Solving (type 1) sign-problems with PIKs

10/09/2026 ECT* Workshop
Bridging Machine Learning and Statistical Field Theory through generative and diffusion processes

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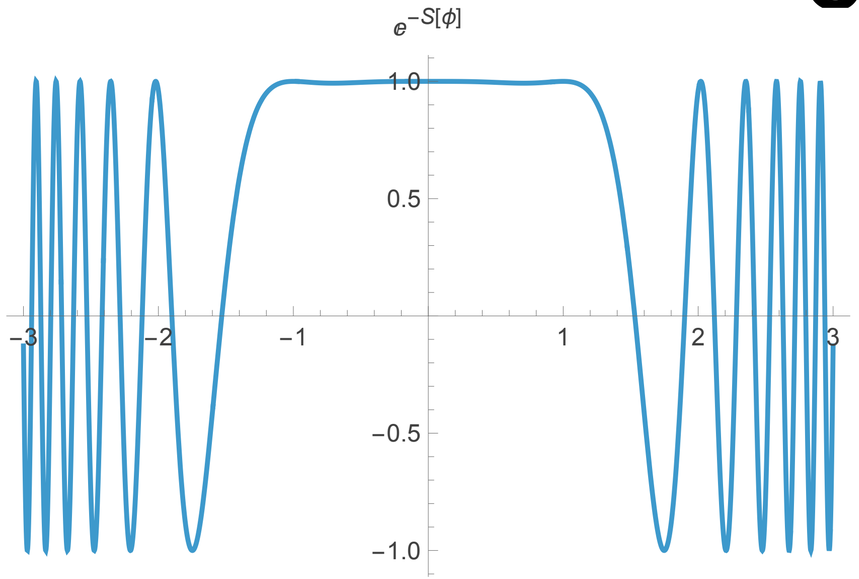
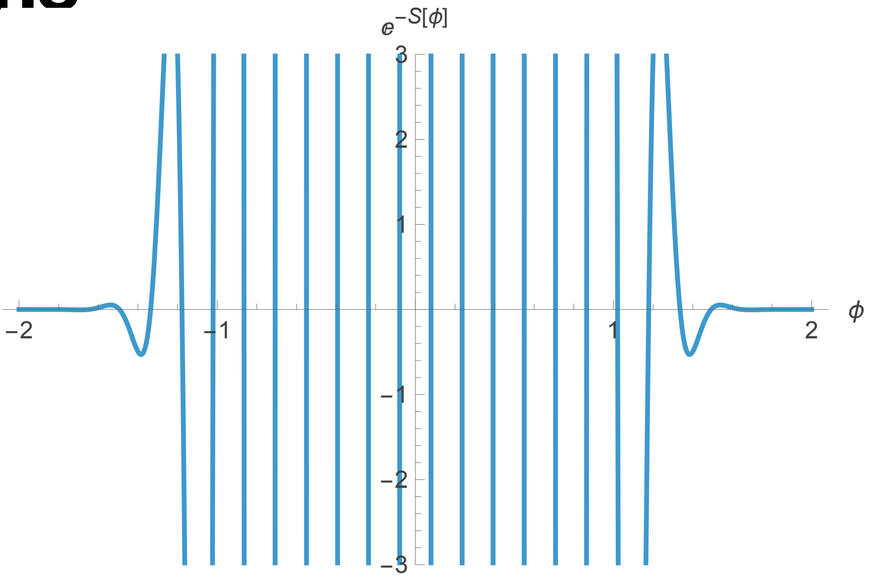
Me :)

Classifying sign problems

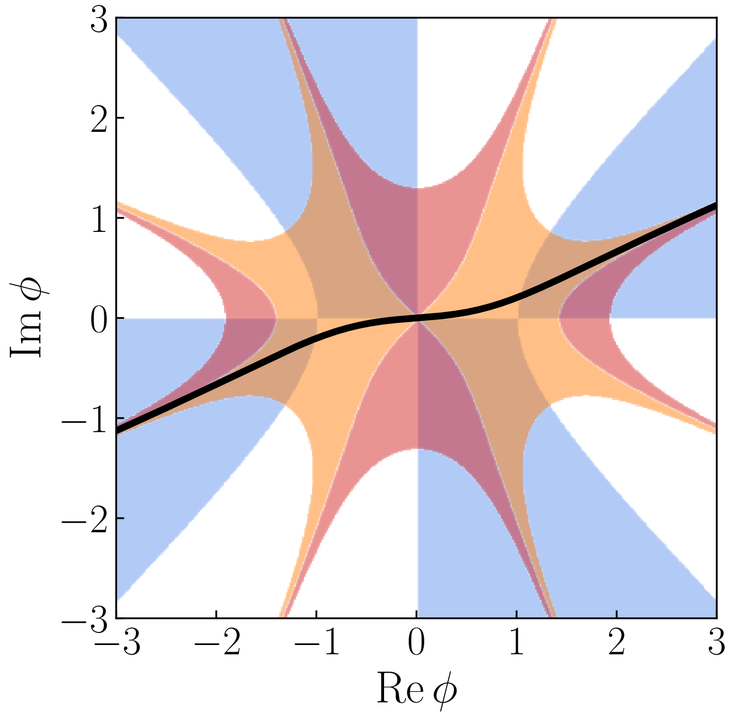




Type 1: Admit a real PIKfold

→ Real-time physics

→ Complex couplings

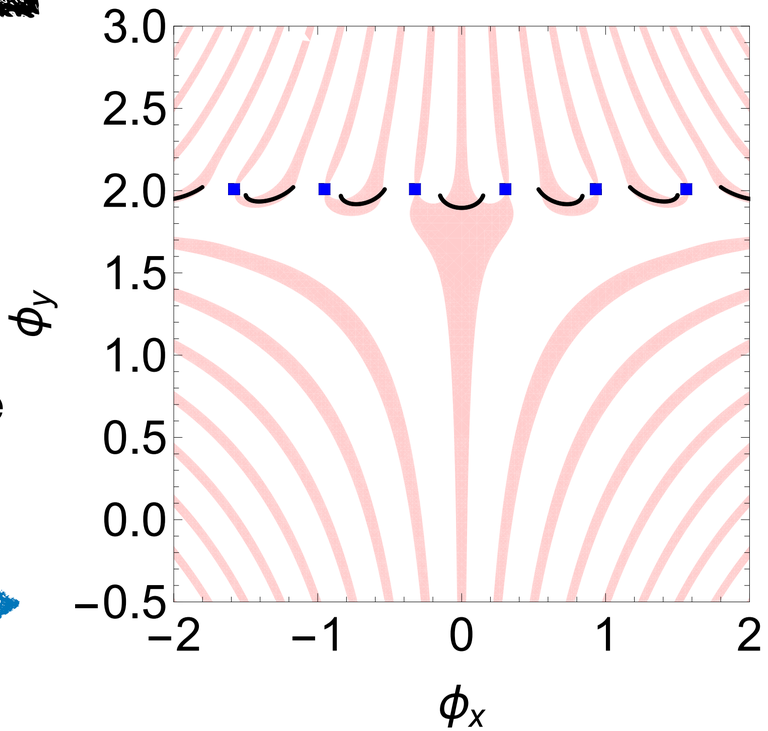


Type 2: No real PIKfold

→ Fermionic sign problem

→ Poles in the complex structure

Renzo's talk this afternoon!



The transport task from an (functional) RG perspective

- Original idea: Compression of information under repeated block-spinning transformations $\dot{\phi} : \phi_i(x) \mapsto \phi_{i+1}(x)$

  → Keep the number of sites intact/infinite lattice

  - Consider infinitesimal field reparametrisations $\phi \to \phi + \delta t \dot{\phi}(\phi)$

    → **Change of variables** in the path integral and changing distribution $p_t \to p_t + \delta t \frac{\partial}{\partial t} p[\phi]$

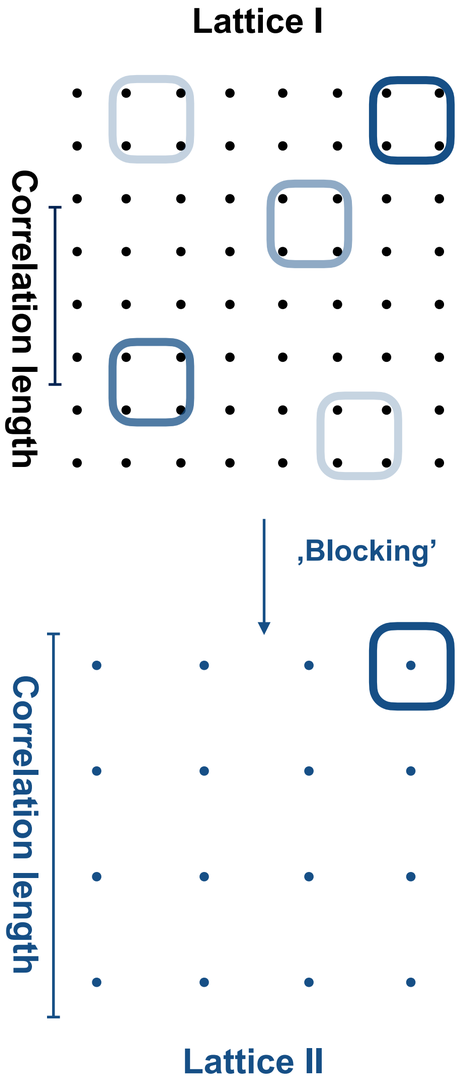
Wegner flow

Wegner, J. Phys. C 7 (1974) 2098

$$\frac{d\,p_t[\phi]}{dt} = -\int_x \frac{\partial}{\partial\phi(x)} \left\{ \dot{\phi}_t[\phi]\, p_t[\phi] \right\} \qquad p_t[\phi] \propto e^{-S_t[\phi]}$$

Diffusion

→ **Continuity equation** for the overall distribution



- Most modern momentum space RG does not use block-spinning anymore:

  → (Non-linear) **Fokker-Planck** dynamics:

  Polchinski-type kernel:

Scale transformations

Additional Reparametrisations

$$\dot{\phi}(\phi) = \frac{1}{2}\mathscr{C}(\phi)\frac{\partial}{\partial\phi}S_t(\phi) + \dot{\phi}_{\mathsf{r}}(\phi)$$

Selection/Filtering of propagating modes, e.g. $\mathscr{C} \propto -G\partial_t R_k G$

Gradient flow

Scale transformations in the path integral

Consider the generating functional for a theory: $Z_t[J] = \int D\phi\, p_t[\phi]\, e^{J\hat{O}_{\phi,t}[\phi]}$

← Current coupled to some generic operator

Some effective degree of freedom

Scale dependent distribution: $p_t[\phi] = \frac{1}{\mathcal{N}} e^{-S_t[\phi]}$

ϕ is in the **co-moving coordinate frame** of the block spinning, i.e. $\partial_t \phi = 0$ (passive transformation)

→ Use the scale dependent (composite) operator: $\hat{O}_{\phi,t}[\phi]$ as a **book-keeping device** $\hat{O}_{\phi,t_0}[\phi] = \phi$

Physics-informed setup

$$\partial_t Z_t[J] = \int D\phi \left\{ \underbrace{\partial_t p_t[\phi]}_{\text{Wegner flow}} + \underbrace{J\, \partial_t \hat{O}_{\phi,t}\, p_t[\phi]}_{\text{Change of the composite operator}} \right\} e^{J\hat{O}_{\phi,t}[\phi]}$$

Two strategies

1) $\partial_t Z_t \equiv 0 \quad \Rightarrow \quad \hat{O}_\phi$ completely absorbs the flow
Sampling

2) $\partial_t Z_t \neq 0 \quad \Rightarrow \quad \hat{O}_\phi$ corrects the existing flow
Modern functional RG

Sampling with the transport maps

Inverse map: $\hat{O}_{\phi,t_0}[\phi]$

t_0 — RG-Flow $\partial_t p_t \neq 0$, with passive transformation $\partial_t \phi = 0$ — t_1

Active transformation: $\phi_{t_1}[\hat{O}_{\phi,t_0}[\phi]] = \phi$

Takeaway:

Integrating $\dot{\phi}[\phi]$ for microscopic transformations gives us a map for the field transformation

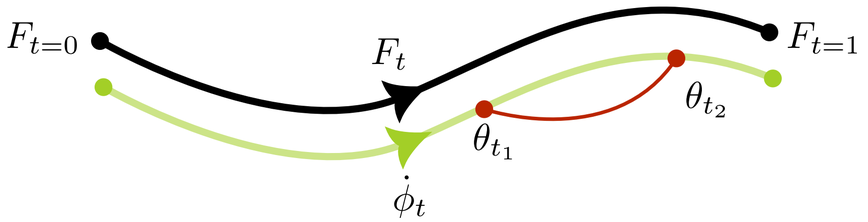
$$\phi_t = \phi_{t_0} + \int_{t_0}^{t} ds\, \dot{\phi}_s[\phi_s]$$

- We use $\partial_t Z = 0$ to **reconstruct the coarse-graining map**
- For general holomorphic operators $O[\phi]$ one can show

Observable in a hard theory — Sample from easy distr. — Map to $t = t_1$

$$\int D\phi\, p_t[\phi] O[\phi] = \int D\varphi\, p_0[\varphi] O[\phi_t[\varphi]]$$

→ This is valid for any pair $\left(p_t[\phi], \dot{\phi}[\phi]\right)$ that fulfills the Wegner equation



→ **Physics-informed kernels**: pick a suitable constraint to fix the pair, e.g.

- Langevin evolution
- Diffusion models: so many people…
- Polchinski kernel: Y. Ashida
- Perfect blocking: A. Hasenfratz

Sample p_t with $\phi_t(\phi_0)$, where $\phi_0 \sim p_0$

FI, Kapust, Pawlowski, arxiv:2510.26678
FI, Kapust, Pawlowski, arxiv:2603.03159

Generalised RG flows: Strategy 2

Scale transformations — Reparametrisations

$$\partial_t Z_t[J] = \int D\phi \left\{ \partial_t p_t[\phi] + J\, \partial_t \hat{O}_{\phi,t}\, p_t[\phi] \right\} e^{J \hat{O}_{\phi,t}[\phi]}$$

Wegner flow — Change of the composite operator

Implement the **scale suppression** explicitly

$$p_t[\phi] = \frac{1}{\mathcal{N}} e^{-S[\phi] - \Delta S_t[\phi]}$$

By allowing $\partial_t Z_t \neq 0$ we remain with a very general reparametrisation freedom

$$\partial_t \hat{O}_{\phi,t}$$

Sampling

Mean fields: $\dot{\phi} = \langle \partial_t \hat{O}_{\phi,t} \rangle\,, \quad \phi = \langle \hat{O}_{\phi,t} \rangle$

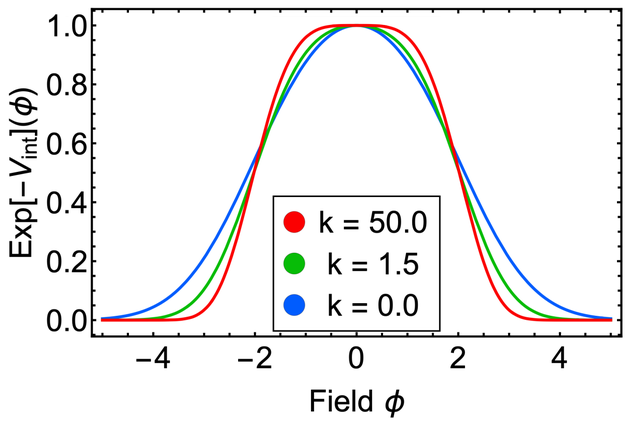
Direct computation of cumulants & optimisation

$$\frac{d\, p_t(\phi)}{dt} = -\int_x \frac{\partial}{\partial \phi(x)} \left[\dot{\phi}_t(\phi)\, p_t(\phi) \right]$$

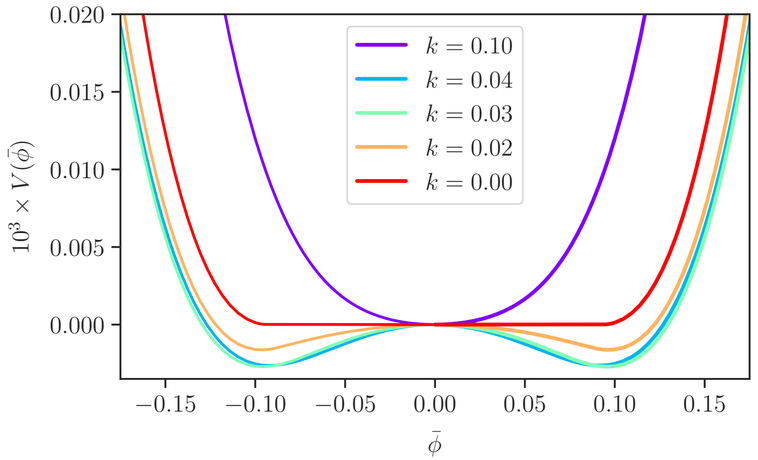
Legendre transform

$$\left(\partial_t + \int_x \dot{\phi}[\phi] \frac{\delta}{\delta \phi} \right) \Gamma_\phi[\phi] = \frac{1}{2} \mathrm{Tr} \left[G_\phi[\phi] \left(\partial_t + 2 \frac{\delta \dot{\phi}}{\delta \phi} \right) R_k \right]$$

Distribution



Rate function



Generalized flow equation

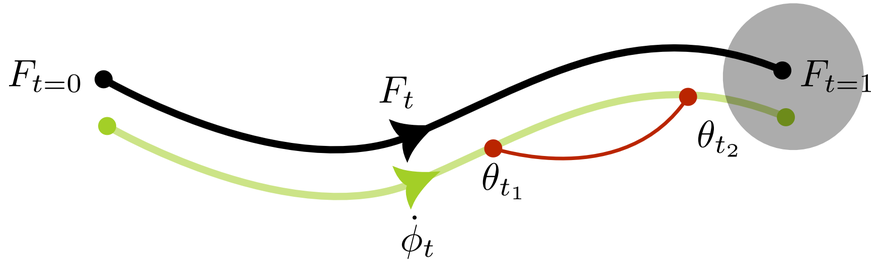
Pawlowski, *Annals Phys.* 322 (2007) 2831

Physics-informed RG

FI, Pawlowski, Annals Phys. 481 (2025) 170177

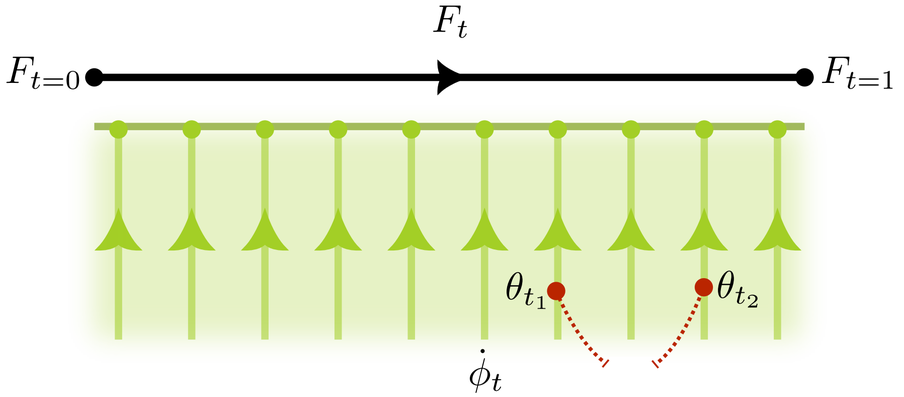
The physics-informed mindset

Allowing general time-dependent paths for distributions
$F_t = \{S_t, p_t, \ldots\}$ does not guarantee the desired physical distribution



…so we impose a trajectory analytically. Then the Wegner equation

$$\partial_t S_t + \dot{\phi}\frac{\delta}{\delta\phi}S_t = \frac{\delta}{\delta\phi}\dot{\phi} \longrightarrow RG_1 + \dot{\phi}RG_2 = \frac{\delta}{\delta\phi}\dot{\phi}$$

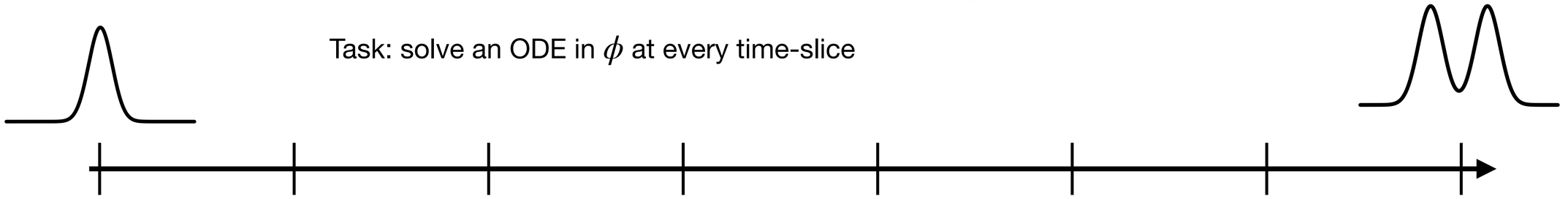


…is a differential equation at every time-slice!

Example: zero-dimension ϕ^4 - theory

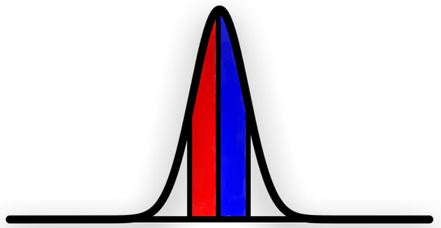
$$\hat{S}_t(\phi) = \frac{1}{2}m^2(t)\phi^2 + \frac{\lambda(t)}{4}\phi^4$$

Task: solve an ODE in ϕ at every time-slice

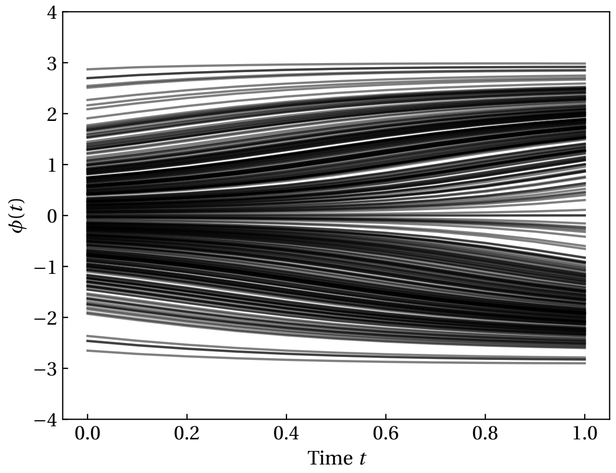


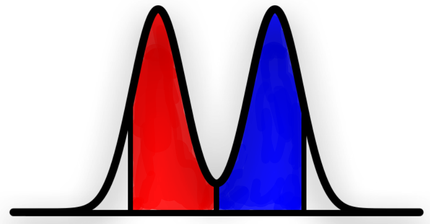
$(m_0^2\,,\,\lambda_0) = (1\,,\,0)$ → Analytic interpolation between couplings → $(m_1^2\,,\,\lambda_1) = (-2\,,\,1/2)$

The weight preserving property



$(m_0^2, \lambda_0) = (1, 0)$





$(m_1^2, \lambda_1) = (-2, 1/2)$

$$\int_{\mathscr{M}_0} p_0(\phi)\, d\mu_0(\phi)$$

Where $\mathscr{M}_0$ is mapped to $\mathscr{M}_1$

$$= \int_{\mathscr{M}_1} p_1(\phi)\, d\mu_1(\phi)$$

…what about complex actions?

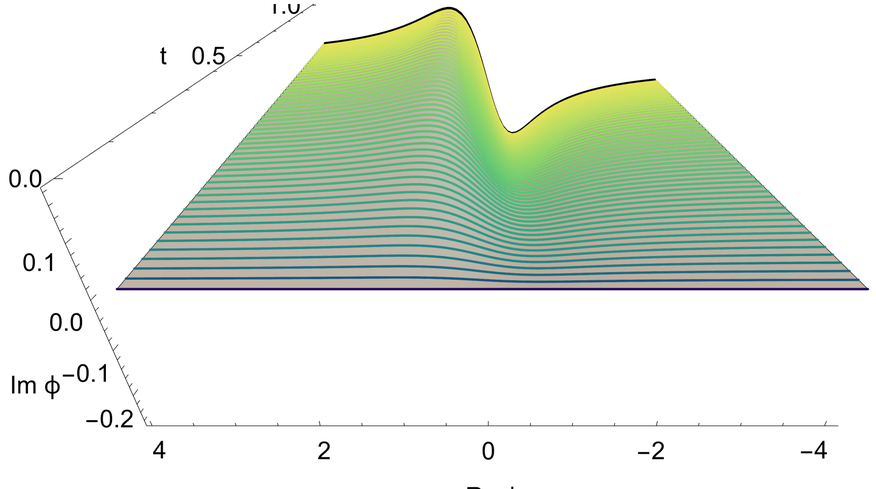
$(m_0^2, \lambda_0) = (1,1)$ → $(m_1^2, \lambda_1) = (1 + 2i, 1)$: $S_t \in \mathbb{C}$

PIK pair becomes complex:

$$\left(p_t[\phi], \dot{\phi}[\phi]\right) \in \mathbb{C}$$

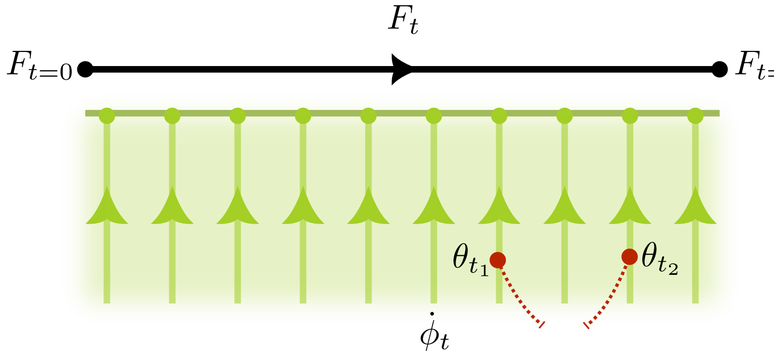
Weight preserving contour deformation with

$$\left[d\phi_R + i\, d\phi_I\right] \left[\mathrm{Re}(p_t(\phi)) + i\mathrm{Im}(p_t(\phi))\right] \in \mathbb{R}$$



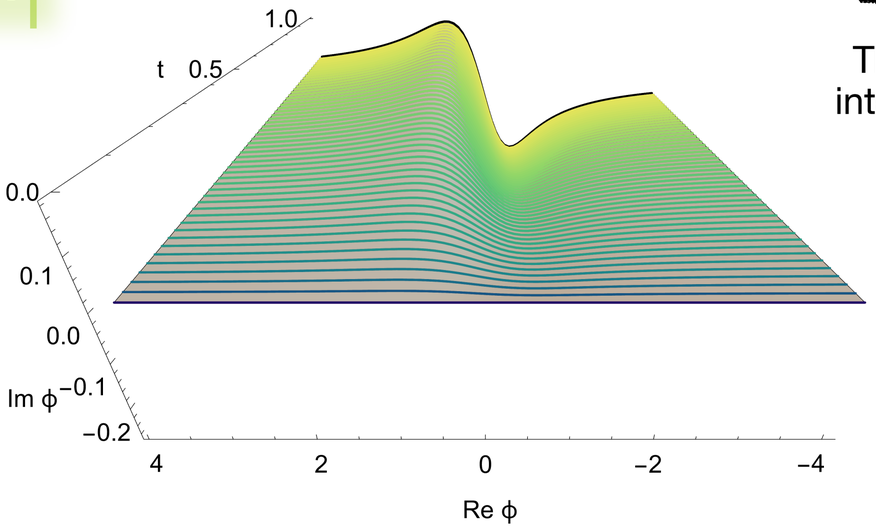
FI, Kapust, Pawlowski, arxiv:2510.26678
FI, Kapust, Pawlowski, arxiv:2603.03159

Summary: Workflow of the PIK



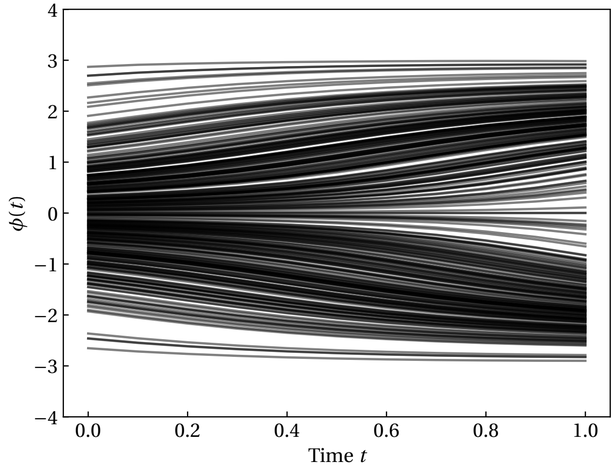
$$\partial_t S_t + \dot{\phi}\frac{\delta}{\delta\phi}S_t = \frac{\delta}{\delta\phi}\dot{\phi}$$

Compute the complex (holomorphic) kernel at each step



This deforms the integration contour

Transport samples on the PIKfold



$$\left(p_0, \mathbb{R}\right) \mapsto \left(p_t, \mathscr{M}_t\right), \quad \mathscr{M}_t \in \mathbb{C}$$

$$\phi_t = \phi_{t_0} + \int_{t_0}^{t} ds\, \dot{\phi}_s[\phi_s]$$

FI, Kapust, Pawlowski, arxiv:2510.26678
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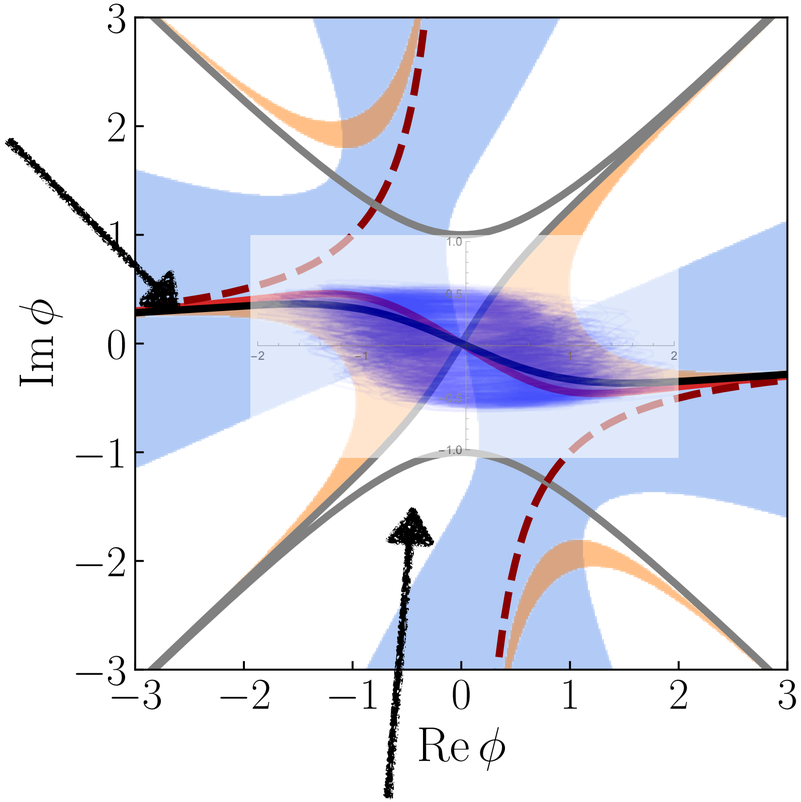
The complex context

PIK sampling manifold at t=1



Consider the complex path

$$(m_0^2, \lambda_0) = (1,1) \longrightarrow (m_1^2, \lambda_1) = (1+2i, 1) \quad : \; S_t \in \mathbb{C}$$

: Areas without sign-problem : Stokes wedges

- **PIK:** Is the solution to the Wegner equation <> **convective transport**

  → Determines a unique **sampling manifold**

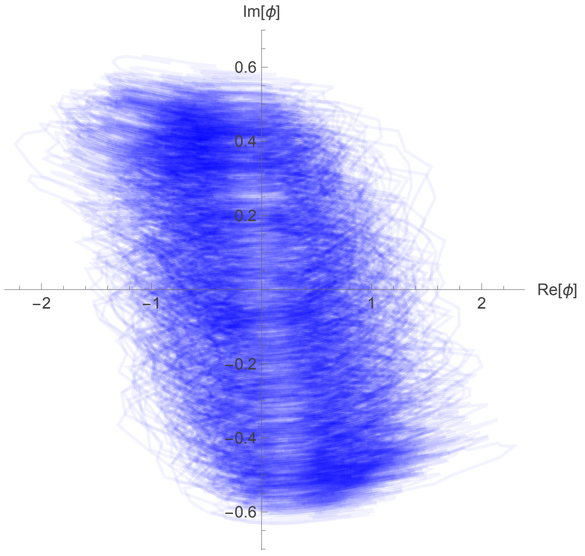
- Distinguishes itself from to **Lefschetz thimbles**

$$\partial_t S_t + \dot{\phi}\frac{\delta}{\delta\phi}S_t = \frac{\delta}{\delta\phi}\dot{\phi} \quad \Longleftrightarrow \quad 0 \stackrel{!}{=} \partial_t \tilde{S}_t(\phi_t) = \partial_t \tilde{S}_t(\phi_t) + \partial_t\phi_t \frac{\delta}{\delta\phi_t}\tilde{S}_t(\phi_t)$$

  → PIK uses curvature in its definition: Thimbles don't and keep a **re-weighting step**

- Stochastic nature of **kernel CLE** creates an area

$$d\phi = -H(\phi)S'(\phi)\,dt + \sqrt{2H(\phi)}\,dW$$



FI, Kapust, Pawlowski, arxiv:2510.26678
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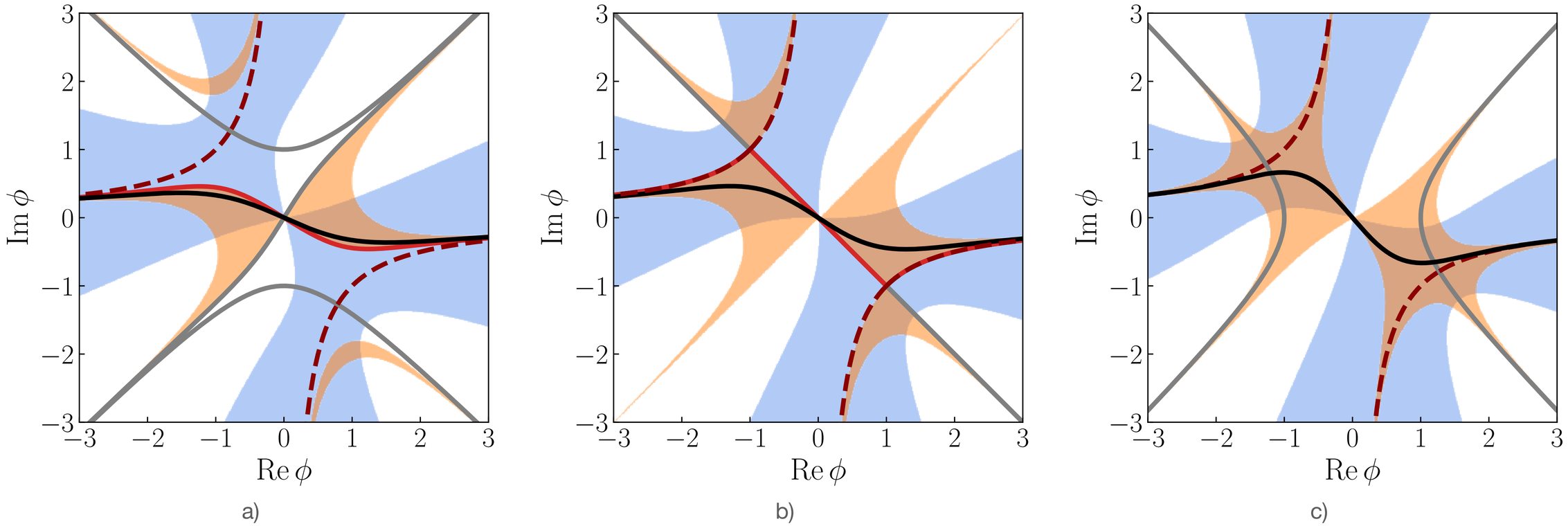
Comparison to the thimble structure



a) b) c)

: Areas without sign-problem

: Stokes wedges

: Thimbles

: Anti-Thimbles

$$\hat{S}_t(\phi) = \frac{1}{2}m^2(t)\phi^2 + \frac{\lambda(t)}{4}\phi^4$$

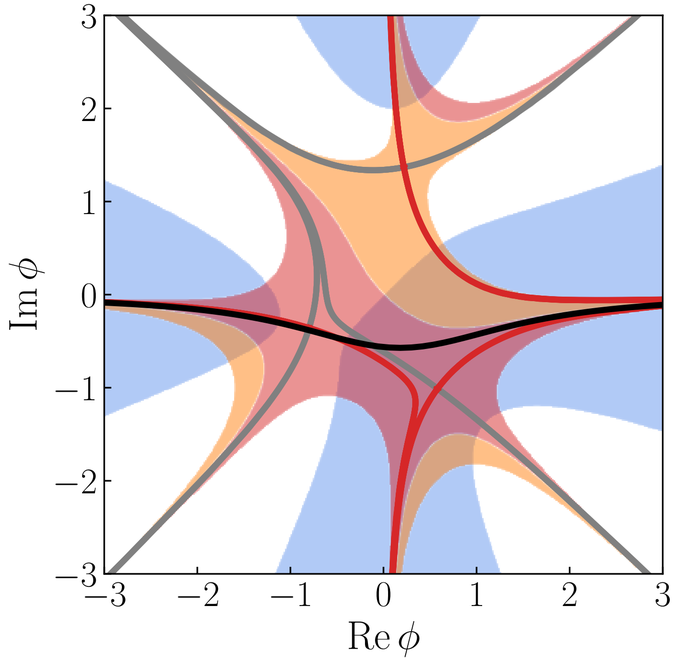
There is a unique PIKfold, but possibly many thimbles

$a)$ $(m_1^2, \lambda) = (1+2i, 1)$

$b)$ $(m_1^2, \lambda) = (2i, 1)$

$c)$ $(m_1^2, \lambda) = (1-2i, 1)$

Sampling

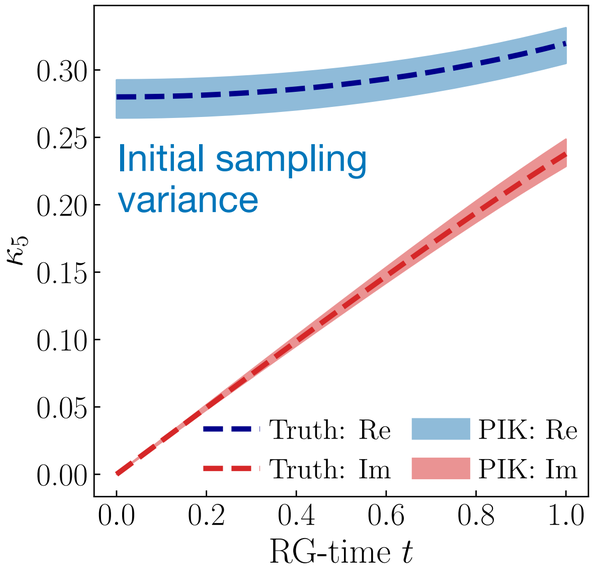


By using Cauchy's theorem, we can directly sample on the PIKfold consider the action

$$S_t(\phi) = \frac{m^2(t)}{2}\phi^2 + \frac{\lambda(t)}{4}\phi^4 + h(t)\phi + \mathcal{N}_t$$

→ No poles between the PIKfold and the real axis

→ Contour within a Stokes-wedge

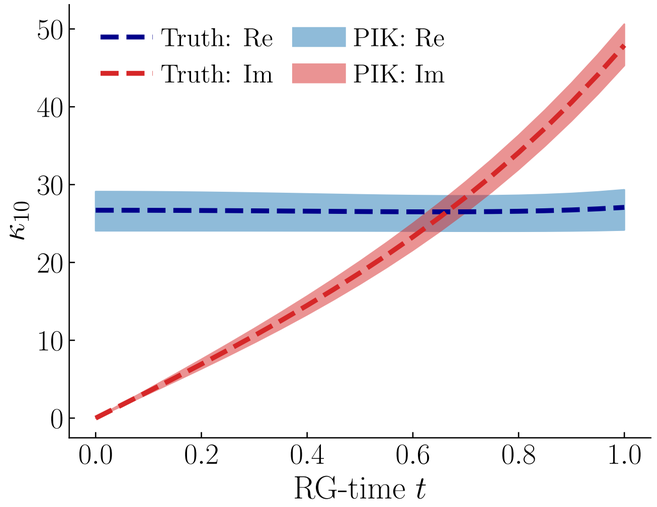


$$\int_{\mathbb{R}} D\phi\, p_t[\phi] O[\phi] = \int_{M_t} D\phi\, p_t[\phi] O[\phi] = \int_{\mathbb{R}} D\varphi\, p_0[\varphi] O[\phi_t[\varphi]]$$

Cauchy: Close contour

Practically, we sum:

$$\langle \hat{O} \rangle = \frac{1}{N} \sum_i \phi_t(\varphi_i), \quad \varphi_i \sim p_0$$



Sampling with $N = 10^5$ samples

Sampling on the contour is real by the weight preserving property:

$$d\phi\, p_0(\phi) = \left[d\phi_R + i\, d\phi_I\right] \left[\mathrm{Re}(p_t(\phi)) + i\mathrm{Im}(p_t(\phi))\right] \in \mathbb{R}$$

The real time harmonic oscillator

Real time theories can be obtained by rotating the euclidean couplings into the complex plane

$$\hat{S}_t(\phi) = \sum_{i\in\mathscr{D}} \frac{\kappa_t}{2}(\phi_{i+1}-\phi_i)^2 + \left(m_t^2+\epsilon\right)\frac{\phi_i^2}{2} \qquad \kappa_t = e^{-i\frac{\pi}{2}t} \qquad m_t^2 = m^2 e^{+i\frac{\pi}{2}t}$$

In case of the harmonic oscillator the PIK can be solved analytically:

→ **Non-sparse transformation** already for a free theory

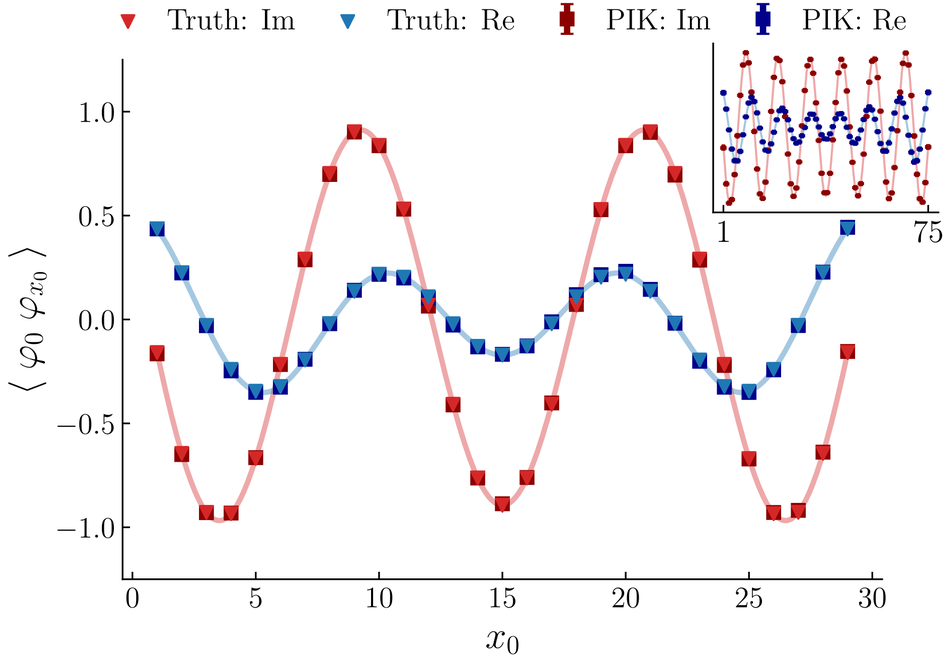
→ Important **first benchmark**

Analytic result, no learning, no guessing

$$\dot{\phi}_t(\phi) = -\frac{1}{2}M_t^{-1}\frac{dM_t}{dt}\phi \qquad \hat{S}_t(\phi) := \frac{1}{2}\phi_i\, M_{t,ij}\phi_j$$

Two point function of the real-time theory in 0+1 dimensions L=30 lattice sites:

FI, Kapust, Pawlowski, arxiv:2603.03159



The remaining task in interacting systems is to solve the **high dimensional PDE**

$$\partial_t S_t + \dot{\phi}\frac{\delta}{\delta\phi}S_t = \frac{\delta}{\delta\phi}\dot{\phi}$$

→ PDE is **linear**

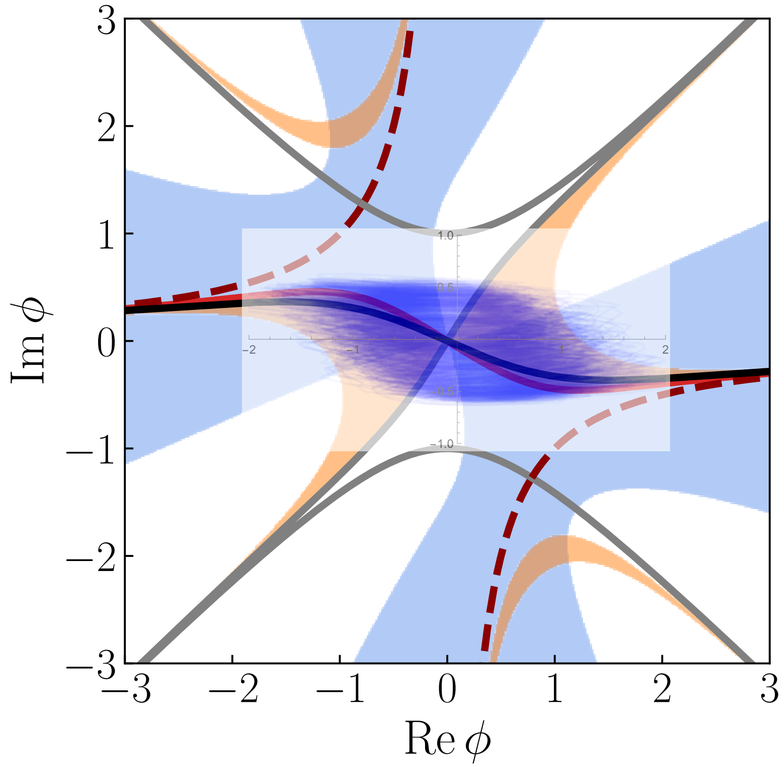
→ Analytic access allows for **systematic improvement** of the PIK

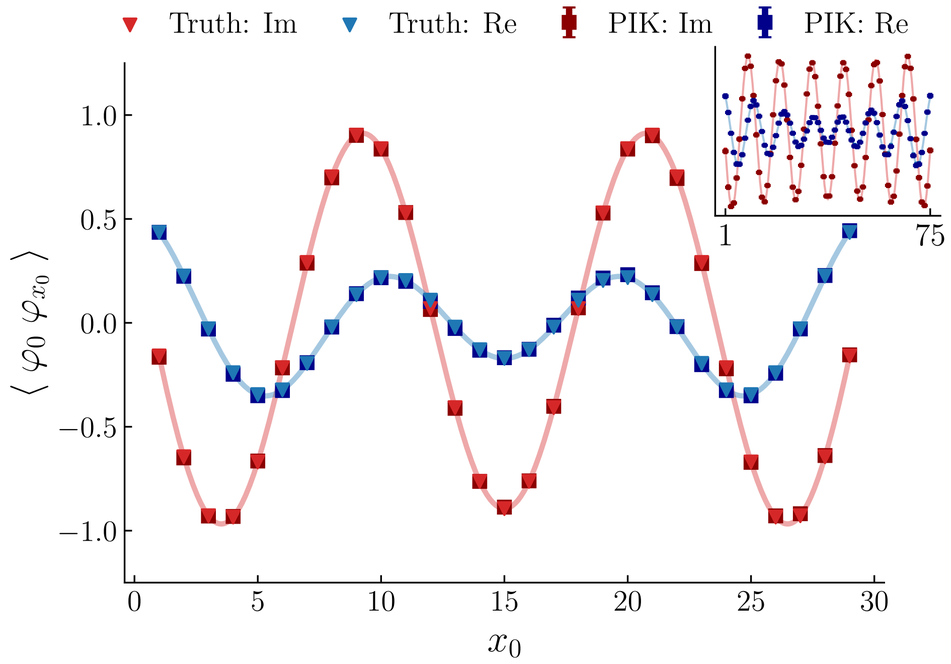
Conclusion

- The Wegner-equation encompasses all **RG schemes** as well as **generative approaches** in lattice field theory
- Sign-problems of the **first type** are solvable with a single real PIKfold

  → **Weight-preserving property** deforms contour and allows for real sampling in the complex plane

  → Comparison to thimbles and kernel CLE
- Outlook: Applications to finite dimensional lattice systems





Thank you for you attention!