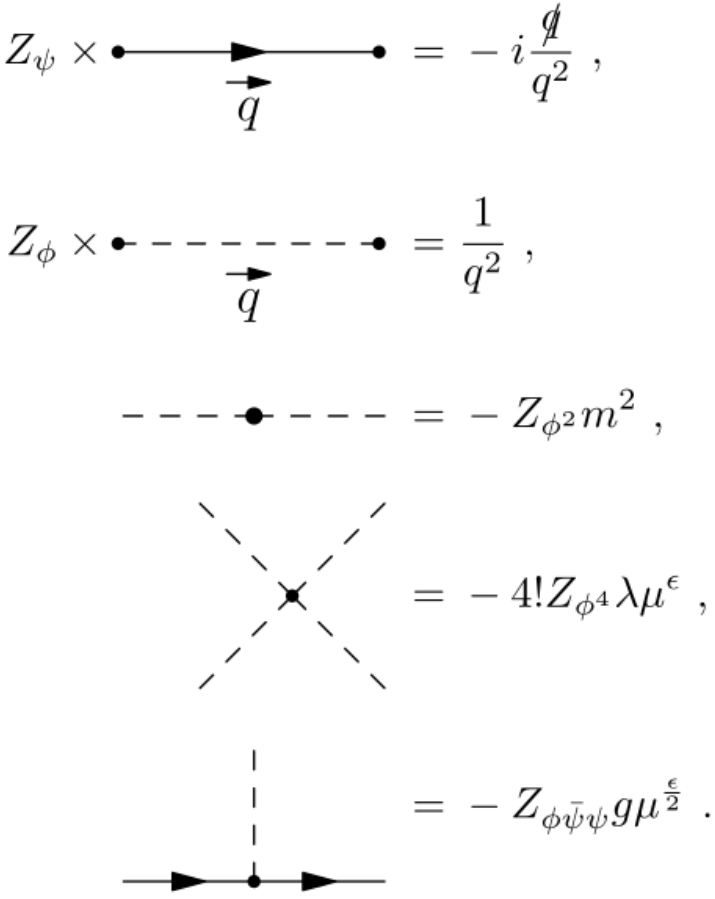
- Gives the Lagrangian and the corresponding Feynman rules the required diagrams can easily be generated, e.g. using `qgraf` [Nogueira]
- The Feynman rules can be inserted using computer algebra programs,



e.g. `FORM` [Vermaseren]

Step-by-stop: Simplification and IR singularities

- In principle, up to four-point functions need to be calculated and their singularities extracted
  $\hookrightarrow$ not really feasible at high-loop order
- way out: UV problem, any kinematic point possible $\rightarrow p_i = 0$
  $\hookrightarrow$ potentially introduces IR singularities
- either use
  - R^* operation to subtract IR singularities
    $\rightarrow L-1$-loop masslees propagators
  - introduce an auxiliary mass to regularize IR singularities
    $\rightarrow L$-loop massive tadpoles

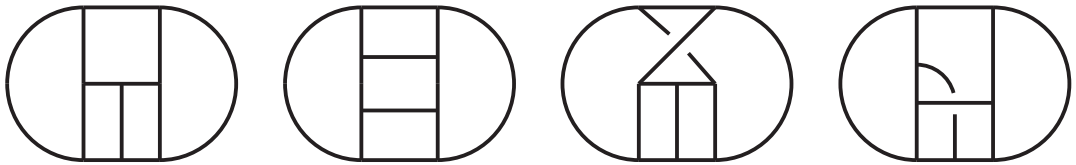
$$\mathcal{D}_\psi \equiv \underset{\vec{q}}{\bullet\!\!\longrightarrow\!\!\bullet} = -i\frac{\not{q}}{q^2+M^2}(1 - i\delta Z_\psi \not{q}\mathcal{D}_\psi),$$

$$\mathcal{D}_\phi \equiv \underset{\vec{q}}{\bullet\!\!- - - -\!\!\bullet} = \frac{1}{q^2+M^2}(1 - q^2\delta Z_\phi \mathcal{D}_\phi - M^2 Z_{M^2}\mathcal{D}_\phi).$$

expand in external momenta

Step-by-stop: Integration-by-parts

- To facilitate further processing the resulting individual Feynman integrals need to be sorted into families according to their numerators. This can be done using graph alogithms.



- Standard procedure is to use Integration-by-parts

$$0 = \int (\Pi_i dk_i) \frac{\partial}{\partial k^\mu} \frac{1}{D_1^{n_1} D_2^{n_2} \cdots D_N^{n_N}}$$

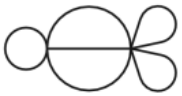
  to drastically reduce the number of integrals needed
  $\mathcal{O}(1,000,000) \to \mathcal{O}(10)$
- implemented in some variation of Laporta's algorithm:
  $\hookrightarrow$ generate large system of equations and solve it
- algebraic operations on rational functions very expensive
  $\hookrightarrow$ supplement by finite fields methods

Massive Tadpoles – 5 loop

$t = 5$:

31744

$t = 6$:



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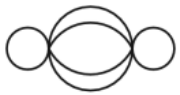

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$t = 7$:

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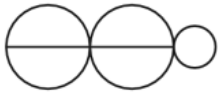

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$t = 8$:

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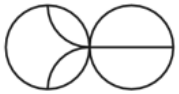

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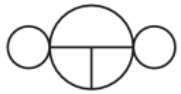

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Massive Tadpoles – 5 loop

$t = 9$:

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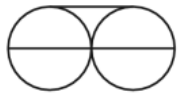
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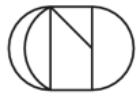
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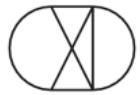
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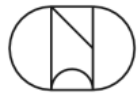
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$t = 10$:

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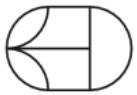
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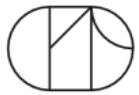
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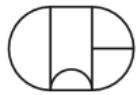
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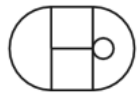
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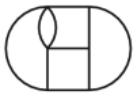
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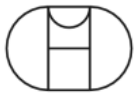
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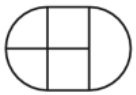
$t = 11$: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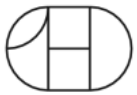
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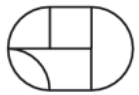
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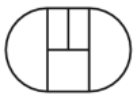
$t = 12$: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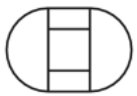
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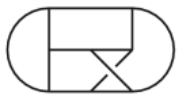
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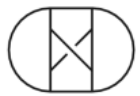
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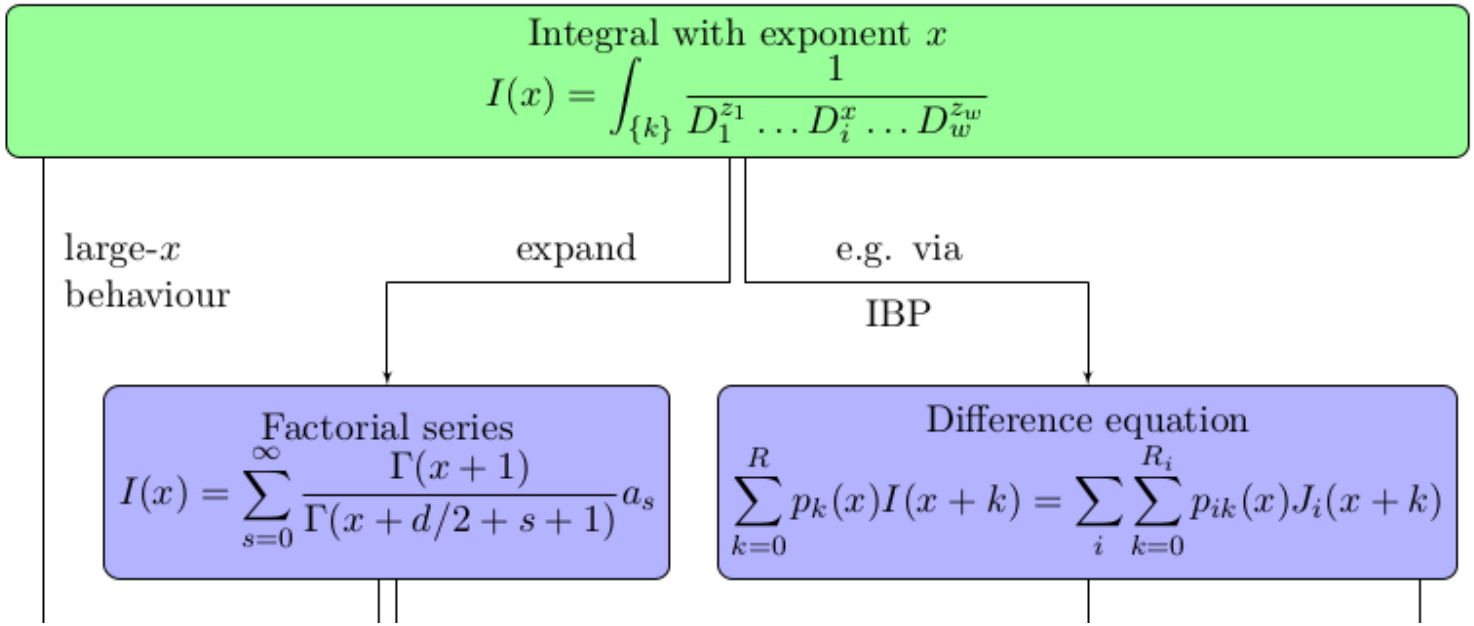
[Luthe '15]

Calculation of master integrals



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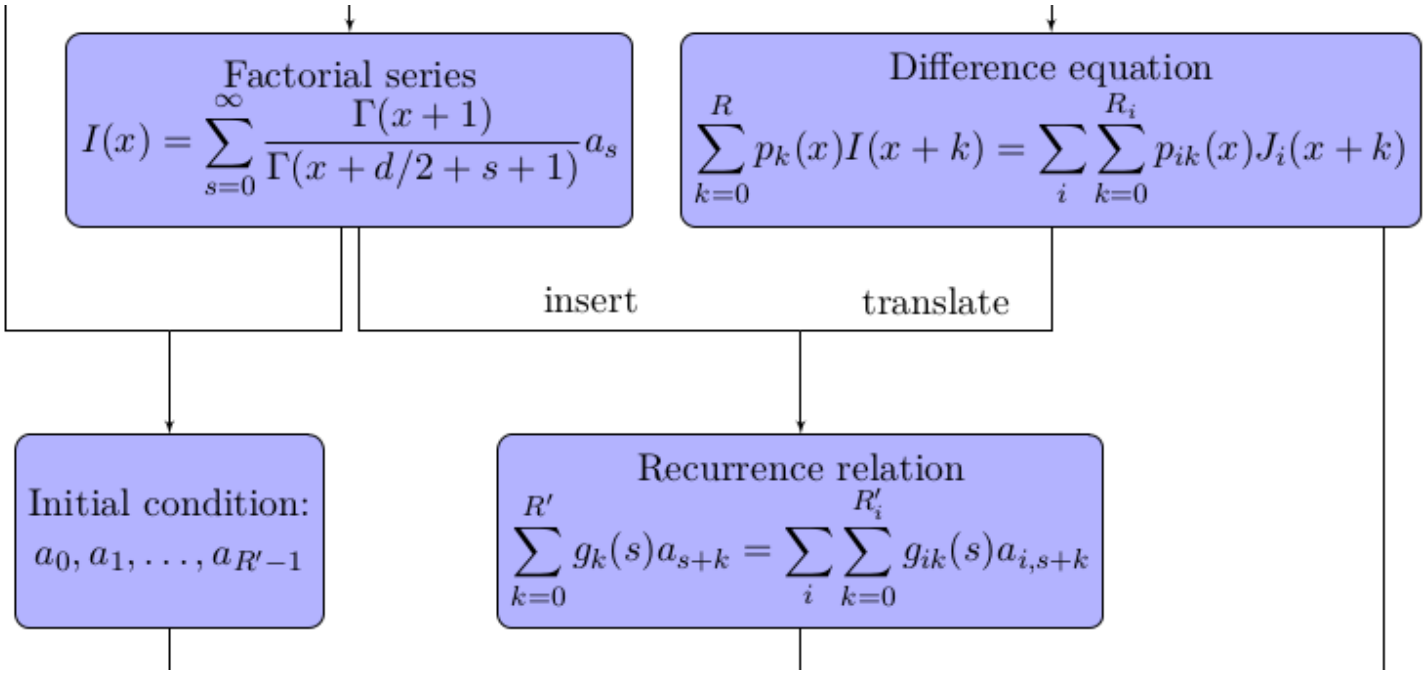

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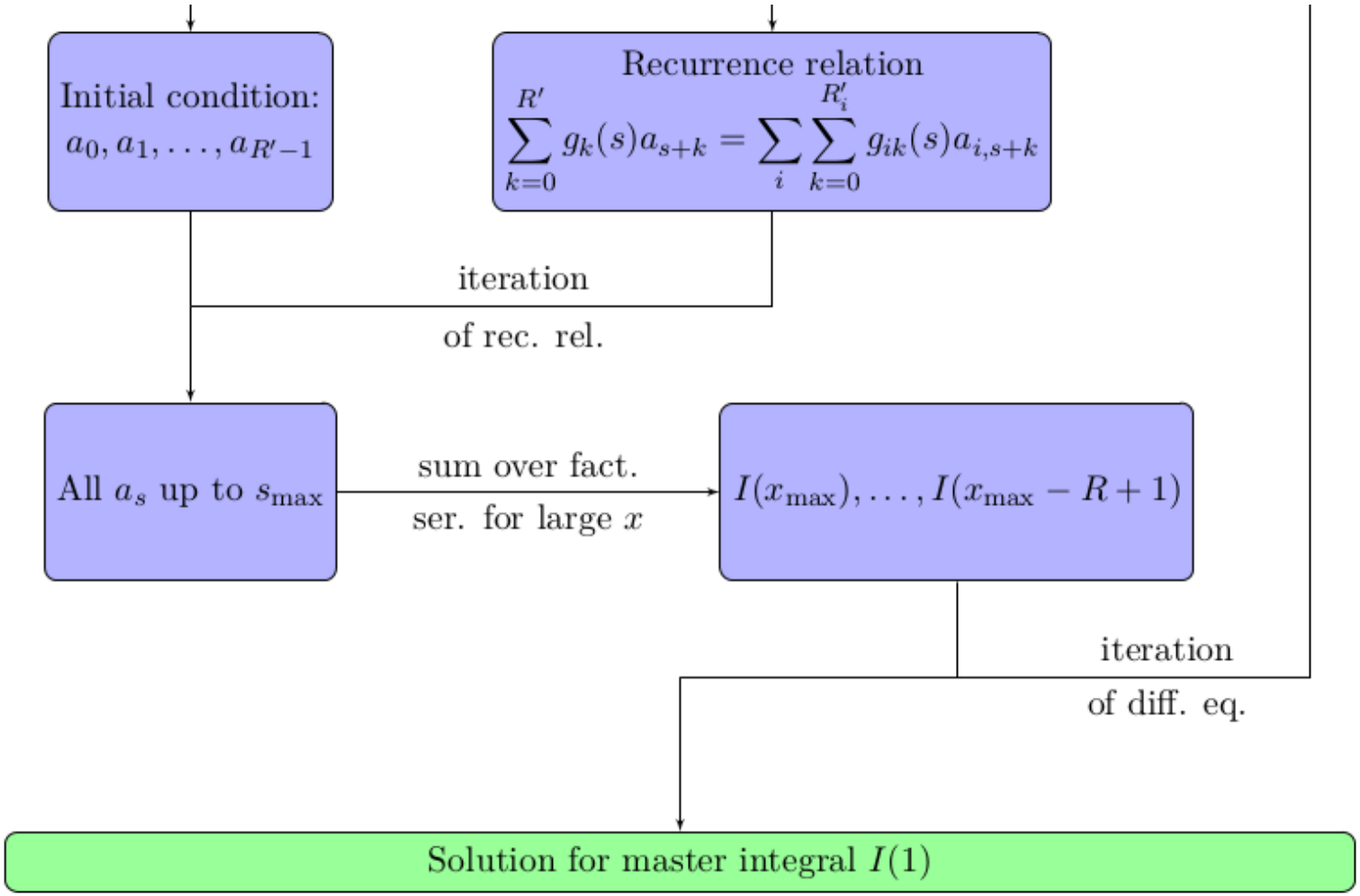

Results

$$\beta_\lambda(\lambda, y) = \frac{d\lambda}{d \ln \mu}, \quad \beta_y(\lambda, y) = \frac{dy}{d \ln \mu}$$

$$\beta_y = -\epsilon y + \sum_{L=1}^{5} \beta_y^{(L)}, \quad \beta_\lambda = -\epsilon \lambda + \sum_{L=1}^{5} \beta_\lambda^{(L)}$$

with one-loop coefficients

$$\beta_\lambda^{(1)} = 36\lambda^2 + 4\lambda N y - N y^2$$

$$\beta_y^{(1)} = 2 N y^2 + 3 y^2$$

value of N corresponds to different physical scenarios

- $N = 2$: graphene on a hexagonal lattice
- $N = 1$: spin-less fermions on a honeycomb lattice
- $N = 5$: for comparison with other methods

Cmp. with large N behaviour

For $d = 2\mu$

$$
\begin{aligned}
\eta_\psi \;=\; & \frac{\eta_1}{N} + \left(\frac{1}{2\mu} + \frac{1}{(\mu-1)^2} + \frac{1}{2(\mu-1)} + \frac{3}{(\mu-2)} + \frac{4}{(2\mu-3)} + \frac{(2\mu-1)}{(\mu-1)} \hat{\Psi}(\mu) \right) \frac{\eta_1^2}{N^2} \\
& + \left(\frac{1}{2\mu^2} - \frac{9}{2} - \frac{7}{2\mu} - \frac{5}{4(\mu-1)^4} - \frac{3}{(\mu-1)^3} - \frac{51}{4(\mu-1)^2} - \frac{79}{4(\mu-1)} + \frac{9}{(\mu-2)^2} + \frac{153}{4(\mu-2)} + \frac{16}{(2\mu-3)^2} \right. \\
& \quad - \frac{52}{(2\mu-3)} - \mu + \left(6 + \frac{5}{4(\mu-1)^2} - \frac{3}{(\mu-1)} + \frac{12}{(\mu-2)} + \frac{27}{2(2\mu-3)} + \frac{1}{4}\mu \right) \hat{\Theta}(\mu) + \frac{3\mu^2}{(\mu-1)} \hat{\Psi}(\mu)\hat{\Theta}(\mu) \\
& \quad + \left(\frac{3}{2\mu} - \frac{3}{2} + \frac{1}{(\mu-1)^3} + \frac{2}{(\mu-1)^2} - \frac{22}{(\mu-1)} + \frac{27}{(\mu-2)} + \frac{48}{(2\mu-3)} - \mu \right) \hat{\Psi}(\mu) + \frac{3\mu^2}{2(\mu-1)} \Xi(\mu)\hat{\Theta}(\mu) \\
& \quad \left. + \frac{(2\mu-1)^2}{2(\mu-1)^2} \left(\hat{\Phi}(\mu) + 3\hat{\Psi}^2(\mu) \right) \right) \frac{\eta_1^3}{N^3} \;+\; \mathcal{O}\left(\frac{1}{N^4} \right) \qquad \text{(A1)}
\end{aligned}
$$

$$
\eta_1 \;=\; -\; \frac{\Gamma(2\mu-1)}{2\mu\Gamma^2(\mu)\Gamma(\mu-1)\Gamma(1-\mu)}
$$

$$
\begin{aligned}
\hat{\Psi}(\mu) &= \psi(2\mu-3) \;-\; \psi(1) \;+\; \psi(3-\mu) \;-\; \psi(\mu-1) \\
\hat{\Phi}(\mu) &= \psi'(2\mu-3) \;-\; \psi'(3-\mu) \;-\; \psi'(\mu-1) \;+\; \psi'(1) \\
\hat{\Theta}(\mu) &= \psi'(\mu-1) \;-\; \psi'(1)
\end{aligned}
$$

Fixed points: $N = 2$

Solve

$$\beta_y(y^*, \lambda^*) = \beta_\lambda(y^*, \lambda^*) = 0$$

$$y^* = \left(-\frac{56967\zeta_3^2}{13176688} - \frac{9370620999331\zeta_3}{755093162831616} + \frac{19976417\zeta_4}{1481167296} - \frac{408151805\zeta_5}{5489031744} + \frac{9525\zeta_6}{537824} - \frac{46467\zeta_7}{2151296} - \frac{2263630018850239}{736498993696887456} \right) \epsilon^5 + \left(\frac{53897407\zeta_3}{2221750944} - \frac{165\zeta_4}{19208} + \frac{635\zeta_5}{33614} + \frac{355798279177}{92460387285504} \right) \epsilon^4 + \left(\frac{2274385}{634785984} - \frac{165\zeta_3}{9604} \right) \epsilon^3 + \frac{533\epsilon^2}{10584} + \frac{\epsilon}{7}$$

$$\lambda^* = \left(-\frac{291071171435\zeta_3^2}{15731107558992} - \frac{701069774301193\zeta_3}{81833221521876384} + \frac{4358213827\zeta_4}{1926258068448} - \frac{46314065257\zeta_5}{793165087008} + \frac{2651375\zeta_6}{130691232} - \frac{9304531\zeta_7}{82287072} - \frac{20192282387582749855}{20433428081126445579264} \right) \epsilon^5 + \left(\frac{2067645667\zeta_3}{240782258556} - \frac{45139\zeta_4}{13224708} + \frac{306275\zeta_5}{8168202} + \frac{17286322925167}{10020394472066496} \right) \epsilon^4 + \left(-\frac{47437\zeta_3}{3306177} - \frac{24550427}{91726574688} \right) \epsilon^3 + \frac{1691\epsilon^2}{134946} + \frac{2\epsilon}{63}$$

insert into anomalous dimensions to get critical exponents

Critical exponents: $N = 2$

$$\eta_\psi = \frac{1}{14}\epsilon - \frac{71}{10584}\epsilon^2 + \left[- \frac{2432695}{158696496} - \frac{18}{2401}\zeta_3 \right]\epsilon^3 + \left[\frac{150}{16807}\zeta_5 - \frac{27}{4802}\zeta_4 + \frac{11109323}{555437736}\zeta_3 - \frac{111266497289}{11557548410688} \right]\epsilon^4$$
$$+ \left[- \frac{89279362217932877}{11783983899150199296} - \frac{1905116200933}{377546581415808}\zeta_3 - \frac{136650473}{2744515872}\zeta_5 - \frac{5643}{537824}\zeta_7 - \frac{2004}{823543}\zeta_3^2 \right.$$
$$\left. + \frac{375}{33614}\zeta_6 + \frac{11109323}{740583648}\zeta_4 \right]\epsilon^5 + \mathcal{O}(\epsilon^6)$$

$$\eta_\phi = \frac{4}{7}\epsilon + \frac{109}{882}\epsilon^2 + \left[\frac{1170245}{26449416} - \frac{144}{2401}\zeta_3 \right]\epsilon^3 + \left[\frac{20491307339}{481564517112} - \frac{108}{2401}\zeta_4 + \frac{1200}{16807}\zeta_5 + \frac{1563532}{23143239}\zeta_3 \right]\epsilon^4$$
$$+ \left[\frac{32079891787774525}{981998658262516608} - \frac{1237285035017}{31462215117984}\zeta_3 - \frac{69575957}{228709656}\zeta_5 - \frac{15885}{823543}\zeta_3^2 - \frac{5643}{67228}\zeta_7 \right.$$
$$\left. + \frac{1500}{16807}\zeta_6 + \frac{390883}{7714413}\zeta_4 \right]\epsilon^5 + \mathcal{O}(\epsilon^6) \, .$$

$$\eta_{\phi^2} = - \frac{8}{21}\epsilon + \frac{2942}{22491}\epsilon^2 + \left[\frac{88644929}{2547960408} - \frac{144404}{1102059}\zeta_3 \right]\epsilon^3$$
$$+ \left[\frac{18303692723219}{835032872672208} - \frac{36101}{367353}\zeta_4 + \frac{155735}{1361367}\zeta_5 + \frac{1707335624}{20065188213}\zeta_3 \right]\epsilon^4$$
$$+ \left[\frac{35319042091625536 49}{212848209178400474784} - \frac{6505866583}{16524272646}\zeta_5 + \frac{24305439575}{327731407479}\zeta_3^2 + \frac{44217078119105}{6819435126823032}\zeta_3 \right.$$
$$\left. + \frac{778675}{5445468}\zeta_6 + \frac{8001521}{13714512}\zeta_7 + \frac{426833906}{6688396071}\zeta_4 \right]\epsilon^5 + \mathcal{O}(\epsilon^6).$$

Critical exponents: $N = 2$ cont'd

$$\frac{1}{\nu} = 2 - \frac{20}{21}\epsilon + \frac{325}{44982}\epsilon^2 + \left[- \frac{36133009}{3821940612} - \frac{78308}{1102059}\zeta_3 \right]\epsilon^3$$
$$+ \left[\frac{58535}{1361367}\zeta_5 - \frac{19577}{367353}\zeta_4 + \frac{351753380}{20065188213}\zeta_3 - \frac{17228234202607}{835032872672208} \right]\epsilon^4$$
$$+ \left[- \frac{136856493433502985 79}{8513928367136018 99136} - \frac{5916014759}{66097090584}\zeta_5 + \frac{30626922980}{327731407479}\zeta_3^2 + \frac{1249594437836159}{27277740507292128}\zeta_3 \right.$$
$$\left. + \frac{292675}{5445468}\zeta_6 + \frac{9152693}{13714512}\zeta_7 + \frac{87938345}{6688396071}\zeta_4 \right]\epsilon^5 + \mathcal{O}(\epsilon^6)$$

$$\eta_\psi = 0.071429\epsilon - 0.006708\epsilon^2 - 0.024341\epsilon^3 + 0.017584\epsilon^4 - 0.051782\epsilon^5 + \mathcal{O}(\epsilon^6)$$
$$\eta_\phi = 0.571429\epsilon + 0.123583\epsilon^2 - 0.027849\epsilon^3 + 0.149112\epsilon^4 - 0.296922\epsilon^5 + \mathcal{O}(\epsilon^6)$$
$$\frac{1}{\nu} = 2 - 0.952381\epsilon + 0.007225\epsilon^2 - 0.094868\epsilon^3 - 0.012653\epsilon^4 + 0.823067\epsilon^5 + \mathcal{O}(\epsilon^6)$$

Going to 3 dimensions

$d = 3 \rightarrow \epsilon = 1$

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better: use Padé approximation

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$d = 3 \rightarrow \epsilon = 1$ maybe a bit too naive

better: use Padé approximation

even better: use two-sided Padé approximation including 2d data

2 dimensional Gross-Neveu model

$$\mathcal{L}^{\text{GN}} = \bar{\psi}\partial\!\!\!/\psi + h\phi\bar{\psi}\psi + \frac{1}{2}\phi^2$$

known up to 4-loop order, five loop still to do

4d GNY and 2d GN models belong to the same universality class

Interpolation

two-sided Padé

$$\mathcal{P}_{[m/n]}(d) \;=\; \frac{\sum_{p=0}^{m} a_p d^p}{1+\sum_{q=1}^{n} b_q d^q}$$

Interpolation

two-sided Padé

$$\mathcal{P}_{[m/n]}(d) \;=\; \frac{\sum_{p=0}^{m} a_p d^p}{1+\sum_{q=1}^{n} b_q d^q}$$

Interpolating polynomial

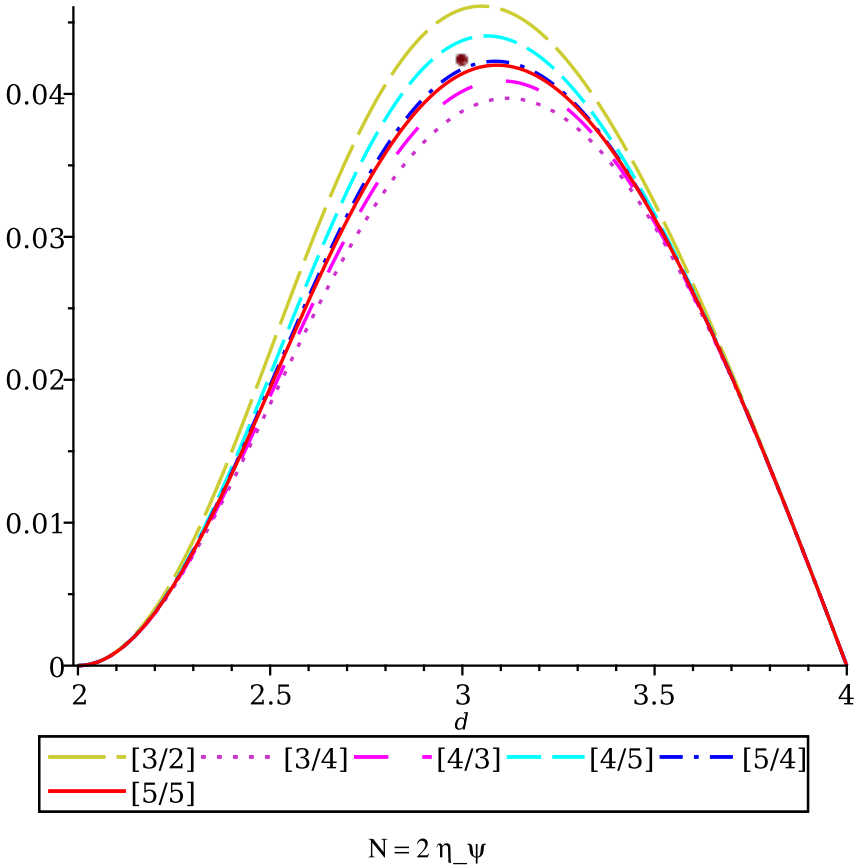
$$\mathcal{I}_{[i,j]}(d) \;=\; \sum_{m=0}^{i} \eta_m^{(2)} (d-2)^m \;+\; \sum_{n=i+1}^{i+j+1} c_n (d-2)^n$$

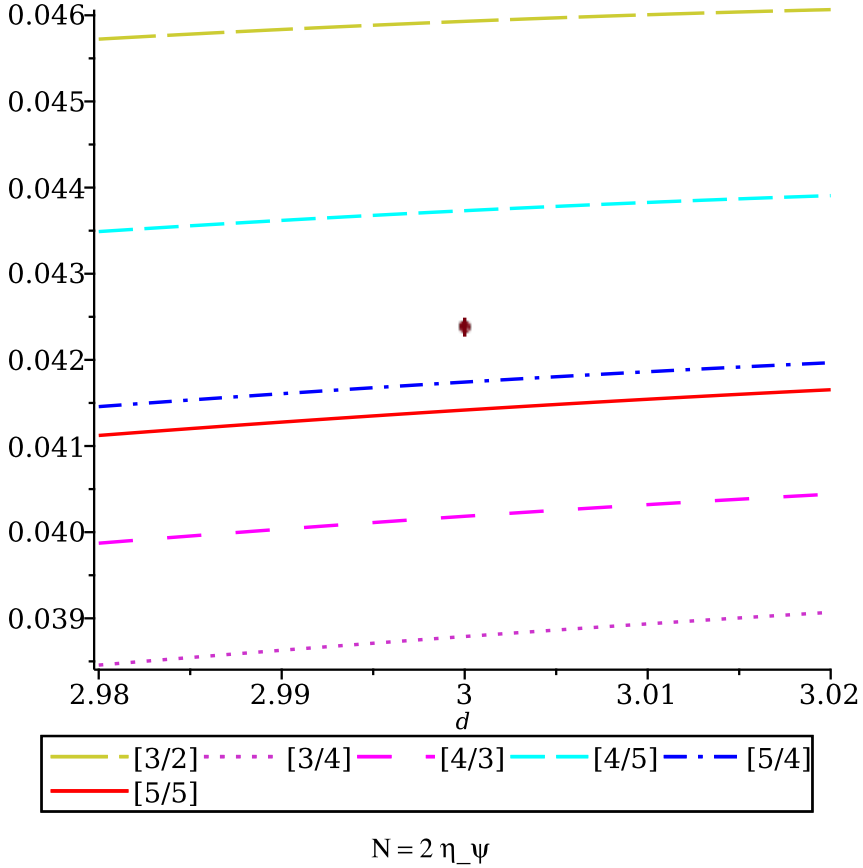
Previous results

Method and source	η_ψ	η_ϕ	$1/\nu$
Large N [17, 31–36]	0.044	0.743	0.952
Monte Carlo [17]	——	0.754(8)	1.00(4)
Monte Carlo [18]	0.38(1)	0.62(1)	1.20(1)
Functional renormalization group [25]	0.032	0.760	0.982
Functional renormalization group [25]	0.033	0.767	0.978
Functional renormalization group [25]	0.032	0.756	0.982
Functional renormalization group [26]	0.0276	0.7765	0.994(2)
Four loop $d = 2$ naive Padé [37]	0.082	0.745	0.931
Three loop $d = 4$ naive Padé [38]	0.0740	0.672	1.048
Conformal bootstrap [28]	0.044	0.742	0.880
Monte Carlo [20]	——	0.65(3)	1.2(1)
Monte Carlo [21]	——	0.54(6)	1.14(2)
Four loop $d = 4$ naive Padé [39]	0.0539	0.7079	0.931
Four loop $d = 4$ naive Padé [39]	0.0506	0.6906	0.945
Four loop two-sided Padé [40]	0.042	0.735	1.004
Four loop interpolating polynomial [40]	0.043	0.731	0.982
Four loop Padé-Borel [40]	0.043(12)	0.704(15)	0.993(27)
Monte Carlo [22]	0.05(2)	0.59(2)	1.0(1)
Conformal bootstrap [29]	0.04238(11)	0.7329(27)	0.998(12)
Monte Carlo [24]	0.043(12)	0.72(6)	1.07(12)
Conformal bootstrap [30]	——	0.7339(26)	0.998(12)
Functional renormalization group [27]	0.032	0.760	0.982

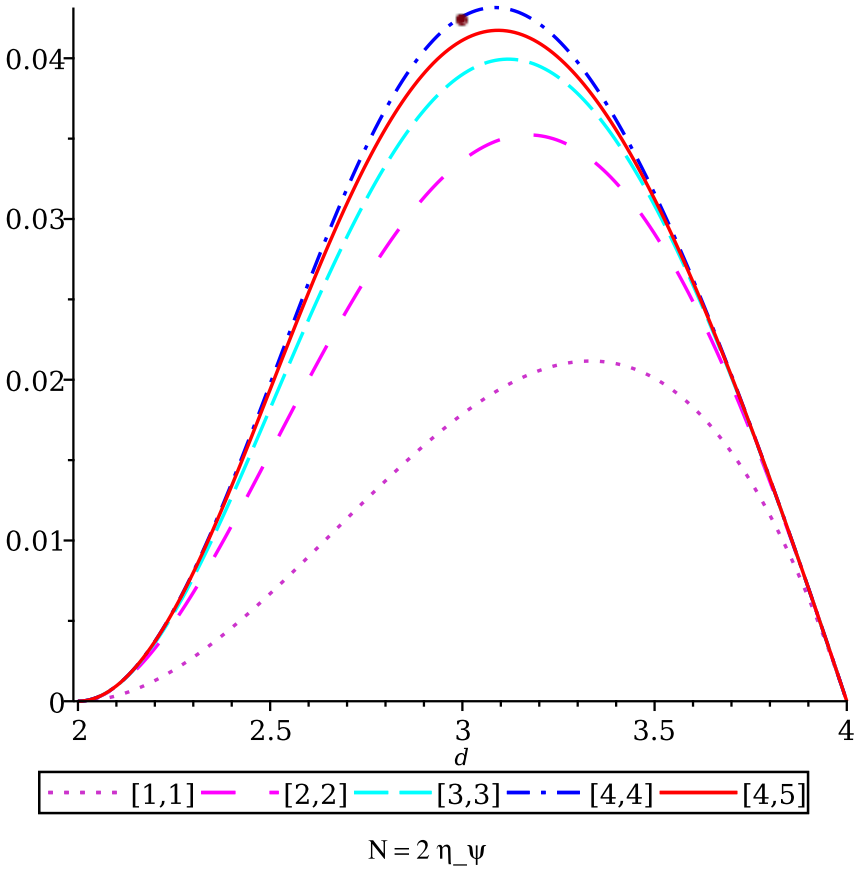
References, see paper

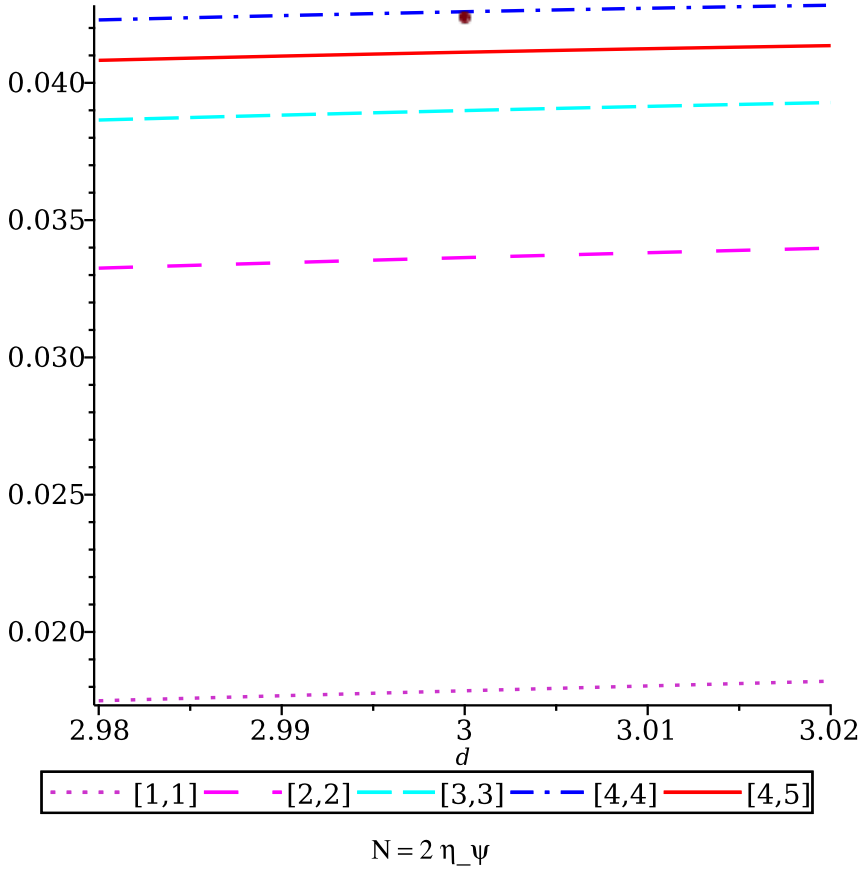
η_ψ: Padé


0.04
0.03
0.02
0.01
0
2
2.5
3
3.5
4
d
[3/2]
[3/4]
[4/3]
[4/5]
[5/4]
[5/5]
N = 2 η_ψ

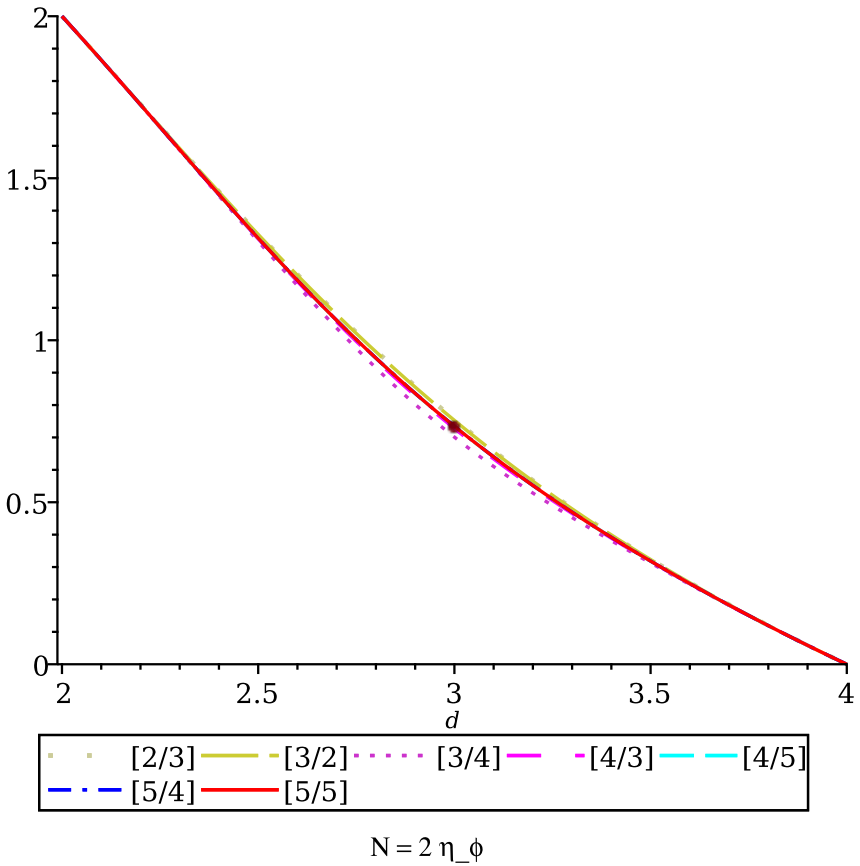

0.046
0.045
0.044
0.043
0.042
0.041
0.040
0.039
2.98
2.99
3
3.01
3.02
d
[3/2]
[3/4]
[4/3]
[4/5]
[5/4]
[5/5]
N = 2 η_ψ

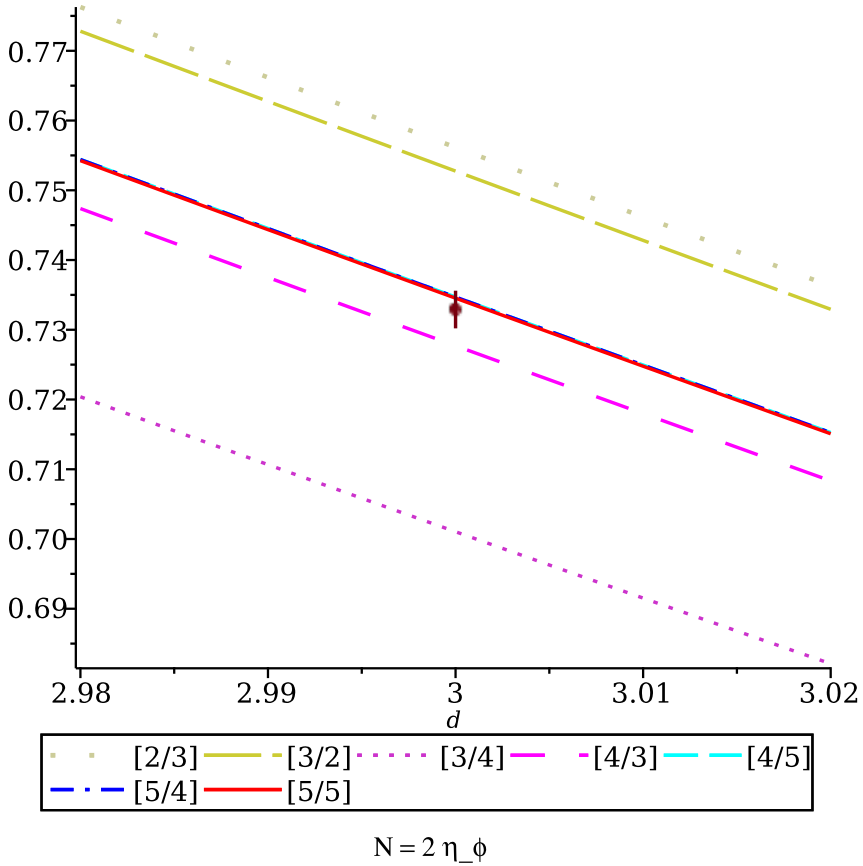
η_ψ: interpolating polynomial





η_ϕ: Padé


2
1.5
1
0.5
0
2
2.5
3
3.5
4
d
[2/3]
[3/2]
[3/4]
[4/3]
[4/5]
[5/4]
[5/5]
N = 2 η_ϕ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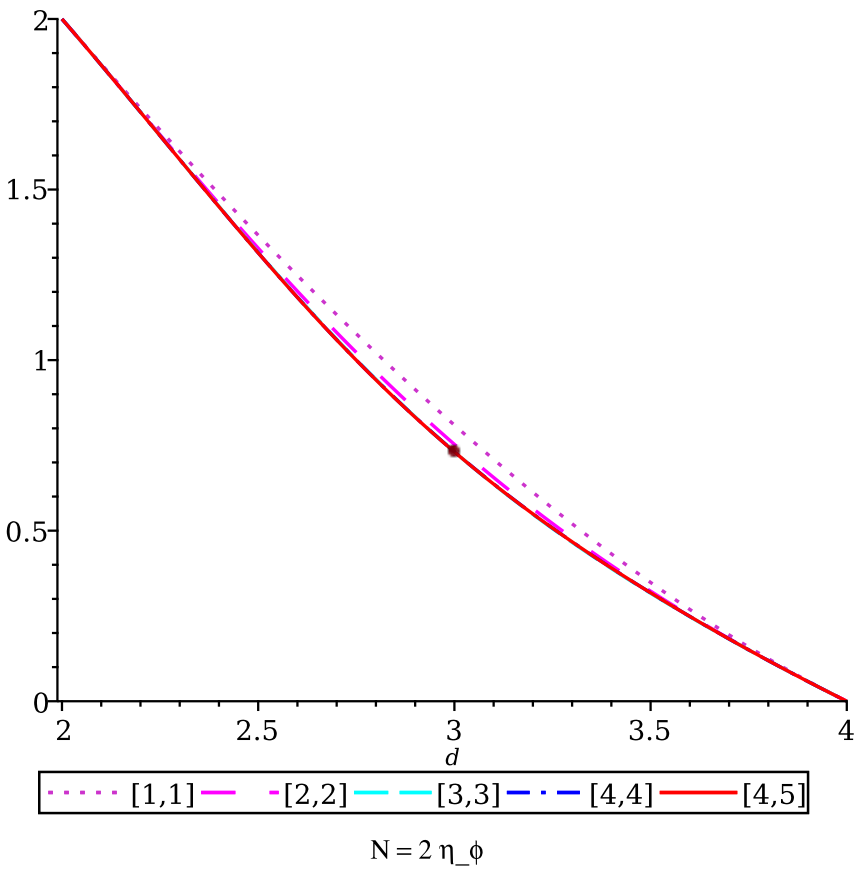
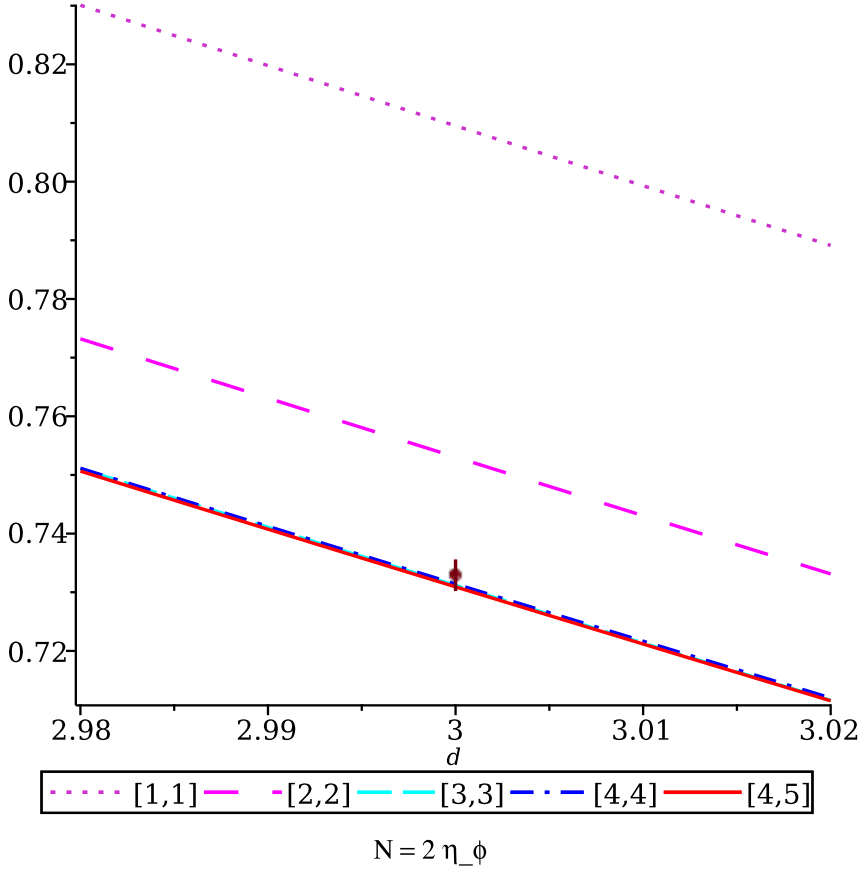
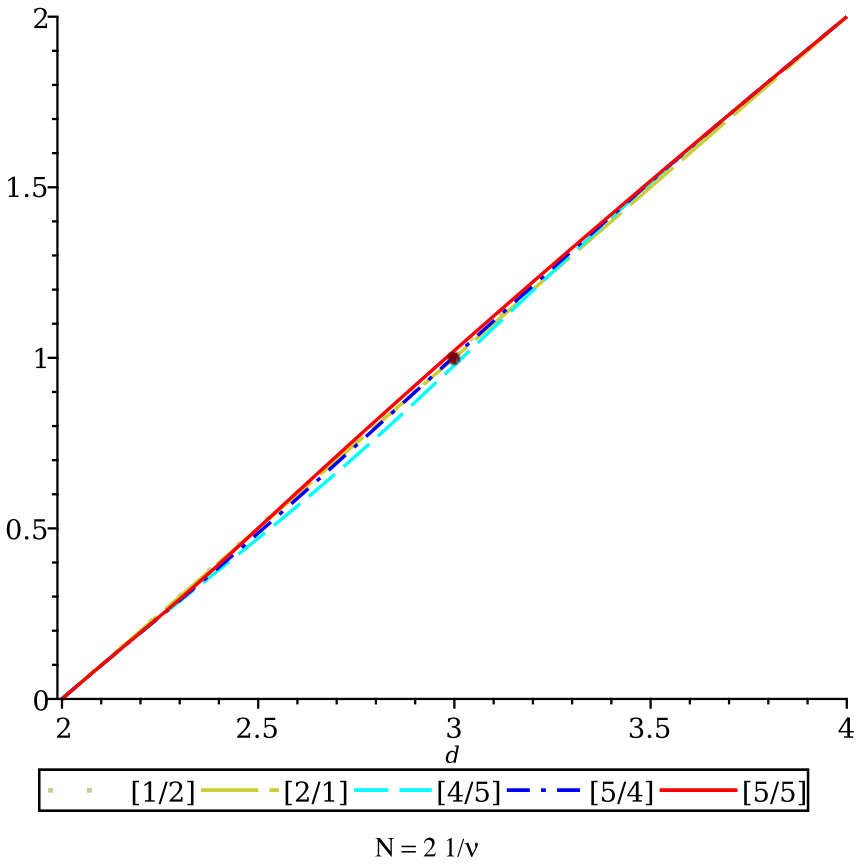
0.74
0.73
0.72
0.71
0.70
0.69
2.98
2.99
3
3.01
3.02
d
[2/3]
[3/2]
[3/4]
[4/3]
[4/5]
[5/4]
[5/5]
N = 2 η_ϕ

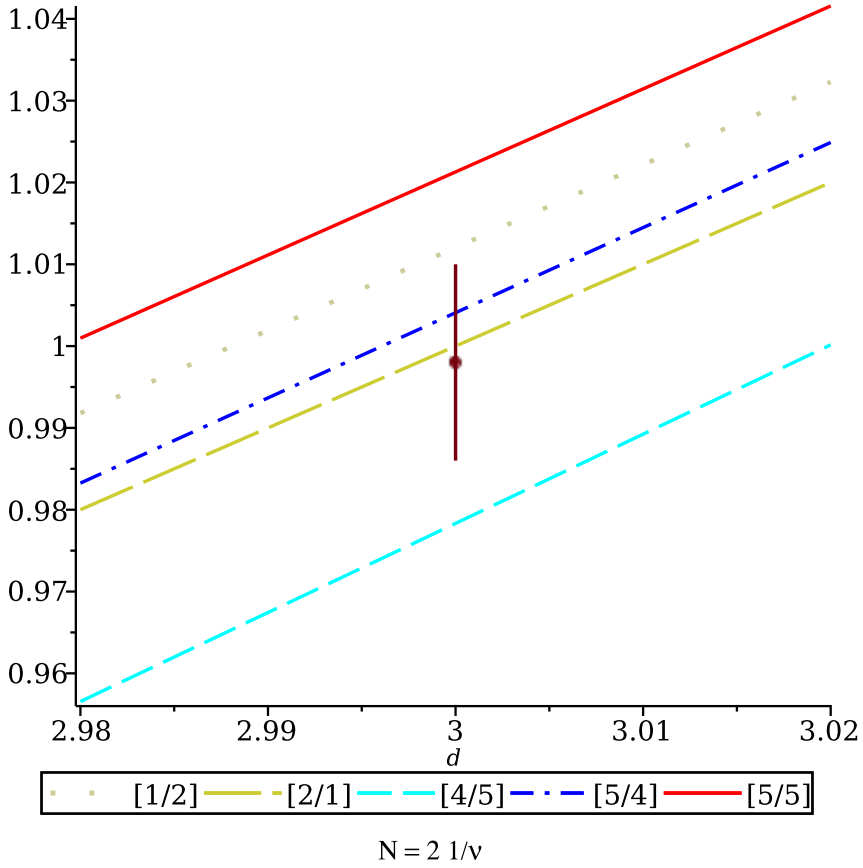
η_ϕ: interpolating polynomial



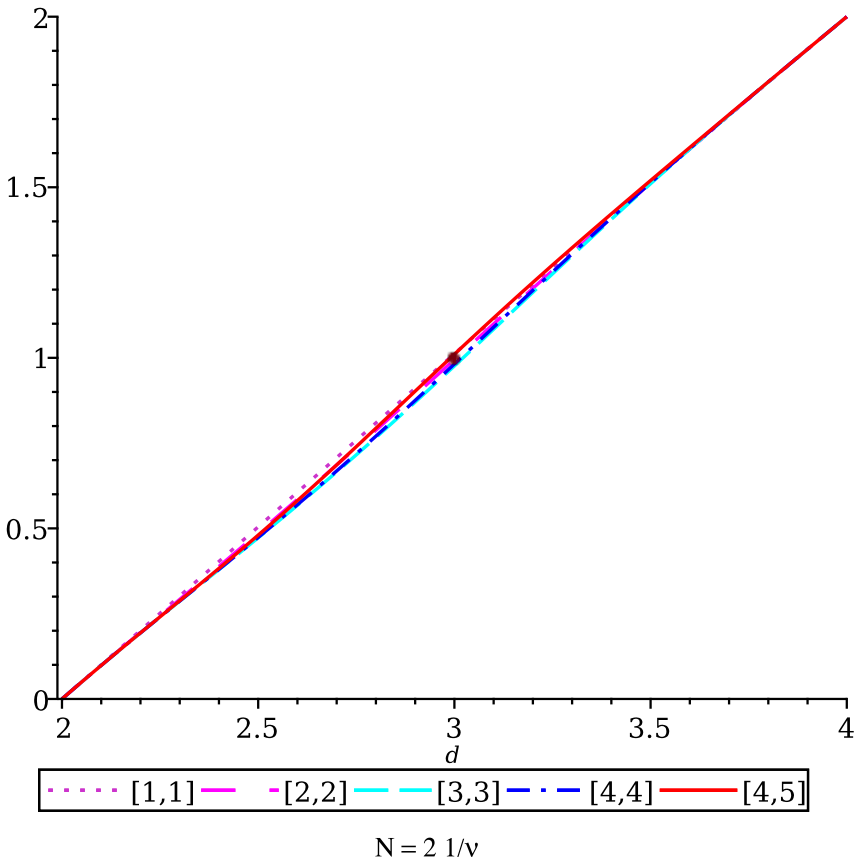


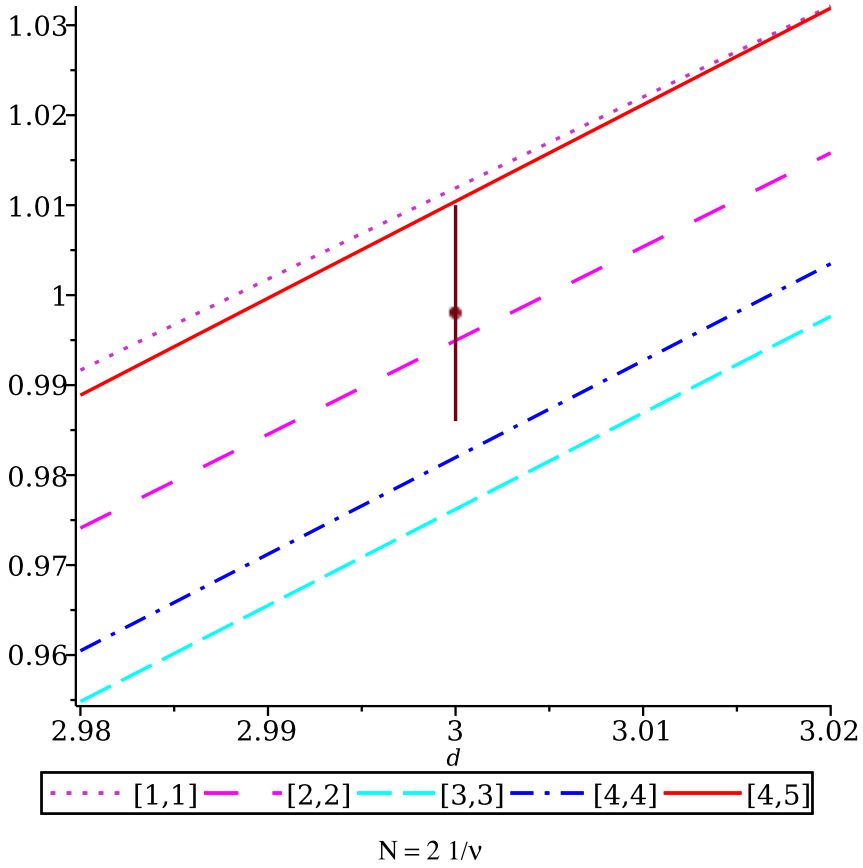
ν^{-1}: Padé


2
1.5
1
0.5
0
2
2.5
3
3.5
4
d
[1/2]
[2/1]
[4/5]
[5/4]
[5/5]
N = 2 1/ν


1.04
1.03
1.02
1.01
1
0.99
0.98
0.97
0.96
2.98
2.99
3
3.01
3.02
d
[1/2]
[2/1]
[4/5]
[5/4]
[5/5]
N = 2 1/ν

ν^{-1}: interpolating polynomial





Results: $N = 2$

Padé	η_ψ	η_ϕ	$1/\nu$
[1/2]	——	0.786885	1.012048
[2/1]	——	0.812500	1.000000
[2/3]	——	0.756156	——
[3/2]	0.045930	0.752752	——
[3/4]	0.038789	0.701041	——
[4/3]	0.040184	0.727706	——
[4/5]	0.043732	0.734702	0.978327
[5/4]	0.041742	0.734696	1.004079
[5/5]	0.041418	0.734530	1.021288

Polynomial	η_ψ	η_ϕ	$1/\nu$
[1, 1]	0.017857	0.809524	1.011905
[1, 2]	0.034037	0.774943	1.025616
[2, 2]	0.033636	0.753047	0.994951
[2, 3]	0.036652	0.746200	1.003792
[3, 3]	0.038991	0.731170	0.976204
[3, 4]	0.040071	0.739556	0.987315
[4, 4]	0.042592	0.731413	0.981959
[4, 5]	0.041117	0.730924	1.010424

Conformal bootstrap [29]	0.04238(11)	0.7329(27)	0.998(12)
Monte Carlo [24]	0.043(12)	0.72(6)	1.07(12)
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Conclusions

- Calculated the five-loop corrections to the renormalisation constants of the GNY model
- Reasonable agreement with other methods
- Need 5-loop 2d GN results to check for convergence and to ascertain the uncertainty
- Further models under investigation