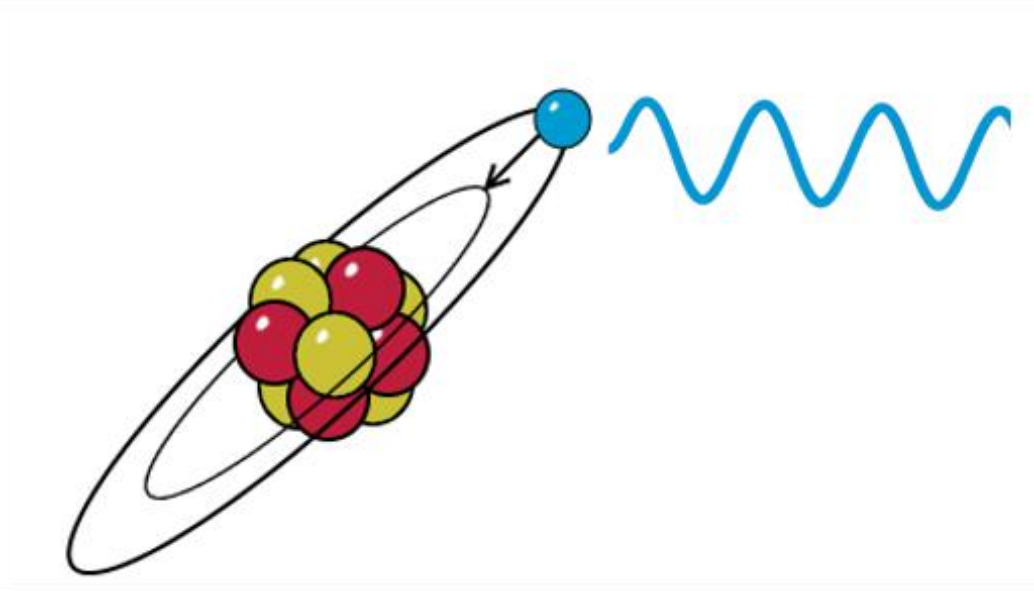
Max-Planck-Institut
für Kernphysik
Heidelberg

Bound-State QED Calculations

Jonas Sommerfeldt, 03.08.2026, ECT*

Atomic Transitions to Measure Nuclear Structure

Atomic Transition



$$E = E_{\mathrm{Dirac}} + E_{\mathrm{Nuclear}} + E_{\mathrm{QED}}$$

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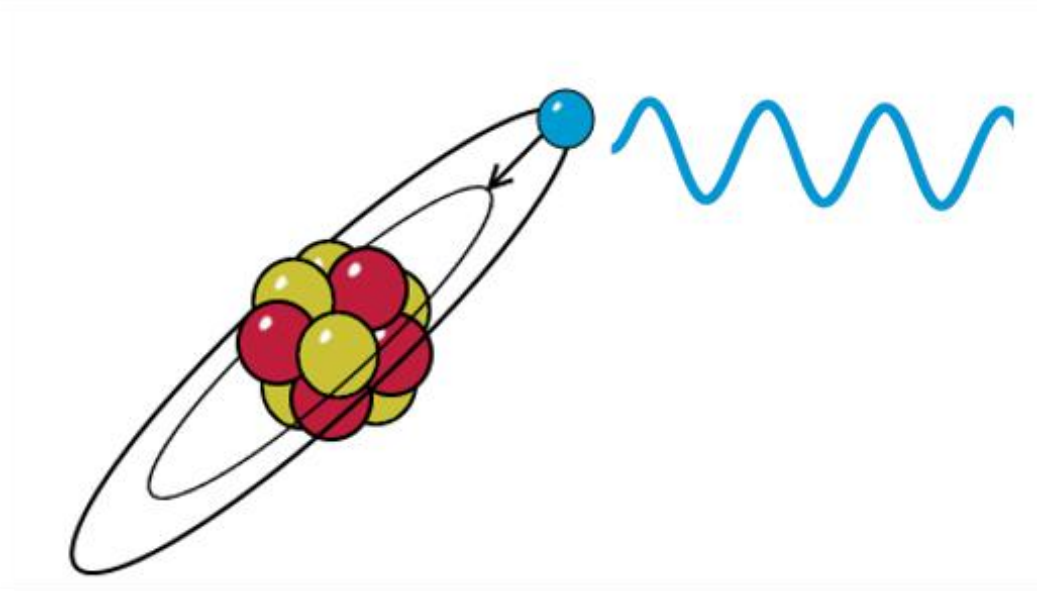
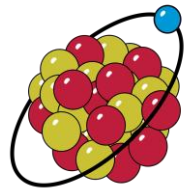


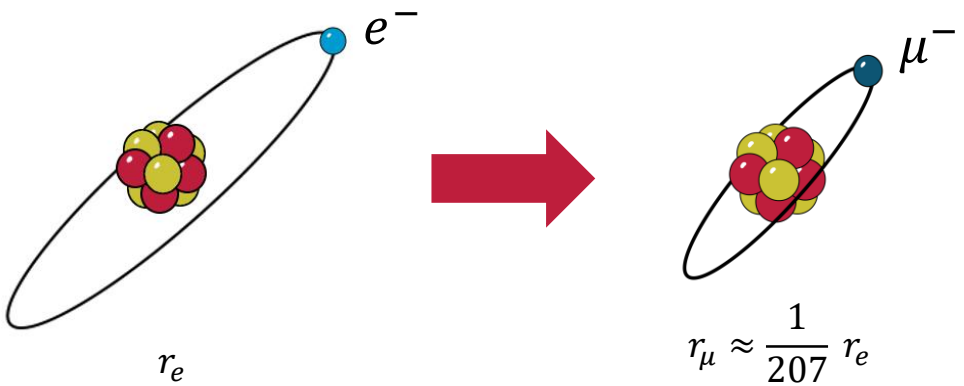
$$E = E_{\text{Dirac}} + \boxed{E_{\text{Nuclear}}} + E_{\text{QED}}$$

Extracting Nuclear Structure

Highly Charged Ions



Exotic Ions



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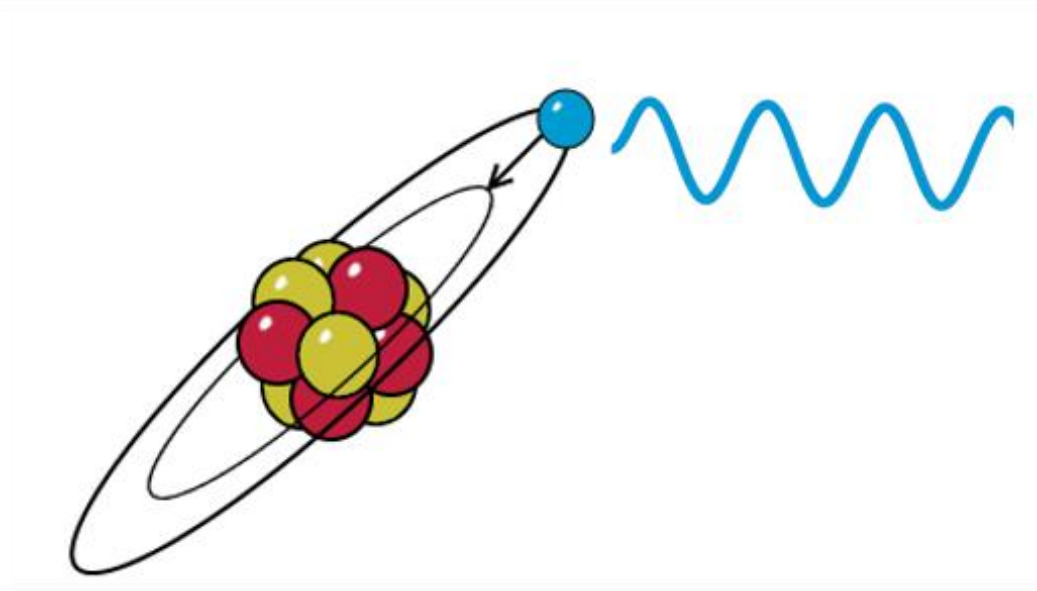
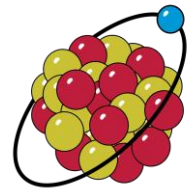


$$E = E_{\text{Dirac}} + E_{\text{Nuclear}} + E_{\text{QED}}$$

QED Calculations need to be sufficiently accurate!

Extracting Nuclear Structure

Highly Charged Ions



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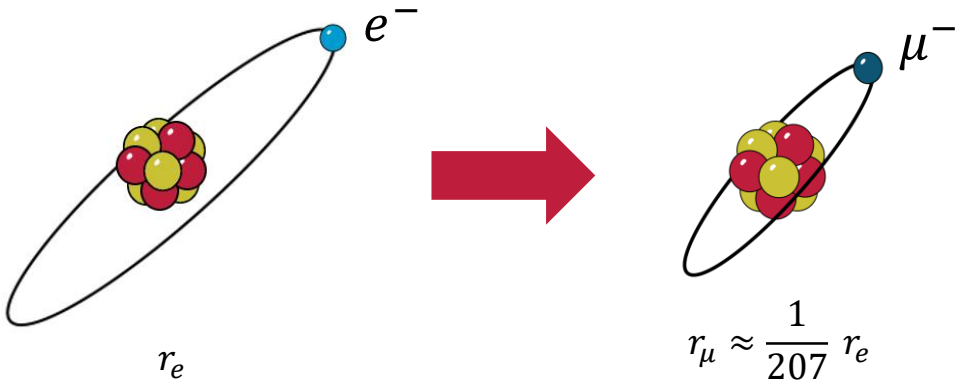
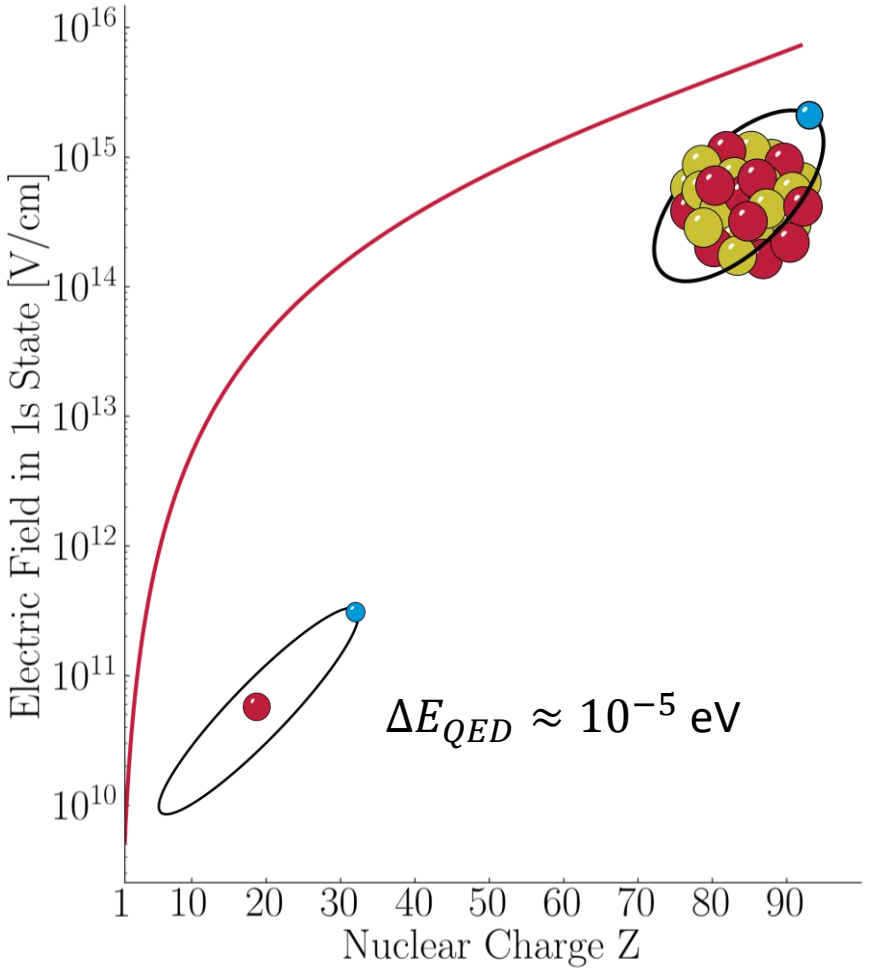


Heavy Ions as a Testbed for Strong-Field QED



$\Delta E_{QED} \approx 500$ eV

$\Delta E_{QED} \approx 10^{-5}$ eV

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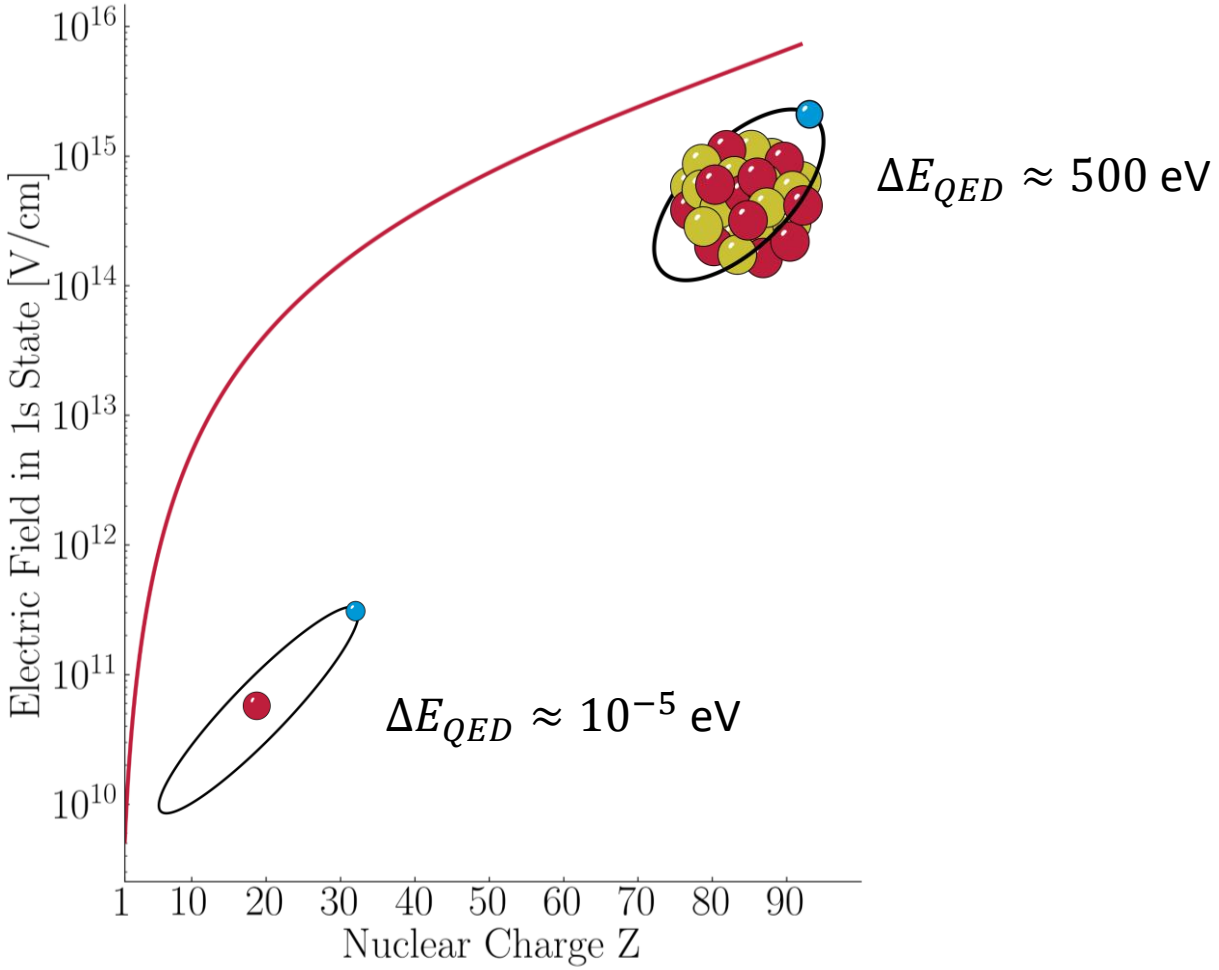

Measuring Atomic Properties in highly charged ions probes QED in the strong field regime!

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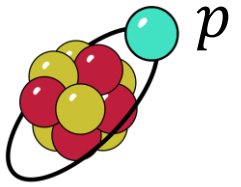
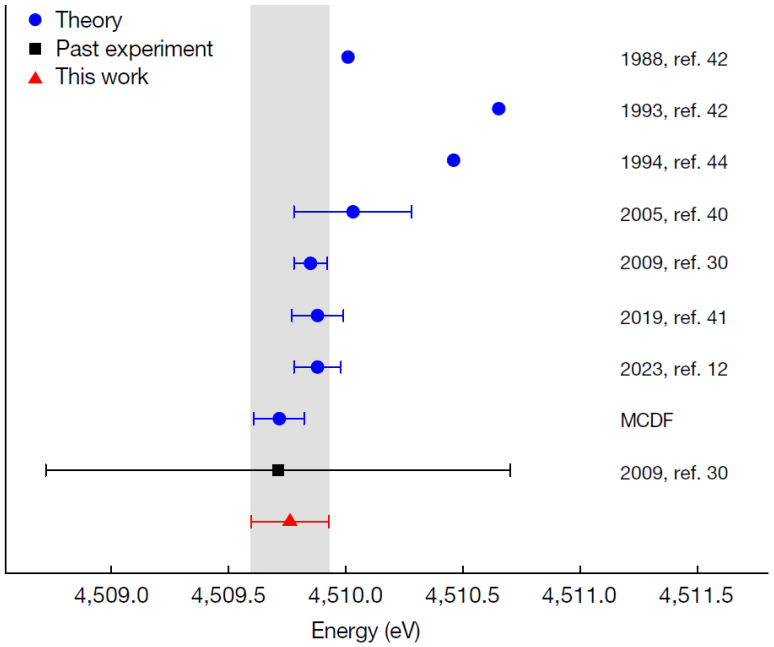
Exotic Atoms are more sensitive to certain QED effects

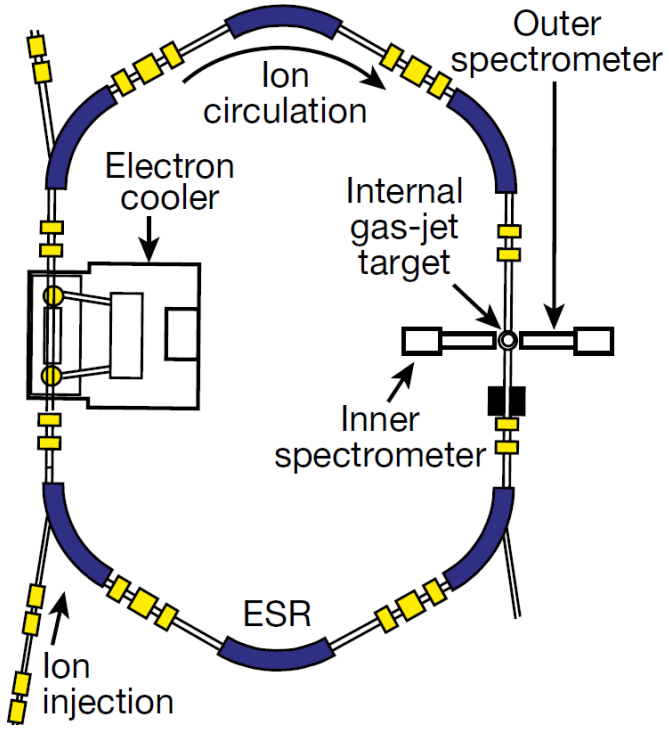
PA
ANTIPROTONIC ATOM X-RAY SPECTROSCOPY



Standard Model Tests in the Strong-Field Regime

QED Tests with Highly Charged Ions





R. Loetzsch et al. Nature 625, 673–678 (2024)

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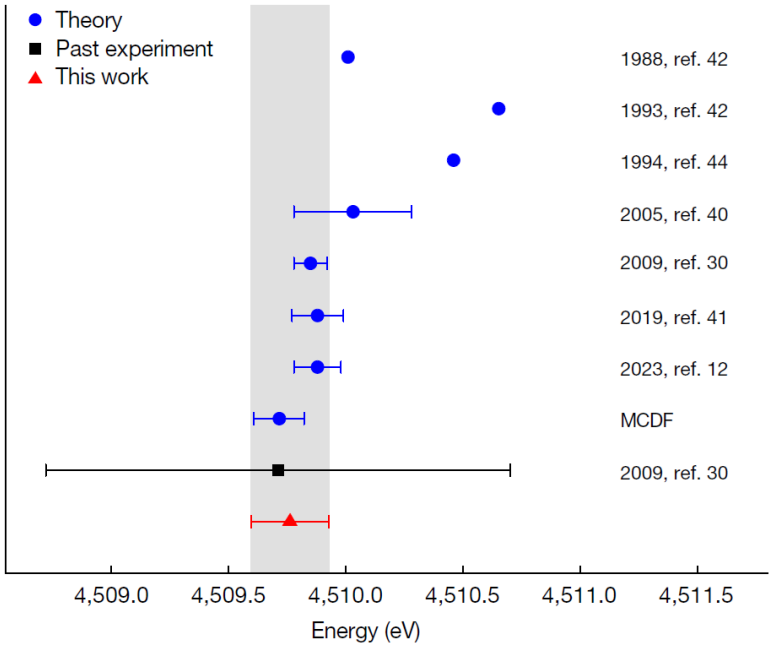

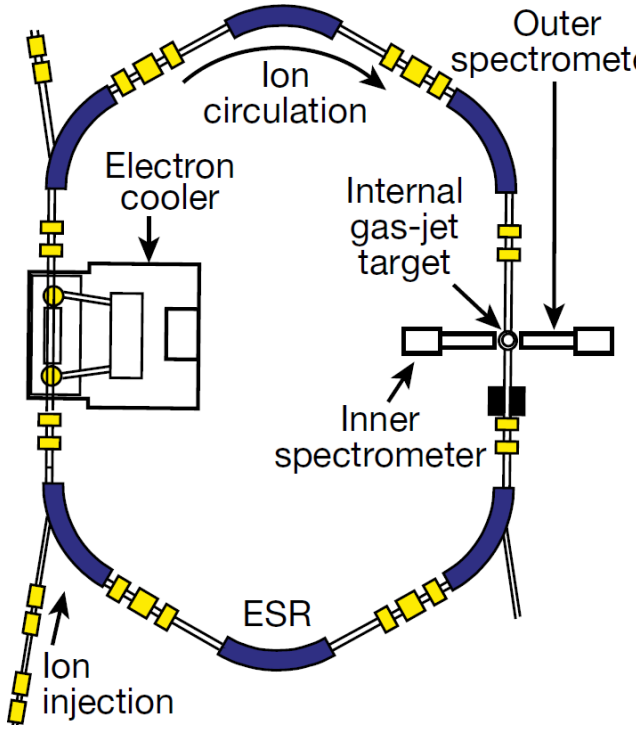

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Theoretical Predictions

Error Budget Uranium [eV]

Source	He	Li	Be
1el-FNS	0.034	0.034	0.034
Nucl. pol.	0.020	0.020	0.020
1el-2loop	0.086	0.086	0.086
H.o. QED, 2loop	0.030	0.030	0.030
H.o. QED, $\geqslant$3ph	0.017	0.018	0.021
H.o. QED, scatter	0.002	0.016	0.021
Other	0.008	0.008	0.009
Total	0.10	0.10	0.10

A. V. Malyshev et al., Phys. Rev. A **107**, 042806 (2023)

Standard Model Tests in the Strong-Field Regime

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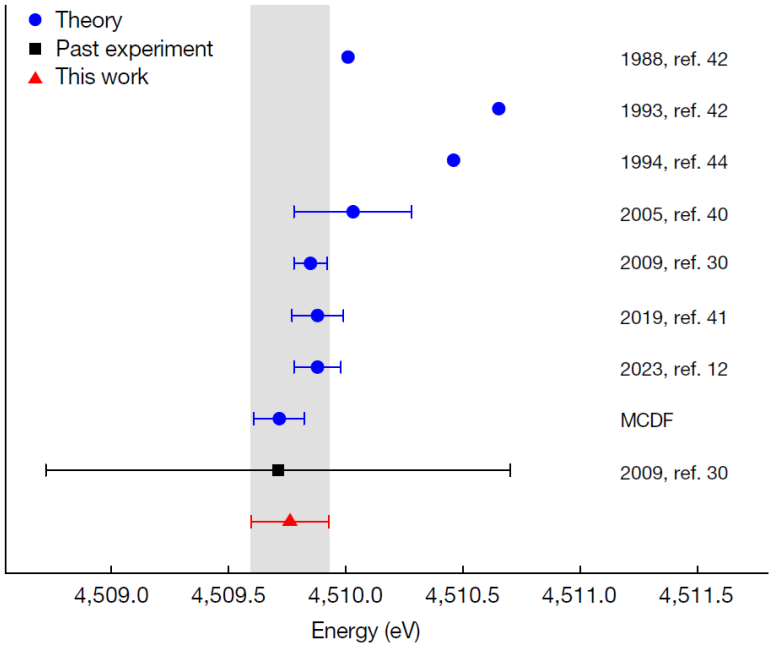

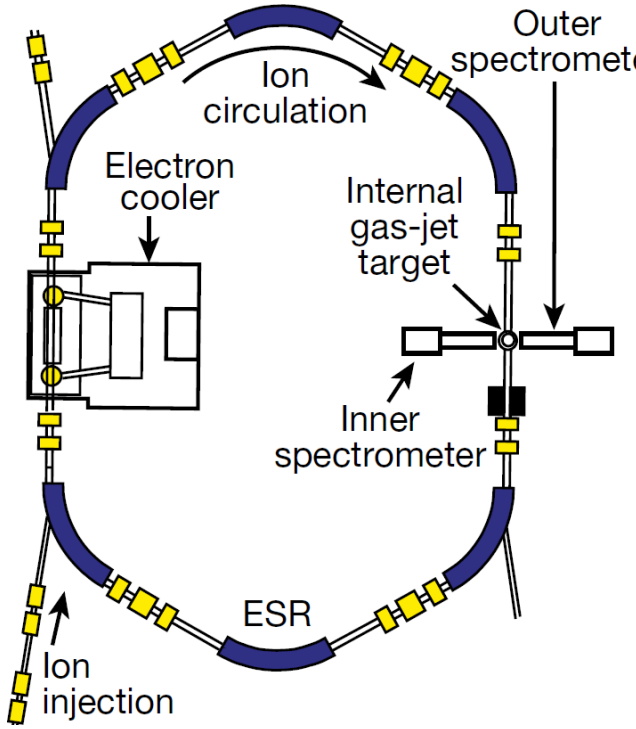

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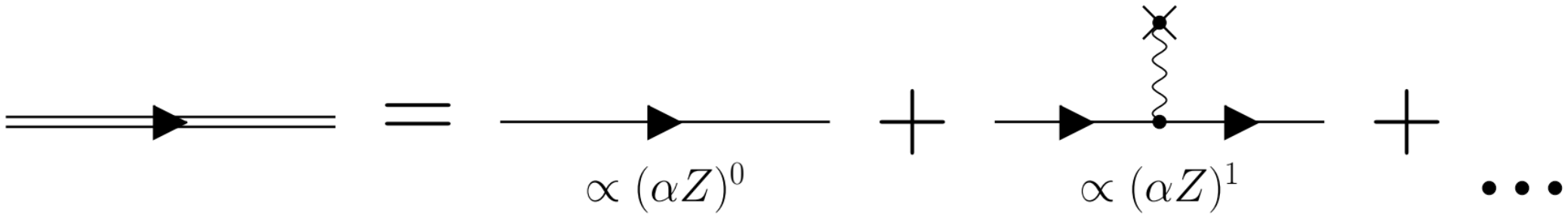
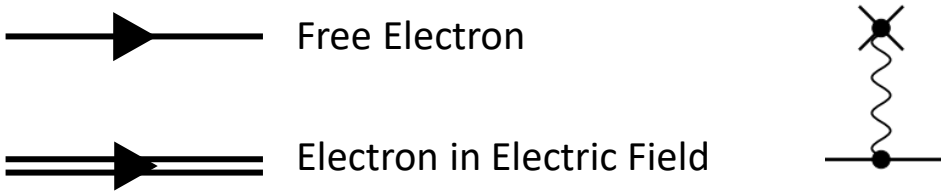
Perturbative vs Non-Perturbative QED

We can treat the interaction of the electron non-perturbatively or perturbatively

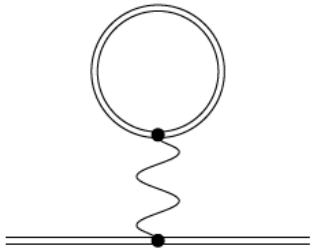




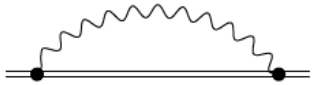
Coulomb Interaction

One- and Two-Loop Bound-State QED Diagrams

1-Electron 1-Loop Diagrams

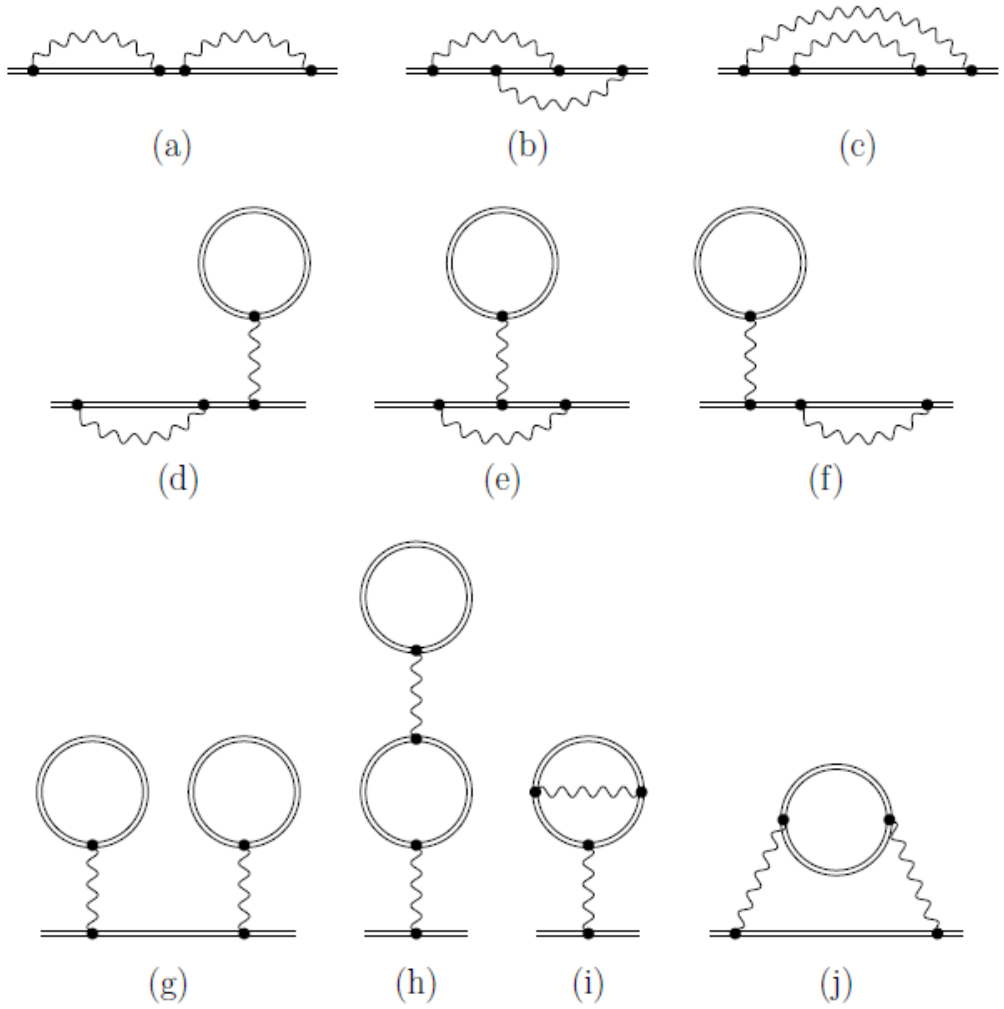


Vacuum Polarization



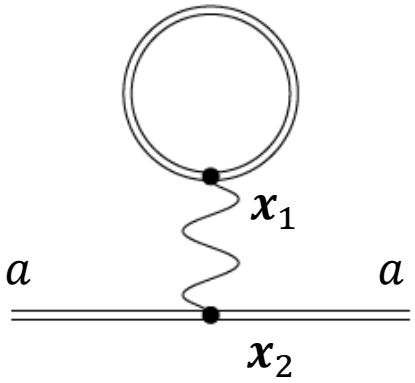
Self-Energy

1-Electron 2-Loop Diagrams



One-Loop Vacuum Polarization

The one-loop vacuum polarization involves an integration over two spatial coordinates (two vertices) and the energy of the loop:



$$E_{VP} = -\frac{\alpha}{2\pi i}\int d^3\boldsymbol{x}_1 \int d^3\boldsymbol{x}_2 \times \int dz\, \mathrm{Tr}[G(\boldsymbol{x}_1, \boldsymbol{x}_1, z)] \times \frac{1}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|} \times \varphi_a^\dagger(\boldsymbol{x}_2)\, \varphi_a(\boldsymbol{x}_2)$$

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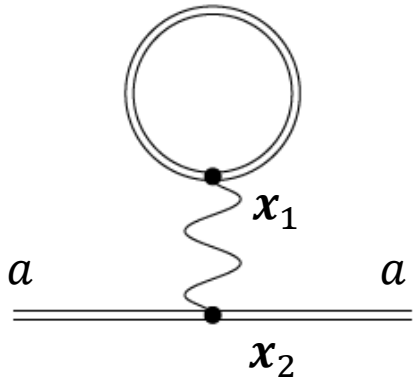

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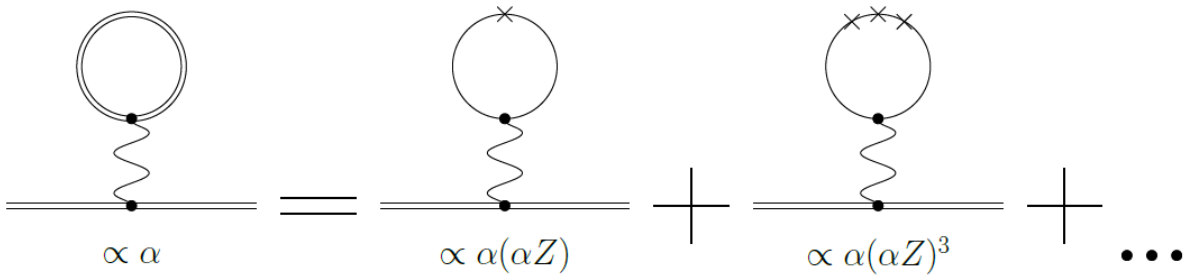
We can write this as an interaction with an electrostatic potential:

$$V_{\mathrm{VP}}(x_2) = -e\int d^3r' \frac{\rho_{\mathrm{VP}}(x_1)}{|x_1 - x_2|} \qquad \rho_{\mathrm{VP}}(\boldsymbol{x}_1) = \frac{e}{2\pi i}\int dz\, \mathrm{Tr}[G(\boldsymbol{x}_1, \boldsymbol{x}_1, z)]$$

One-Loop Vacuum Polarization: Divergent Terms

The vacuum polarization correction can be split into divergent and finite terms:



All-Order VP (Divergent)

Uehling Term (Divergent)

Wichmann-Kroll Term (Finite)

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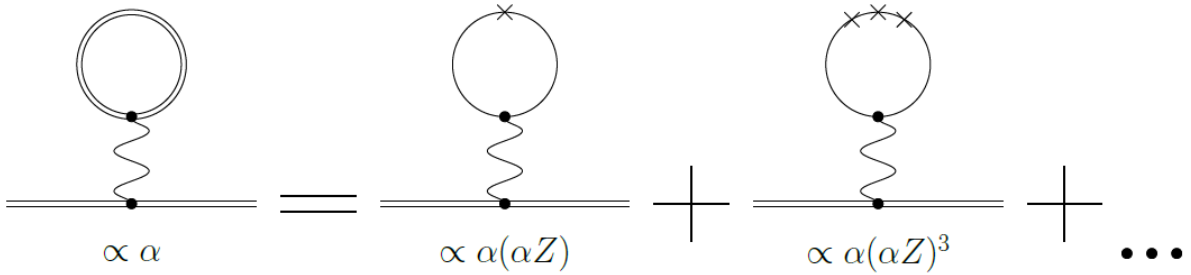

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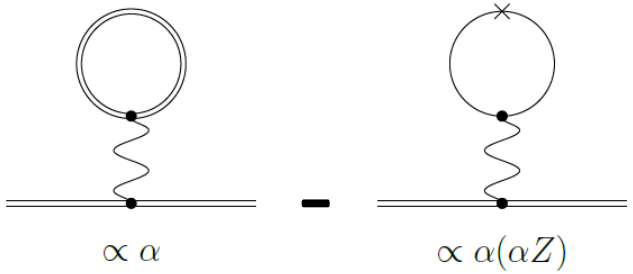
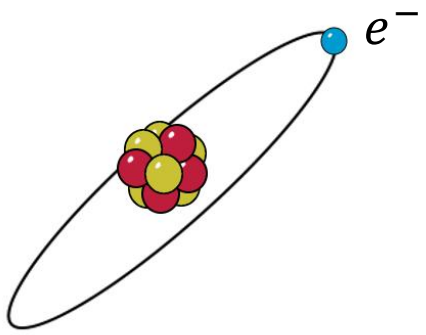


$$\rho_{\mathrm{WK}}(x_1) \sim \int_0^\infty du\, \mathrm{Re}\Big[\sum\nolimits_n G^{nn}(x_1, x_1, iu) + \int dx'\, x'^2 V(x') \sum\nolimits_{n,m} F^{nm}(x_1, x', iu)\Big]$$

$$V_{\mathrm{Ueh}}(x_1) = \frac{-2\alpha(\alpha Z)}{3x_1} \int_0^\infty dx'\, x' \rho(x') [\chi_2(2|x_1 - x'|) - \chi_2(2|x_1 + x'|)]$$

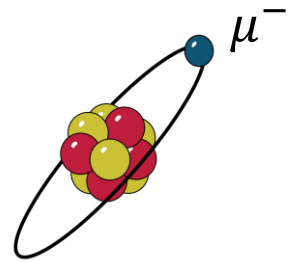
One-Loop Vacuum Polarization: Size of the Lamb Shift Contribution

Hydrogen-Like Uranium



State	E_{Uehl} [eV]	E_{WK} [eV]
$1s_{1/2}$	-93.60	4.99
$2s_{1/2}$	-16.46	0.82
$2p_{1/2}$	-2.91	0.21
$2p_{3/2}$	-0.13	0.02

Muonic Uranium

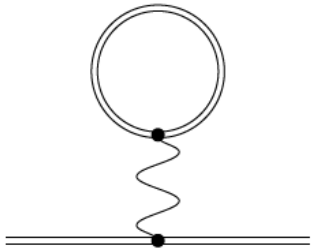


State	E_{Uehl} [eV]	E_{WK} [eV]
$1s_{1/2}$	-74 314.8	698.3
$2s_{1/2}$	-23 266.2	371.2
$2p_{1/2}$	-39845.5	523.8
$2p_{3/2}$	-36 970.1	506.9
$9l_{17/2}$	-138.0	11.8

J. Sommerfeldt and P. Indelicato, Phys. Rev. A **113**, 022806 (2026)

One- and Two-Loop Bound-State QED Diagrams

1-Electron 1-Loop Diagrams



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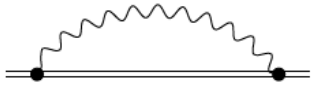

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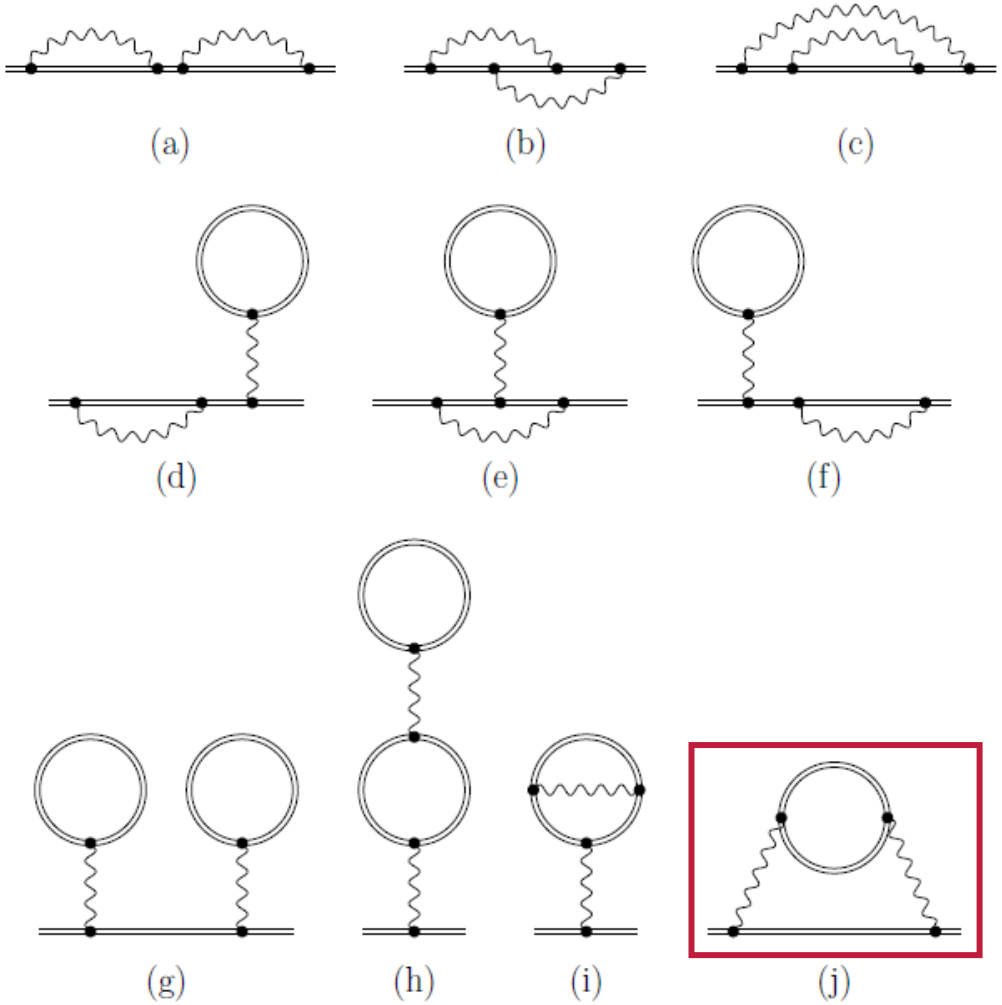
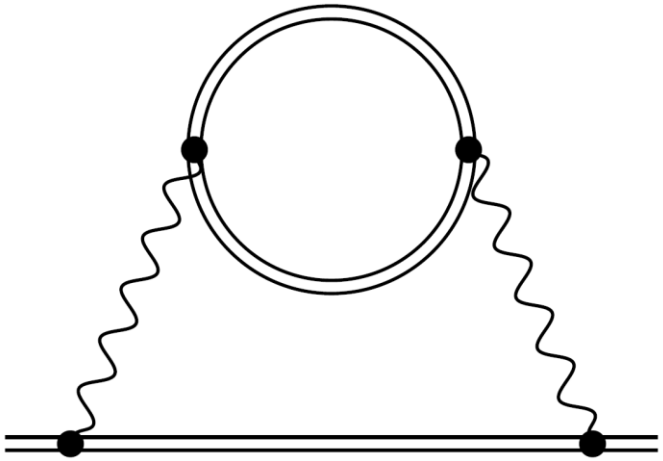
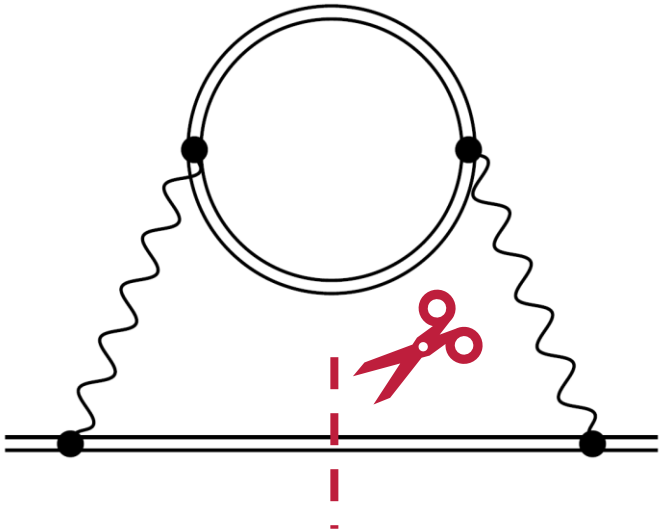


The SVPE Diagram



- 4 Vertices (4 Radial Integrals)
- 2 Loops (2 Energy Integrals)
- 3 Electron Propagators (2 Infinite Sums)

The SVPE Diagram



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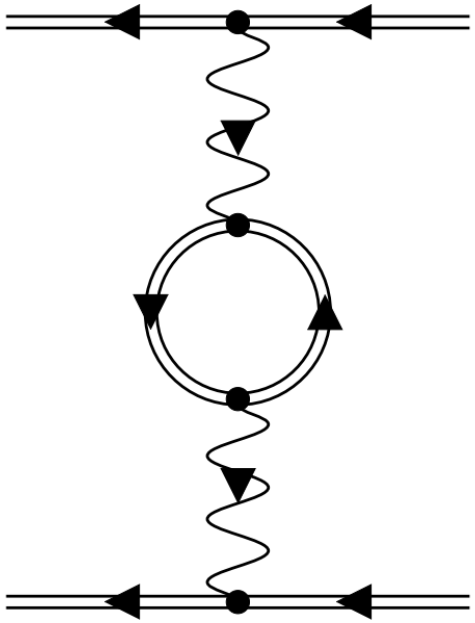
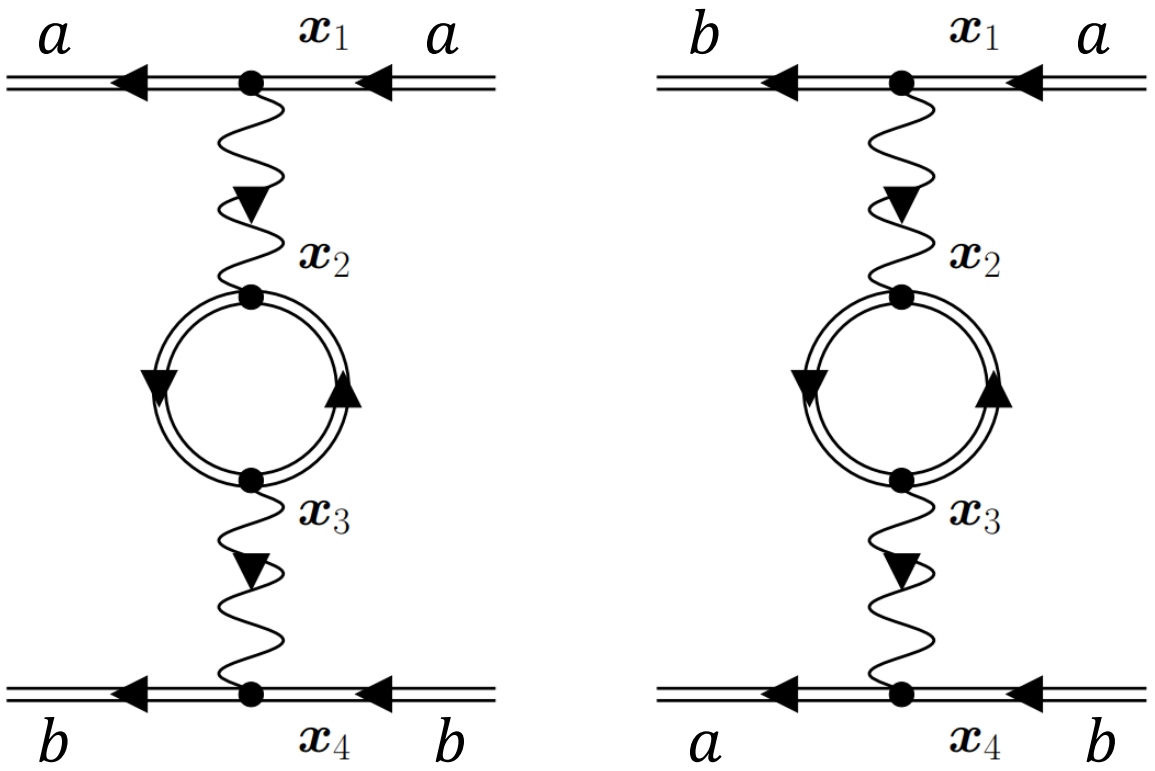


- 4 Vertices (4 Radial Integrals)
- 1 Loop (1 Energy Integral)
- 2 Electron Propagators (1 Infinite Sum)

Two-Electron QED: eVPe

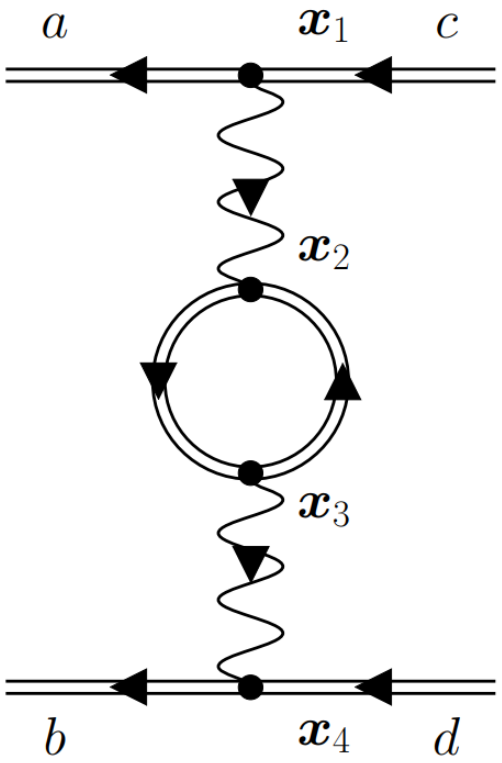
$$E_{e\mathrm{VP}e} = M(ab, ab; 0) - M(ba, ab; \omega)$$



Two-Electron QED: eVPe

$$E_{e\mathrm{VP}e} = M(ab, ab; 0) - M(ba, ab; \omega)$$

$$M(ab, cd; \omega) = \frac{(4\pi\alpha)^2}{2\pi i} \int d^3\boldsymbol{x}_1 \int d^3\boldsymbol{x}_2 \int d^3\boldsymbol{x}_3 \int d^3\boldsymbol{x}_4$$



$$\times \varphi_a^\dagger(\boldsymbol{x}_1)\alpha_\mu\varphi_c(\boldsymbol{x}_1)$$

$$\times D^{\mu\nu}(\omega, \boldsymbol{x}_{12})$$

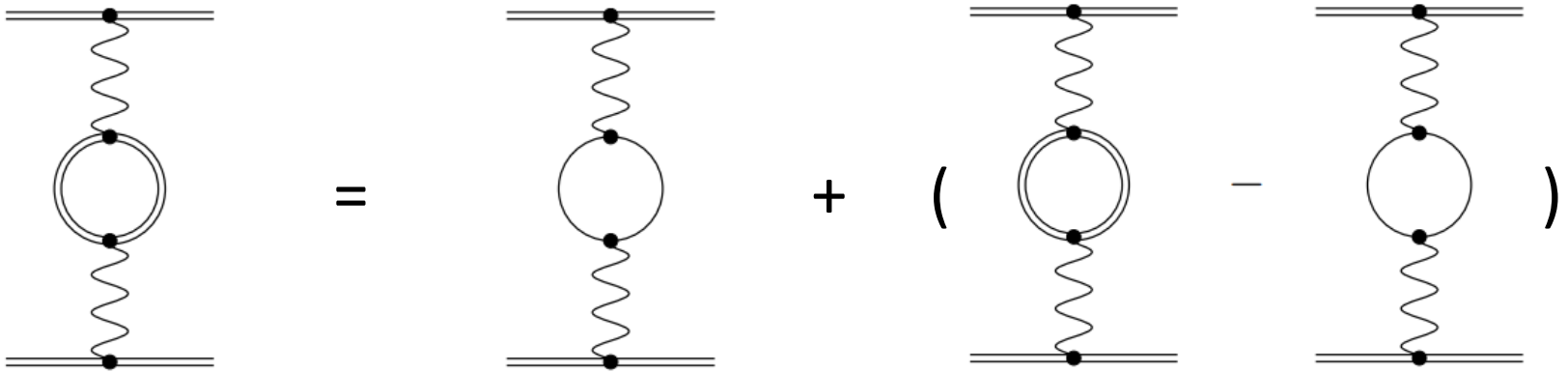
$$\times \int dz \ \mathrm{Tr}[\alpha_\nu \mathrm{G}(\boldsymbol{x}_2, \boldsymbol{x}_3, \mathrm{z} - \omega/2)\alpha_\rho \mathrm{G}(\boldsymbol{x}_3, \boldsymbol{x}_2, \mathrm{z} + \omega/2)]$$

$$\times D^{\rho\sigma}(\omega, \boldsymbol{x}_{34})$$

$$\times \varphi_b^\dagger(\boldsymbol{x}_4)\alpha_\sigma\varphi_d(\boldsymbol{x}_4)$$

eVPe Uehling Term

The vacuum polarization correction can be split into divergent and finite terms:



All-Order eVPe
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Uehling Term
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Wichmann-Kroll Term
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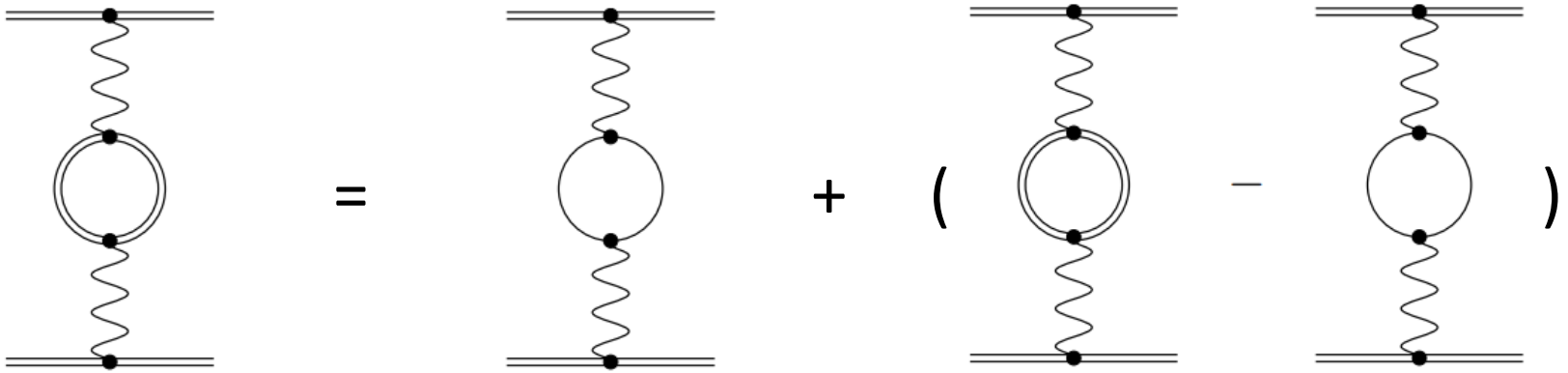


All-Order eVPe (Divergent)

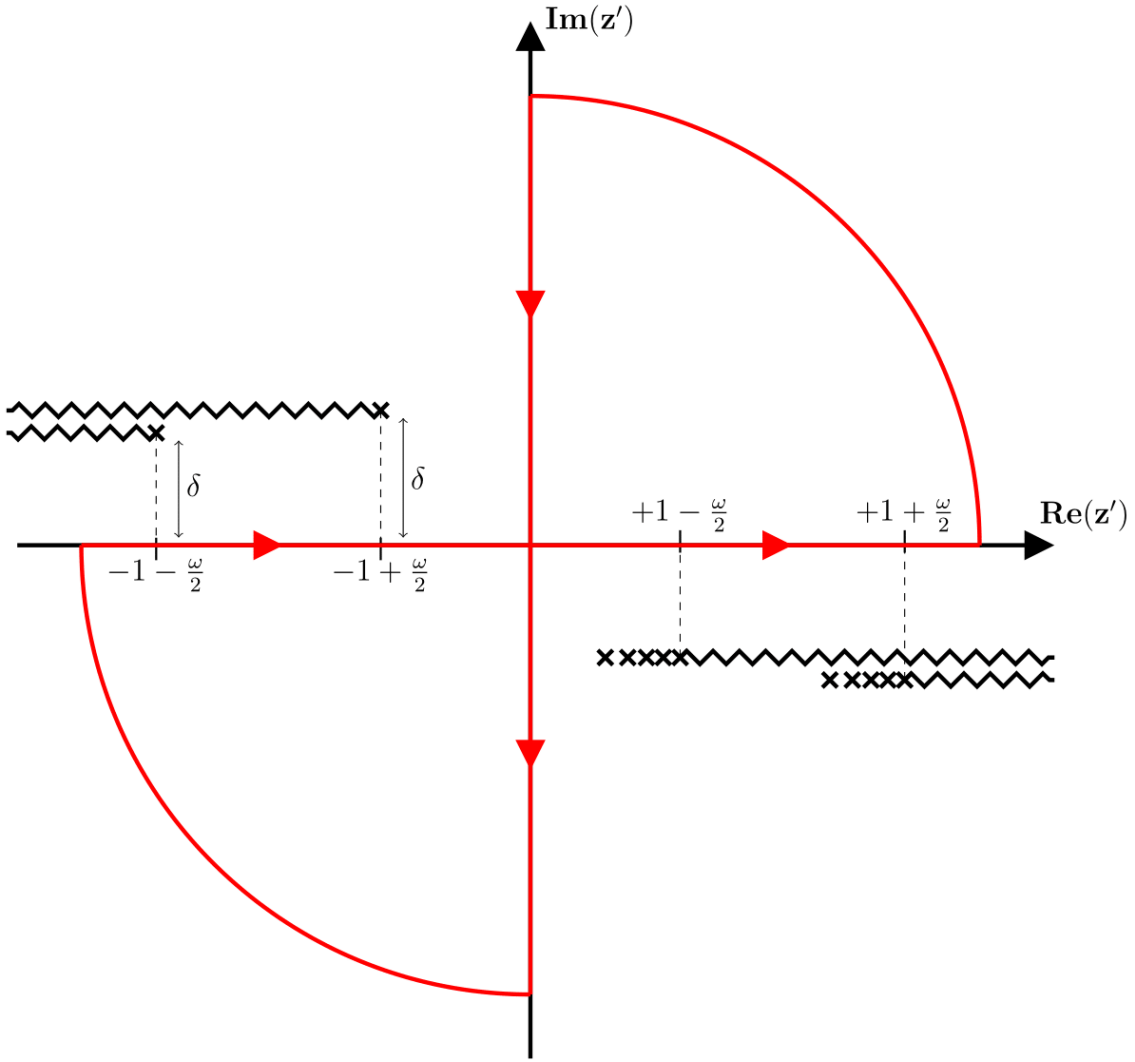
Uehling Term (Divergent)

Wichmann-Kroll Term (Finite)

$$M^{(0)}(ab, cd; \omega) = \langle ab|U^{(0)}|cd\rangle$$

$$U^{(0)}(\omega, \boldsymbol{x}_1, \boldsymbol{x}_2) = \frac{2\alpha^2}{3\pi} \frac{\alpha_{1\mu}\alpha_2^{\mu}}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|} \int_1^{\infty} dt \left(1 + \frac{1}{2t^2}\right) \frac{\sqrt{t^2-1}}{t^2} e^{-\sqrt{(2t)^2-\omega^2}|\boldsymbol{x}_1 - \boldsymbol{x}_2|}$$

Wick Rotation



We can turn the integration along the imaginary axis because of

$$\oint dz' \, f(z') = 0$$

If $\mathrm{Re}[\omega]$ is below the pair production threshold

eVPe: Results

He-like HCI

$(1s_{1/2})^2$

Z	E_{eVPe} [meV]	Ref. [1] [meV]
20	-0.00051(1)	-0.
50	-0.0908(1)	-0.
74	-0.047(1)	-0.
80	+0.366(1)	0.
92	+3.126(3)	3.
100	+7.948(8)	8.

[1] Artemyev et al., Phys. Rev. A 56, 3529 (1997)

J. Sommerfeldt, V. A. Yerokhin, Z. Harman, and C. H. Keitel, arXiv:2607.12168

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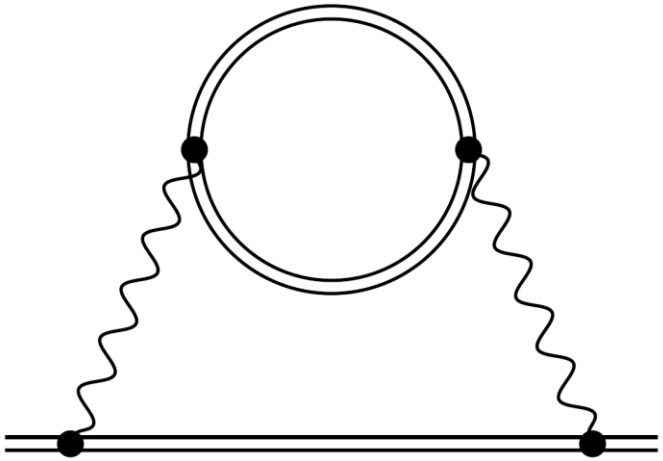
Li-like HCI

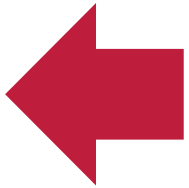
	$(1s_{1/2})^2 2s_{1/2}$	$(1s_{1/2})^2 2p_{1/2}$	$(1s_{1/2})^2 2p_{3/2}$
Z	E_{eVPe} [meV]	E_{eVPe} [meV]	E_{eVPe} [meV]
20	-0.00006(1)	+0.00004(1)	+0.00005(1)
50	-0.0100(1)	-0.0045(1)	+0.0020(1)
74	+0.0056(1)	-0.274(2)	-0.0439(1)
80	+0.0600(1)	-0.580(1)	-0.0943(1)
92	+0.373(3)	-2.356(1)	-0.347(1)
92 [2]	-2.1	-1.3	-0.2
100	+0.820(1)	-5.758(1)	-0.738(1)

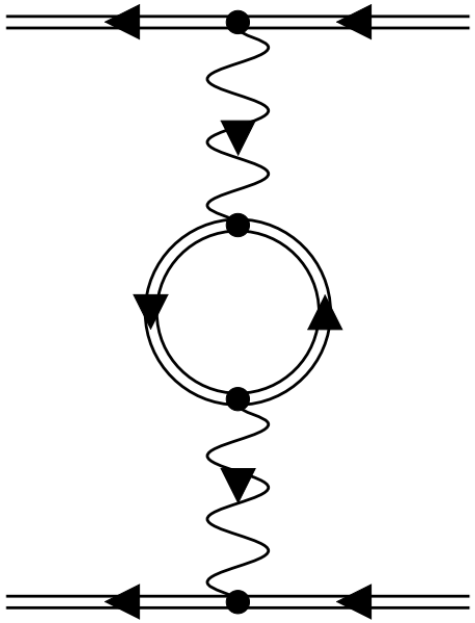
[2] Artemyev et al., Phys. Rev. A 60, 45 (1999)

J. Sommerfeldt, V. A. Yerokhin, Z. Harman, and C. H. Keitel, arXiv:2607.12168

The SVPE Diagram





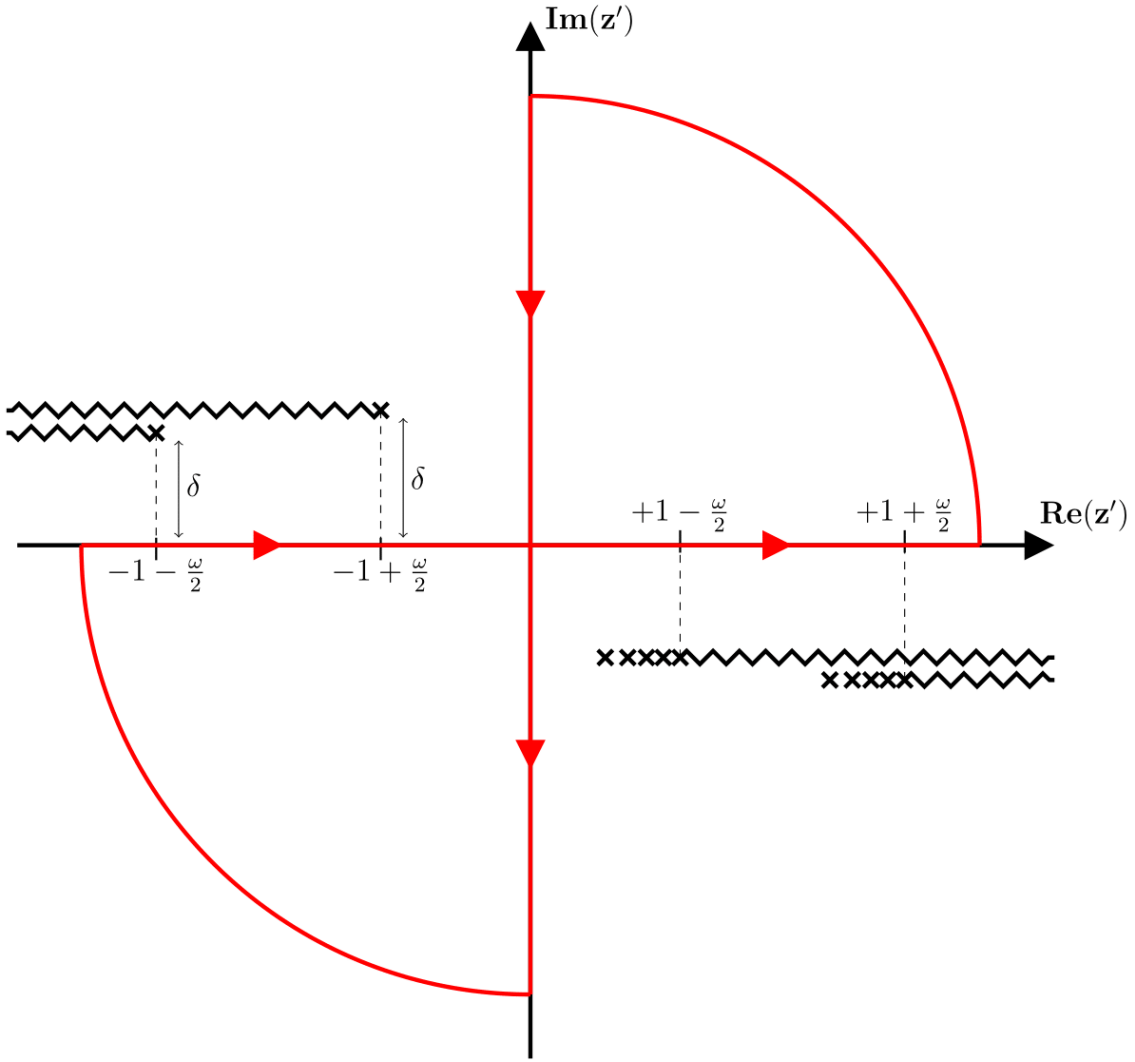


- 4 Vertices (4 Radial Integrals)
- 2 Loops (2 Energy Integrals)
- 3 Electron Propagators (2 Infinite Sums)

- 4 Vertices (4 Radial Integrals)
- 1 Loop (1 Energy Integral)
- 2 Electron Propagators (1 Infinite Sum)

Preliminary calculations show that the evaluation of the SVPE diagram is feasible!

Outlook: Muonic Atoms



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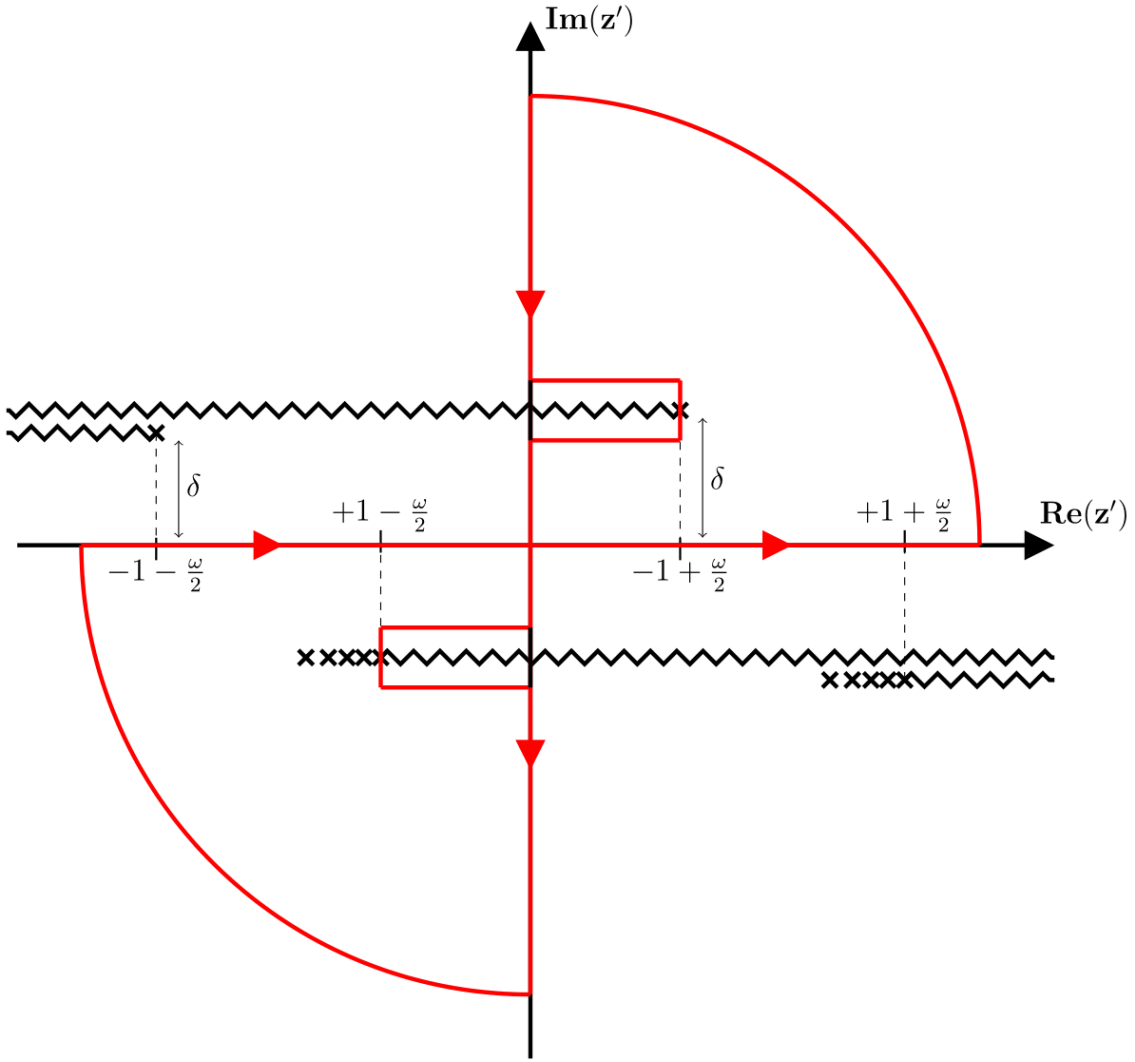

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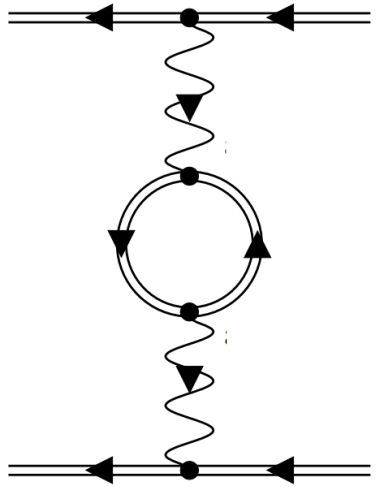
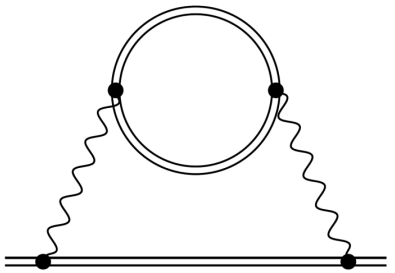
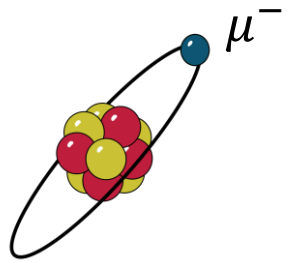
In muonic systems, $\mathrm{Re}[\omega]$ can be above the pair production threshold, so we need to modify the integration contour

$$\oint dz'\, f(z') = 2\pi i \sum_k \mathrm{Res}(f, z'_k)$$

Summary and Outlook

- Improved calculations of the one-loop vacuum polarization and the two-electron eVPe diagram
- Discrepancy for Li-like ions with previous calculations by Artemyev et al.
- New computation methods can be applied to the SVPE diagram to reduce theoretical uncertainty in HCIs
- Future application to muonic atoms and the bound electron g-factor