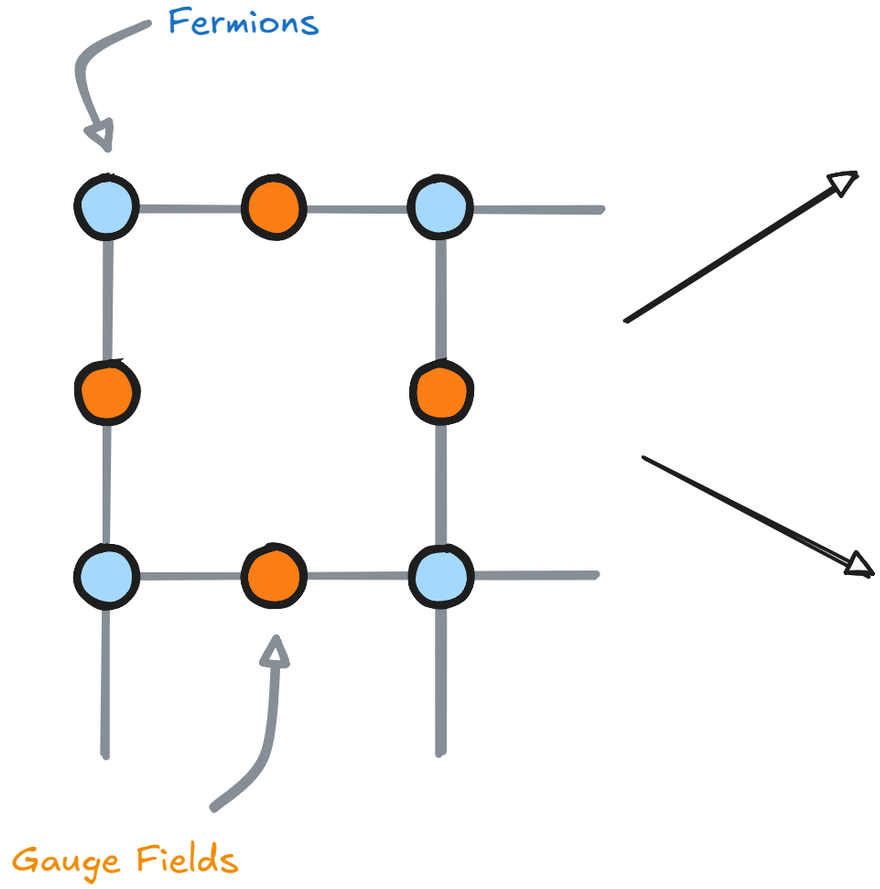
Neural Network States for SU(2) Theories with Dynamical Fermions

Patrick Emonts

ECT* Trento | 29.09.2026

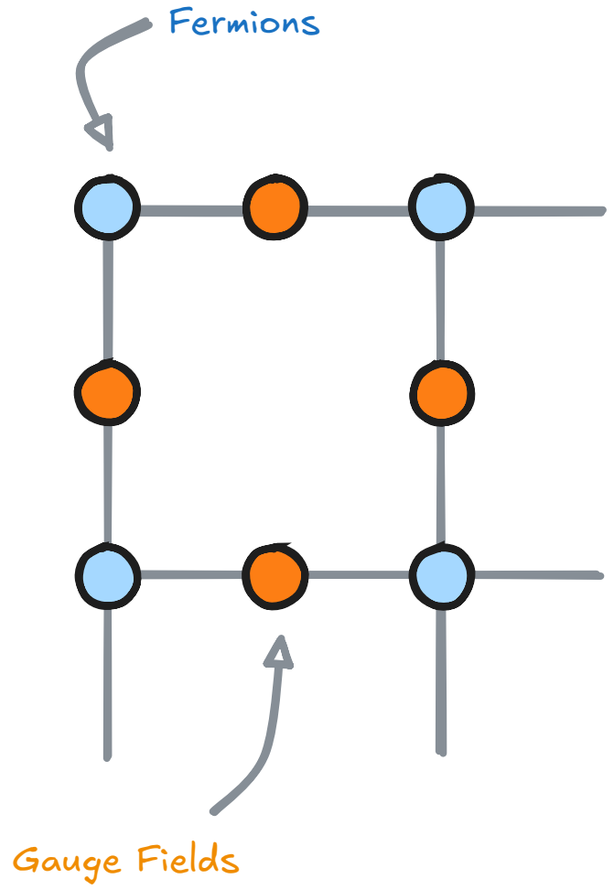
Lattice Gauge Theories



Action Formalism

Hamiltonian Formalism

Hamiltonian Formalism



Hilbert Space

$$\mathcal{H} \subset \mathcal{H}_{\text{fermions}} \otimes \mathcal{H}_{\text{fields}}$$

The Gauss Law

$$\hat{G}_{\mathbf{x}} = \sum_{i=1}^{2} (\hat{E}_{\mathbf{x},i} - \hat{E}_{\mathbf{x}-\mathbf{e}_i,i}) - \hat{Q}_{\text{stag},\mathbf{x}}$$

$$\hat{G}_{\mathbf{x}} |\text{phys}\rangle = q_x |\text{phys}\rangle \quad \forall\, \mathbf{x}$$

The Physical State

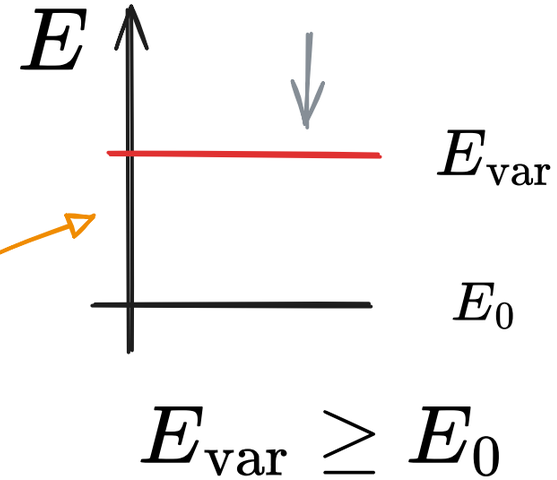
$$|\Psi\rangle = \int D\theta \, \Psi_G(\theta) |\Psi_F(\theta)\rangle |\theta\rangle$$

The Variational Principle

$$H|\psi\rangle = E|\psi\rangle$$

Idea: Use an ansatz

$$E_{\text{var}} = \min_{\alpha} \frac{\langle\psi(\alpha)|H|\psi(\alpha)\rangle}{\langle\psi(\alpha)|\psi(\alpha)\rangle}$$



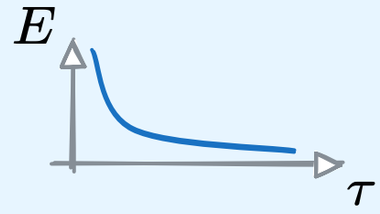
How do we choose the ansatz?

How large is this gap?

Choosing a Good Ansatz

Criteria

- high expressivity



- computationally cheap

- analytically controllable

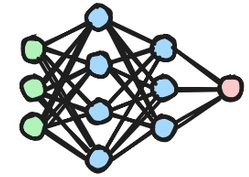
$$G|\psi\rangle = |\psi\rangle$$

Superposition of Gaussian States

Bender et al. PRR 2020

Bender et al. PRR 2023

Neural Network States



Luo et al., PRR. 5, 013216 (2023)

Luo et al. arXiv:2211 .03198

Luo et al. PRL 127, 276402 (2021)

Tensor Networks

Magnifico et al., Nat. Comm. 12 (2021)

Felser et al., PRX 10 (2020)

Robaina et al,. PRL 126 (2021)

Bymes PRD (2002)

Suihara NPB (2004)

Tagliacozzo PRB (2011)

Sugihara JHEP (2005)

Meurice PRB (2013)

and many more

The Storyline

U(1) with dynamical fermions
in 2+1 d

J. Bender, P.Emonts, J.I. Cirac, PRR 5, 043128 (2023)

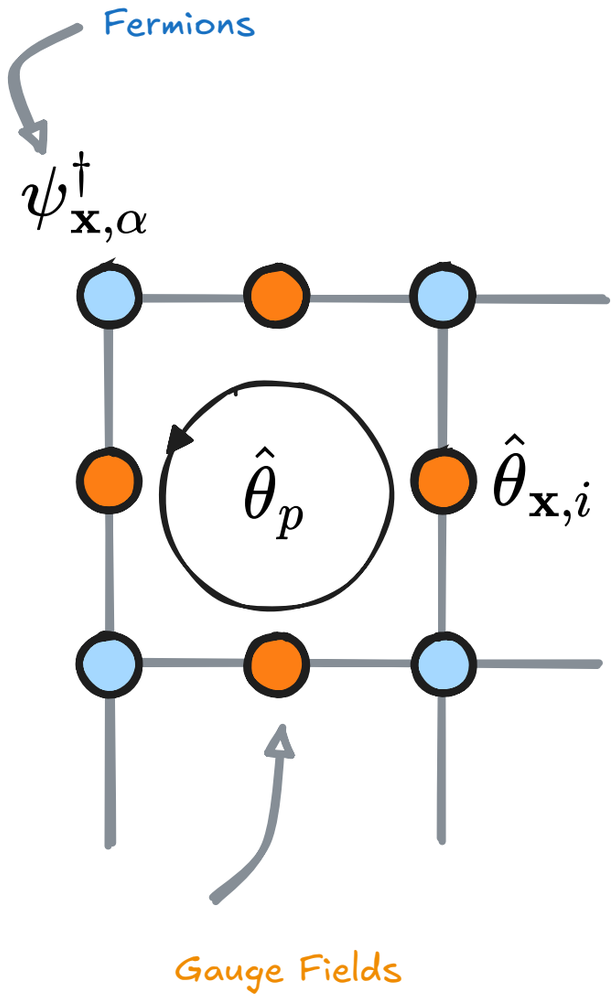
SU(2) pure gauge
in 2+1 and 3+1 d

T. Spriggs, E. Greplova, J. Carrasquilla, J. Nys,
PRL 136, 10190 (2026)

SU(2) with dynamical fermions
in 2+1 d

arXiv:2607.26131
arXiv:2607.26132

Simulating U(1) in 2 Spatial Dimensions

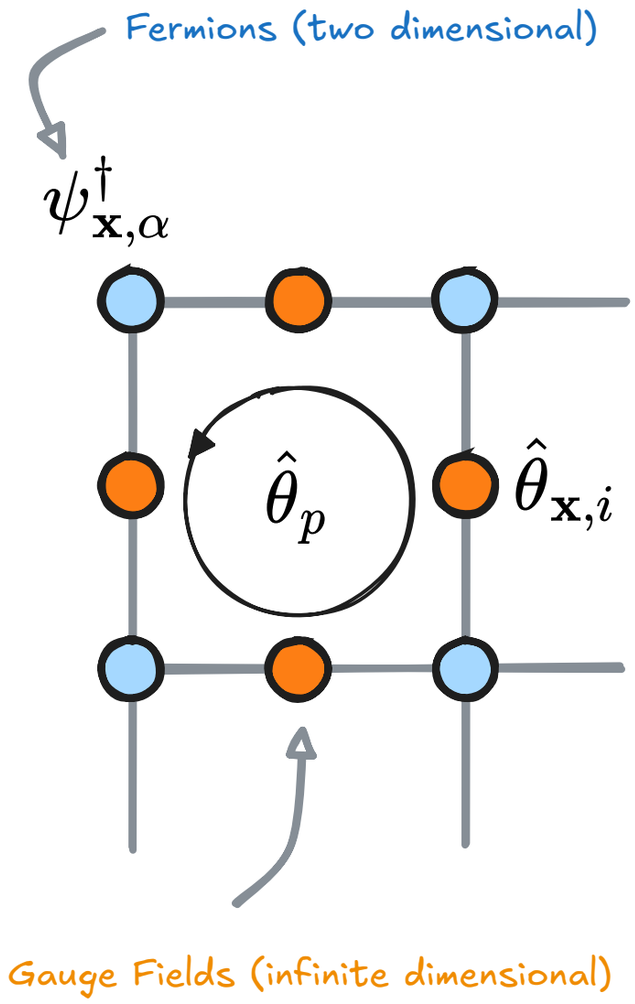


$$H = H_{\mathrm{E}} + H_{\mathrm{B}} + H_{\mathrm{GM}} + H_{\mathrm{M}}$$

$$\begin{aligned} H = & \frac{g^2}{2} \sum_{\mathbf{x},i} \hat{E}^2_{\mathbf{x},i} + g_{\mathrm{mag}} \sum_{\mathbf{p}} (1 - \cos(\hat{\theta}_p)) \\ & - t \sum_{\mathbf{x},i,\alpha} \psi^\dagger_{\mathbf{x},\alpha} e^{i\hat{\theta}_{\mathbf{x},i}} \psi_{\mathbf{x}+\mathbf{e}_i,\alpha} + \mathrm{H.\,c.} \\ & + \sum_{\mathbf{x},\alpha} [m(-1)^x + \mu_\alpha] \psi^\dagger_{\mathbf{x},\alpha} \psi_{\mathbf{x},\alpha} \end{aligned}$$

J. Kogut and L. Susskind, Phys. Rev. D 11, 395 (1975).

Simulating U(1) in 2 Spatial Dimensions



Coping with Link Dimension

Discretize the gauge field

$$U(1) \to Z_N$$

Cut the electric field

$$\{|n\rangle\}_n \to \{|-1\rangle, |0\rangle, |1\rangle\}$$

Do we really need a finite dimension here?

Truncation-Free Simulation

$$|\Psi\rangle = \prod_{\mathbf{x},i} \int_0^{2\pi} d\theta_{\mathbf{x},i}\, \Psi_G(\{\theta_{\mathbf{x},i}\})\Psi_F(\{\theta_{\mathbf{x},i}\})|\{\theta_{\mathbf{x},i}\}\rangle$$

$$\equiv \int D\theta\, \Psi_G(\theta)|\Psi_F(\theta)\rangle|\theta\rangle$$

gauge invariant

superposition of Gaussian states

Idea: We do not store the full wavefunction, but we sample it

The Gauge Field Part

$$|\Psi\rangle = \int D\theta \, \Psi_G(\theta) |\Psi_F(\theta)\rangle |\theta\rangle$$

Approach 1

$$\phi_G(\theta) = \sum_{N_{\mathbf{p}}=-\infty}^{+\infty} e^{-\frac{1}{2}(\theta_{\mathbf{p}}-2\pi N_{\mathbf{p}})\alpha_{\mathbf{p}\mathbf{p}'}(\theta_{\mathbf{p}'}-2\pi N_{\mathbf{p}'})}$$

Villain approximation

$$e^{\gamma[1-\cos(\theta)]} \to \sum_N e^{-\frac{1}{2}\gamma(\theta-2\pi N)^2}$$

J. Bender, P. Emonts, E. Zohar, and J. I. Cirac,
Phys. Rev. Research 2, 043145 (2020).

Approach 2

$$\Psi_G(\theta) = e^{-\sum_{\mathbf{p}\mathbf{p}'}\cos(\theta_{\mathbf{p}})\alpha_{\mathbf{p},\mathbf{p}'}\cos(\theta_{\mathbf{p}'})+\sum_{\mathbf{p}}\beta_{\mathbf{p}}\cos(\theta_{\mathbf{p}})}$$

$$= e^{-\frac{1}{2}b(\theta)^T\alpha b(\theta)-\beta^T b(\theta)}$$

J. Bender, P. Emonts, and J. I. Cirac,
Phys. Rev. Research 5, 043128 (2023).

The Fermionic Part

$$|\Psi\rangle = \int D\theta \, \Psi_G(\theta) |\Psi_F(\theta)\rangle |\theta\rangle$$

$$|\Psi_F(\theta)\rangle = U_{\mathrm{GS}}(\theta) |\Psi_0\rangle$$

reference state

$$|\Psi_0\rangle = \prod_{\mathbf{x} \in \mathcal{O}} \psi^\dagger_{\mathbf{x}} |0\rangle$$

How do we choose the unitary transformation?

The Fermionic Part

$$|\Psi\rangle = \int D\theta \, \Psi_G(\theta) |\Psi_F(\theta)\rangle |\theta\rangle$$

$$|\Psi_F(\theta)\rangle = U_{\mathrm{GS}}(\theta)|\Psi_0\rangle$$

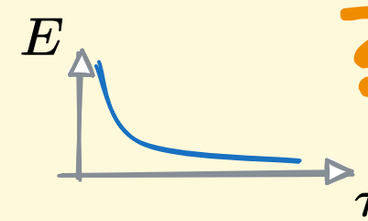
$$H_{\mathrm{GM}} = \int D\theta \, |\theta\rangle\langle\theta| \vec{\psi}^\dagger_{\mathbf{x}} h_{\mathrm{GM}}(\theta)_{\mathbf{xy}} \vec{\psi}_{\mathbf{y}}$$

Use the eigenvectors as building blocks

$$|\Psi_F(\theta)\rangle = \exp\left(\frac{i}{2} \sum_{\mathbf{x},\mathbf{y}} \psi^\dagger_{\mathbf{x}} V(\theta)_{\mathbf{x}i} \, \tilde{\xi}_{ij} \, V(\theta)^\dagger_{j\mathbf{y}} \psi_{\mathbf{y}} \right) |D\rangle$$

Evaluating the Ansatz

$$|\Psi\rangle = \int D\theta \; \Psi_G(\theta) |\Psi_F(\theta)\rangle |\theta\rangle$$



$G|\psi\rangle = |\psi\rangle$

$$\Psi_G(\theta) = e^{-\frac{1}{2} b(\theta)^T \alpha b(\theta) - \beta^T b(\theta)}$$

2Nx2N matrix

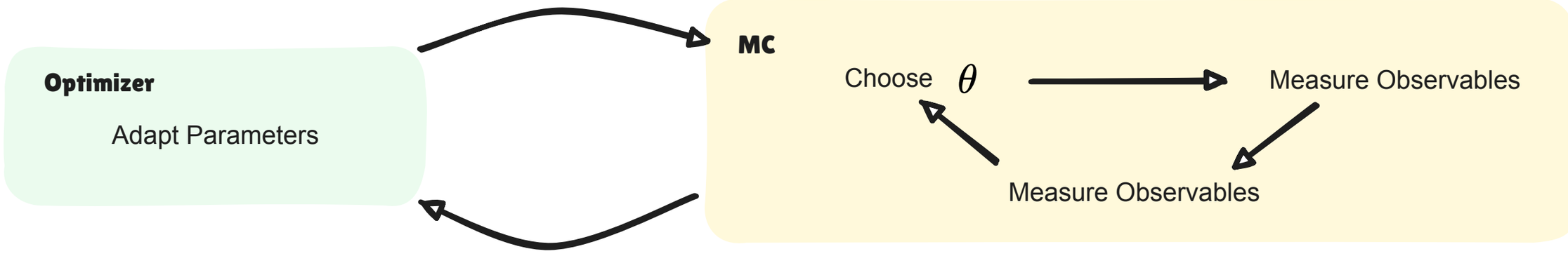
$$|\Psi_F(\theta)\rangle = \exp\left(\frac{i}{2} \sum_{\mathbf{x},\mathbf{y}} \psi^\dagger_{\mathbf{x}} V(\theta)_{\mathbf{x}i} \, \tilde{\xi}_{ij} \, V(\theta)^\dagger_{j\mathbf{y}} \psi_{\mathbf{y}} \right) |D\rangle$$

NxN matrix

Evaluating the Ansatz

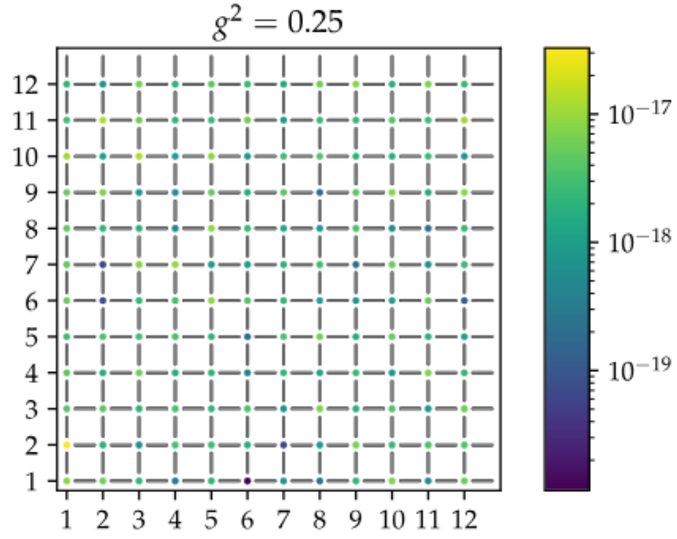
Variational Monte Carlo

$$\frac{\langle\Psi|O|\Psi\rangle}{\langle\Psi|\Psi\rangle} = \frac{\int D\theta\, O_{\mathrm{loc}}(\theta)|\Psi_G(\theta)|^2}{\int D\theta\, |\Psi_G(\theta)|^2} = \int D\theta\, O_{\mathrm{loc}}(\theta)p(\theta)$$

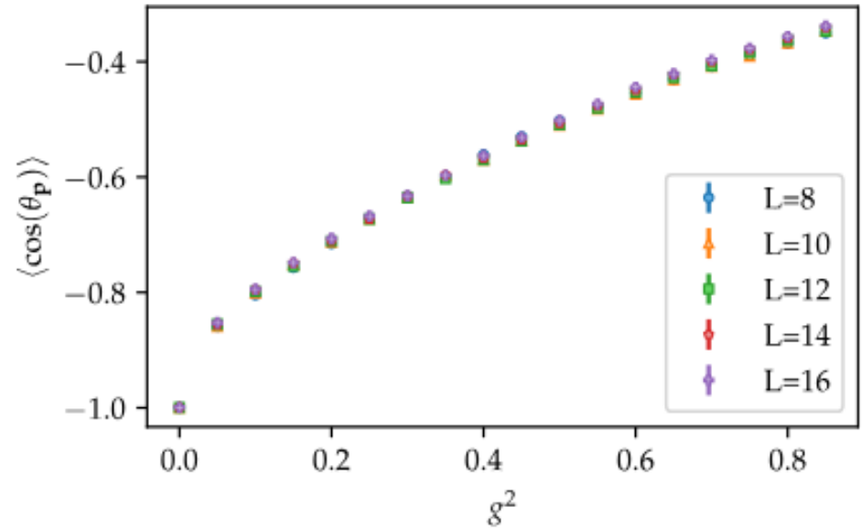


J. Bender, P. Emonts, and J. I. Cirac, Phys. Rev. Research 5, 043128 (2023).

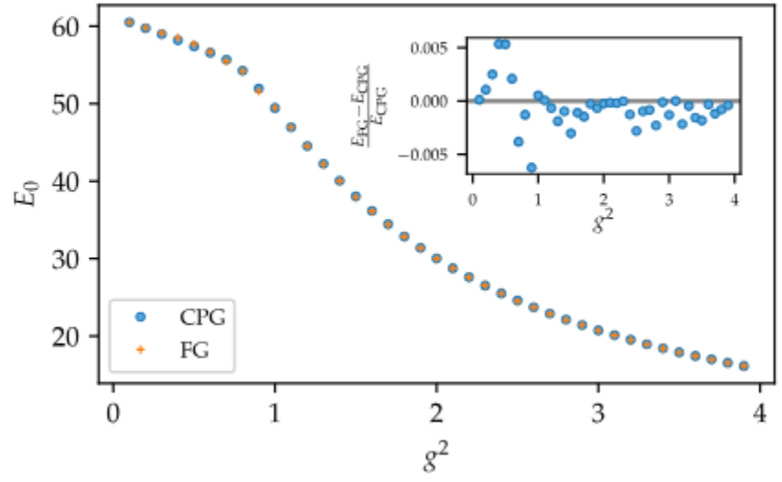
Does it work?



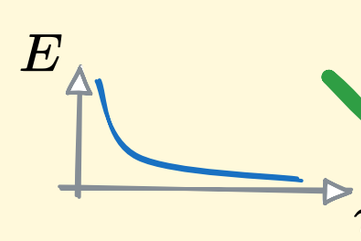
Gauss law



Comparison to MC



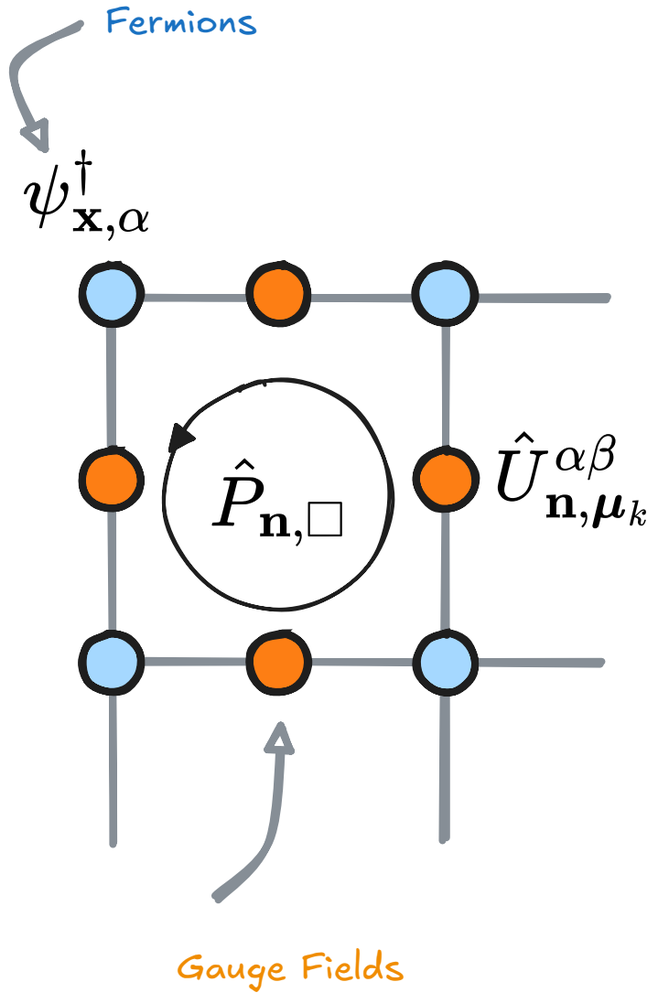
Comparison to pure gauge



$$G|\psi\rangle = |\psi\rangle$$

J. Bender, P. Emonts, and J. I. Cirac, Phys. Rev. Research 5, 043128 (2023).

Simulating SU(2) in 2 Spatial Dimensions



$$\hat{H} = \hat{H}_E + \hat{H}_B + \hat{H}_m + \hat{H}_t,$$

$$\hat{H}_E = \frac{g^2}{2a_s} \sum_{\mathbf{n},k,a} \hat{E}^a_{\mathbf{n},\boldsymbol{\mu}_k} \hat{E}^a_{\mathbf{n},\boldsymbol{\mu}_k}$$

$$\hat{H}_B = \frac{\lambda}{a_s} \sum_{\mathbf{n}} \left(1 - \frac{1}{2} \mathrm{Tr}\, \hat{P}_{\mathbf{n},\square} \right)$$

$$\hat{H}_m = m \sum_{\mathbf{n},\alpha} (-1)^{n_x+n_y} \hat{\psi}^\dagger_{\mathbf{n},\alpha} \hat{\psi}_{\mathbf{n},\alpha}$$

$$\hat{H}_t = -\frac{it}{2a_s} \sum_{\mathbf{n},k,\alpha,\beta} \left(\hat{\psi}^\dagger_{\mathbf{n},\alpha} \hat{U}^{\alpha\beta}_{\mathbf{n},\boldsymbol{\mu}_k} \hat{\psi}_{\mathbf{n}+\boldsymbol{\mu}_k,\beta} \eta_{\mathbf{n},\boldsymbol{\mu}_k} - \mathrm{H.c.} \right)$$

Simulating SU(2) in 2 Spatial Dimensions

The same idea

$$|\Psi\rangle = \int DU\ \Psi_G(U)|\Psi_F(U)\rangle|U\rangle$$

Gauge Part

Fermionic Part

T. Spriggs, E. Greplova, J. Carrasquilla, J. Nys,
PRL 136, 10190 (2026)

G. Rouxinol, J. Bender, M. Grossi, P. Emonts, and J. C. Halimeh,
arXiv:2607.26131.

G. Rouxinol, J. Bender, P. Emonts, M. Grossi, and J. C. Halimeh,
arXiv:2607.26132.

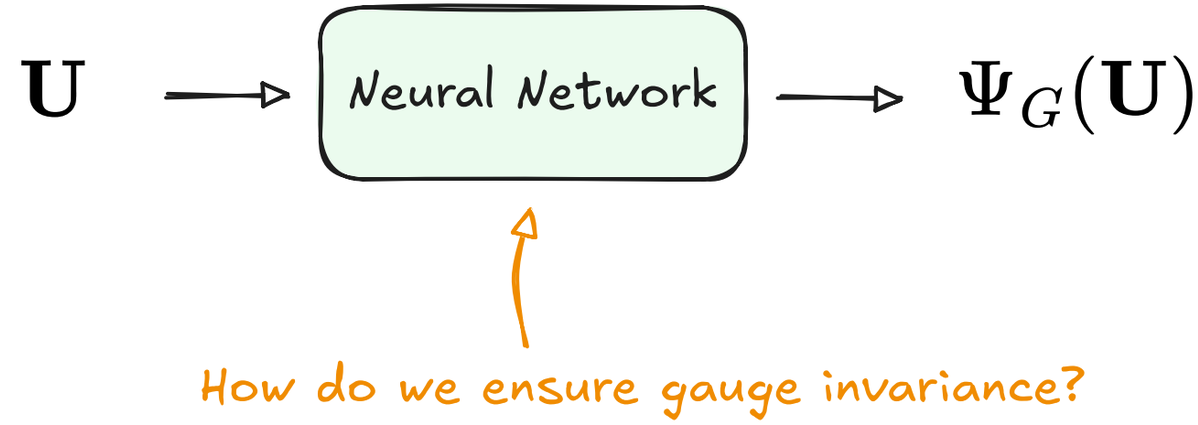
Interlude: Neural-Network Quantum States

Pure Gauge

$$|\Psi\rangle = \int DU\ \Psi_G(U)|U\rangle$$

Loss Function: Energy of the System

Simplest Setup



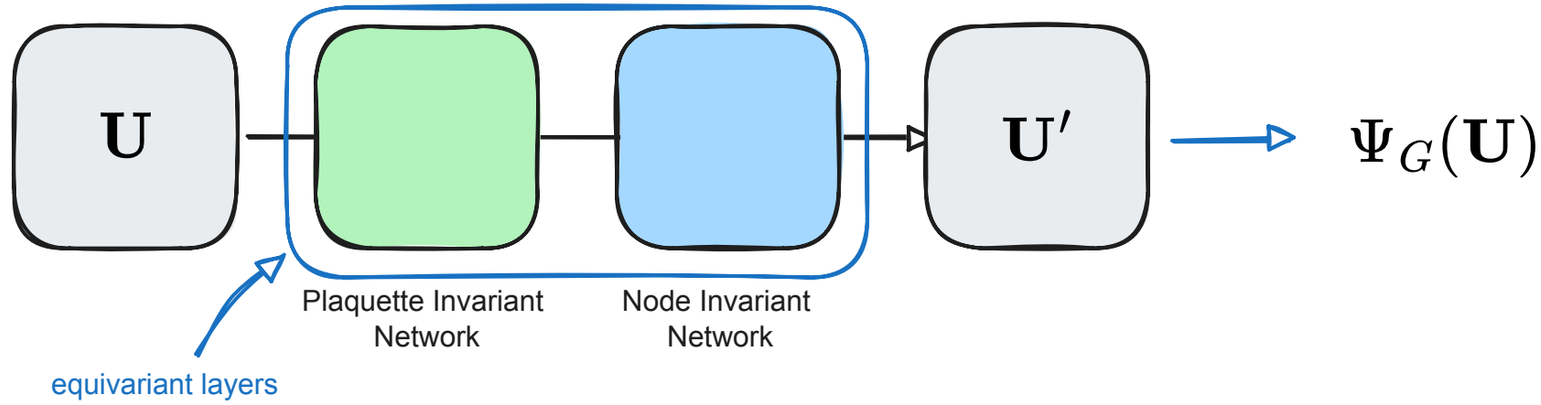
T. Spriggs, E. Greplova, J. Carrasquilla, J. Nys,PRL 136, 10190 (2026)

D. Luo, Z. Chen, K. Hu, Z. Zhao, et al. Phys. Rev. Res. 5, 013216 (2023).

R. G. Melko, G. Carleo, J. Carrasquilla, and J. I. Cirac, Nat. Phys. 15, 887 (2019).

J. Carrasquilla and G. Torlai, PRX Quantum 2, 040201 (2021).

The Gauge Field Part



The Wave Function Ansatz

$$\Psi(\boldsymbol{U}) = \prod_{\mu<\nu,x} e^{\alpha_\mu W_{\mu,\nu}(x)} \prod_{\mu'<\nu',x'} e^{\beta^{\mu,\mu',\nu,\nu'}_{d(x,x')}} W_{\mu,\nu}(x) W_{\mu',\nu'}(x')$$

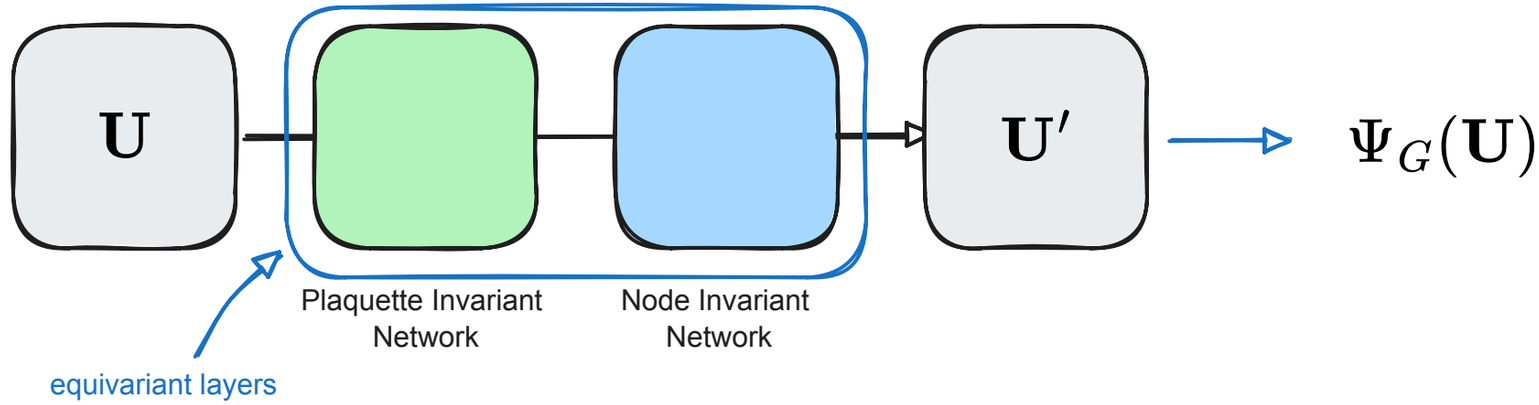
$$W_{\mu,\nu}(x) = \frac{1}{2}\mathrm{Tr} P_{\mu,\nu}(x).$$

Reminder U(1)

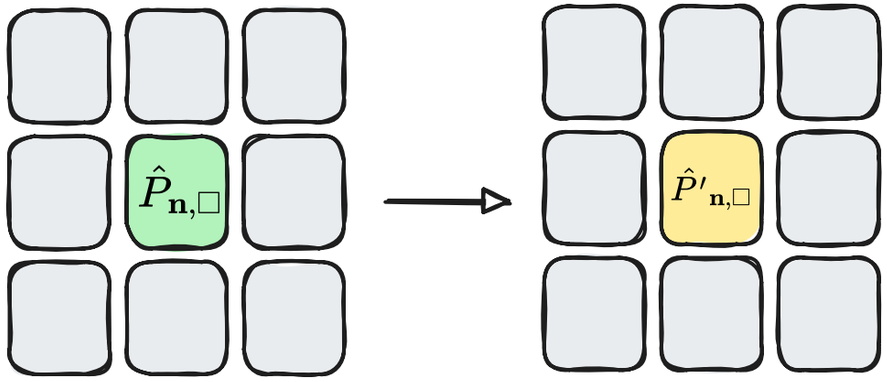
$$\Psi_G(\theta) = e^{-\frac{1}{2}b(\theta)^T \alpha b(\theta) - \beta^T b(\theta)}$$

T. Spriggs, E. Greplova, J. Carrasquilla, J. Nys,PRL 136, 10190 (2026)

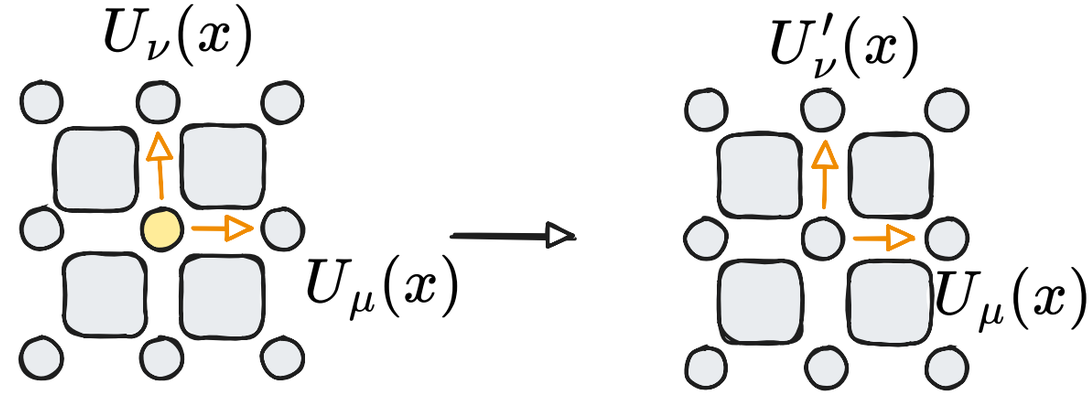
The Gauge Field Part



Plaquette Invariant Layer



Node Invariant Layer



T. Spriggs, E. Greplova, J. Carrasquilla, J. Nys,PRL 136, 10190 (2026)

The Fermionic Part

$$|\Psi_F(\mathbf{U})\rangle = \hat{U}_{\text{corr}}(\mathbf{U})|\Psi_0\rangle$$

similar idea as before

$$\hat{U}_{\text{corr}}(\mathbf{U}) = \exp\left(i \sum_{\mathbf{n},\alpha,\mathbf{n}',\beta} \hat{\psi}^\dagger_{\mathbf{n},\alpha} [H_{\text{full}}(\mathbf{U})]^{\alpha\beta}_{\mathbf{n},\mathbf{n}'} \hat{\psi}_{\mathbf{n}',\beta} \right)$$

+information of short Wilson loops

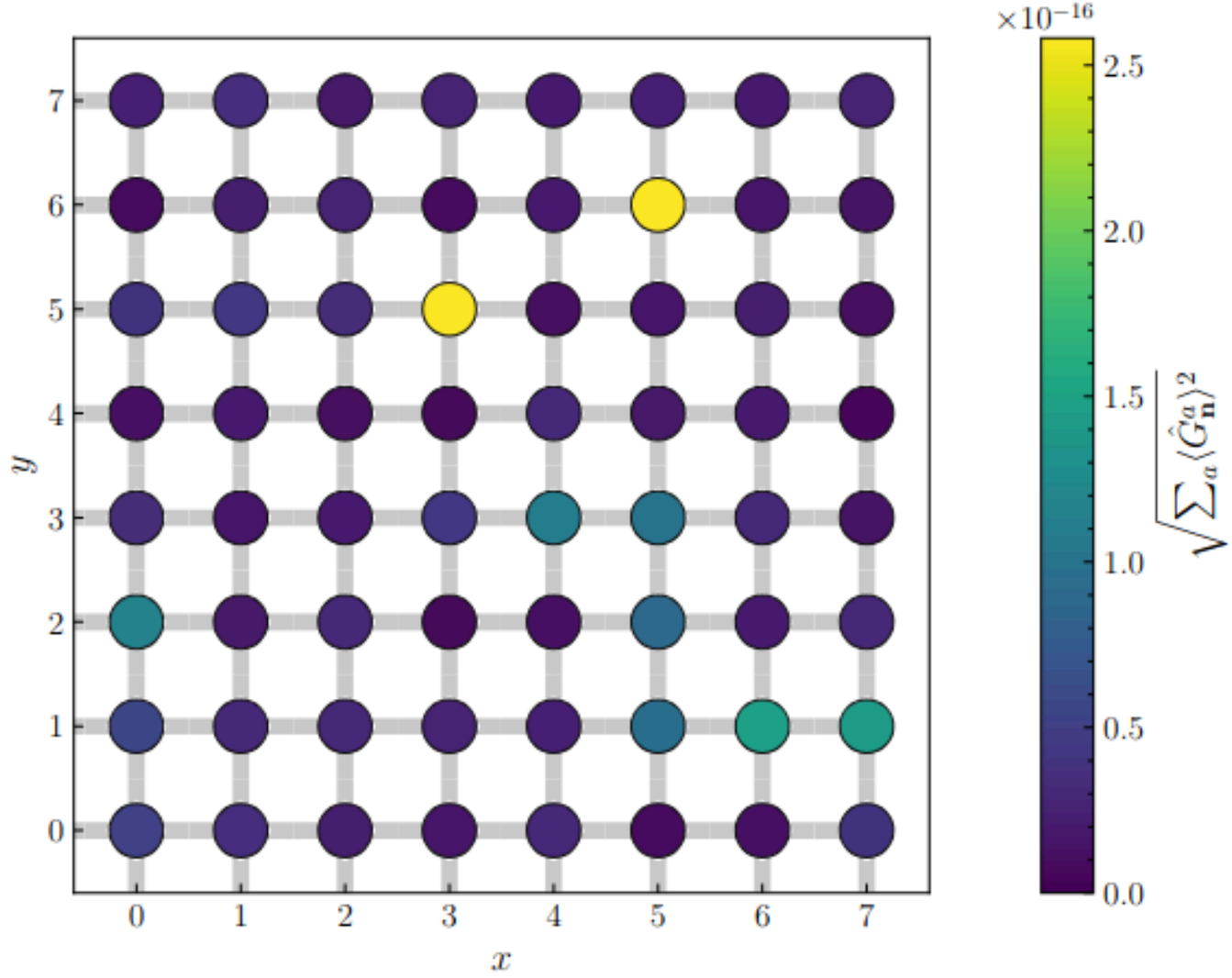
Reminder U(1)

$$|\Psi_F(\theta)\rangle = U_{\text{GS}}(\theta)|\Psi_0\rangle$$

G. Rouxinol, J. Bender, M. Grossi, P. Emonts, and J. C. Halimeh, arXiv:2607.26131.

G. Rouxinol, J. Bender, P. Emonts, M. Grossi, and J. C. Halimeh, arXiv:2607.26132.

Does it work?



No Gauss law violation

$g = 8, t = 1, m = 0.5$

G. Rouxinol, J. Bender, M. Grossi, P. Emonts, and J. C. Halimeh, arXiv:2607.26131.

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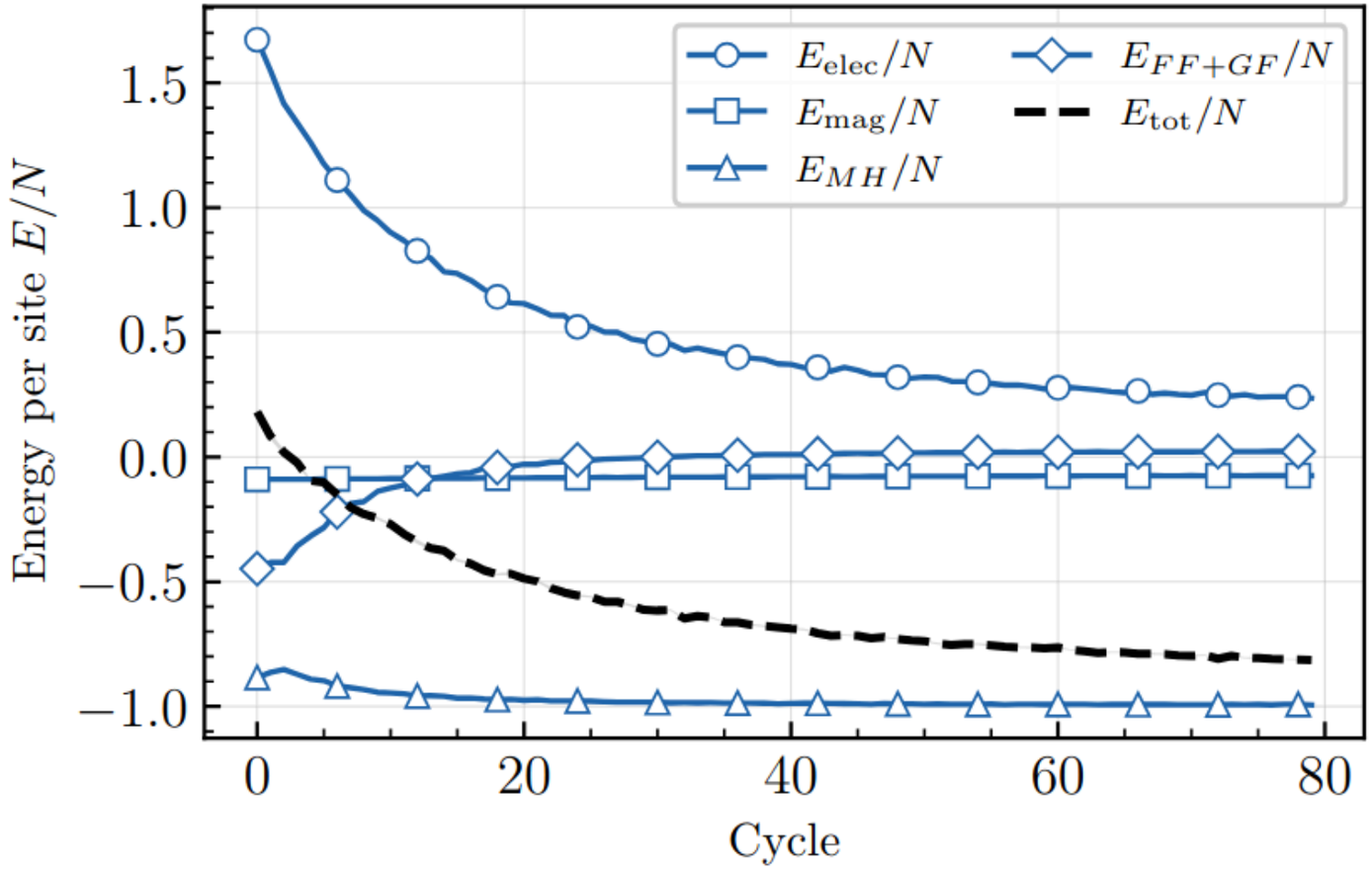


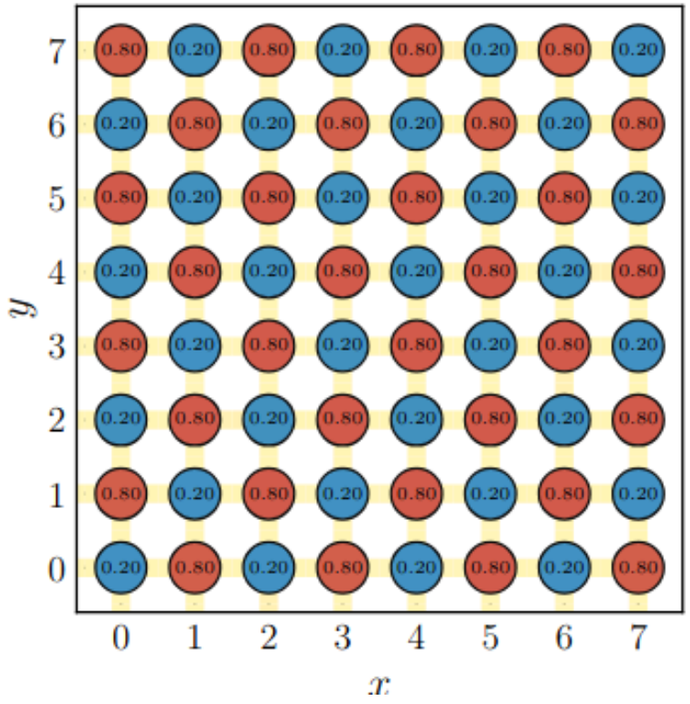
Energy is decreasing

$g^2 = 0.5, \lambda = -0.05, m = 0.5$

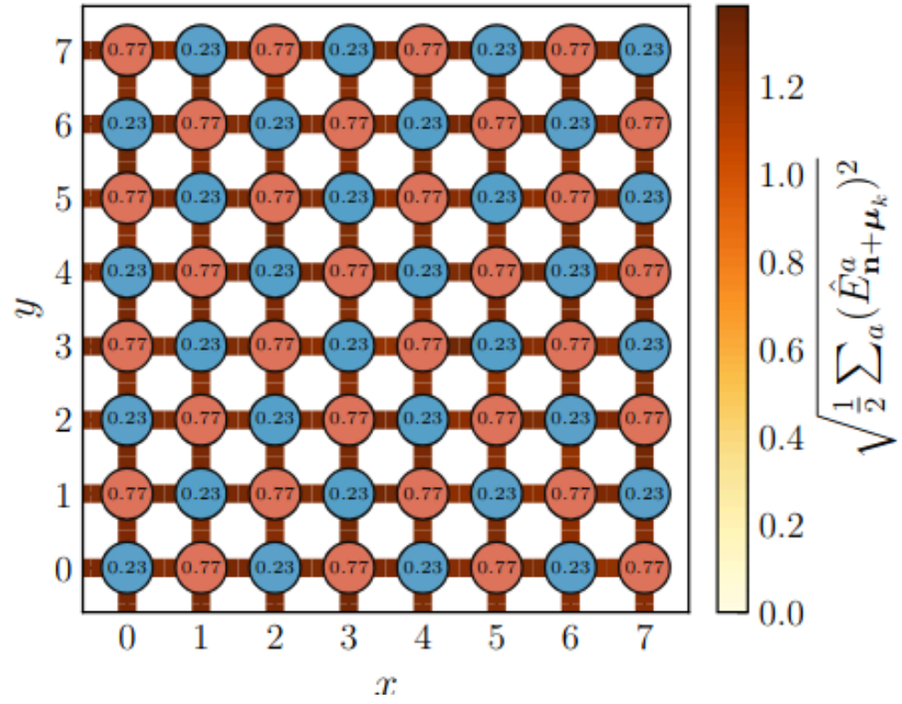
G. Rouxinol, J. Bender, M. Grossi, P. Emonts, and J. C. Halimeh, arXiv:2607.26131.

G. Rouxinol, J. Bender, P. Emonts, M. Grossi, and J. C. Halimeh, arXiv:2607.26132.

Can we study physics with it?



$g = 2.828$



$g = 0.632$

$$\hat{H} = \hat{H}_E + \hat{H}_B + \hat{H}_m + \hat{H}_t$$

$$\hat{H}_E = \frac{g^2}{2a_s} \sum_{\mathbf{n},k,a} \hat{E}^a_{\mathbf{n},\boldsymbol{\mu}_k} \hat{E}^a_{\mathbf{n},\boldsymbol{\mu}_k}$$

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$$\lambda = 4/g^2$$

$$m = 0.5, t = 1.0$$

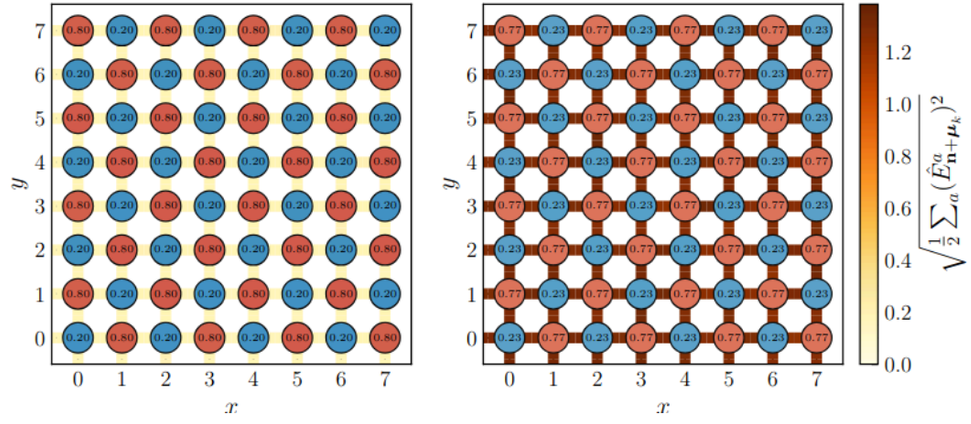
G. Rouxinol, J. Bender, M. Grossi, P. Emonts, and J. C. Halimeh, arXiv:2607.26131.

G. Rouxinol, J. Bender, P. Emonts, M. Grossi, and J. C. Halimeh, arXiv:2607.26132.

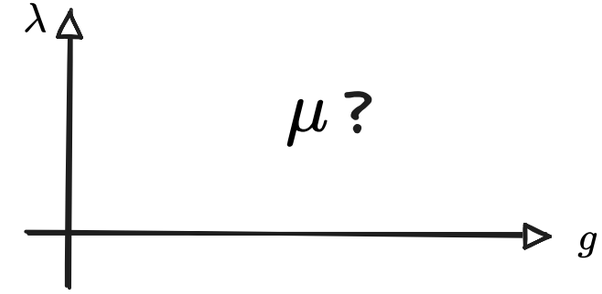
Conclusions and Future Ideas

NQS Ansatz for SU(2) with dynamical fermions

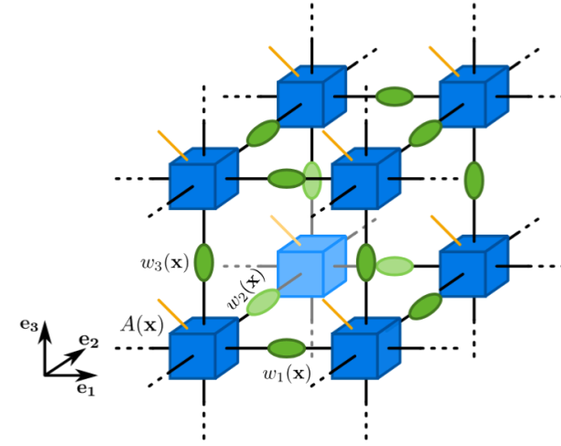
$$|\Psi\rangle = \int D\theta \, \Psi_G(\theta) |\Psi_F(\theta)\rangle |\theta\rangle$$



Finite chemical potential



Scaling to (3+1) D lattices



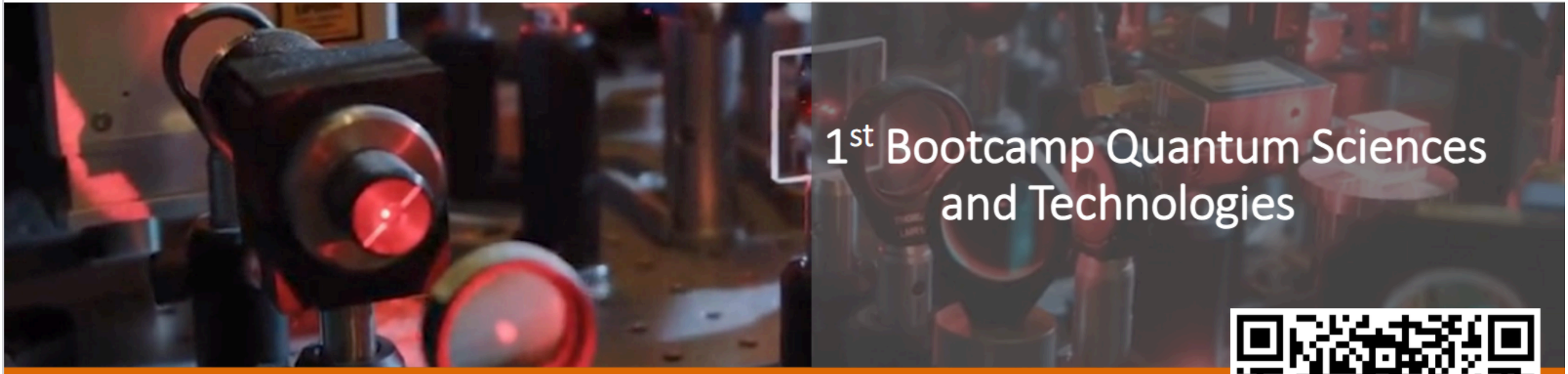
Science is Team Work

Gabriele Rouxinol

Jad Halimeh

Julian Bender

Michele Grossi



1st Bootcamp Quantum Sciences and Technologies

March 15th-24th, 2027

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Neural Network States for SU(2) Theories with Dynamical Fermions

Patrick Emonts

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