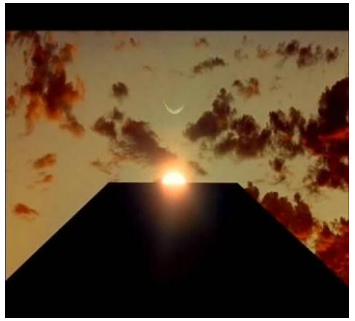
Dirac fermions in two (spatial) dimensions: a brief review and some general remarks

Igor Herbut

(Simon Fraser University, Vancouver, Canada)

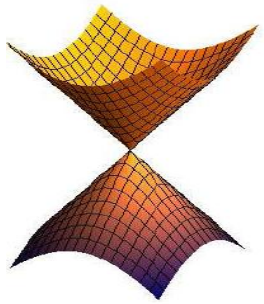
Relativistic fermions in Flatland, ECT, Trento, 28/9/2026

Relativistic (Dirac) fermions: a rough timeline



0) The dawn of relativistic field theory:

a) Dirac theory of electrons (1928) and the massive four-component Dirac equation in 3+1
b) Weyl equation (1929) for massless two-component spinor



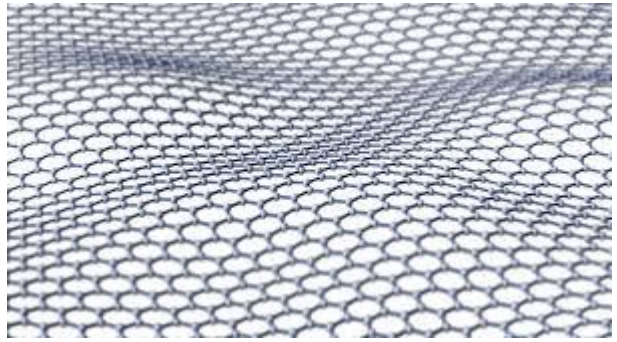
1) Early appearances of massless fermions in Flatland (D=2+1):

a) honeycomb lattice (Semenoff 1984)
a) Affleck-Marston large-N theory of Heisenberg model: spinons on pi-flux lattice (1988)
b) Bogoliubov quasiparticles in d-wave superconductors (1993)

d) compact and non-compact QED_3 effective theory (Pisarski 1984)

2) Renaissance: Dirac fermions take the center stage in 2+1

a) graphene (honeycomb lattice) (Geim and Novoselov 2004)
b) surface states of 3D topological insulators (2005)



3) Modern period: Dirac-like fermions galore

a) quadratic band touching in bilayer graphene and the generalizations
b) Weyl fermions in 3+1: surface arcs, disorder effects
d) Luttinger fermions (quadratic band touching) in 3+1, non-Fermi liquid
e) mass generation without symmetry breaking
f) enhanced superconductivity by Gross-Neveu criticality
g) real (Majorana) fermions in Kitaev model
h) moire lattices

. . . .

What we talk about when we talk about Dirac fermions

In two spatial dimensions:

$$H(\mathbf{q}) = v_x q_x \Gamma_1 + v_y q_y \Gamma_2 + \cdots$$

two Gamma-matrices of certain size are

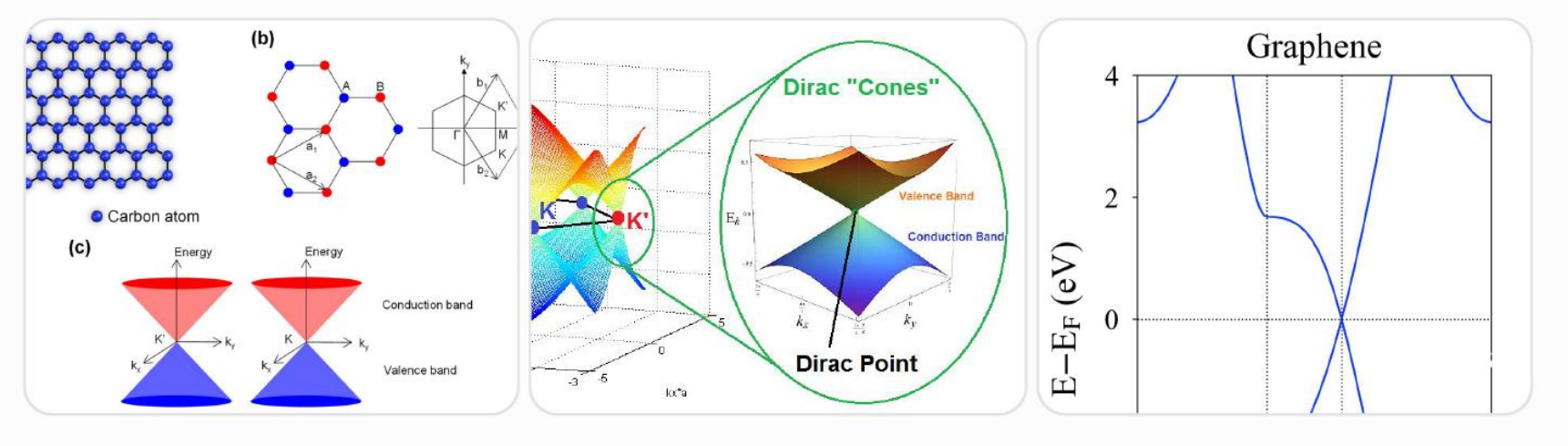
a) Hermitian

b) square to unity

c) mutually anticommuting

i. e. form two-dimensional Clifford algebra. Unique irrep: (any) two Pauli matrices, 2 x 2.

Classic example: graphene



Honeycomb lattice implies Dirac points and no Fermi surface:

$$H_K \simeq v_F(q_x\sigma_x + q_y\sigma_y),$$

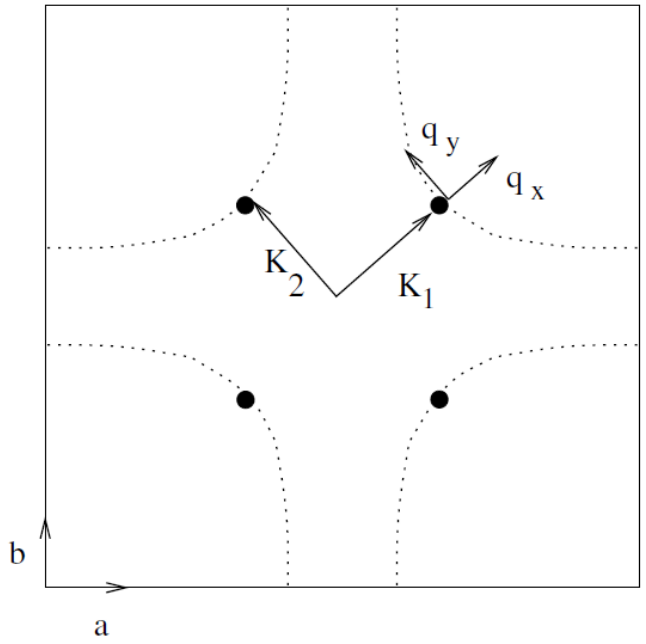
$$H_{K'} \simeq v_F(-q_x\sigma_x + q_y\sigma_y).$$

$$\Psi^\dagger_\sigma(\vec{x},\tau) = T\sum_{\omega_n}\int^\Lambda \frac{d\vec{q}}{(2\pi a)^2} e^{i\omega_n\tau + i\vec{q}\cdot\vec{x}}(u^\dagger_\sigma(\vec{K}+\vec{q},\omega_n)$$

$$v^\dagger_\sigma(\vec{K}+\vec{q},\omega_n), u^\dagger_\sigma(-\vec{K}+\vec{q},\omega_n), v^\dagger_\sigma(-\vec{K}+\vec{q},\omega_n))$$

Doublings: 2 (spin) x 2 (Dirac) x 2 (sublattice) = 8 fermion components.

Earlier and different example: d-wave superconductor



$$\Psi_i'^{\dagger}(\vec{q},\omega_n)=(c_+^{\dagger}(\vec{k},\omega_n),c_-(-\vec{k},-\omega_n),$$
$$c_+^{\dagger}(\vec{k}-\vec{Q}_i,\omega_n),c_-(-\vec{k}+\vec{Q}_i,-\omega_n))$$

with $\vec{Q}_i=2\vec{K}_i$

At low energies:

$$S[\Psi']=\int d^2\vec{r}\int_0^{\beta}d\tau\Psi_1'^{\dagger}[\partial_\tau+M_1 v_f\partial_x+M_2 v_\Delta\partial_y]\Psi_1'$$
$$+(1\rightarrow 2,x\leftrightarrow y)+O(\partial\Psi'^{\dagger}\partial\Psi',\Psi'^4),$$

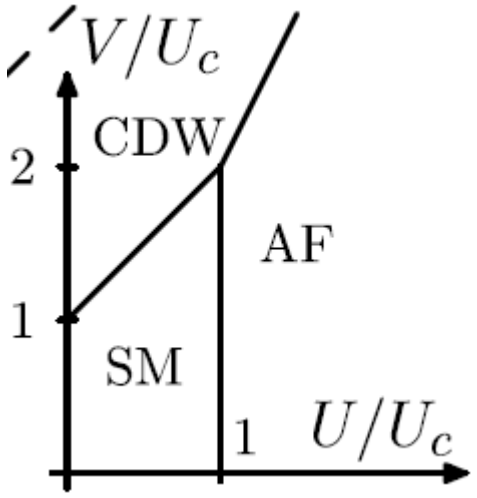
with matrices $M_1=i\sigma_3\otimes\sigma_3$ and $M_2=-i\sigma_3\otimes\sigma_1$. Again 8 fermion components!

Why are Dirac fermions here massless ?

Band crossing is symmetry-protected.

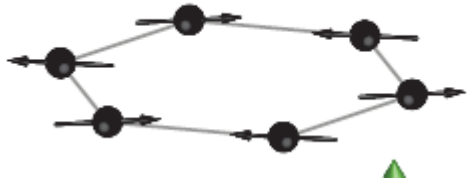
Some symmetry needs to break to open a “relativistic mass” gap.

Canonical example: Hubbard model on honeycomb (or pi-flux) lattice



(IH, 2006)

$U > U_c$: magnetic (Neel) Mott insulator



Chiral-Heisenberg universality class: (IH, Juricic, Vafek 2009)

Gross-Neveu-Yukawa:

$$S = \int d\tau d^{D-1}x \left(\mathcal{L}_\psi + \mathcal{L}_{\psi\phi} + \mathcal{L}_\phi\right)$$

where

$$\mathcal{L}_\psi = \bar{\psi}(x)\partial\!\!\!/\psi(x)\,,$$

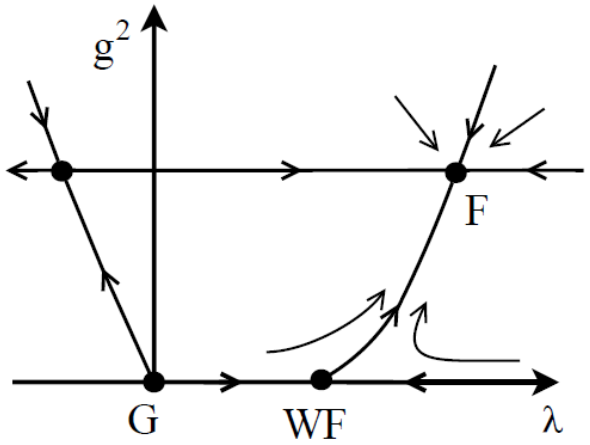
with Yukawa term

$$\mathcal{L}_{\psi\phi,\chi\mathrm{H}} = g\vec{\phi}\,\bar{\psi}\left(\vec{\sigma}\otimes\mathbb{1}_{2N}\right)\psi\,.$$

and O(3) order parameter action

$$\mathcal{L}_{\phi,\chi\mathrm{H}} = \frac{1}{2}\vec{\phi}\left(m^2 - \partial_\mu^2\right)\vec{\phi} + \lambda\left(\vec{\phi}\cdot\vec{\phi}\right)^2$$

Critical surface:



U=Uc : "relativistic Mott transition" (Assad, IH 2013, Toldin et al 2014, Otsuka et al 2016, Scherer, Marquard et al 2017, Gracey 2018, Wang et al 2026, ….)

Experiment:

Hubbard ratio U/t is too weak in monolayer graphene, but hopping depends strongly on the twist-angle in multilayers

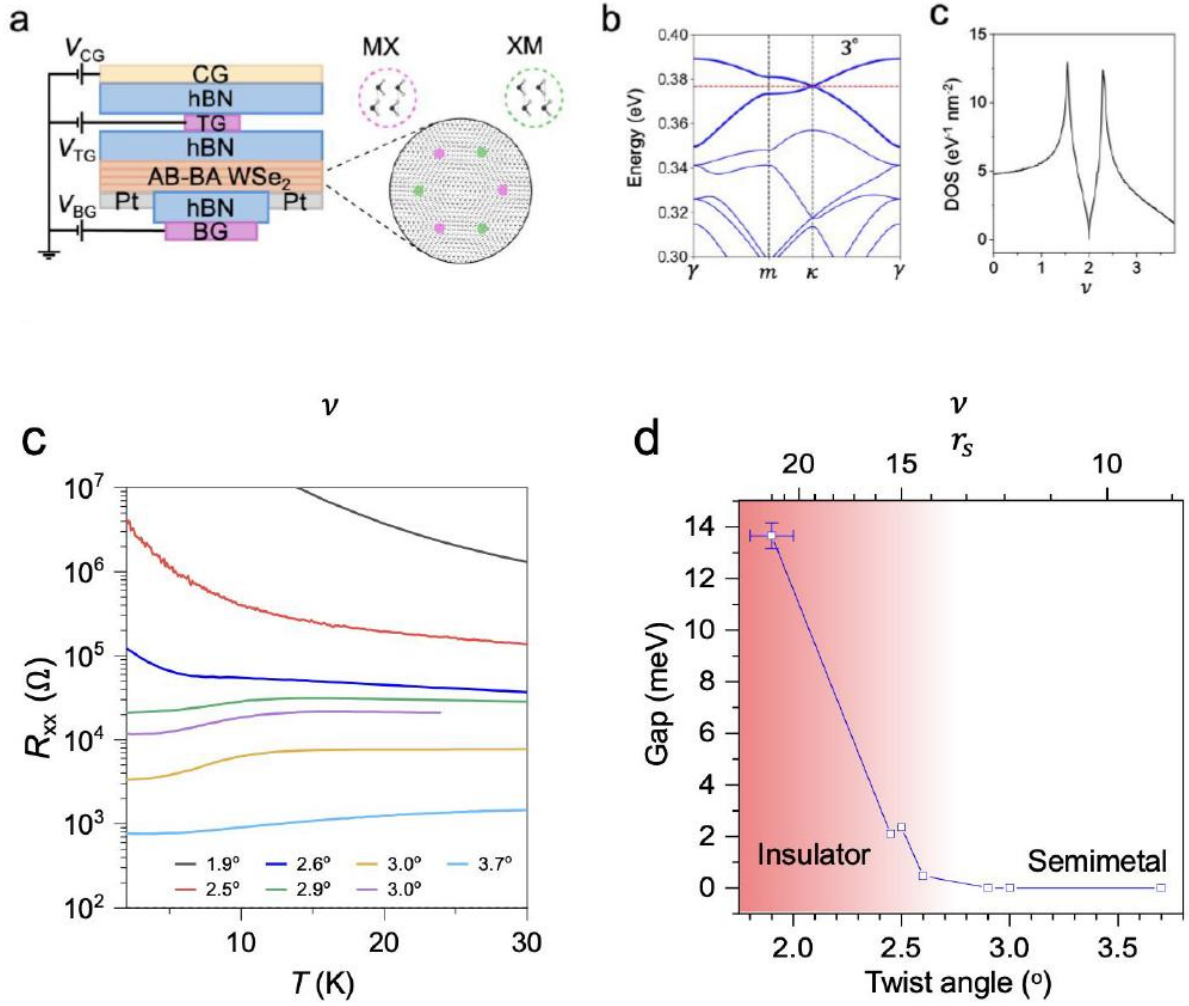
In tetralayer WSe_2:

Relativistic Mott transition in strongly correlated artificial graphene

Liguo Ma[1,2], Raghav Chaturvedi[1], Phuong X. Nguyen[1,3], Kenji Watanabe[4], Takashi Taniguchi[4], Kin Fai Mak[1,3,5*], Jie Shan[1,3,5*]

[1]School of Applied and Engineering Physics, Cornell University, Ithaca, NY, USA
[2]Zhejiang Key Laboratory of Micro-nano Quantum Chips and Quantum Control and School of Physics, Zhejiang University, Hangzhou, China
[3]Kavli Institute at Cornell for Nanoscale Science, Ithaca, NY, USA
[4]National Institute for Materials Science, Tsukuba, Japan
[5]Laboratory of Atomic and Solid State Physics, Cornell University, Ithaca, NY, USA

Figures



(experiment: Ma et al 2025, theory: Biedermann, Janssen 2026)

Remarks on symmetry: (IH, Mandal 2023)

Dirac theory, universal form: N copies of minimal ("Weyl") fermions

$$L_0 = \psi^\dagger (1_4 \otimes (1_2 \partial_\tau - i\sigma_1 \partial_1 - i\sigma_2 \partial_2))\psi$$

Besides space-time symmetries, there is a flavor SU(N):

$$\psi \to (U \otimes 1_2)\psi, \quad \psi^\dagger \to \psi^\dagger (U^\dagger \otimes 1_2),$$

In graphene: SU(2) (spin) x SU(2) (Dirac points) = SO(4) -> SU(4)

Masses: q. anom. Hall, all other insulators, and all gapped superconductors form irreps

1, 15, 10, and 10*

respectively, of flavor SU(4).

Hidden O(2N):

After rotation of σ_2 into σ_3 Dirac Hamiltonian becomes real and symmetric.

Defining Majorana fields as:

$$\chi = \chi_1 - i\chi_2 \qquad \chi^\dagger = \chi_1^T + i\chi_2^T$$

They decouple and the Lagrangian becomes

$$L = \phi^T [1_{2N} \otimes (1_2 \partial_\tau - i\sigma_1 \partial_1 - i\sigma_3 \partial_2)]\phi$$

For N=4 the Majorana field $\phi^T = (\chi_1^T, \chi_2^T)$ now has 16 components.

Enlarged symmetry follows from compatibility of reality and minimal ("Weyl") size conditions in D=2+1!

The result: grand unification of order parameters

$$10 + 10^* + 15 + 1 \rightarrow 35 + 1 \quad (\text{for } O(8))$$

35: symmetric 2^{nd}-rank tensor irrep.

QAH mass remains a singlet. (-> O(2N) is not anomalous)

Nematic orders form the irrep 28.

Interacting theory

(with O(2N) and Lorentz symmetries)

O(2N) allows only a unique interaction term (IH, Mandal 2023)

$$L_1 = g_1(\bar{\psi}(1_4 \otimes 1_2)\psi)^2$$

Gross-Neveu : when $g_1 < g_{1c} = -\pi/(4N\Lambda)$ O(2N)-singlet condenses:

$$\langle \bar{\psi}\psi \rangle \neq 0$$

Result: O(2N) preserved, inversion and time-reversal broken.

(Gross, Neveu 1974, . . . Gracey 1994, Eramilli 2023)

Gross-Neveu interaction, rewritten

$$(\phi^\intercal(\mathbb{I}_{2N}\otimes\sigma_2)\phi)^2 = \frac{-1}{N+1}(\phi^\intercal(\mathbb{S}_b\otimes\sigma_2)\phi)(\phi^\intercal(\mathbb{S}_b\otimes\sigma_2)\phi)$$ (IH, Mandal 2023)

For $g > g_{c2} \sim (N+1)/(N\Lambda)$ one finds

$$\langle\phi^\intercal(\mathbb{S}_b\otimes\sigma_2)\phi\rangle \neq 0$$

and O(2N) spontaneously broken as

$$\mathrm{SO}(2N) \to \mathrm{SO}(N)\times\mathrm{SO}(N)$$

New phase transition in the middle-aged Gross-Neveu model ?

Yes, but probably (fluctuation-induced) first-order!

Gross-Neveu-Yukawa theory (for symmetric M x M real tensor order parameter)

$$\bar{\mathcal{S}}_{\rm GNY} = \int d\tau d^d x [\frac{1}{2}(\phi^\intercal(\mathbb{I}_M \otimes (\partial_\tau - iv_F\sigma_1\partial_1 - iv_F\sigma_3\partial_2)\phi) + \frac{\bar{g}}{2}(\phi^\intercal(S\otimes\sigma_2)\phi)]$$
$$+ \int d\tau d^d x [\frac{1}{\bar{M}_{\rm Tr}}[\frac{1}{2}({\rm Tr}[(\partial_\tau S)^2 + v_B^2(\nabla S)^2 + rS^2] + \frac{\bar{\lambda}_1}{4\bar{M}_{\rm Tr}}({\rm Tr}[S^2])^2 + \frac{\bar{\lambda}_2}{4}{\rm Tr}[S^4])]$$

There are two quartic couplings, and the Yukawa coupling. (Han, IH, 2024)

Close to D= 3+1: introduce $\epsilon = 3 - d$

$$\frac{d\alpha_g}{d\ell} = \epsilon\alpha_g - \frac{\alpha_g^2}{8}[2M_s + MN_f + 2(M-2)],$$
$$\frac{d\lambda_1}{d\ell} = \epsilon\lambda_1 - \frac{MN_f\alpha_g\lambda_1}{4} - \frac{(M^2+M+14)}{8}\lambda_1^2 - \frac{(2M^2+3M-6)}{4}\lambda_1\lambda_2 - \frac{3(M^2+6)}{8}\lambda_2^2$$
$$\frac{d\lambda_2}{d\ell} = \epsilon\lambda_2 - \frac{MN_f\alpha_g\lambda_2}{4} - 3\lambda_1\lambda_2 - \frac{(2M^2+9M-36)}{8}\lambda_2^2 + \alpha_g^2\frac{MN_f}{4},$$

Runaway flow without additional fermions!

(Pelissetto, Tripodo, Vicari, 2018)

Conclusion:

1) Ultra-relativistic (i. e. massless) fermions emerge at low energies in diverse condensed matter systems.

2) Strong (local or gauge-mediated) interactions may (or may not!) break some of their rich symmetry and introduce a gap.

3) Such quantum phase transitions define new “chiral” universality classes and can often be studied efficiently numerically on a lattice as well.